Inverse Spectral Results on Even Dimensional Tori

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Inverse Spectral Results on Even Dimensional Tori

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Erklärung

Ich erkläre, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe. Weiterhin erkläre ich mich einverstanden, dass ein Exemplar dieser Diplomarbeit in der Bibliothek des Institutes für Mathematik der Humboldt-Universität zu Berlin verbleibt.

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The introductory quotation has been cited by Tito Tonietti in a letter to the editor of The Mathematical Intelligencer, volume 7 (1985), number 4, page 8 from a letter by Hermann Weyl to Freeman Dyson.

Typography and design by Tillmann Berg.

This work is typeset in Hermann Zapf's beautiful Palatino font. The spread is designed to be compatible with the A4 paper format. A version of this work with a better spread can be found on my webpage. Produced using LaTeX together with the memoir-class.

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Abstract

This thesis is concerned with Schrödinger operators acting on the sections of a Hermitian line bundle over an even-dimensional flat torus. Schrödinger operators are constructed from a connection on the bundle and a function on the torus. Restricting to *translation-invariant* connections and line bundles with *nondegenerate* Chern class I give statements concerning the extent to which the spectrum of the Schrödinger operator determines the torus, line bundle or potential.

I show that the potential is determined by the collection of spectra of translation-invariant connections. This collection is a canonical generalisation of the Bloch spectrum of the torus. For *weakly* $\mathbb{Z}_2$*-invariant* connections one can recover the even part of the potential from the corresponding spectrum provided that the lattice has a *nondegenerate length spectrum.* Counterexamples show that this condition cannot be dropped and also that neither potentials, tori nor line bundles are spectral invariants.

Those results were obtained by Gordon, Guerini, Kappeler and Webb. However, I streamlined their work by constructing explicit transplantations for the counterexamples. Also, this thesis includes a classification of line bundles.

Contents

Introduction

Let M be an n-dimensional *flat torus* $\mathcal{L}\backslash\mathbb{R}^n$ given by some *lattice* $\mathcal{L}$. Every *potential* $Q \in C^\infty(M)$ can be interpreted as a smooth $\mathcal{L}$-periodic function $Q\colon \mathbb{R}^n \to \mathbb{R}$. It gives rise to a Schrödinger operator $\Delta + Q$ on $\mathbb{R}^n$. The periodic spectrum of Q is defined to be the spectrum of this Schrödinger operator acting on the $\mathcal{L}$-periodic functions on $\mathbb{R}^n$. The (classical) Bloch spectrum is the collection of all spectra $(\operatorname{Spec}_\alpha Q)_{\alpha\in(\mathbb{R}^n)'}$ of the Schrödinger operator acting on functions that satisfy $f(x+l) = \mathrm{e}^{2\pi\mathrm{i}\alpha(l)} f(x)$ for all $l \in \mathcal{L}$. In [ERT84a] and [ERT84b] Gregory Eskin, James Ralston and Eugene Trubowitz describe the extend to which the potential Q is determined by its periodic spectrum or by its Bloch spectrum.

From every connection ∇ on a given Hermitian line bundle λ one can construct a *Laplacian* acting on the smooth sections of λ. Let $\operatorname{Spec}(Q,\lambda,\nabla)$ denote the spectrum of the corresponding Schrödinger operator. This spectrum depends on the potential and on the connection. In [Gui90] Victor Guillemin noted that in the case of two-dimensional tori the Bloch spectrum of a potential is also given by the collection of spectra $(\operatorname{Spec}(Q,\lambda,\nabla))_{(\lambda,\nabla)}$, where (λ,∇) ranges over all line bundle-connection pairs with vanishing curvature form. In particular, if λ is the trivial complex line bundle over M and ∇ is the trivial connection, then the periodic spectrum coincides with $\operatorname{Spec}(Q,\lambda,\nabla)$.

Guillemin considered only Hermitian line bundles with Chern number ± 1 over two-dimensional tori with *nondegenerate length spectrum* and assumes that one is given a $\mathbb{Z}_2$*-invariant* bundle-connection pair and a $\mathbb{Z}_2$-invariant potential Q, i. e. a connection and a potential invariant under the isometry induced on the torus by the map $x \mapsto -x$. With those assumptions he proves that, if ∇ satisfies some curvature bounds, then both the connection ∇ and the potential Q are determined by $\operatorname{Spec}(Q,\lambda,\nabla)$.

This thesis largely follows the paper [GGKW08] in which Carolyn S. Gordon, Pierre Guerini, Thomas Kappeler and David L. Web extend Guillemins considerations to higher-dimensional tori. The connections used have a "constant" curvature in the following sense: A connection is called *translation-invariant* if its curvature 2-form is invariant under the group of isometries induced on M by the translations. The *λ-Bloch spectrum* of a potential Q is defined to be the map $\nabla \mapsto \operatorname{Spec}(Q,\lambda,\nabla)$, where ∇ ranges over the translation-invariant connections.

Every line bundle over a torus is uniquely, up to isomorphism, determined by its Chern class, which can be understood as a translation-invariant 2-form. A Chern class is called nondegenerate if it is nondegenerate as an antisymmetric bilinear form at one and therefore at every point. Only *nondegenerate* line bundles ω will be considered, that is line bundles with a nondegenerate Chern class. This condition requires the dimension of the torus to be even, say $n = 2m$, and associates a unique ordered m-tuple $(r_1,\ldots,r_m)$ of positive integers with $r_1 \mid \cdots \mid r_m$ to every line bundle. The r_i are called *Chern invariant factors*. On those line bundles the map $\check{x} := -x$ gives rise to a map $\nabla \mapsto \check{\nabla}$ acting on the set of connections such that ∇ is $\mathbb{Z}_2$-invariant if and only if $\nabla = \check{\nabla}$. If this equation holds only up to gauge equivalence, $\nabla \sim \check{\nabla}$, then ∇ is called *weakly*

$\mathbb{Z}_2$-*invariant*. The following questions are addressed in [GGKW08]:

– Does $\mathrm{Spec}(Q,\omega,\nabla)$ determine the potential Q when ∇ is a fixed translation- and weakly $\mathbb{Z}_2$-invariant connection?

– Does the entire ω-Bloch spectrum determine the potential?

If all Chern invariant factors are equal to 1, then the second question can be answered positively even without assuming nondegeneracy of the length spectrum. Concerning the first question, one can recover the even part of the potential from the (single) spectrum of a translation- and weakly $\mathbb{Z}_2$-invariant connection provided that the length spectrum is nondegenerate and the Chern invariant factors equal 1. A counterexample shows that this assumption cannot be dropped. The positive results are shown by constructing wave invariants. The interested reader can find those in the Appendix of [GGKW08].

The focus of this text lies on the negative results. Those can already be found in [GGKW08]. However, I use the main proof of the negative results [GGKW08, Theorem 3.4] to calculate explicit *transplantations*, i. e. isomorphisms of function spaces mapping the eigenfunctions of one differential operator to eigenfunctions with the same eigenvalue of another differential operator, thereby establishing bijections between the eigenspaces of the two operators. The technique of transplantations was first introduced by Peter Buser in [Bus86].

Using this method it will be shown that the spectrum of the vanishing potential $\mathrm{Spec}(0,\omega,\nabla)$ is independent of the choice of translation-invariant connection ∇, thereby defining a *spectrum* of the line bundle ω. Examples show that neither the isometry class of the underlying torus nor the line bundle ω are determined by this spectrum.

Also, I prove that line bundles are classified by their Chern class and I will describe all proofs in more detail than in [GGKW08]. On the other hand, the technique of transplantations is more direct than the proof of [GGKW08, Theorem 3.4] and therefore I do not need to consider principal circle bundles and their connections [GGKW08, Section 2C] to prove Theorem 6.13. This makes the construction of a nilpotent Lie group and of actions of this group on some function spaces superfluous. The organization of this work is outlined in the following section.

Finally, note that there are two small errors in [GGKW08, Notation 3.3 (ii) and Theorem 3.4 (2)]: One should have $q_{r,c,\mu}(s) := q(s_1 + (\mu_1 - c_1)/r_1, \ldots, s_m + (\mu_m - c_m)/r_m)$ and the spectrum $\mathrm{Spec}_\alpha(Q, L_r)$ of $Q(u,v) = q(v)$ coincides with $S(q,r,b,a,-\mu)$. Confer Definition 6.6 and Definition 6.16.

Acknowledgment. I want to thank Professor Dr. Dorothee Schüth for support and proofreading. Also, I thank Peter Herbrich for suggesting many improvements for this work.

Results and Overview

Sheaves and Cohomology & Line Bundles The Hermitian line bundles over a fixed closed manifold M are classified, up to an isomorphism, by the second Čech cohomology group $\check{H}^2(M, \mathbb{Z})$ of the constant sheaf $\mathbb{Z}$ on M. The Čech cohomology class corresponding to an Hermitian line bundle is called its *Chern class* and can be interpreted as a de Rham cohomology class, which contains exactly one harmonic representative Ω.

Flat Tori I will not consider arbitrary manifolds but only *flat tori* which are the quotient manifolds $\mathbb{R}^n$ modulo some *lattice* $\mathcal{L}$. On flat tori the harmonic representative is given by a bilinear, antisymmetric map on $\mathbb{R}^n \times \mathbb{R}^n$. If it is nondegenerate, then the dimension of the torus must be even and there is a normal form $\Omega = 2\pi \mathrm{i} \sum_{i=1}^{n/2} r_i \mathrm{d}u^i \wedge \mathrm{d}v^i$, where the $\{u^i, v^i\}$ are coordinates with respect to some lattice basis and r_i are integers, called *Chern invariant factors*. Line bundles with such nondegenerate representatives will be called *nondegenerate*. The normal form will be used to construct an explicit representative ω for every class of nondegenerate line bundles over M.

Connections The sections and connections of this bundle ω are in close relationship to the corresponding objects of the trivial line bundle θ^1 over $\mathbb{R}^n$. The sections of ω are isomorphic to the $\mathcal{L}$-invariant functions $C^\infty(\mathbb{R}^n, \mathbb{C})^{\mathcal{L}}$ and the connections of ω descend from connections on the trivial bundle of the form $\mathrm{d} + A$, where A is an appropriate imaginary-valued 1-form. I will only regard *translation-invariant* connections up to gauge equivalence. In every gauge equivalence class of translation-invariant connections there is a connection $\nabla^{\mathrm{D}} + a$, where a is a harmonic 1-form and ∇^{D} a *distinguished connection* constructed from Ω. The form a can be interpreted as a functional on $\mathbb{R}^n$. Also *(weak)* $\mathbb{Z}_2$*-invariance* will be introduced in this section.

The Laplacian and Spectra From each connection one can construct a *Laplacian* $\Delta := \operatorname{trace} \nabla^2$ acting on sections. For the connections $\nabla^{\mathrm{D}} + a$ the Laplacian shall be denoted by Δ^{D}_a. The *spectrum* $\operatorname{Spec}_a(Q, \omega)$ is the set of eigenvalues of the Schrödinger operator $\Delta^{\mathrm{D}}_a + Q$, where $Q \in C^\infty(M)$ is a *potential*. The *ω-Bloch spectrum* is the collection of all such spectra.

Transplantations The negative results will be obtained constructively by giving explicit *transplantations*. In Theorem 6.13 a map from $(\mathrm{L}^2(\omega), \Delta^{\mathrm{D}}_a + Q)$ with $Q(u, v) = p(u)$ to some simpler space $(\mathcal{H}, D_{\mathfrak{U},p})$ is given provided that the line bundle is *rectangular*. This transplantation maps every section to finitely many of its Fourier coefficients with respect to the v-coordinate. Those are sufficient to reconstruct any section. In Lemma 6.19 two of those transplantations are joint together to yield a transplantation from one torus to another.

Negative Results In this section I use the transplantations to prove the following Corollaries of Theorem 6.13:

Corollary 7.1. For every even-dimensional rectangular flat torus and every nondegenerate rectangular line bundle ω over this torus $\operatorname{Spec}(\omega) := \operatorname{Spec}_a(0, \omega)$ is independent of a and is called the *spectrum of* ω.

Corollary 7.3. For every even integer $n \geq 4$ there exists an n-dimensional flat torus M and two topologically distinct but isospectral nondegenerate rectangular line bundles over M.

Corollary 7.4. For every even-dimensional rectangular flat torus M and every nondegenerate rectangular line bundle ω over M there exists another nonisometric rectangular flat torus $\tilde{M}$ and a nondegenerate rectangular line bundle $\tilde{\omega}$ over this torus with the same Chern invariant factors such that (M, ω) and $(\tilde{M}, \tilde{\omega})$ are isospectral.

Corollary 7.7. Let ω be a nondegenerate rectangular line bundle over a flat torus M and let $\{U_i, V_i\}$ be an orthogonal Chern basis. If there is a j such that $\|V_j\|/\|U_j\|$ is rational but not equal to 1, then for every translation-invariant connection there exist two isospectral but noncongruent potentials on M. Those potentials can be chosen to be analytic and $\mathbb{Z}_2$-invariant. In particular, the statement holds for any nondegenerate line bundle over any nonsquare, two-dimensional torus with a rational ratio of side lengths.

The expression of the transplantation in Corollary 7.7 can be simplified for a $\mathbb{Z}_2$-invariant connection and $r_1 = \cdots = r_m = 1$ such that it does not contain any infinite sums.

Positive Results To contrast the negative results I will cite, without proof, some spectral invariants which are used to show the following results. Although a potential is in general not determined by a single spectrum by Corollary 7.7, it is determined for line bundles with $r_1 = \cdots = r_m = 1$ by the entire ω-Bloch spectrum. However, parts of the potential are determined by a single spectrum provided that the corresponding connection is chosen appropriately:

Theorem 8.4. Every smooth potential on a fixed even-dimensional torus with line bundle ω whose Chern invariant factors are $r_1 = \cdots = r_m = 1$ is uniquely determined by its ω-Bloch spectrum.

Theorem 8.8. Let M be an even-dimensional torus with *nondegenerate length spectrum*, let ω be a line bundle with Chern invariant factors $r_1 = \cdots = r_m = 1$ and assume a translation- and weakly $\mathbb{Z}_2$-invariant connection is given, then the even part of a potential is spectrally determined.

With sufficiently strong restrictions on the potential one may conclude that the potential is uniquely determined by its spectrum with respect to any translation- and weakly $\mathbb{Z}_2$-invariant connection.

If one restricts to two-dimensional tori with nondegenerate length spectrum, to line bundles with Chern invariant factor $r_1 = 1$ and to the distinguished connection, then one can improve this result: There are no nontrivial continuous isospectral deformations within the space of smooth potentials. Also, if one is given two isospectral potentials P and Q with real analytic odd parts such that the odd part of one of them is *one-dimensional*, then $P = Q$ or $P = \check{Q}$, where $\check{Q}(x) := Q(-x)$.

My work always tried to unite the truth with the beautiful; but when I had to chose one or the other, I usually chose the beautiful.

Hermann Weyl

1 Sheaves and Cohomology

In this section I will briefly describe the language of sheaves used in the next section to classify line bundles. In those two sections I largely follow [GH94].

Definition 1.1. A *sheaf* $\mathcal{F}$ on a topological space M associates to each nonempty open set $U \subset M$ an abelian group $\mathcal{F}(U)$ and to each pair of nonempty open sets $V \subset U$ a group homomorphism $\rho_{U,V} \colon \mathcal{F}(U) \to \mathcal{F}(V)$ such that

(a) for any triple $W \subset V \subset U$ of nonempty open sets $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$.

An element $s \in \mathcal{F}(U)$ is called a *section* of $\mathcal{F}$ over U. The maps $\rho_{U,V}$ are called *restrictions* and are written as $s|_V := \rho_{U,V}(s)$. For any two nonempty open sets $U, V \subset M$ another two axioms are postulated:

(b) For all $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ with $s|_{U\cap V} = t|_{U\cap V}$ there is an $r \in \mathcal{F}(U \cup V)$ with $r|_U = s$ and $r|_V = t$ and

(c) if $s \in \mathcal{F}(U \cup V)$ with $s|_U = 0$ and $s|_V = 0$ then $s = 0$.

Definition 1.2. A *sheaf map* $f \colon \mathcal{F} \to \mathcal{G}$ on M is a collection of group homomorphisms $\{f_U \colon \mathcal{F}(U) \to \mathcal{G}(U) \mid \emptyset \neq U \subset M \text{ open}\}$ that commute with the restrictions. A short sequence of sheaf maps

$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \to 0$$

is called *exact* if for every open $\emptyset \neq U \subset M$

(a) f_U is injective,

(b) for every $s \in \ker g_U$ and all $p \in U$ there is a neighbourhood $p \in V \subset U$ such that $s|_V \in \operatorname{im} f_V$ and

(c) for every $s \in \mathcal{H}(U)$ and all $p \in U$ there is a neighbourhood $p \in V \subset U$ such that $s|_V \in \operatorname{im} g_V$.

Definition 1.3. Let $\mathcal{F}$ be a sheaf on M and let $\mathfrak{U} = \{U_j\}_{j\in J}$ be a locally finite open cover of M. Then for $p \in \mathbb{N}_0$ the group of *p-cochains* is defined as

$$C^p(\mathfrak{U}, \mathcal{F}) := \prod_{i_0 \neq \cdots \neq i_p} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}) \,.$$

Denote the p-cochains $s = \left\{s_I \in \mathcal{F}(\bigcap_{i\in I} U_i)\right\}_{\#I=p+1}$, like sections, by Latin letters. The *coboundary operator* is a map

$$\delta\colon C^p(\mathfrak{U},\mathcal{F}) \to C^{p+1}(\mathfrak{U},\mathcal{F}) \qquad \text{with}$$
$$(\delta s)_{i_0\cdots i_{p+1}} := \sum_{k=0}^{p+1} (-1)^k s_{i_0\cdots\hat{i}_k\cdots i_{p+1}}|_{U_{i_0}\cap\cdots\cap U_{i_{p+1}}} .$$

Every $s \in \operatorname{im}\delta$ is a *coboundary* and the group of *cocycles* is

$$Z^p(\mathfrak{U},\mathcal{F}) := \ker\delta .$$

A straightforward calculation shows that $\delta\circ\delta = 0$ and therefore that $\delta C^{p-1}(\mathfrak{U},\mathcal{F}) \subset Z^p(\mathfrak{U},\mathcal{F})$. The factor group

$$\check{\mathrm{H}}^p(\mathfrak{U},\mathcal{F}) := Z^p(\mathfrak{U},\mathcal{F})/\delta C^{p-1}(\mathfrak{U},\mathcal{F})$$

is the *pth Čech cohomology of the cover* $\mathfrak{U}$ *and the sheaf* $\mathcal{F}$.

Remark 1.4. An open cover $\mathfrak{V} = \{V_j\}_{j\in J}$ is called *refinement* of the cover $\mathfrak{U} = \{U_i\}_{i\in I}$, written $\mathfrak{V} \leq \mathfrak{U}$, if there is map $\phi\colon J \to I$ such that $V_j \subset U_{\phi j}$ for all $j \in J$. This map ϕ induces a map

$$\phi^\#\colon C^p(\mathfrak{U},\mathcal{F}) \to C^p(\mathfrak{V},\mathcal{F}) \quad \text{with} \quad (\phi^\# s)_{j_0\cdots j_p} := s_{\phi j_0\cdots\phi j_p}|_{V_{j_0}\cap\cdots\cap V_{j_p}} .$$

Two calculations show that this map is a chain map, which means that it commutes with the coboundary operator, $\phi^\#\circ\delta = \delta\circ\phi^\#$, and that for another such map ψ one has a homotopy operator $K\colon C^p(\mathfrak{U},\mathcal{F}) \to C^{p-1}(\mathfrak{V},\mathcal{F})$ with

$$(Ks)_{j_0\cdots j_{p-1}} = \sum_{k=0}^{p-1} (-1)^k s_{\phi j_0\cdots\phi j_k \psi j_k\cdots\psi j_{p-1}}$$

between $\phi^\#$ and $\psi^\#$, that is $\delta K + K\delta = \phi^\# - \psi^\#$. Thus, both maps induce the same map in cohomology, which we denote by $\rho_{\mathfrak{U},\mathfrak{V}}\colon \check{\mathrm{H}}^p(\mathfrak{U},\mathcal{F}) \to \check{\mathrm{H}}^p(\mathfrak{V},\mathcal{F})$.

Definition 1.5. The set of locally finite open covers is directed by the notion of refinement. The set of Čech cohomology groups is indexed by this directed set and together with the maps $\rho_{\mathfrak{U},\mathfrak{V}}$ it forms a *direct system* of groups, which means

$$\rho_{\mathfrak{U},\mathfrak{U}} = \mathrm{id}_{\check{\mathrm{H}}^p(\mathfrak{U},\mathcal{F})} \qquad \text{and} \qquad \rho_{\mathfrak{U},\mathfrak{W}} = \rho_{\mathfrak{V},\mathfrak{W}}\circ\rho_{\mathfrak{U},\mathfrak{V}} \qquad \text{for } \mathfrak{W} \leq \mathfrak{V} \leq \mathfrak{U} .$$

The *pth Čech cohomology of the sheaf* $\mathcal{F}$ is defined to be the direct limit of this direct system:

$$\check{\mathrm{H}}^p(M,\mathcal{F}) := \varinjlim_{\mathfrak{U}} \check{\mathrm{H}}^p(\mathfrak{U},\mathcal{F}) := \coprod_{\mathfrak{U}} \check{\mathrm{H}}^p(\mathfrak{U},\mathcal{F})\Big/\sim$$

with $\check{\mathrm{H}}^p(\mathfrak{U},\mathcal{F}) \ni s \sim t \in \check{\mathrm{H}}^p(\mathfrak{V},\mathcal{F})$ if there is a refinement $\mathfrak{W}$ of $\mathfrak{V}$ and $\mathfrak{U}$ such that $\rho_{\mathfrak{U},\mathfrak{W}} s = \rho_{\mathfrak{V},\mathfrak{W}} t$. Heuristically, the maps ρ are restrictions of classes of cocycles to finer open covers of M and two such classes are considered equal if they are equal on a sufficiently fine cover. The group operation on the direct limit is defined by

$$[s] + [t] := [\rho_{\mathfrak{U},\mathfrak{W}} s + \rho_{\mathfrak{V},\mathfrak{W}} t] ,$$

where $s \in \check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F})$ and $t \in \check{\mathrm{H}}^p(\mathfrak{V}, \mathcal{F})$ are representatives of the corresponding classes in $\check{\mathrm{H}}^p(M, \mathcal{F})$ and $\mathfrak{W}$ is a common refinement of the locally finite open covers $\mathfrak{U}$ and $\mathfrak{V}$.

Remark 1.6. A sheaf map $f : \mathcal{F} \to \mathcal{G}$ induces a map of cochains

$$f : C^p(\mathfrak{U}, \mathcal{F}) \to C^p(\mathfrak{U}, \mathcal{G}) ,$$

which commutes not only with restrictions but also with the coboundary operator δ and thus induces a map in cohomology

$$f^* : \check{\mathrm{H}}^p(M, \mathcal{F}) \to \check{\mathrm{H}}^p(M, \mathcal{G}) .$$

Definition 1.7 (Another Coboundary Operator).
Given an exact sequence of sheaves

$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \to 0$$

one can define another coboundary operator

$$\delta^* : \check{\mathrm{H}}^p(M, \mathcal{H}) \to \check{\mathrm{H}}^{p+1}(M, \mathcal{F})$$

through diagram chasing:

$$\begin{array}{ccccc} C^p(\mathfrak{U}, \mathcal{F}) & \xrightarrow{f} & C^p(\mathfrak{U}, \mathcal{G}) & \xrightarrow{g} & C^p(\mathfrak{U}, \mathcal{H}) \\ \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\ C^{p+1}(\mathfrak{U}, \mathcal{F}) & \xrightarrow{f} & C^{p+1}(\mathfrak{U}, \mathcal{G}) & \xrightarrow{g} & C^{p+1}(\mathfrak{U}, \mathcal{H}) \end{array}$$

For each $s \in Z^p(\mathfrak{U}, \mathcal{H})$ one can find a refinement $\mathfrak{V}$ of $\mathfrak{U}$ and a cochain $t \in C^p(\mathfrak{V}, \mathcal{G})$ for which $gt = \rho_{\mathfrak{U},\mathfrak{V}} s$ and thus $g\delta t = \delta g t = \delta \rho_{\mathfrak{U},\mathfrak{V}} s = 0$ holds. This means $\delta t \in \ker g \subset C^{p+1}(\mathfrak{V}, \mathcal{G})$. After passing to another refinement $\mathfrak{W}$ one can find a cochain $u \in C^{p+1}(\mathfrak{W}, \mathcal{F})$ with $fu = \rho_{\mathfrak{V},\mathfrak{W}} \delta t$. In particular, $\delta f u = \delta \rho_{\mathfrak{V},\mathfrak{W}} \delta t = 0$ and, since f is injective, $u \in Z^{p+1}(\mathfrak{W}, \mathcal{F})$. Define

$$\delta^*[s] := [u] \in \check{\mathrm{H}}^{p+1}(M, \mathcal{F}) .$$

Considering all three choices made (that of a particular s, t and u) one finds that δ^* is well-defined.

Remark 1.8. The following sequence is exact:

$$0 \to \check{\mathrm{H}}^0(M, \mathcal{F}) \xrightarrow{f^*} \check{\mathrm{H}}^0(M, \mathcal{G}) \xrightarrow{g^*} \check{\mathrm{H}}^0(M, \mathcal{H}) \xrightarrow{\delta^*} \cdots$$
$$\cdots \xrightarrow{\delta^*} \check{\mathrm{H}}^p(M, \mathcal{F}) \xrightarrow{f^*} \check{\mathrm{H}}^p(M, \mathcal{G}) \xrightarrow{g^*} \check{\mathrm{H}}^p(M, \mathcal{H}) \xrightarrow{\delta^*} \cdots .$$

Proof. For example, $\operatorname{im} f^* \subset \ker g^*$ follows from $g^* \circ f^* = (g \circ f)^* = 0$ because the short exact sheaf sequence is exact. For any $[t] \in \check{\mathrm{H}}^p(M, \mathcal{G})$ one has $\delta^* \circ g^*([t]) = \delta^*[g(t)] = [u]$ with $fu = \delta t = 0$. Due to the exactness f is injective, $u = 0$ and $\operatorname{im} g^* \subset \ker \delta^*$. Similarly, $f^* \circ \delta^*[s] = f^*[u] = [fu] = [\delta t] = 0$. The converse inclusions follow similarly. □

2 Line Bundles

In this section I will use the notion of sheaves and cohomology to introduce the Chern class of a line bundle and I will show that every isomorphism class of line bundles over a fixed manifold is uniquely determined by its Chern class. Again, I follow [GH94]. For more information about fibre bundles in general and vector bundles in particular, see [Hus94].

Definition 2.1. Given a smooth real manifold M let $\mathcal{A}^p$ be the sheaf of smooth complex-valued p-forms on M and $\mathcal{A}^*$ the sheaf of the multiplicative group of smooth nowhere vanishing complex-valued functions on M, which means for every nonempty open set $U \subset M$

$$\mathcal{A}^p(U) := \left(\Omega^p(U, \mathbb{C}), +\right) \qquad \text{and} \qquad \mathcal{A}^*(U) := \left(C^\infty(U, \mathbb{C}\backslash\{0\}), \cdot\right).$$

Lemma 2.2. $\check{\mathrm{H}}^q(M, \mathcal{A}^p) = 0$ for $q > 0$.

Proof. For every locally finite cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of M there is a partition of unity $\{\rho_i\}_{i\in I}$ subordinate to $\mathfrak{U}$, i.e. $\operatorname{supp} \rho_i \subset U_i$ and $\sum_{i\in I} \rho_i \equiv 1$. For $s \in Z^q(\mathfrak{U}, \mathcal{A}^p)$ define

$$t_{i_0\cdots i_{q-1}} := \sum_{i\in I} \rho_i s_{i\, i_0\cdots i_{q-1}} \in \mathcal{A}^p(U_{i_0} \cap \cdots \cap U_{i_{q-1}}),$$

where every section $\rho_i s_{i\, i_0\cdots i_{q-1}}$ is extended onto $U_{i_0} \cap \cdots \cap U_{i_{q-1}}$ by zero. Note that the sum is well-defined as $\mathfrak{U}$ is locally finite. By substituting

$$(\delta s)_{i\, i_0\cdots i_q} = s_{i_0\cdots i_q} + \sum_{j=0}^{q} (-1)^{j+1} s_{i\, i_0\cdots \hat{i}_j \cdots i_q} = 0 \qquad \text{into}$$

$$\delta t_{i_0\cdots i_q} = \sum_{i,j} (-1)^j \rho_i s_{i\, i_0\cdots \hat{i}_j\cdots i_q} \qquad \text{one obtains } \delta t = \sum_{i\in I} \rho_i s = s.$$ □

Definition 2.3 (Picard group).
A complex line bundle $\lambda = (L, \pi, M)$ is a rank 1 complex vector bundle $\pi\colon L \to M$. The *Picard group*

$$\left(\operatorname{Pic} M, \otimes\right)$$

is defined as the set of isomorphism classes of line bundles with the group operation given by the tensor product. This group is abelian, the trivial bundle is the neutral element and the inverse of a line bundle λ is the dual bundle λ^*.

Definition 2.4. If $\mathfrak{U} = \{U_i\}$ is an open cover of M with trivialisations

$$\varphi_i \colon \pi^{-1}(U_i) \to U_i \times \mathbb{C} \qquad \text{of } \lambda\text{, then}$$

the transition functions $g = \{g_{ij}\} \in C^1(\mathfrak{U}, \mathcal{A}^*)$ are defined by

$$\big(\varphi_i \circ \varphi_j^{-1}\big)(p, z) = \big(p, g_{ij}(p) z\big)$$

for all $p \in U_i \cap U_j$ and $z \in \mathbb{C}$.

Lemma 2.5. The assignment $\lambda \mapsto g$ mapping line bundles to their transition functions induces a group isomorphism

$$\Phi \colon \operatorname{Pic} M \to \check{\mathrm{H}}^1(M, \mathcal{A}^*) .$$

Proof. First, Φ is well-defined: For an arbitrary isomorphism class of line bundles in $\operatorname{Pic} M$ choose a representative λ and a trivialising cover $\mathfrak{U}$ of M. The corresponding transition functions g always satisfy

$$g_{ij} \cdot g_{ji} = 1 \qquad \text{and} \qquad g_{jk} \cdot g_{ki} \cdot g_{ij} = 1 ,$$

which means that $\delta g = 1$. Therefore, these transition functions form a cocycle and represent a class in $\check{\mathrm{H}}^1(M, \mathcal{A}^*)$. If one chooses another set of trivialisations $\{\tilde{\varphi}_i\}$ over $\mathfrak{U}$, then there is an $f \in C^0(\mathfrak{U}, \mathcal{A}^*)$ with $\tilde{\varphi}_i = f_i \cdot \varphi_i$ and the transition functions are given by

$$\tilde{g}_{ij} = \frac{f_i}{f_j} \cdot g_{ij} \qquad \text{meaning} \qquad \tilde{g} \cdot \delta f = g .$$

The two cocycles thus represent the same cohomology class.

For transition functions given over different covers one can use the same argument after passing to a common refinement. To see that isomorphic line bundles give the same transition functions just note that, if $F \colon \lambda \to \lambda'$ is a bundle isomorphism, then $\varphi_i \circ F^{-1}$ are trivialisations for λ' with the transition functions g.

Second, Φ is a group homomorphism: It is well-known that, if λ and λ' are line bundles with transition functions g and g' over a common trivialising cover, then $\lambda \otimes \lambda'$ has the transition functions $\{g_{ij} \cdot g'_{ij}\}$ and the dual bundle λ^* has the transition functions $\{g_{ij}^{-1}\}$.

Third, Φ is surjective: For a collection of functions $\{g_{ij}\} \in Z^1(\mathfrak{U}, \mathcal{A}^*)$ over a cover $\mathfrak{U}$ of M one can construct a line bundle by setting

$$L := \Big(\coprod_i U_i \times \mathbb{C}\Big)\Big/\sim ,$$

with $(U_i \cap U_j) \times \mathbb{C} \ni (p, z) \sim (p, g_{ij} z) \in (U_i \cap U_j) \times \mathbb{C}$. This yields a line bundle with the given cocycle as its transition functions.

Fourth, Φ is injective: If all the components of a given cocycle are the constant function 1, then the above construction gives a line bundle isomorphic to the trivial bundle. $\square$

Corollary 2.6. Since the Picard group is the image of the set $\check{\mathrm{H}}^1(M, \mathcal{A}^*)$ under the function Φ^{-1}, the axiom schema of replacement of the set theory of Zermelo and Fraenkel states that $\operatorname{Pic} M$ is indeed a set.

Theorem 2.7. There is a group isomorphism between the isomorphism classes of line bundles over M and the cohomology group of the constant sheaf $\mathbb{Z}$ on M

$$c\colon \operatorname{Pic} M \to \check{\mathrm{H}}^2(M, \mathbb{Z}) .$$

Proof. It is sufficient to prove that there is a group isomorphism between $\check{\mathrm{H}}^1(M, \mathcal{A}^*)$ and $\check{\mathrm{H}}^2(M, \mathbb{Z})$. Such a map is obtained from the short exact sequence of sheaves

$$0 \to \mathbb{Z} \hookrightarrow \mathcal{A}^0 \overset{\exp(2\pi i \cdot)}{\to} \mathcal{A}^* \to 0 , \qquad \text{which gives the exact sequence}$$

$$\cdots \to \check{\mathrm{H}}^1(M, \mathcal{A}^0) \to \check{\mathrm{H}}^1(M, \mathcal{A}^*) \overset{\delta^*}{\to} \check{\mathrm{H}}^2(M, \mathbb{Z}) \to \check{\mathrm{H}}^2(M, \mathcal{A}^0) \to \cdots .$$

Since $\check{\mathrm{H}}^q(M, \mathcal{A}^p) = 0$ for all $q > 0$ by Lemma 2.2, the obtained function $\Psi := \delta^*$ must be an isomorphism. Set $c := \Psi \circ \Phi$ with Φ as in Lemma 2.5. □

Definition 2.8 (Chern class).
The first and only *Chern class* of a line bundle λ is defined as

$$c(\lambda) \in \check{\mathrm{H}}^2(M, \mathbb{Z}) .$$

Since c is a homomorphism, one has for any two line bundles λ and λ' and the trivial line bundle θ^1_M (all over M) that

$$c(\lambda \otimes \lambda') = c(\lambda) + c(\lambda') , \qquad c(\lambda^*) = -c(\lambda) \qquad \text{and} \qquad c(\theta^1_M) = 0 .$$

Also, the Chern class is natural, which means that if $f\colon M \to N$ is a smooth map between two manifolds and λ is a line bundle over N, then

$$c(f^*\lambda) = f^*c(\lambda) \qquad \text{because the diagram}$$

$$\begin{array}{ccc} \check{\mathrm{H}}^1(M, \mathcal{A}^*) & \xrightarrow{\delta^*} & \check{\mathrm{H}}^2(M, \mathbb{Z}) \\ \uparrow f^* & & \uparrow f^* \\ \check{\mathrm{H}}^1(N, \mathcal{A}^*) & \xrightarrow{\delta^*} & \check{\mathrm{H}}^2(N, \mathbb{Z}) \end{array}$$

is commutative.

Theorem 2.7 shows that the line bundles over any smooth manifold are classified by their Chern classes. The next Lemma gives an interpretation of the Chern classes as de Rham cohomology classes and it yields a connection of those classes to the geometry of the line bundle. To formulate this Lemma the following Definition is needed.

Definition 2.9 (Curvature form).
If ξ is a vector bundle over M, let $\mathcal{A}^p(\xi)$ be the sheaf of smooth ξ-valued p-forms. Every section of $\mathcal{A}^p(\xi)$ over some open set $U \subset M$ is a linear combination of sections of the form $\alpha \otimes s$, where $\alpha \in \mathcal{A}^p(U)$ is a p-form on U and $s \in \mathcal{E}(\xi)(U)$ a section of ξ on U. A connection ∇ can be interpreted as a sheaf map

$$\mathrm{d}^\nabla := \nabla\colon \mathcal{A}^0(\xi) \to \mathcal{A}^1(\xi)\,.$$

This sheaf map can be extended to a sequence of sheaf maps

$$\mathrm{d}^\nabla : \mathcal{A}^p(\xi) \to \mathcal{A}^{p+1}(\xi) \qquad \text{by demanding that}$$
$$\mathrm{d}^\nabla(\alpha \otimes s) := \mathrm{d}\alpha \otimes s + (-1)^p \alpha \wedge \nabla s$$

holds for every open set U and all p-forms α and all ξ-sections s on U.

In contrast to the usual exterior derivative d this sequence need not be exact. If $e := \{e_i\}$ is a local frame of the vector bundle ξ over some open set $U \subset M$, then one can locally define a curvature matrix κ on U by $\mathrm{d}^\nabla \circ \mathrm{d}^\nabla e_i = \sum_j \kappa_{ij} \cdot e_j$. Each component of this matrix is a 2-form, $\kappa_{ij} \in \mathcal{A}^2(U)$. If $\tilde{e}$ is another frame over another open set $\tilde{U} \subset M$ and if g is the corresponding matrix of transition functions, i. e. $\tilde{e}_i = \sum_j g_{ij} \cdot e_j$, on $\tilde{U} \cap U$, then $\tilde{\kappa} = g \cdot \kappa \cdot g^{-1}$ on $\tilde{U} \cap U$. If ξ is a line bundle, then those local curvature matrices consist of only one 2-form satisfying $\tilde{\kappa} = \kappa$. Thus, the local curvature forms patch together to give one global *curvature form* of the connection ∇.

The curvature form can be calculated from the connection as follows:

Remark 2.10. If ξ is a vector bundle over M and e a frame over some open subset $U \subset M$, then the *connection matrix* θ of a connection ∇ is defined by $\nabla e_i = \sum_j \theta_{ij} \otimes e_j$ and one has the *Cartan structure equation*

$$\kappa = \mathrm{d}\theta + \theta \wedge \theta \qquad \text{with } (\theta \wedge \theta)_{ij} := \textstyle\sum_k \theta_{ik} \wedge \theta_{kj}\,.$$

On line bundles the connection matrix consists of a single 1-form and this equation simplifies to $\kappa = \mathrm{d}\theta$. Analogously, if $\tilde{e}$ is another frame over $\tilde{U} \subset M$ and g the corresponding matrix of transition functions, then

$$\tilde{\theta} = \mathrm{d}g \cdot g^{-1} + g \cdot \theta \cdot g^{-1},$$

which simplifies to $\tilde{\theta} - \theta = \mathrm{d}g/g$ in the case of line bundles. See [GH94, Chapter 0.5] for details.

Lemma 2.11 (Čech-de Rham isomorphism).
There is an isomorphism $\check{\mathrm{H}}^p(M, \mathbb{R}) \cong \mathrm{H}^p_{\mathrm{dR}}(M)$ and thus an inclusion

$$\check{\mathrm{H}}^p(M, \mathbb{Z}) \hookrightarrow \mathrm{H}^p_{\mathrm{dR}}(M)\,.$$

Under this inclusion one may write

$$c(\lambda) = \left[-\frac{1}{2\pi \mathrm{i}}\kappa\right],$$

where $\mathrm{H}^p_{\mathrm{dR}}(M)$ is the pth de Rham cohomology on M and κ is the curvature form of any connection on λ. In particular, the curvature forms of any two connections are cohomologous.

Proof. In analogy to the definition of $\mathcal{A}^p$ let $\mathcal{Z}^p$ be the sheaf of closed differential p-forms. Thus, $\mathrm{H}^p_{\mathrm{dR}}(M) := \mathcal{Z}^p(M)/\mathrm{d}\mathcal{A}^{p-1}(M)$ and one has $\mathcal{Z}^p(M) \cong \check{\mathrm{H}}^0(M, \mathcal{Z}^p)$: If $s = \{s_i\}$ is a cocycle over an open cover $\{U_i\}$ representing an element of $\check{\mathrm{H}}^0(M, \mathcal{Z}^p)$, then the closed differential forms s_i on U_i patch together to give a closed differential form on M,

$$0 = (\delta s)_{ij} = s_j|_{U_i \cap U_j} - s_i|_{U_i \cap U_j} .$$

Conversely, every element of $\mathcal{Z}^p(M)$ gives a cocycle over every open cover and all those cocycles represent the same element of $\check{\mathrm{H}}^0(M, \mathcal{Z}^p)$.

Together with the analogous isomorphism for $\mathcal{A}^p$ one obtains that $\mathrm{H}^p_{\mathrm{dR}}(M) \cong \check{\mathrm{H}}^0(M, \mathcal{Z}^p)/\mathrm{d}^*\check{\mathrm{H}}^0(M, \mathcal{A}^{p-1})$. Therefore, it suffices to show that the last group is isomorphic to the Čech cohomology of the constant sheaf $\mathbb{R}$ on M.

For every $p > 0$ the sheaf sequences

$$0 \to \mathcal{Z}^p \overset{\mathrm{incl}}{\hookrightarrow} \mathcal{A}^p \overset{\mathrm{d}}{\to} \mathcal{Z}^{p+1} \to 0$$

are exact by the Poincaré lemma, which yields a long exact sequence

$$\cdots \to \check{\mathrm{H}}^q(M, \mathcal{A}^{p-q-1}) \overset{\mathrm{d}^*}{\to} \check{\mathrm{H}}^q(M, \mathcal{Z}^{p-q}) \to \\ \check{\mathrm{H}}^{q+1}(M, \mathcal{Z}^{p-q-1}) \overset{\mathrm{incl}^*}{\to} \check{\mathrm{H}}^{q+1}(M, \mathcal{A}^{p-q-1}) \to \cdots .$$

Since $\check{\mathrm{H}}^k(\mathcal{A}^l) = 0$ for any $k > 0$, the following groups are isomorphic:

$$\check{\mathrm{H}}^0(M, \mathcal{Z}^p)/\mathrm{d}^*\check{\mathrm{H}}^0(M, \mathcal{A}^{p-1}) \cong \check{\mathrm{H}}^1(M, \mathcal{Z}^{p-1}) \cong \cdots \cong \check{\mathrm{H}}^p(M, \mathcal{Z}^0),$$

if $p > 0$. But for $p = 0$ this is trivially true as $\mathcal{A}^{-1} := 0$. Thus, $\mathrm{H}^p_{\mathrm{dR}}(M) \cong \check{\mathrm{H}}^p(M, \mathcal{Z}^0)$ for $p > 0$.

For an open cover $\mathfrak{U}$ consisting of connected open sets one has $C^p(\mathfrak{U}, \mathcal{Z}^0) = C^p(\mathfrak{U}, \mathbb{R})$. Every open cover has such a cover as a refinement, which shows that $\check{\mathrm{H}}^p(M, \mathbb{R}) \cong \check{\mathrm{H}}^p(M, \mathcal{Z}^0) \cong \mathrm{H}^p_{\mathrm{dR}}(M)$.

To see the form of the inclusion Remark 2.10 is needed. Let λ be a line bundle over M with a trivialising cover $\mathfrak{U} = \{U_i\}$ consisting of connected U_i and trivialisations φ_i over those U_i. Every $e_i(x) := \varphi_i^{-1}(x, 1)$ for $x \in U_i$ is a frame of λ over U_i and each transition function g_{ij} of λ is also the transition function of the corresponding frames e_i and e_j, i. e. $g_{ij} e_i = e_j$. Further, every connection ∇ on λ is given by its connection forms θ_i. Those θ_i are 1-forms on the corresponding $U_i \subset M$ and satisfy

$$\theta_j - \theta_i = -g_{ij}^{-1}\mathrm{d}g_{ij} = -\mathrm{d}(\log g_{ij}) \qquad \text{on } U_j \cap U_i.$$

Remark 2.10 also states that $\kappa = \mathrm{d}\theta_i$ on U_i.

In the case $p = 2$ the Čech-de Rham isomorphism is induced by the two exact sequences

$$0 \to \mathcal{Z}^1 \hookrightarrow \mathcal{A}^1 \overset{\mathrm{d}}{\to} \mathcal{Z}^2 \to 0 \quad \text{and} \quad 0 \to \mathcal{Z}^0 \hookrightarrow \mathcal{A}^0 \overset{\mathrm{d}}{\to} \mathcal{Z}^1 \to 0,$$

which give two coboundary operators

$$\check{\mathrm{H}}^0(M, \mathcal{Z}^2)/\mathrm{d}^*\check{\mathrm{H}}^0(M, \mathcal{A}^1) \overset{\delta_1^*}{\to} \check{\mathrm{H}}^1(M, \mathcal{Z}^1) \quad \text{and} \quad \check{\mathrm{H}}^1(M, \mathcal{Z}^1) \overset{\delta_2^*}{\to} \check{\mathrm{H}}^2(M, \mathbb{R}).$$

Using $\theta_j - \theta_i = -\mathrm{d}(\log g_{ij})$, $\kappa = \mathrm{d}\theta_i$ and (see the proof of Theorem 2.7)

$$c(\lambda)_{ijk} = \Psi \circ \Phi(\lambda)_{ijk} = \left(\delta \circ \exp(2\pi \mathrm{i} \cdot .)^{-1} g\right)_{ijk} = \frac{1}{2\pi \mathrm{i}}(\log g_{ij} - \log g_{ik} + \log g_{jk})$$

those coboundary operators evaluate as follows:

$$\delta_2^* \delta_1^*(\kappa) = \delta_2^*(\delta \circ \mathrm{d}^{-1}\kappa) = \delta_2^*(\{\theta_j - \theta_i\}) = \\ \delta \circ \mathrm{d}^{-1}(\{\theta_j - \theta_i\}) = \{-(\log g_{jk} - \log g_{ik} + \log g_{ij})\} = -2\pi \mathrm{i} \cdot c(\lambda)\,.$$

Here $f^{-1}(y)$ for a nonbijective map $f\colon X \to Y$ is not the preimage of y but any element $x \in X$ with $f(x) = y$. □

The Chern class seen as a de Rham cohomology class is not only given by the curvature of an arbitrary connection but is also represented by a unique harmonic 2-form:

Definition 2.12. If the manifold M is closed and oriented, then there exists the Hodge-operator $*$ and one can define an operator $\mathrm{d}^* := (-1)^{n \cdot p + n + 1} * \mathrm{d} *$ mapping p-forms to $p-1$-forms. The *Laplace-Beltrami operator* is given by

$$\Delta_{\mathrm{LB}} := \mathrm{d}^* \circ \mathrm{d} + \mathrm{d} \circ \mathrm{d}^*\,.$$

A p-form ω is called *harmonic* if $\Delta_{\mathrm{LB}}\omega = 0$. Details can be found in [War83, Chapter 6].

Conclusion 2.13. The line bundles over a given closed and oriented manifold M are classified by their Chern classes in $\check{\mathrm{H}}^2(M, \mathbb{Z}) \hookrightarrow \mathrm{H}^2_{\mathrm{dR}}(M)$ and by Hodge decomposition, [War83, Theorem 6.8], every de Rham cohomology class contains one and only one harmonic representative. This means that $c(\lambda) = [\Omega]$, where Ω is a harmonic 2-form.

3 Flat Tori

To facilitate calculations I will from now on consider only very simple closed manifolds: flat tori. Additionally, the line bundles over those tori will be assumed to have nondegenerate Chern classes. This demand will require the dimensions of the tori to be even.

Definition 3.1 (Lattice).
If $(W_1, \dots, W_n)$ is a basis of $\mathbb{R}^n$, then $\mathcal{L} := \mathbb{Z}(W_1, \dots, W_n)$ is called a *lattice*. The *unit cell* of the lattice $\mathcal{L}$ with respect to a given basis is the compact set $\{\sum_i \alpha_i W_i \mid \alpha_i \in [0,1]\}$.

Definition 3.2 (Flat torus).
A lattice $\mathcal{L}$ forms a group and acts on $\mathbb{R}^n$ via the addition $+$. Every *translation* by some $l \in \mathcal{L}$

$$T_l \colon \mathbb{R}^n \ni x \mapsto l + x \in \mathbb{R}^n$$

is an isometry and thus $\mathcal{L}$ is a group of isometries acting on $\mathbb{R}^n$. Let

$$\pi_{\mathcal{L}} \colon \mathbb{R}^n \to \mathcal{L} \backslash \mathbb{R}^n =: M$$

be the projection. There is exactly one metric $\langle . , . \rangle_{\mathcal{L}}$ on M such that $\pi_{\mathcal{L}}$ is a Riemannian covering, where $\mathbb{R}^n$ is equipped with the standard metric $\langle . , . \rangle$. The Riemannian manifold $(M, \langle . , . \rangle_{\mathcal{L}})$ is called *flat torus*.

Proof. M is a manifold because $\mathcal{L}$ acts smoothly, freely and properly on $\mathbb{R}^n$, confer [Lee02, Theorem 9.19].

The map $\pi_{\mathcal{L}}$ is a smooth covering and therefore there is a neighbourhood U for every $[x] \in M$ such that U is diffeomorphic via $\pi_{\mathcal{L}}$ to an open set V in $\mathbb{R}^n$. The choice of V is not important because the translations are isometries. Then, the metric $\langle . , . \rangle_{\mathcal{L}}$ can and must be defined as

$$\langle X, Y \rangle_{\mathcal{L}} := \left\langle (\mathrm{d}\pi_{\mathcal{L}})^{-1}_{[x]} X, (\mathrm{d}\pi_{\mathcal{L}})^{-1}_{[x]} Y \right\rangle \qquad \text{for } X, Y \in \mathrm{T}_{[x]} M.$$

This metric is smooth because $\pi_{\mathcal{L}}$ is a diffeomorphism on V, in particular it is a chart for M. In this chart the coordinate vector fields are $\frac{\partial}{\partial x^i{}_M} = \mathrm{d}\pi_{\mathcal{L}} \frac{\partial}{\partial x^i}$ and thus

$$\left\langle \frac{\partial}{\partial x^i{}_M}, \frac{\partial}{\partial x^j{}_M} \right\rangle_{\mathcal{L}} = \delta_{ij} \qquad \text{for all } i, j = 1, \dots, n.$$

Hence, $\langle . , . \rangle_{\mathcal{L}}$ is indeed a flat metric. □

Remark 3.3. Every flat torus is closed and orientable. Therefore, the Laplace-Beltrami operator Δ_{LB} exists and by Conclusion 2.13 every line bundle over a flat torus is represented by an harmonic 2-form.

Definition 3.4. Every translation T_x on $\mathbb{R}^n$ by an $x \in \mathbb{R}^n$ descends to an isometry on the torus. A differential form is called *translation-invariant* if it is invariant under the group of those isometries.

Lemma 3.5. A differentiable form on a flat torus is translation-invariant if and only if it is harmonic.

Proof. Obviously, every translation-invariant form is also harmonic. Since M is closed, Green's formula reads

$$\int_M \langle \operatorname{grad} h, \operatorname{grad} f \rangle - h \cdot \Delta_{\mathrm{LB}} f \, \mathrm{d}V = 0 \qquad \text{for all } h, f \in C^\infty(M).$$

For a harmonic function f and $h := f$ one obtains $\int_M \|\operatorname{grad} f\|^2 \,\mathrm{d}V = 0$ and thus $\operatorname{grad} f = 0$. Hence, f is translation-invariant.

Now, let $\omega = \sum_{i_1 < \cdots < i_p} \omega_{i_1,\ldots,i_p} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}$ be a p-form in the standard local coordinates. One can check by a tedious calculation that

$$\Delta_{\mathrm{LB}} \omega = \sum_{i_1 < \cdots < i_p} (\Delta_{\mathrm{LB}} \omega_{i_1,\ldots,i_p}) \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} \, .$$

Thus, if ω is harmonic, so are its components. This means that the components and therefore ω itself are translation-invariant. □

Remark 3.6. Let $\{e_i\}$ denote the standard basis of $\mathbb{R}^n$ and $(x^1, \ldots, x^n)$ the corresponding coordinates. If $U \subset \mathbb{R}^n$ is the interior of a unit cell, then the standard coordinates descend to coordinates on $\pi_{\mathcal{L}}(U)$ and thus one obtains a basis $\{\frac{\partial}{\partial x^i}\}$ on every $\mathrm{T}_x M$ with $x \in U$.

A translation-invariant p-form A on M is already defined by any single $A_x \colon \mathrm{T}_x M^p \to \mathbb{R}$. When identifying the tangent space $\mathrm{T}_x M$ with $\mathbb{R}^n$ via the map $\frac{\partial}{\partial x^i} \mapsto e_i$, one can identify the translation-invariant p-form A with an antisymmetric p-linear map on $\mathbb{R}^n$.

This identification will be of great importance later in this work because I will study connections which are determined by harmonic forms. Before that the following observation about the harmonic representative of a Chern class will be very useful.

Lemma 3.7. If Ω is the harmonic representative of $[\Omega] \in \check{\mathrm{H}}^2(M, \mathbb{Z})$, then Ω can be considered as an antisymmetric bilinear map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with $\Omega(\mathcal{L} \times \mathcal{L}) \subset \mathbb{Z}$.

Proof. Two linearly independent lattice vectors $l, k \in \mathcal{L}$ span a parallelepiped in $\mathbb{R}^n$. Its projection $P \subset M$ is a closed 2-chain in the singular homology of M. The class $[\Omega]$ can be seen as an element of the corresponding cohomology and one can show that the de Rham isomorphism is given by integration. Thus:

$$\Omega(l, k) = \int_P [\Omega] = [\Omega](P) \in \mathbb{Z}$$

Confer [Lee02, Chapter 16] or [War83, Chapter 5]. □

Lemma 3.8. If $\Omega \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a nondegenerate antisymmetric bilinear map with $\Omega(\mathcal{L} \times \mathcal{L}) \subset \mathbb{Z}$, then $n = 2m$ is an even integer and there exists one and only one tuple $(r_1, \ldots, r_m) \in \mathbb{N}^m$ such that

(a) there exists a lattice basis $\mathfrak{B} := \{U_1, \ldots, U_m, V_1, \ldots, V_m\}$ of $\mathcal{L}$ with $\Omega(U_i, V_j) = r_i \cdot \delta_{ij}$ and $\Omega(U_i, U_j) = \Omega(V_i, V_j) = 0$ for $i, j = 1, \ldots, m$ and

(b) $r_1 \mid r_2 \mid \cdots \mid r_m$.

If $\{u^1, \ldots, u^m, v^1, \ldots, v^m\}$ are the coordinates corresponding to the basis $\mathfrak{B}$, then $\Omega = \sum_i r_i \, du^i \wedge dv^i$. The integers r_i are called *Chern invariant factors* and every such basis $\mathfrak{B}$ shall be called *Chern basis*.

Proof. For every $U \in \mathcal{L}$ the set $\mathfrak{a}_U := \{\Omega(U, V) \mid V \in \mathcal{L}\} \subset \mathbb{Z}$ is an ideal and, since $\mathbb{Z}$ is a principal ideal domain, $\mathfrak{a}_U = d_U\mathbb{Z}$. Since Ω is nondegenerate, one can assume $d_U > 0$ for $U \neq 0$ and thus $r_1 := \min\left\{d_U \mid U \in \mathcal{L}\backslash\{0\}\right\} > 0$. Choose U_1 and V_1 such that $\Omega(U_1, V_1) = r_1$.

By construction r_1 divides both $\Omega(U, U_1)$ and $\Omega(U, V_1)$ for every $U \in \mathcal{L}$; in particular

$$U + \frac{\Omega(U, U_1)}{r_1} V_1 - \frac{\Omega(U, V_1)}{r_1} U_1 \in \mathcal{L}_{n-2}\,,$$

where $\mathcal{L}_{n-2}$ is the orthogonal complement with respect to Ω of $\mathbb{Z}(U_1, V_1)$ in $\mathcal{L}$. This means that $\mathcal{L} = \mathbb{Z}(U_1, V_1) \oplus \mathcal{L}_{n-2}$ and that $\mathcal{L}_{n-2}$ is a lattice of dimension $n-2$ in $\mathbb{Z}(U_1, V_1)^{\perp} = \mathbb{R} \cdot \mathcal{L}_{n-2}$, where $\perp$ is to be understood with respect to Ω.

The restriction of Ω to $\mathbb{R} \cdot \mathcal{L}_{n-2}$ is again nondegenerate, and repeating this process with $\mathcal{L}_{n-2}$ inductively gives a tuple $(r_1, \ldots, r_m)$ of integers with the corresponding lattice basis $(U_1, \ldots, U_m, V_1, \ldots, V_m)$. In particular, $n = 2m$ is even.

Now assume (without loss of generality) that $r_1 \nmid r_2$, i. e. there is an $a \in \mathbb{Z}$ such that $0 < r_2 - ar_1 < r_1$. This implies

$$\begin{aligned} 0 < \Omega(U_2, V_2) - a\Omega(U_1, V_1) &= \\ \Omega(U_2, V_1 + V_2) - a\Omega(U_1, V_1 + V_2) &= \Omega(U_2 - aU_1, V_1 + V_2) < r_1\,, \end{aligned}$$

which contradicts the choice of r_1.

It remains to show that the tuple r is unique. Let $\Omega^i := \Omega \wedge \cdots \wedge \Omega$. This is an alternating $2i$-linear form and, if one chooses pairwise distinct indices $k_1, \ldots, k_i \in \{1, \ldots, m\}$, then

$$\Omega^i(U_{k_1}, V_{k_1}, \ldots, U_{k_i}, V_{k_i}) = i! \cdot r_{k_1} \cdots r_{k_i}\,.$$

When applied to any other combination of $2i$ vectors in $\{U_i, V_i\}$ the form vanishes. Therefore, for every $2i$-tuple $(W_1, \ldots, W_{2i}) \in \mathcal{L}^{2i}$ there are integers $\beta_{k_1 \cdots k_i}$ with

$$\Omega^i(W_1, \ldots, W_{2i}) = \sum_{k_1 < \cdots < k_i} \beta_{k_1 \cdots k_i} \cdot i! \cdot r_{k_1} \cdots r_{k_i}\,.$$

Since $r_1 \mid \cdots \mid r_m$, the product $i! \cdot r_1 \cdots r_i$ divides all summands and there is a suitable $N \in \mathbb{N}$ with $|\Omega^i(W_1, \ldots, W_{2i})| = i! \cdot r_1 \cdots r_i \cdot N$.

This means $i! \cdot r_1 \cdots r_i$ is the minimum of the nonzero values of $|\Omega^i|$ on $\mathcal{L}^{2i}$. This characterises the invariants r_i of Ω. □

Conclusion 3.9. Line bundles over a given manifold M are classified, up to isomorphism, by their Chern classes $c(\lambda) \in \check{H}^2(M, \mathbb{Z})$, and on flat tori those classes can be identified with antisymmetric bilinear maps Ω on $\mathbb{R}^n \times \mathbb{R}^n$ taking integer values on the lattice. If this map is nondegenerate, then the line bundle is said to have a *nondegenerate Chern class* or just to be *nondegenerate*. In this case one also obtains a normal form of Ω.

In the following this normal form is used to construct an explicit representative for every isomorphism class of nondegenerate line bundles over a given flat torus.

Definition 3.10. For any given class $[\Omega] \in \check{H}^2(M, \mathbb{Z})$ use its Chern invariant factors r_i and a fixed Chern basis $\mathfrak{B}$ to define for $x, y \in \mathbb{R}^n$:

$$w_x(y) := \sum_{i=1}^{m} r_i u^i(x) v^i(y) \qquad \text{and} \qquad e_x(y) := \mathrm{e}^{2\pi \mathrm{i} \cdot w_x(y)} \in \mathrm{U}(1)\,,$$

where $\{u^i, v^i\}$ are the coordinates corresponding to the Chern basis. Also define an action of the group $\mathcal{L}$ on the total space $\mathbb{R}^n \times \mathbb{C}$ of the trivial line bundle θ^1 over $\mathbb{R}^n$ via

$$l\,.\,(x, z) := (l + x,\, e_l(x) \cdot z) \qquad \text{for } l \in \mathcal{L},\, x \in \mathbb{R}^n \text{ and } z \in \mathbb{C}.$$

Define a bundle $\omega := (L_\omega, \pi_\omega, M)$ over M by setting $L_\omega := \mathcal{L}\backslash(\mathbb{R}^n \times \mathbb{C})$ and $\pi_\omega \colon L_\omega \ni [x, z] \mapsto [x] \in \mathcal{L}\backslash\mathbb{R}^n = M$.

Lemma 3.11. ω is a Hermitian complex line bundle over M, where the Hermitian structure is induced by the standard Hermitian product of the trivial bundle. Moreover, ω pulls back to the trivial bundle over $\mathbb{R}^n$ under the canonical projection $\mathbb{R}^n \to \mathcal{L}\backslash\mathbb{R}^n$. The Chern class of ω is $[\Omega]$.

Proof. There are local trivialisations such that the structure group is the unitary group $\mathrm{U}(1)$: For every $[x] \in M$ there is a neighbourhood U which is evenly covered by $\pi_{\mathcal{L}}$, i. e. $\pi_{\mathcal{L}}^{-1}(U) = \bigcup_{l \in \mathcal{L}} l + V$, where V is diffeomorphic to U via $\pi_{\mathcal{L}}$ and all $l + V$ are disjoint. Now, for every class $[y] \in U$ there is a unique lift $y \in V$. Define the trivialisations as

$$t_V \colon L_\omega|_U \ni [y, z] \mapsto ([y], z) \in U \times \mathbb{C}\,.$$

For another trivialisation $t_{V'}$ one may assume that $\pi_{\mathcal{L}}(V) = \pi_{\mathcal{L}}(V') = U$ and thus that $V + l = V'$ for an $l \in \mathcal{L}$. Therefore, one obtains

$$t_{V'} \circ t_V{}^{-1}([y], z) = ([y],\, e_l(y) \cdot z)\,,$$

which proves that the transition functions are given by

$$\tau_{V'V} \colon U \cap U' \ni [y] \mapsto e_l(y) \in \mathrm{U}(1)\,.$$

Also, one can define a Hermitian metric on ω through the local trivialisations and those local metrics are independent of the choice of the trivialisation function and thus give a global metric.

The claim that ω pulls back to the trivial bundle means ${\pi_{\mathcal{L}}}^*(\omega) = \theta^1$ and is obvious from the commuting diagram

$$\begin{array}{ccc} \mathbb{R}^n \times \mathbb{C} & \xrightarrow{\pi} & \mathcal{L}\backslash(\mathbb{R}^n \times \mathbb{C}) \\ \downarrow{\scriptstyle \pi_{\theta^1}} & & \downarrow{\scriptstyle \pi_\omega} \\ \mathbb{R}^n & \xrightarrow{\pi_{\mathcal{L}}} & \mathcal{L}\backslash\mathbb{R}^n \end{array}$$

and from

$$\begin{aligned} {\pi_{\mathcal{L}}}^*(\mathcal{L}\backslash(\mathbb{R}^n \times \mathbb{C})) = \{(x, [y, z]) \mid \pi_{\mathcal{L}}(x) = \pi_\omega([y, z])\} = \\ \{(x, [x, z]) \mid x \in \mathbb{R}^n, z \in \mathbb{C}\} \cong \mathbb{R}^n \times \mathbb{C} \,. \end{aligned}$$

The Chern class $c(\omega)$ will be computed at the end of this section. □

To clarify calculations I will introduce some notations relating sections to functions. Also, a model connection on ω will be distinguished.

Definition 3.12. A section $s \in \mathcal{E}(\theta^1)$ of the trivial line bundle θ^1 over $\mathbb{R}^n$ has the form $s(x) = (x, f(x))$ with a function $f \in C^\infty(\mathbb{R}^n, \mathbb{C})$. Define an isomorphism

$$\begin{aligned} &\mathfrak{f}\colon \mathcal{E}(\theta^1) \ni s \mapsto f \in C^\infty(\mathbb{R}^n, \mathbb{C}) \qquad \text{with inverse} \\ &\mathfrak{s}\colon C^\infty(\mathbb{R}^n, \mathbb{C}) \ni f \mapsto \big(.\,, f(.)\big) \in \mathcal{E}(\theta^1)\,. \end{aligned}$$

Remark 3.13. Given a group G acting on two spaces X and Y a function $f\colon X \to Y$ is called G-equivariant if $g \cdot f(x) = f(g \cdot x)$ for all $x \in X$ and $g \in G$. The group also acts on the set of functions $\mathrm{Fun}(X, Y) = \{f\colon X \to Y\}$ via

$$\big(g \lrcorner f\big)(x) := g \cdot f(g^{-1} \cdot x)\,.$$

Thus, a function is G-equivariant if and only if it is G-invariant under this action. If $F \subset \mathrm{Fun}(X, Y)$ is an arbitrary set of functions from X to Y, denote the subset of G-invariant functions by

$$\begin{aligned} F^G := \{f \in F \mid g \lrcorner f = f \text{ for all } g \in G\} = \\ \{f \in F \mid g \cdot f(x) = f(g \cdot x) \text{ for all } g \in G, x \in X\}\,. \end{aligned}$$

Definition 3.14. The group $\mathcal{L}$ acts on the vector fields $\mathcal{X}(\mathbb{R}^n)$ via

$$l * X := T_{l*} X_{T_{-l}}\,.$$

A vector field on $\mathbb{R}^n$ descends to a vector field on M if and only if it is $\mathcal{L}$-invariant. There is an isomorphism

$$\mathcal{X}(M) \to \mathcal{X}(\mathbb{R}^n)^{\mathcal{L}}\,.$$

Remark 3.15. Note that the group $\mathcal{L}$ acts on the sections of θ^1 via

$$\big(l \lrcorner s\big)(x) = l\,.\,s(x - l) = \big(x,\, e_l(x) \cdot \mathfrak{f}s(x - l)\big)\,,$$

because $w_l(x-l) - w_l(x) = -w_l(l) \in \mathbb{Z}$ and thus $e_l(x-l) = e_l(x)$. Also, one can define an action of $\mathcal{L}$ on $C^\infty(\mathbb{R}^n, \mathbb{C})$ by setting

$$\big(l \lrcorner f\big)(x) := e_l(x) \cdot f(x-l)\,.$$

Those two $\mathcal{L}$-actions are the same actions in the following sense: Sections of ω pull back to $\mathcal{L}$-invariant sections of θ^1 under $\pi_{\mathcal{L}}$ and those correspond, via $\mathfrak{f}$, to $\mathcal{L}$-invariant functions. Conversely, every such function gives an $\mathcal{L}$-invariant section via $\mathfrak{s}$ and thus a section of ω. There is an isomorphism

$$\mathfrak{f}_\omega := \mathfrak{f} \circ {\pi_{\mathcal{L}}}^* \colon \mathcal{E}(\omega) \to C^\infty(\mathbb{R}^n, \mathbb{C})^{\mathcal{L}}\,.$$

Denote its inverse by $\mathfrak{s}_\omega$.

Proof. The diagram on page 14 shows that the pullback s^* of a given section $s \in \mathcal{E}(\omega)$ is the unique section of θ^1 with $\pi \circ s^* = s \circ \pi_{\mathcal{L}}$. Thus, $\pi \circ s^*(x) = \pi \circ s^*(x+l)$. This means that there exists a $k \in \mathcal{L}$ with $k \,.\, s^*(x) = s^*(x+l)$. Since s^* is a section and satisfies $\pi_{\theta^1} \circ s^* = \mathrm{id}$, this k must be l: $l \,.\, s^*(x) = s^*(x+l)$. Hence, s^* is $\mathcal{L}$-invariant.

If one is given an $\mathcal{L}$-invariant section $s^* \in \mathcal{E}(\theta^1)$, then one obtains a section $s \in \mathcal{E}(\omega)$ by setting $s([x]) := \pi \circ s^*(x)$. This is well-defined:

$$s([x+l]) = \pi \circ s^*(x+l) = \pi(l \,.\, s^*(x)) = \pi \circ s^*(x) = s([x])\,. \qquad \square$$

Definition 3.16. For every complex-valued 1-form A on $\mathbb{R}^n$ one obtains a *connection* on the trivial bundle θ^1 over $\mathbb{R}^n$ by setting

$$\nabla := \mathrm{d} + A\,, \qquad \text{which means that for any vector field } X \in \mathcal{X}(\mathbb{R}^n)$$

$$\big(\nabla_X s\big)(x) = \big(x, X_x(\mathfrak{f}s) + A_x(X) \cdot \mathfrak{f}s(x)\big) \qquad \text{for every } s \in \mathcal{E}(\theta^1).$$

Also, every connection on θ^1 has this form. A connection ∇ on θ^1 descends to a connection on ω, also called ∇, if and only if it maps $\mathcal{L}$-invariant sections to $\mathcal{L}$-invariant sections. More precisely, for every $s \in \mathcal{E}(\theta^1)^{\mathcal{L}}$ and $X \in \mathcal{X}(\mathbb{R}^n)^{\mathcal{L}}$ the connection must satisfy

$$\big(\nabla_X s\big)(l+x) = l \,.\, \big(\nabla_X s\big)(x) \qquad \text{for all } x \in \mathbb{R}^n \text{ and } l \in \mathcal{L}.$$

Lemma 3.17. ∇ descends to a connection on ω if and only if

$$T_l^* A = A - 2\pi \mathrm{i} w_l \qquad \text{for all } l \in \mathcal{L}.$$

Proof. Note that using the canonical isomorphism identifying $\mathbb{R}^n$ with all its tangent spaces one can consider w_l as a function and as a 1-form on $\mathbb{R}^n$ and for $X \in \mathcal{X}(\mathbb{R}^n)$ one has $X(w_l) = w_l(X)$. Assuming A satisfies the given equation one has for $s \in \mathcal{E}(\theta^1)^{\mathcal{L}}$, $X \in \mathcal{X}(\mathbb{R}^n)^{\mathcal{L}}$ and $x \in \mathbb{R}^n$:

$$A_{T_l(x)}(X) \cdot (\mathfrak{f}s \circ T_l(x)) = (T_l^* A)_x(X) \cdot \mathfrak{f}s(x) \cdot e_l(x) =$$
$$e_l(x) \cdot \big(A_x(X) \cdot \mathfrak{f}s(x) - \mathfrak{f}s(x) \cdot 2\pi \mathrm{i} w_l(X_x)\big)$$

and with

$$X_{T_l(x)}(\mathfrak{f}s) = X_x(\mathfrak{f}s \circ T_l) = X_x(e_l \cdot \mathfrak{f}s) = e_l(x) \cdot \Big(X_x(\mathfrak{f}s) + \mathfrak{f}s(x) \cdot 2\pi \mathrm{i} w_l(X_x)\Big)$$

it follows that $\nabla_X s$ is $\mathcal{L}$-equivariant:

$$\mathfrak{f}((\nabla_X s) \circ T_l) = X_{T_l}(\mathfrak{f}s) + A_{T_l}(X) \cdot (\mathfrak{f}s \circ T_l) = e_l \cdot \Big(X(\mathfrak{f}s) + A(X) \cdot \mathfrak{f}s\Big) = \mathfrak{f}(l \,.\, \nabla_X s)\,.$$

Reasoning backwards gives the converse. □

Lemma 3.18. A connection is compatible with the Hermitian product on the trivial bundle θ^1 if and only if the 1-form A is purely imaginary, i. e. $A_x(X) \in \mathrm{i}\mathbb{R}$ for all $x \in \mathbb{R}^n$ and all $X \in \mathrm{T}_x\mathbb{R}^n$.

Proof. Compatibility with the Hermitian product means

$$X.\langle s,t\rangle = \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle \qquad \text{for all } s,t \in \mathcal{E}(\theta^1) \text{ and } X \in \mathcal{X}(\mathbb{R}^n).$$

The right hand side is equal to $X(\mathfrak{f}s) \cdot \overline{\mathfrak{f}t} + \mathfrak{f}s \cdot X(\overline{\mathfrak{f}t}) + \Big(A(X) + \overline{A(X)}\Big) \cdot \mathfrak{f}s\overline{\mathfrak{f}t} = X.\langle s,t\rangle + \mathfrak{f}s\overline{\mathfrak{f}t} \cdot \Big(A(X) + \overline{A(X)}\Big)$. Thus, compatibility is equivalent to

$$A(X) + \overline{A(X)} = 0\,.$$ □

Definition 3.19. For convenience let us define a *distinguished connection* ∇^{D} on ω through

$$A^{\mathrm{D}}_x := -2\pi\mathrm{i}\, w_x \qquad \text{for all } x \in \mathbb{R}^n.$$

This connection is Hermitian.

Proof. Since all tangent spaces of $\mathbb{R}^n$ can be identified, the equations

$$T_l^* A^{\mathrm{D}}_x = -2\pi\mathrm{i} w_{T_l x} = -2\pi\mathrm{i} w_x - 2\pi\mathrm{i} w_l = A^{\mathrm{D}}_x - 2\pi\mathrm{i} w_l$$

are meaningful and thus ∇^{D} induces a connection on ω by Lemma 3.17. □

This connection is very useful as every connection on ω is equal to this connection plus some 1-form on the torus. But before studying this, the distinguished connection will be used to calculate the harmonic representative of the Chern class of ω.

Lemma 3.20. The Chern class $c(\omega)$ is represented by Ω.

Proof. Under the Čech-de Rham isomorphism (Lemma 2.11) the Chern class is represented by $-\frac{1}{2\pi\mathrm{i}}\kappa$, where κ is the curvature form of any connection on ω. Let e be a local frame of the trivial bundle θ^1 over $\mathbb{R}^n$ such that $\mathrm{d}(\mathfrak{f}e) = 0$. The connection form of the distinguished connection with respect to this frame is A^{D} and the Cartan structure equation (see Lemma 2.10) yields

$$\kappa = \mathrm{d}A^{\mathrm{D}} = -2\pi\mathrm{i} \sum_{i=1}^{m} r_i\, \mathrm{d}u^i \wedge \mathrm{d}v^i = -2\pi\mathrm{i}\Omega\,.$$ □

4 Connections

The Laplacians and their spectra considered later will be constructed from connections on ω. Thus, it is important to know those connections. Their number can be reduced by excluding some of them from our considerations and by introducing an equivalence relation on the remaining ones. The classes of this equivalence relation will have very simple representatives.

Remark 4.1. Every connection ∇ on the line bundle ω has the form

$$\nabla = \nabla^{\mathrm{D}} + B \qquad \text{with} \qquad B = a + \mathrm{d}b + \mathrm{d}^*c\,,$$

where ∇^{D} is the distinguished connection and B is a 1-form on M split up by Hodge decomposition into a direct sum of a harmonic 1-form a, an exact part given by a function $b \in C^\infty(M,\mathbb{C})$ and a coexact part given by a 2-form c. If the connection is Hermitian, then B as well as a, b and c are imaginary-valued.

Proof. The Leibniz rule yields $\big(\nabla - \nabla^{\mathrm{D}}\big)(fs) = f\big(\nabla - \nabla^{\mathrm{D}}\big)s$ for every $f \in C^\infty(M,\mathbb{C})$ and every section s of ω. Hence, $\nabla - \nabla^{\mathrm{D}}$ is a C^∞-linear operator and thus $(\nabla - \nabla^{\mathrm{D}}) \in \Omega^1(\operatorname{End} L_\omega)$. Since ω is a line bundle, every endomorphism on the fibres is just a complex number, i. e. $\nabla - \nabla^{\mathrm{D}} = B \in \Omega^1(M,\mathbb{C})$. If ∇ is Hermitian, B must be imaginary. As the Hodge decomposition is valid for the real and imaginary parts separately, a, b and c must also be imaginary. □

Corollary 4.2. In particular, the form B can be pulled back to a form in $\Omega^1(\mathbb{R}^n,\mathbb{C})$ with $T_l^*B = B$. Conversely, for every such form $A^{\mathrm{D}} + B$ is a form with $T_l^*(A^{\mathrm{D}} + B) = A^{\mathrm{D}} + B - 2\pi \mathrm{i} w_l$ and this means that the connection $\mathrm{d} + A^{\mathrm{D}} + B$ on θ^1 descends to a connection on ω, namely ∇. Therefore, there is a bijection between the connections on ω and the $\mathcal{L}$-invariant complex-valued 1-forms on $\mathbb{R}^n$.

Definition 4.3. A connection ∇ on a line bundle λ over M is called *translation-invariant* if its curvature form is invariant under translations.

This is not a meaningless definition; there are translation-invariant connections on every nondegenerate line bundle over any even-dimensional torus: At the end of the previous section it was shown that the curvature form of the distinguished connection is $-2\pi\mathrm{i}\Omega$, which is by definition harmonic and hence translation-invariant.

Remark 4.4. All translation-invariant connections have the same curvature form.

Proof. If κ is the curvature form of a translation-invariant connection ∇ and κ^{D} the one belonging to ∇^{D}, then $\kappa = \kappa^{\mathrm{D}} + \mathrm{d}\alpha$ with some $\alpha \in \Omega^1(M,\mathbb{C})$ since all curvature forms are cohomologous. As both sides are translation-invariant, they are harmonic by Lemma 3.5 and thus $\Delta_{\mathrm{LB}}(\kappa^{\mathrm{D}} + \mathrm{d}\alpha) = 0$. Since the curvature form of the distinguished connection is $-2\pi\mathrm{i}\Omega$, which is by definition harmonic, this gives $\Delta_{\mathrm{LB}}\mathrm{d}\alpha = 0$ and hence $\mathrm{d}^*\mathrm{d}\alpha = 0$ and $\langle \mathrm{d}\alpha, \mathrm{d}\alpha\rangle = 0$, because d^* is the adjoint of d with respect to the inner product $\langle \alpha, \beta\rangle := \int_M \alpha \wedge *\beta$, see [War83, Chapter 6]. Therefore, $\mathrm{d}\alpha = 0$. □

Proposition 4.5. A connection $\nabla = \nabla^{\mathrm{D}} + B$ on ω is translation-invariant if and only if the coexact part of B vanishes, $\mathrm{d}^* c = 0$.

Proof. By the Cartan structure equation (Remark 2.10) both curvature forms differ by $\mathrm{d}B = \mathrm{dd}^* c$. If the coexact part of B vanishes, both curvature forms are equal and ∇ is translation-invariant, because ∇^{D} is. Conversely, if both connections are translation-invariant, they have the same curvature form and thus $\mathrm{dd}^* c = 0$. From this and $\mathrm{d}^*\mathrm{d}^* c = 0$ it follows that $\mathrm{d}^* c$ is harmonic. By the Hodge decomposition $\mathrm{d}^* c = 0$. □

Definition 4.6. A *bundle automorphism* F of a line bundle λ is a bundle map such that: If $s \in \mathcal{E}(\lambda)$ is a smooth section, then $F \circ s$ a smooth section and F is a linear automorphism on each fibre of λ. If λ is an Hermitian line bundle, then a bundle automorphism F of λ is called *Hermitian* if F is an isometric automorphism on each fibre.

Definition 4.7. Two connections ∇^1 and ∇^2 on a Hermitian line bundle λ over a manifold M are *gauge equivalent* if there is a Hermitian bundle automorphism F which *intertwines* the two connections:

$$\nabla^1_X \circ F = F \circ \nabla^2_X \qquad \text{for all } X \in \mathcal{X}(M).$$

Gauge equivalent connections will have the same spectrum and it will therefore be sufficient to consider only one preferably simple representative of each gauge equivalence class.

Definition 4.8. The *dual lattice* $\mathcal{L}' \subset (\mathbb{R}^n)'$ of the lattice $\mathcal{L}$ is the set of all linear functionals on $\mathbb{R}^n$ with integer values on $\mathcal{L}$. If $(W^1, \ldots, W^n)$ is the dual basis of a basis $(W_1, \ldots, W_n)$ of $\mathcal{L}$, i. e. $W^i(W_j) = \delta_{ij}$, then

$$\mathcal{L}' = \mathbb{Z}(W^1, \ldots, W^n)$$

and therefore $\mathcal{L}'$ is indeed a lattice in $(\mathbb{R}^n)'$.

Proposition 4.9. The gauge equivalence class of a translation-invariant Hermitian connection $\nabla = \nabla^{\mathrm{D}} + B = \nabla^{\mathrm{D}} + a + \mathrm{d}b$ is independent of the function $b \in C^\infty(M, \mathrm{i}\mathbb{R})$ and depends solely on the class $[a] \in (2\pi\mathrm{i}\mathcal{L}')\backslash\mathrm{i}(\mathbb{R}^n)'$.

Proof. Given a connection $\nabla^b = \nabla^{\mathrm{D}} + a + \mathrm{d}b$ define a bundle map $F_b \colon \omega \to \omega$ through $L_\omega \ni [x, z] \mapsto [x, \mathrm{e}^{-b(x)} z] \in L_\omega$, which is a well-defined Hermitian bundle automorphism with inverse F_{-b}. Using this automorphism one can construct a new connection $\nabla := F_b^{-1} \circ \nabla^b \circ F_b$ satisfying

$$\nabla s = F_b^{-1} \circ \nabla^b(\mathrm{e}^{-b} s) = F_b^{-1}(\mathrm{de}^{-b} \cdot s + \mathrm{e}^{-b} \nabla^b s) = -\mathrm{d}b \cdot s + \nabla^b s = \nabla^{\mathrm{D}} + a$$

for every $s \in \mathcal{E}(\omega)$. Thus, every translation-invariant connection ∇^b is gauge equivalent to a translation-invariant connection whose form has vanishing exact part.

As $\mathcal{L}'$ are the functionals on $\mathbb{R}^n$ with integer values on $\mathcal{L}$, one can argue similarly: For $a \in 2\pi\mathrm{i}\mathcal{L}'$ there is a well-defined Hermitian bundle automorphism F_a via $L_\omega \ni [x, z] \mapsto [x, \mathrm{e}^{-a(x)} z] \in L_\omega$ and a new connection $F_a^{-1} \circ \nabla^b \circ F_a$, which differs from ∇^b by $\mathrm{d}a = a$. (The differential of the function $a \colon \mathbb{R}^n \to \mathbb{R}$ is equal to the harmonic form a.) □

Definition 4.10. Define an involutive isometry $\check{}\colon \mathbb{R}^n \to \mathbb{R}^n$ by setting $\check{x} := -x$. This map descends to an involutive isometry $\check{}\colon M \to M$ of the torus M. Define a map on the smooth functions $C^\infty(\mathbb{R}^n, \mathbb{C})$ by setting $\check{f}(x) := f(\check{x})$. This map descends to a map on the functions on the torus M and can also be used to construct an analogous map on the smooth sections $\mathcal{E}(\omega)$ of the line bundle ω by setting $\check{s} := \mathfrak{s}_\omega((\mathfrak{f}_\omega s)\check{})$. Further, denote the push-forward of $X \in \mathcal{X}(M)$ under the isometry $\check{}$ by $\check{X}$, which means $\check{X}_x(f) := X_{\check{x}}\check{f}$.

Using the involutions on $\mathcal{X}(M)$ and $\mathcal{E}(\omega)$ one can construct an involution on the set of connections on ω: For any connection ∇ on the line bundle ω set

$$\check{\nabla}_X s := (\nabla_{\check{X}}\check{s})\check{} \qquad \text{for all } X \in \mathcal{X}(M) \text{ and every section } s \in \mathcal{E}(\lambda).$$

A connection ∇ on the line bundle ω over M is called $\mathbb{Z}_2$*-invariant* if $\nabla = \check{\nabla}$. A connection is called *weakly* $\mathbb{Z}_2$*-invariant* if $\nabla \sim \check{\nabla}$ with respect to gauge equivalence.

Proof. The map $\mathcal{E}(\omega) \ni s \mapsto \check{s} \in \mathcal{E}(\omega)$ is well-defined because $(\mathfrak{f}_\omega s)\check{}$ is $\mathcal{L}$-invariant: For all $x \in \mathbb{R}^n$ and $l \in \mathcal{L}$ one has

$$(\mathfrak{f}_\omega s)\check{}\,(x+l) = \mathfrak{f}_\omega s(-x-l) = \mathfrak{f}_\omega s(-x) \cdot e_{-l}(-x) = (\mathfrak{f}_\omega s)\check{}(x) \cdot e_l(x)$$

Also, one has to show that $\check{\nabla}$ is indeed a connection. All involutions $\check{}$ are $\mathbb{C}$-linear and thus $\check{\nabla}$ is $\mathbb{C}$-bilinear. For all $f \in C^\infty(M, \mathbb{C})$ and $s \in \mathcal{E}(\omega)$ one has $(fs)\check{} = \check{f}\check{s}$ and $(\check{X}\check{f})\check{} = X(f)$. If f is real-valued and $X \in \mathcal{X}(M)$, then $(fX)\check{} = \check{f}\check{X}$ and hence

$$\check{\nabla}_X fs = (\nabla_{\check{X}}\check{f}\check{s})\check{} = (\check{X}\check{f})\check{} + \check{\check{f}}(\nabla_{\check{X}}\check{s})\check{} = X(f) + f \cdot \check{\nabla}_X s \qquad \text{and}$$
$$\check{\nabla}_{fX} s = (\nabla_{\check{f}\check{X}}\check{s})\check{} = (\check{f}\nabla_{\check{X}}\check{s})\check{} = f \cdot \check{\nabla}_X s \qquad \text{for all real-valued } f.$$

Thus, $\check{\nabla}$ is indeed a connection. ◻

Lemma 4.11. *A translation-invariant connection $\nabla^{\mathrm{D}} + a$ with a harmonic imaginary-valued 1-form a is $\mathbb{Z}_2$-invariant if and only if $a = 0$. A translation-invariant connection $\nabla^{\mathrm{D}} + a$ is weakly $\mathbb{Z}_2$-invariant if and only if $a(\mathcal{L}) \subset \pi\mathrm{i}\mathbb{Z}$.*

Proof. By Remark 4.1 every connection on the line bundle ω has the form $\nabla^{\mathrm{D}} + a + \mathrm{d}b + \mathrm{d}^*c$, where a is a harmonic imaginary-valued 1-form. For a translation-invariant connection d^*c vanishes by Propositions 4.5 and the gauge equivalence class of the connection is independent of b by Proposition 4.9.

Assume one is given a connection $\nabla = \mathrm{d} + A$ with some 1-form A on M and an arbitrary tangent vector $X_x \in \mathrm{T}_x M$ at some point $x \in M$. Any tangent vector can be extended to a translation-invariant vector field on M: $X_y := T_{y-x*}X_x$, where T_{y-x} is one of the translations acting on M with $x \mapsto y$. With this vector field one has for any section $s = \mathfrak{s}_\omega f \in \mathcal{E}(\omega)$ that $\mathfrak{f}_\omega(\nabla_X s)(x) = X_x(f) + A_x(X)f(x)$ and

$$\mathfrak{f}_\omega(\check{\nabla}_X s)(x) = \mathfrak{f}_\omega(\nabla_{\check{X}}\check{s})(\check{x}) = \check{X}_{\check{x}}(\check{f}) + A_{\check{x}}(\check{X})\check{f}(\check{x}) = X_x(f) + A_{-x}(-X)f(x).$$

Therefore, the connection $\mathrm{d} + A$ is $\mathbb{Z}_2$-invariant if and only if $A_x(X) = A_{-x}(-X)$. By Definition 3.10 and Definition 3.19 one has $A^{\mathrm{D}}_x(X) = A^{\mathrm{D}}_{-x}(-X)$, which implies that the distinguished connection is $\mathbb{Z}_2$-invariant. Thus, a connection $\nabla^{\mathrm{D}} + a$ for some harmonic imaginary-valued 1-form a is $\mathbb{Z}_2$-invariant if and only if $a_x(X) = a_{-x}(-X) = a_x(-X)$, because every harmonic 1-form on a torus M is translation-invariant. Hence, $\nabla^{\mathrm{D}} + a$ is $\mathbb{Z}_2$-invariant if and only if $a = 0$.

To prove the second claim note first that $(\nabla^{\mathrm{D}} + a)\check{}_X s = \nabla^{\mathrm{D}}_{\check{X}} s + a(\check{X})s$. This implies with $a(\check{X}) = -a(X)$ that a connection $\nabla^{\mathrm{D}} + a$ is weakly $\mathbb{Z}_2$-invariant if and only if there is an Hermitian bundle automorphism $F\colon \omega \to \omega$ with

$$F^{-1} \circ (\nabla^{\mathrm{D}} + a) \circ F = \nabla^{\mathrm{D}} - a\,.$$

Every Hermitian bundle automorphism on ω has the form

$$F[x, z] = [x, \mathrm{e}^{2\pi \mathrm{i}(\alpha(x)+h(x))} \cdot z]$$

for all elements $[x, z]$ in the total space L_ω of ω, see Definition 3.10. Here $\alpha \in \mathcal{L}'$ and h is an $\mathcal{L}$-periodic function on $\mathbb{R}^n$ and

$$F^{-1} \circ (\nabla^{\mathrm{D}} + a) \circ Fs = F^{-1}\mathfrak{s}_\omega\Big(\mathrm{d}(\mathrm{e}^{2\pi\mathrm{i}(\alpha+h)} f) + (A^{\mathrm{D}} + a)\mathrm{e}^{2\pi\mathrm{i}(\alpha+h)} f\Big) = \\ \mathfrak{s}_\omega\Big(\mathrm{d}f + 2\pi\mathrm{i}(\mathrm{d}\alpha + \mathrm{d}h)f + (A^{\mathrm{D}} + a)f\Big)\,,$$

where $f = \mathfrak{f}_\omega s$. Since the exterior differential of the function $\alpha \in C^\infty(\mathbb{R}^n)$ is equal to the harmonic 1-form α, $\mathrm{d}\alpha = \alpha$, the automorphism F intertwines the connection $\nabla^{\mathrm{D}} + a$ with the connection $\nabla^{\mathrm{D}} + a + 2\pi\mathrm{i}(\alpha + \mathrm{d}h)$. It follows that $\nabla^{\mathrm{D}} + a$ is weakly $\mathbb{Z}_2$-invariant if there are an $\alpha \in \mathcal{L}'$ and an $\mathcal{L}$-periodic function $h \in C^\infty(\mathbb{R}^n)$ with $a + 2\pi\mathrm{i}(\alpha + \mathrm{d}h) = -a$. By Hodge decomposition $\mathrm{d}h$ must vanish and therefore $\nabla^{\mathrm{D}} + a$ is weakly $\mathbb{Z}_2$-invariant if and only if $a = -\pi\mathrm{i}\alpha \in \pi\mathrm{i}\mathcal{L}'$. □

Remark 4.12. Note that the distinguished connection ∇^{D} is the unique (up to gauge equivalence) translation- and $\mathbb{Z}_2$-invariant connection on ω. However, there are 2^n different gauge equivalence classes of translation- and weakly $\mathbb{Z}_2$-invariant connections.

5 The Laplacian and Spectra

In this section I will construct the Laplacian of a connection, introduce potentials and define their spectra.

Remark 5.1. Let ∇^{LC} denote the Levi-Civita connection over the flat torus $(M, \langle . , . \rangle_{\mathcal{L}})$. Generally, a connection on a vector bundle ξ over M may be seen as a map

$$\nabla^{\xi} : \mathcal{E}(\xi) \to \mathcal{E}(\mathrm{T}^{*}M \otimes \xi)\,.$$

With the help of the Levi-Civita connection one can construct a connection $\nabla^{\mathrm{T}^{*}M}$ on the cotangent bundle $\mathrm{T}^{*}M$ via

$$\left(\nabla_X^{\mathrm{T}^{*}M}\mu\right)(Y) := X.\mu(Y) - \mu(\nabla_X^{\mathrm{LC}}Y)$$

and a connection $\nabla^{\mathrm{T}^{*}M\otimes\xi}$ on the product bundle $\mathrm{T}^{*}M \otimes \xi$ by setting

$$\nabla_X^{\mathrm{T}^{*}M\otimes\xi}(\mu\otimes\eta) := (\nabla_X^{\mathrm{T}^{*}M}\mu)\otimes\eta + \mu\otimes(\nabla_X^{\xi}\eta)\,.$$

With those connections one has a "second derivative"

$$\nabla^2 = \nabla^{\mathrm{T}^{*}M\otimes\xi}\circ\nabla^{\xi}\,, \qquad \text{which satisfies}$$

$$\left(\nabla^{\mathrm{T}^{*}M\otimes\xi}\circ\nabla^{\xi}s\right)(X,Y) = \nabla_X^{\xi}\circ\nabla_Y^{\xi}s - \nabla^{\xi}_{\nabla_X^{\mathrm{LC}}Y}s$$

for all $X, Y \in \mathcal{X}(M)$ and every section $s \in \mathcal{E}(\xi)$.

Proof. If $\{X_i\}$ is a local frame of $\mathrm{T}M$ and $\{\omega_i\}$ its dual frame, then $\nabla^{\xi}s = \sum_i \omega_i \otimes \nabla^{\xi}_{X_i}s$. The product rule for the covariant derivative on tensor products yields

$$\nabla^{\mathrm{T}^{*}M\otimes\xi}\circ\nabla^{\xi}s = \sum_{i=1}^{n}\left(\left(\nabla^{\mathrm{T}^{*}M}\omega_i\right)\otimes\nabla^{\xi}_{X_i}s + \omega_i\otimes\nabla^{\xi}\left(\nabla^{\xi}_{X_i}s\right)\right) \qquad \text{and thus}$$

$$\begin{aligned}\left(\nabla^{\mathrm{T}^{*}M\otimes\xi}\circ\nabla^{\xi}s\right)(X_k, X_l) &= \sum_{i=1}^{n} -\omega_i\left(\nabla^{\mathrm{LC}}_{X_k}X_l\right)\nabla^{\xi}_{X_i}s + \nabla^{\xi}_{X_k}\circ\nabla^{\xi}_{X_l}s \\ &= \nabla^{\xi}_{X_k}\circ\nabla^{\xi}_{X_l}s - \sum_{i=1}^{n}\left(\omega_i\otimes\nabla^{\xi}_{X_i}s\right)\left(\nabla^{\mathrm{LC}}_{X_k}X_l\right) = \nabla^{\xi}_{X_k}\circ\nabla^{\xi}_{X_l}s - \nabla^{\xi}_{\nabla^{\mathrm{LC}}_{X_k}X_l}s\end{aligned}$$

Both sides of this equation are $C^{\infty}(M)$-linear in X_k and in X_l. Hence, this equation holds not only on the local frame $\{X_i\}$ but for all $X, Y \in \mathcal{X}(M)$. □

Remark 5.2. Given a connection ∇ on the line bundle ξ let

$$D_X^2 := \left(\nabla^2(X_i, X_j)\right)_{i,j=1\cdots n}$$

be the matrix of ∇^2 with respect to the local frame $X = \{X_i\}$. The entries of this matrix are the maps

$$\nabla^{\xi}_{X_k}\circ\nabla^{\xi}_{X_l} - \nabla^{\xi}_{\nabla^{\mathrm{LC}}_{X_k}X_l} : \mathcal{E}(\xi) \to \mathcal{E}(\xi)\,.$$

If $Y = \{Y_i\}$ is another local frame with transition matrix A, i.e. A is a matrix-valued function on M with $X_x = A_x Y_x$ for all $x \in M$, then the $C^\infty(M)$-bilinearity of the map $(X, Y) \mapsto \nabla^2(X, Y)$ yields $D_X^2 = AD_Y^2 A^T$. In particular, $\operatorname{trace} D_X^2 = \operatorname{trace} D_Y^2 A^T A$ and thus there is a well-defined trace of ∇^2 for all $\mathrm{O}(n)$-classes of frames.

Definition 5.3. Every connection ∇^λ on a line bundle λ over a flat torus M yields a *Laplacian* Δ acting on the sections of the line bundle λ. The Laplacian

$$\Delta\colon \mathcal{E}(\lambda) \to \mathcal{E}(\lambda) \qquad \text{is defined by} \qquad \Delta := -\operatorname{trace} \nabla^2$$

with respect to the class of orthonormal frames. If X is such an orthonormal frame, then the Laplacian satisfies

$$\Delta = -\sum_{i=1}^{n} \Big(\nabla^\lambda_{X_i} \circ \nabla^\lambda_{X_i} - \nabla^\lambda_{\nabla^{\mathrm{LC}}_{X_i} X_i} \Big).$$

If one additionally assumes that the frame field is translation-invariant, then $\nabla^{\mathrm{LC}}_{X_i} X_i = 0$, since $(M, \langle .\,, . \rangle_{\mathcal{L}})$ is flat, and thus

$$\Delta = -\sum_{i=1}^{n} \nabla^\lambda_{X_i} \circ \nabla^\lambda_{X_i}\,.$$

Remark 5.4. The definition of the Laplacian via orthonormal frames will be the reason for considering only rectangular tori and only line bundles with rectangular Chern basis over those tori in the section about negative inverse spectral results.

Definition 5.5. Every Hermitian connection ∇ on the line bundle ω constructed in Definition 3.10 can be pulled back to a connection $\mathrm{d} + A$ on the trivial bundle θ^1 with an imaginary-valued 1-form A. The corresponding Laplacian on ω shall be denoted by Δ_A. If $A = A^{\mathrm{D}} + a$, where A^{D} is the form of the distinguished connection and a a harmonic 1-form (see Definition 3.19 and Remark 4.1), abbreviate $\Delta_a^{\mathrm{D}} := \Delta_{A^{\mathrm{D}}+a}$.

Remark 5.6. The notions of Laplacians on ω and θ^1 are compatible in the following sense: If one pulls back a connection ∇ on ω to the trivial bundle θ^1, then one obtains a Laplacian acting on $\mathcal{E}(\theta^1)$ by setting $\Delta^{\theta^1} := -\operatorname{trace} \nabla^2 = -\sum_{i=1}^{n} \nabla_{X_i} \circ \nabla_{X_i}$, where $\{X_i\}$ is a translation-invariant orthonormal frame field. Since the connection comes from ω and the X_i are translation-invariant, ∇ commutes with the action $\lrcorner$ of $\mathcal{L}$ on the sections $\mathcal{E}(\theta^1)$. Thus, Δ^{θ^1} commutes with that action and descends to a Laplacian on ω, namely Δ.

The following proposition illustrates how the Laplacians of connections differ from the usual Laplacian acting on functions.

Proposition 5.7. For every Hermitian connection $\nabla = \mathrm{d} + A$ on the line bundle ω over the flat torus $(M, \langle .\,, . \rangle_{\mathcal{L}})$ one has on $C^\infty(\mathbb{R}^n, \mathbb{C})^{\mathcal{L}}$

$$\mathfrak{f}_\omega \circ \Delta_A \circ \mathfrak{s}_\omega = -\Big(\operatorname{div} \circ \operatorname{grad} + \operatorname{div}(A^\#) + 2A \circ \operatorname{grad} - \|A\|^2\Big),$$

where the index-raising musical isomorphism $\#\colon \mathrm{T}^*M \to \mathrm{T}M$ is the inverse of the duality isomorphism $\flat\colon \mathrm{T}M \ni X \mapsto \langle X, .\rangle \in \mathrm{T}^*M$.

Proof. By definition one has for every $f \in C^\infty(\mathbb{R}^n, \mathbb{C})^{\mathcal{L}}$ and $X \in \mathcal{X}(M)$

$$-\nabla_X \circ \nabla_X \mathfrak{s}_\omega f = -\nabla_X \mathfrak{s}_\omega\big(X(f) + A(X)f\big) = \\ -\mathfrak{s}_\omega\Big(X\big(\mathrm{d}f(X)\big) + X\big(A(X)\big)f + 2A(X)X(f) + A(X)^2 f\Big),$$

where both A and X are pulled back to $\mathbb{R}^n$. By evaluating the four summands one obtains the desired formula:

Note that the gradient is defined via $\langle \operatorname{grad} f, X\rangle_{\mathcal{L}} = X(f)$, which means $\operatorname{grad} f = (\mathrm{d}f)^\#$. Thus, for the first two terms it suffices to show that $\operatorname{div} Y = \sum_i X_i\big(Y^\flat(X_i)\big)$ for any vector field Y and any translation-invariant orthonormal frame $\{X_i\}$.

The divergence is defined by $(\operatorname{div} Y)\mathrm{d}V = \mathrm{d}(Y \lrcorner\, \mathrm{d}V)$ and $\mathrm{d}V = X_1^\flat \wedge \cdots \wedge X_n^\flat$. With $Y = \sum_i Y^i X_i$

$$Y \lrcorner\, \mathrm{d}V = \sum_{i=1}^n Y^i (-1)^{i-1} X_1^\flat \wedge \cdots \wedge \widehat{X_i^\flat} \wedge \cdots \wedge X_n^\flat\,.$$

Since the $\{X_i\}$ are translation-invariant, $[X_i, X_j] = 0$ and hence

$$\operatorname{div} Y = \mathrm{d}(Y \lrcorner\, \mathrm{d}V)(X_1, \ldots, X_n) \\ = \sum_{i,l=1}^n (-1)^{i+l} X_l\Big(Y^i X_1^\flat \wedge \cdots \wedge \widehat{X_i^\flat} \wedge \cdots \wedge X_n^\flat\,(X_1, \ldots, \widehat{X_l}, \ldots, X_n)\Big) = \sum_{i=1}^n X_i\big(Y^\flat(X_i)\big)$$

as $Y^\flat(X_i) = \langle Y, X_i\rangle = Y^i$. The third summand is given by

$$2\sum_i A(X_i)X_i(f) = 2A\Big(\sum_i \langle \operatorname{grad} f, X_i\rangle_{\mathcal{L}} X_i\Big) = 2A \circ \operatorname{grad} f\,.$$

Since A is imaginary-valued and satisfies $A = \sum_i A(X_i)X_i{}^\flat$, one has for the fourth summand that $\sum_i A(X_i)^2 = -\|A\|^2$. □

Definition 5.8. A *potential* is a real-valued function $Q \in C^\infty(M)$. For $a \in \mathrm{i}(\mathbb{R}^n)'$ let $\Delta_a^{\mathrm{D}} + Q$ denote the *Schrödinger operator* and define the *spectrum* $\operatorname{Spec}_a(Q, \omega)$ of the translation-invariant connection $\nabla^{\mathrm{D}} + a$ on ω and the potential Q as the set of eigenvalues with multiplicities of the Schrödinger operator $\Delta_a^{\mathrm{D}} + Q$ acting on smooth sections $\mathcal{E}(\omega)$ of the line bundle ω. Here, an eigenvalue λ is a complex number with the property that there is a nonvanishing eigensection $s \in \mathcal{E}(\omega)$ with $(\Delta_a^{\mathrm{D}} + Q)s = \lambda s$.

The map that assigns the spectrum of the corresponding Schrödinger operator to each translation-invariant connection

$$(2\pi\mathrm{i}\mathcal{L}')\backslash \mathrm{i}(\mathbb{R}^n)' \ni [a] \mapsto \operatorname{Spec}_a(Q, \omega)\,,$$

is called the *ω-Bloch spectrum* of Q.

Proof. The ω-Bloch spectrum is well-defined because Hermitian translation-invariant connections differing by an element of $2\pi\mathrm{i}\mathcal{L}'$ are gauge equivalent and gauge equivalent connections on ω have the same spectrum: If one is given two connections intertwined by an ω-automorphism F, $\nabla_1 \circ F = F \circ \nabla_2$, and an eigenvalue $\lambda \in \mathbb{C}$ of the Schrödinger operator belonging to ∇_2 with eigensection $s \in \mathcal{E}(\omega)$, $(\Delta_2 + Q)s = \lambda s$, then $F \circ s$ is an eigensection with eigenvalue λ of the Schrödinger operator belonging to ∇_1. □

Remark 5.9. Every Schrödinger operator $\Delta_a^{\mathrm{D}} + Q$ is symmetric with respect to the inner products on the fibres of ω because every connection $\nabla^{\mathrm{D}} + a$ is Hermitian and every potential Q is real-valued. Therefore, every eigenvalue λ is actually a real number.

The ω-Bloch spectrum contains the spectra of all translation-invariant connections since gauge equivalent connections have the same spectrum. Before constructing transplantations the relation to the classical Bloch spectrum shall be displayed.

Proposition 5.10. For every $a \in \mathrm{i}(\mathbb{R}^n)'$ and every potential Q on M the spectrum $\mathrm{Spec}_a(Q, \omega)$ coincides with the spectrum of the operator $\Delta_0^{\mathrm{D}} + Q$ acting on the space of all smooth sections $s \in \mathcal{E}(\theta^1)$ of the trivial bundle θ^1 satisfying

$$\mathfrak{s}(x + l) = \mathrm{e}^{a(l)} e_l(x)\mathfrak{s}(x) \qquad \text{for all } x \in M \text{ and } l \in \mathcal{L}.$$

Proof. In analogy to the definition of the line bundle ω construct a new line bundle ω_a with the total space $L_a := \mathbb{R}^n \times \mathbb{C}/{\sim_a}$, where the equivalence relation is given by $(x, z) \sim_a (x + l, \mathrm{e}^{a(l)} e_l(x)z)$ for $l \in \mathcal{L}$. Sections of ω_a can be considered as sections of the trivial bundle satisfying the above condition.

Again, there is a bundle automorphism $F\colon \theta^1 \to \theta^1$ given by $\mathbb{R}^n \times \mathbb{C} \ni (x, z) \mapsto (x, \mathrm{e}^{a(x)} z) \in \mathbb{R}^n \times \mathbb{C}$, which intertwines the connections $\nabla^{\mathrm{D}} + a$ and ∇^{D}. This automorphism of θ^1 induces an isomorphism between ω and ω_a, which intertwines the connection $\nabla^{\mathrm{D}} + a$ on ω with the connection ∇^{D} on ω_a. (Note that ∇^{D} does indeed descend to a connection on ω_a.)

Thus, this isomorphism also intertwines the Schrödinger operator belonging to $\nabla^{\mathrm{D}} + a$ and Q on ω with the operator $\Delta_0^{\mathrm{D}} + Q$ acting on the specified set of sections of θ^1. □

Example 5.11. The Chern class of the trivial bundle $\pi\colon M \times \mathbb{C} \to M$ over M is 0 and thus its harmonic representative also vanishes. Hence, $\Omega = 0$ and, arguing analogously to the case with a nondegenerate Ω, $w_l = 0$ for all $l \in \mathcal{L}$. The distinguished connection is given by $\nabla^{\mathrm{D}} = \mathrm{d}$ and the Laplacian is just the Euclidean Laplacian defined by the Euclidean metric. The spectrum $\mathrm{Spec}_a(Q)$ is the spectrum of the Schrödinger operator $\Delta + Q$ acting on smooth functions $f \in C^\infty(\mathbb{R}^n, \mathbb{C})$ with $f(x + l) = \mathrm{e}^{a(l)} f(x)$. Thus, the definition of the Bloch spectrum agrees in this case with the classical definition of the Bloch spectrum.

6 Transplantations

Definition 6.1 (Transplantation).
For two sets of functions C and $\tilde{C}$ with differential operators H and $\tilde{H}$ a *transplantation* from (C, H) to $(\tilde{C}, \tilde{H})$ is a linear isomorphism $\Psi\colon C \to \tilde{C}$ which maps eigenfunctions of H to eigenfunctions of $\tilde{H}$ with the same eigenvalue, thereby establishing bijections between all eigenspaces of H and $\tilde{H}$. A transplantation between two manifolds is a transplantation between their sets of smooth functions and two given differential operators acting on those. The first transplantations were constructed by Peter Buser in [Bus86].

Note that if such a transplantation exists, then the two sets with their differential operators are *isospectral*, i. e. they have equal spectra including multiplicities. Two potentials Q_1 and Q_2 are called *isospectral* with respect to some Laplacian Δ if the corresponding Schrödinger operators $\Delta + Q_1$ and $\Delta + Q_2$ are isospectral.

The aim of this and the next section is to find distinct yet isospectral tori, potentials, line bundles and Laplacians. To this end the concepts of the previous sections will be used to construct a transplantation and in the next section this transplantation will be applied to various examples to give the promised negative results (see section "Results and Overview").

Example 6.2. With the notation of Definition 4.10 one has for any $a \in \mathrm{i}(\mathbb{R}^n)'$ that the map $s \mapsto \check{s}$ is a transplantation from $(\mathcal{E}(\omega), \Delta^{\mathrm{D}}_a + Q)$ to $(\mathcal{E}(\omega), \Delta^{\mathrm{D}}_{-a} + \check{Q})$. Thus, $\mathrm{Spec}_a(Q, \omega) = \mathrm{Spec}_{-a}(\check{Q}, \omega)$. In particular, Q and $\check{Q}$ are isospectral potentials with respect to the Laplacian Δ^{D}_0 on ω.

Proof. Let s be an eigensection of $\Delta^{\mathrm{D}}_a + Q$ with eigenvalue λ. Then

$$\begin{aligned}\mathfrak{f}_\omega \nabla^{A^{\mathrm{D}}-a}_X \check{s}(x) &= X_x(\mathfrak{f}_\omega \check{s}) + \big(A^{\mathrm{D}}_x(X) - a(X)\big)\mathfrak{f}_\omega \check{s}(x) \\ &= -X_{-x}(\mathfrak{f}_\omega s) + \big(A^{\mathrm{D}}_{-x}(-X) + a(-X)\big)\mathfrak{f}_\omega s(-x) = \big(\mathfrak{f}_\omega \nabla^{A^{\mathrm{D}}+a}_{-X} s\big)\check{}\,(x)\end{aligned}$$

gives for any translation-invariant orthonormal frame $\{X_i\}$ that

$$\begin{aligned}\big(\Delta^{\mathrm{D}}_{-a} + \check{Q}\big)\check{s} &= -\sum_{i=1}^{n} \nabla^{A^{\mathrm{D}}-a}_{X_i} \circ \nabla^{A^{\mathrm{D}}-a}_{X_i} \check{s} + \check{Q}\check{s} \\ &= \Big(-\sum_{i=1}^{n} \nabla^{A^{\mathrm{D}}+a}_{-X_i} \circ \nabla^{A^{\mathrm{D}}+a}_{-X_i} s + Qs\Big)\check{} = \big((\Delta^{\mathrm{D}}_a + Q)s\big)\check{} = \lambda \check{s}\,,\end{aligned}$$

since $\{-X_i\}$ is also a translation-invariant orthonormal frame. □

In this section assume that an even integer $n = 2m$ and a *rectangular lattice* $\mathcal{L} \subset \mathbb{R}^n$, i. e. a lattice with an orthogonal basis, are given. Let $M = \mathcal{L}\backslash\mathbb{R}^n$ denote the corresponding rectangular flat torus and fix a nondegenerate Hermitian line bundle ω over M such that the harmonic representative of its Chern class has an orthogonal Chern basis $\{U_1, \dots, U_m, V_1, \dots, V_m\}$ of $\mathcal{L}$. Call such line bundles *rectangular*.

Let $(u,v) = (u^1,\dots,u^m, v^1,\dots,v^m)$ denote the coordinates corresponding to this basis and let $r = (r_1,\dots,r_m) \in \mathbb{N}^m$ denote the tuple with $r_1 \mid \cdots \mid r_m$ such that $\Omega = \sum_i r_i \,\mathrm{d}u^i \wedge \mathrm{d}v^i$.

Finally, assume that a harmonic imaginary 1-form a on M is given. It can be viewed as a linear functional on $\mathbb{R}^{2m}$ and can be expressed as $a(u,v) = 2\pi\mathrm{i}\sum_i(\mu_i u^i + \nu_i v^i)$ with $\mu,\nu \in \mathbb{R}^m$.

Definition 6.3. Let $\mathfrak{U} := \{U_1,\dots,U_m\}$ and $\mathfrak{V} := \{V_1,\dots,V_m\}$. Denote by $\mathbb{R}\mathfrak{U}$ and $\mathbb{R}\mathfrak{V}$ the two m-dimensional subspaces of $\mathbb{R}^n$ spanned by $\mathfrak{U}$ and $\mathfrak{V}$. Analogously, define two lattices $\mathcal{L}_\mathfrak{U} := \mathbb{Z}\mathfrak{U}$ and $\mathcal{L}_\mathfrak{V} := \mathbb{Z}\mathfrak{V}$ in those two subspaces and define two unit cells $U := [0,1]\mathfrak{U}$ and $V := [0,1]\mathfrak{V}$.

With those definitions one can consider μ and ν as linear forms on $\mathbb{R}\mathfrak{U}$ and $\mathbb{R}\mathfrak{V}$ such that $a(u,v) = 2\pi\mathrm{i}\big(\mu(u)+\nu(v)\big)$. One also has $\mathbb{R}^n = \mathbb{R}\mathfrak{U} \oplus \mathbb{R}\mathfrak{V}$, $\mathcal{L} = \mathcal{L}_\mathfrak{U} \oplus \mathcal{L}_\mathfrak{V}$ and the unit cell of $\mathcal{L}$ with respect to the basis $\mathfrak{U} \cup \mathfrak{V}$ is $U \times V$.

The transplantation constructed in this section depends on the calculation of Fourier coefficients, which are coefficients of vectors in a Hilbert space with respect to a Hilbert space basis. Thus, it is necessary to extend the sets of smooth functions considered until now to L^2-spaces.

Definition 6.4. Let $\mathrm{L}^2(\mathbb{R}^n,\mathbb{C})^\mathcal{L}$ be the completion of the set of $\mathcal{L}$-invariant smooth functions $C^\infty(\mathbb{R}^n,\mathbb{C})^\mathcal{L}$ with respect to the norm

$$\|f\|^2_{\mathrm{L}^2(\mathbb{R}^n,\mathbb{C})^\mathcal{L}} := \int_{U\times V} |f(u,v)|^2 \,\mathrm{d}u\mathrm{d}v$$

where $\mathrm{d}u := \mathrm{d}u^1\cdots\mathrm{d}u^m$ and $\mathrm{d}v := \mathrm{d}v^1\cdots\mathrm{d}v^m$ are given by coordinates corresponding to the Chern basis $\mathfrak{U}\cup\mathfrak{V}$ of $\mathcal{L}$. This norm can be used to define a norm on the smooth $\mathcal{E}(\omega)$ by demanding that

$$\mathfrak{f}_\omega \colon \mathcal{E}(\omega) \to C^\infty(\mathbb{R}^n,\mathbb{C})^\mathcal{L}$$

and its inverse $\mathfrak{s}_\omega$ are unitary isomorphisms. This implies that

$$\|s\|^2_{\mathrm{L}^2(\omega)} = \int_{U\times V} |\mathfrak{f}_\omega s|^2(u,v)\,\mathrm{d}u\mathrm{d}v = \int_{U\times V} \|s\|^2_\omega(u,v)\,\mathrm{d}u\mathrm{d}v \quad \text{for all } s \in \mathcal{E}(\omega),$$

where $\|s\|^2_\omega(u,v) = \langle s[u,v], s[u,v]\rangle_\omega$ is the norm on the fibres of ω and $\mathrm{L}^2(\omega)$ the completion of $\mathcal{E}(\omega)$ with respect to $\|.\|_{\mathrm{L}^2(\omega)}$. $\mathrm{L}^2(\omega)$ can be regarded as the set of measurable sections s of ω for which the norm $\|s\|_{\mathrm{L}^2(\omega)}$ is finite and $\mathfrak{f}_\omega$ can be extended to a unitary isomorphism $\mathrm{L}^2(\omega) \to \mathrm{L}^2(\mathbb{R}^n,\mathbb{C})^\mathcal{L}$.

Finally, let $\mathrm{L}^2(\mathbb{R}\mathfrak{U},\mathbb{C})$ denote the completion of the set $C^\infty(\mathbb{R}\mathfrak{U},\mathbb{C})$ by the norm

$$\|f\|^2_{\mathrm{L}^2(\mathbb{R}\mathfrak{U},\mathbb{C})} := \int_{\mathbb{R}\mathfrak{U}} |f(u)|^2\,\mathrm{d}u\,.$$

Definition 6.5. Define a map $F\colon \mathbb{R}^n \to (\mathbb{R}^n)'$ by $x \mapsto \Omega(x,.)$ and write $x' := F(x)$ for $x \in \mathbb{R}^n$. This map is a linear isomorphism, denote its inverse by G. The restriction of F to $\mathcal{L}_\mathfrak{U}$ is an injective map from $\mathcal{L}_\mathfrak{U}$ into $\mathcal{L}_\mathfrak{V}{}'$, where $\mathcal{L}_\mathfrak{V}{}'$ is the subset of the dual lattice $\mathcal{L}'$ containing the linear functionals vanishing on $\mathfrak{U}$.

Using the decomposition of $\mathbb{R}^n$ into $\mathbb{R}\mathfrak{U} \oplus \mathbb{R}\mathfrak{V}$ one has $u' = w_u$ for $u \in \mathbb{R}\mathfrak{U}$. Also, $G(\mathrm{d}v^i) = U_i/r_i$ and $G(\mathrm{d}u^i) = -V_i/r_i$. In particular, $F\colon \mathcal{L}_\mathfrak{U} \to \mathcal{L}_\mathfrak{V}'$ is surjective if and only if $r = (1, \ldots, 1)$.

Proof. The linearity follows from the bilinearity of Ω and the injectivity from the nondegeneracy of Ω. Since $\mathbb{R}^n$ and its dual space have the same dimension, F must be an isomorphism. Because of $\Omega(\mathcal{L} \times \mathcal{L}) \subset \mathbb{Z}$, one has $F(\mathcal{L}_\mathfrak{U}) \subset \mathcal{L}_\mathfrak{V}'$. Also, for $u \in \mathbb{R}\mathfrak{U}$ and $v \in \mathbb{R}\mathfrak{V}$ one has $F(u)(v) = \Omega(u, v) = w_u(v) - w_v(u)$ and $w_v = 0$. It follows that $F(u) = w_u$. □

Definition 6.6. Define an operator $D_\mathfrak{U}$ on the dense subspace of smooth functions in $\mathrm{L}^2(\mathbb{R}\mathfrak{U}, \mathbb{C})$ by

$$D_\mathfrak{U} := \sum_{i=1}^{m} \Big((2\pi r_i u^i / b_i)^2 - \frac{1}{a_i{}^2} \frac{\partial^2}{\partial u^{i2}} \Big),$$

where $a_i := \|U_i\|$ and $b_i := \|V_i\|$ denote the lengths of the basis vectors. For a system of representatives S of the finite quotient space $F(\mathcal{L}_\mathfrak{U}) \backslash \mathcal{L}_\mathfrak{V}'$ define a Hilbert space

$$\mathcal{H} := \bigoplus_{c \in S} \mathrm{L}^2(\mathbb{R}\mathfrak{U}, \mathbb{C}),$$

where the inner product is induced by the product on $\mathrm{L}^2(\mathbb{R}\mathfrak{U}, \mathbb{C})$. With an $\mathcal{L}_\mathfrak{U}$-periodic function $p \in C^\infty(\mathbb{R}\mathfrak{U})$ and $a(u, v) = 2\pi\mathrm{i}(\mu(u) + \nu(v))$ let

$$D_{\mathfrak{U},p} := (D_\mathfrak{U} + p^c)_{c \in S} \qquad \text{with} \qquad p^c := p\Big(\,.\, + G(\nu - c)\Big)$$

act componentwise on the tuples of smooth functions in $\mathcal{H}$.

Remark 6.7. By Definition 6.3 $\nu \in (\mathbb{R}\mathfrak{V})'$ and with Definition 6.5 one has $G(\nu) \in \mathbb{R}\mathfrak{U}$. Also, $\mathcal{H}$ and $D_{\mathfrak{U},p}$ are independent of the choice of systems of representatives: If R is another set of representatives of $F(\mathcal{L}_\mathfrak{U}) \backslash \mathcal{L}_\mathfrak{V}'$, then $\bigoplus_{c \in R} \mathrm{L}^2(\mathbb{R}\mathfrak{U}, \mathbb{C})$ is canonically isomorphic to $\mathcal{H}$ and the operator $D_{\mathfrak{U},p}$ stays the same because $G(c) \in \mathcal{L}_\mathfrak{U}$ for $c \in F(\mathcal{L}_\mathfrak{U})$ and p is $\mathcal{L}_\mathfrak{U}$-periodic.

The function p can be extended to a function $(u, v) \mapsto p(u)$ on $\mathbb{R}\mathfrak{U} \oplus \mathbb{R}\mathfrak{V} = \mathbb{R}^n$, which descends to a function on M. This function shall also be called p. It will now be shown that there is a transplantation from $(\mathrm{L}^2(\omega), \Delta_a^\mathrm{D} + p)$ to $(\mathcal{H}, D_{\mathfrak{U},p})$. Since the latter space and operator are simpler than the Laplacian, one can find other tori, Laplacians and potentials such that the associated $(\tilde{\mathcal{H}}, \tilde{D}_{\tilde{\mathfrak{U}},\tilde{p}})$ is equal or isospectral to $(\mathcal{H}, D_{\mathfrak{U},p})$.

Note that the definition $q_{r,c,\mu}(s) := q(s_1 + \mu_1 - c_1/r_1, \ldots, s_m + \mu_m - c_m/r_m)$ in [GGKW08, Notation 3.3 (ii)] is not correct. Definition 6.6 implies that one should have $q_{r,c,\mu}(s) := q(s_1 + (\mu_1 - c_1)/r_1, \ldots, s_m + (\mu_m - c_m)/r_m)$.

Definition 6.8. For any $c \in (\mathbb{R}^n)'$ let $E_c := \mathrm{e}^{-2\pi\mathrm{i}c}$. If $c \in \mathcal{L}'$, then E_c is called a trigonometric monomial and descends to a function on the torus $\mathcal{L} \backslash \mathbb{R}^n = M$.

Remark 6.9. The trigonometric monomials $\{E_c\}_{c\in\mathcal{L}_{\mathfrak{V}}'}$ form a complete orthonormal system in the Hilbert space $\mathrm{L}^2(V,\mathbb{C})$ of $\mathcal{L}_{\mathfrak{V}}$-periodic square-integrable functions on $\mathbb{R}\mathfrak{V}$, where the inner product is given by

$$\langle f,g\rangle_{\mathrm{L}^2(V,\mathbb{C})} := \int_V \big(f\cdot\overline{g}\big)(v)\,\mathrm{d}v \qquad \text{with a unit cell } V := [0,1]\mathfrak{V}.$$

Confer [War83, 6.16].

Lemma 6.10. Given a function $f \in \mathrm{L}^2(\mathbb{R}^n,\mathbb{C})^{\mathcal{L}}$ and any $u \in \mathbb{R}\mathfrak{U}$ let $f_c(u) := \langle f(u,.),E_c\rangle_{\mathrm{L}^2(V,\mathbb{C})}$ denote the Fourier coefficients of $f(u,.)$ with respect to the Hilbert space basis $\{E_c\}_{c\in\mathcal{L}_{\mathfrak{V}}'}$. Those Fourier coefficients are well-defined and satisfy (with $k' := \Omega(k,.) = Fk$)

$$f_{c+k'}(u) = f_c(u+k) \qquad \text{for all } k \in \mathcal{L}_{\mathfrak{U}}.$$

Proof. Let $(k,l) \in \mathcal{L}_{\mathfrak{U}} \oplus \mathcal{L}_{\mathfrak{V}}$ and recall that any $\mathcal{L}$-invariant function f satisfies $f(u+k,v+l) = e_{(k,l)}(u,v)f(u,v) = E_{k'}(-v)f(u,v)$. In particular, every square-integrable $\mathcal{L}$-invariant function f on $\mathbb{R}^n$ is $\mathcal{L}_{\mathfrak{V}}$-periodic and has well-defined Fourier coefficients. Also:

$$f_{c+k'}(u) = \int_V f(u,v)E_{c+k'}(-v)\,\mathrm{d}v = \int_V f(u+k,v)E_c(-v)\,\mathrm{d}v = f_c(u+k)\,. \qquad \square$$

This lemma shows that the Fourier coefficients with respect to dual vectors of the same class in $F(\mathcal{L}_{\mathfrak{U}})\backslash\mathcal{L}_{\mathfrak{V}}'$ contain the same information. This is used to construct the first part of the transplantation to $\mathcal{H}$:

Lemma 6.11. The map

$$\mathcal{F}_S\colon \mathrm{L}^2(\mathbb{R}^n,\mathbb{C})^{\mathcal{L}} \to \mathcal{H} \qquad \text{with} \qquad f \mapsto (f_c)_{c\in S}$$

is a well-defined unitary isomorphism. It maps smooth maps to tuples of smooth maps. Its inverse is given by

$$\big(\mathcal{F}_S^{-1}(f_c)_{c\in S}\big)(u,v) := \sum_{c\in S}\sum_{k\in\mathcal{L}_{\mathfrak{U}}} f_c(u+k)\cdot E_{c+k'}(v)\,.$$

Proof. Lemma 6.10 shows that the formula gives the inverse of $\mathcal{F}_S$ if it is well-defined. If $f_c \in \mathrm{L}^2(\mathbb{R}\mathfrak{U},\mathbb{C})$, then $(f_c(u+k))_{k\in\mathcal{L}_{\mathfrak{U}}}$ must be a square-summable sequence for almost all $u \in \mathbb{R}\mathfrak{U}$. By Hilbert space theory the sums $\sum_{k\in\mathcal{L}_{\mathfrak{U}}} f_c(u+k)E_{c+k'}$ converge to $\mathcal{L}_{\mathfrak{V}}$-periodic functions on $\mathbb{R}\mathfrak{V}$ for almost all fixed $u \in \mathbb{R}\mathfrak{U}$.

Thus, the formula gives an almost everywhere defined function $g := \mathcal{F}_S^{-1}(f_c)_{c\in S}$ on $\mathbb{R}^n$, which is $\mathcal{L}$-invariant since

$$\begin{aligned} g\big((u,v)+(k,l)\big) &= \sum_{c\in S}\sum_{h\in\mathcal{L}_{\mathfrak{U}}} f_c(u+k+h)\cdot E_{c+h'}(l+v) \\ &= \sum_{c\in S}\sum_{h\in\mathcal{L}_{\mathfrak{U}}} f_c(u+h)E_{c+h'-k'}(v) = E_{-k'}(v)g(u,v) = e_{(k,l)}(u,v)g(u,v)\,. \end{aligned}$$

The orthogonality of the $E_{c+k'}$ and $\int_V |E_{c+k'}(v)|^2 \,\mathrm{d}v = 1$ imply that

$$\|\mathcal{F}_S{}^{-1}(f_c)_{c\in S}\|^2_{\mathrm{L}^2(\mathbb{R}^n,\mathbb{C})} = \Big\|(u,v) \mapsto \sum_{c\in S}\sum_{k\in\mathcal{L}_\mathfrak{U}} f_c(u+k)E_{c+k'}(v)\Big\|^2_{\mathrm{L}^2(\mathbb{R}^n,\mathbb{C})^\mathcal{L}} =$$

$$\sum_{c\in S}\sum_{k\in\mathcal{L}_\mathfrak{U}} \int_U |f_c(u+k)|^2 \,\mathrm{d}u = \sum_{c\in S}\int_{\mathbb{R}\mathfrak{U}} |f_c(u)|^2 \,\mathrm{d}u = \sum_{c\in S} \|f_c\|^2_{\mathrm{L}^2(\mathbb{R}\mathfrak{U},\mathbb{C})}\,,$$

confer Definition 6.4. Since $\sum_{c\in S}\|f_c\|^2_{\mathrm{L}^2(\mathbb{R}\mathfrak{U},\mathbb{C})} = \|(f_c)_{c\in S}\|^2_\mathcal{H} < \infty$, it follows that $\mathcal{F}_S{}^{-1}$ is a unitary isomorphism. In particular, $\mathcal{F}_S{}^{-1}(f_c)_{c\in S}$ is a well-defined element of $\mathrm{L}^2(\mathbb{R}^n,\mathbb{C})^\mathcal{L}$ and $\mathcal{F}_S$ is also a unitary isomorphism. □

The above Lemma yields a unitary isomorphism

$$\mathcal{F}_S \circ \mathfrak{f}_\omega \colon \mathrm{L}^2(\omega) \to \mathcal{H}\,.$$

However, this map is not yet a transplantation between $\Delta^\mathrm{D}_a + p$ and $D_{\mathfrak{U},p}$. To obtain one, a small twist by the following unitary isomorphism is needed.

Definition 6.12. With $a(u,v) = 2\pi\mathrm{i}(\mu(u)+\nu(v))$ define a unitary isomorphism by

$$\mathcal{T}_S \colon \mathcal{H} \ni (f_c)_{c\in S} \mapsto \Big(E_{-\mu}\cdot f_c(.+G(\nu-c))\Big)_{c\in S} \in \mathcal{H}\,.$$

Theorem 6.13. Every $\mathcal{L}_\mathfrak{U}$-periodic function $p \in C^\infty(\mathbb{R}\mathfrak{U})$ descends to a potential on the torus M. If ω is a nondegenerate rectangular line bundle over the torus M, then the map

$$\Psi_S := \mathcal{T}_S \circ \mathcal{F}_S \circ \mathfrak{f}_\omega \colon (\mathrm{L}^2(\omega), \Delta^\mathrm{D}_a + p) \to (\mathcal{H}, D_{\mathfrak{U},p})$$

is a transplantation for any chosen set of representatives S of $F(\mathcal{L}_\mathfrak{U})\backslash\mathcal{L}_\mathfrak{V}{}'$. In particular,

$$\mathrm{Spec}_a(p,\omega) = \mathrm{Spec}(D_{\mathfrak{U},p})\,.$$

Proof. The function $p \in C^\infty(\mathbb{R}\mathfrak{U})$ can be extended to a function $(u,v) \mapsto p(u)$ on $\mathbb{R}\mathfrak{U} \oplus \mathbb{R}\mathfrak{V} = \mathbb{R}^n$, which descends to a function on M. This potential shall also be called p and is used to construct the Schrödinger operator $\Delta^\mathrm{D}_a + p$ as explained in Section 5.

The first step of the proof will be to show that Ψ_S intertwines the Schrödinger operator $\Delta^\mathrm{D}_a + p$ and

$$D_{\mathfrak{U},p} = \Big(\sum_{i=1}^m\Big(-\frac{1}{a_i{}^2}\frac{\partial^2}{\partial u^{i2}} + (2\pi r_i u^i/b_i)^2\Big) + p\Big(.+G(\nu-c)\Big)\Big)_{c\in S}$$

on the smooth sections in $\mathrm{L}^2(\omega)$ and their images under Ψ_S. Since ω is rectangular, the Chern basis $\mathfrak{U}\cup\mathfrak{V}$ consists of orthogonal tangent vectors $U_i = \frac{\partial}{\partial u^i}$ and $V_i = \frac{\partial}{\partial v^i}$. By Definition 5.3 normalising those gives

$$\Delta^\mathrm{D}_a + p = -\sum_{i=1}^m\Big(\frac{1}{a_i{}^2}\nabla_{U_i}\circ\nabla_{U_i} + \frac{1}{b_i{}^2}\nabla_{V_i}\circ\nabla_{V_i}\Big) + p\,.$$

The map Ψ_S intertwines the three types of summands pairwise. For an arbitrary smooth section $s \in \mathcal{E}(\omega)$ and $f := \mathfrak{f}_\omega s$ one has for the first summand $\nabla_{U_i} \circ \nabla_{U_i}$ that

$$\mathfrak{f}_\omega(\nabla_{U_i} s) = U_i(f) + 2\pi \mathrm{i} \mu_i f$$

because $(A^{\mathrm{D}} + a)(U_i) = 0 + 2\pi\mathrm{i}\mu_i$. For smooth f one can interchange differentiation and integration, $(U_i f)_c = U_i f_c$, and differentiation also commutes with the translation by $G(\nu - c)$. Thus, for $c \in S$ the c-component of $\Psi_S(\nabla_{U_i} s)$ in $\mathcal{H} = \bigoplus_{c \in S} \mathrm{L}^2(\mathbb{R}\mathfrak{U}, \mathbb{C})$ is

$$\begin{aligned}(\Psi_S \nabla_{U_i} s)_c = \mathcal{T}_S\Big((U_i f)_c + 2\pi\mathrm{i}\mu_i f_c\Big) &= E_{-\mu} \cdot \Big(U_i f_c + 2\pi\mathrm{i}\mu_i f_c\Big)(. + G(\nu - c)) \\ &= U_i\Big(E_{-\mu} \cdot f_c(. + G(\nu - c))\Big) = U_i(\Psi_S s)_c \,.\end{aligned}$$

This means that Ψ_S intertwines $-\nabla_{U_i} \circ \nabla_{U_i} / a_i{}^2$ and $-\frac{1}{a_i{}^2} U_i^2 = -\frac{1}{a_i{}^2} \frac{\partial^2}{\partial u^{i2}}$.

To calculate the second summand, a fact about the Fourier coefficients is needed: Since all $f \in \mathrm{L}^2(\mathbb{R}^n, \mathbb{C})^{\mathcal{L}}$ are $\mathcal{L}_{\mathfrak{V}}$-periodic, one can integrate $V_i f \cdot \overline{E_c}$ by parts to obtain $(V_i f)_c = -2\pi\mathrm{i} c_i f_c$, where $c_i := c(V_i)$. Also, $(A^{\mathrm{D}} + a)(V_i) = 2\pi\mathrm{i}(\nu_i - r_i u^i)$ and hence

$$\begin{aligned}(\Psi_S \nabla_{V_i} s)_c = \mathcal{T}_S\Big((V_i f)_c + 2\pi\mathrm{i}(\nu_i - r_i u^i) f_c\Big) &= \\ E_{-\mu} \cdot 2\pi\mathrm{i}\Big(\nu_i - c_i - r_i(G(\nu - c))^i - r_i u^i\Big) &\cdot f_c(. + G(\nu - c)) \,.\end{aligned}$$

At this point the nondegeneracy of the Chern class of the line bundle is crucial because it is the nondegeneracy, which guarantees the existence of the map G, which satisfies $\nu_i - r_i(G(\nu - c))^i = c_i$ by definition. This removes the dependence on $a = 2\pi\mathrm{i}(\mu + \nu)$ from the operator. The expression above thus equals

$$E_{-\mu} \cdot \Big(-2\pi\mathrm{i} r_i u^i \cdot f_c(. + G(\nu - c))\Big) = -2\pi\mathrm{i} r_i u^i (\Psi_S s)_c \,.$$

This shows: Ψ_S intertwines $-\nabla_{V_i} \circ \nabla_{V_i} / b_i^2$ and $(2\pi r_i u^i / b_i)^2$.

Therefore, Ψ_S intertwines the operators Δ_a^{D} and $D_{\mathfrak{U}}$. It remains to show that the map intertwines p and $p^c = p(. + G(\nu - c))$. Since p depends only on u, it follows that

$$\begin{aligned}(\Psi_S(p \cdot s))_c = E_{-\mu} \cdot (p \cdot f)_c(. + G(\nu - c)) & \\ = E_{-\mu} \cdot (p \cdot f_c)(. + G(\nu - c)) &= p^c \cdot (\Psi_S s)_c \,.\end{aligned}$$

Thus, we have $D_{\mathfrak{U},p} \circ \Psi_S = \Psi_S \circ (\Delta_a^{\mathrm{D}} + p)$ on smooth sections $s = \mathfrak{s}_\omega f$.

Since Ψ_S intertwines the two operators, it maps eigensections of one operator to eigenfunctions of the other: If $s \in \mathcal{E}(\omega)$ is a smooth eigensection of the Schrödinger operator $\Delta_a^{\mathrm{D}} + p$ with eigenvalue λ, then $\Psi_S(s)$ is a tuple of eigenfunction of $D_{\mathfrak{U},p}$ with eigenvalue λ, $D_{\mathfrak{U},p}\Psi_S(f) = \Psi_S((\Delta_a^{\mathrm{D}} + p)f) = \lambda \Psi_S(f)$.

Unfortunately, this does not yet show that Ψ_S is a transplantation. If $(f_c)_{c \in S}$ is a tuple of smooth functions in $\mathcal{H}$, it is not clear whether $\Psi_S^{-1}(f_c)_{c \in S}$ is smooth and thus one might not be able to apply the Schrödinger operator to this section. To work around this, some theory about differential operators is needed:

A *complete spectral resolution* for a partial differential operator P acting on sufficiently smooth sections of ω is defined as a complete orthonormal basis $\{s_n\}_{n\in\mathbb{N}}$ of $\mathrm{L}^2(\omega)$ such that every section is an eigensection of P: $Ps_n = \lambda_n s_n$. By [Gil95, Lemma 1.6.3] such a spectral resolution exists for every symmetric and elliptic differential operator, like $\Delta_a^{\mathrm{D}} + p$, and it consists of smooth eigensections $s_n \in \mathcal{E}(\omega)$.

Since $\mathcal{T}_S$, $\mathcal{F}_S$ and $\mathfrak{f}_\omega$ are unitary isomorphisms, Ψ_S is also one. Thus, it maps the complete spectral resolution of the Schrödinger operator to an orthonormal basis of $\mathcal{H}$, which again consists—as was shown earlier in this proof—of smooth eigenfunctions of $D_{\mathfrak{U},p}$. In particular, Ψ_S maps the eigenspaces in $\mathrm{L}^2(\omega)$ bijectively to the eigenspaces in $\mathcal{H}$. It is a transplantation. □

Theorem 6.13 has an analogue for potentials given by an $\mathcal{L}_{\mathfrak{V}}$-periodic function in $C^\infty(\mathbb{R}\mathfrak{V})$. However, it is not possible to apply Theorem 6.13 to this setting immediately because the functions representing sections of the line bundle ω are periodic only in the v-variables. Thus, one cannot calculate the Fourier coefficients with respect to the u-variables. However, the asymmetry between the roles of u and v in our choice of coordinates is arbitrary:

Definition 6.14. Let ω be a nondegenerate rectangular line bundle over the torus $\mathcal{L}\backslash\mathbb{R}^n = M$. Let $\mathfrak{U}\cup\mathfrak{V}$ denote a Chern basis of this line bundle. Set $\hat{\mathcal{L}} := \mathcal{L}$, $\hat{\mathfrak{U}} := \mathfrak{V}$ and $\hat{\mathfrak{V}} := \mathfrak{U}$. Construct a new line bundle $\hat{\omega}$ over M from the Chern invariant factors $r_1,\dots,r_m$ of ω but use $\hat{\mathfrak{U}}\cup\hat{\mathfrak{V}} = \mathfrak{V}\cup\mathfrak{U}$ (in this order) as a Chern basis, see Definition 3.10. This amounts to replacing Ω by $\hat{\Omega} := -\Omega$ and switching the roles of the u- and v-coordinates in the definition of $\mathcal{L}$-invariant functions $\mathbb{R}^n$, write

$$\hat{\varpi}_x(y) := \sum_{i=1}^m r_i v^i(x) u^i(y) \qquad \text{and} \qquad \hat{e}_x(y) := \mathrm{e}^{2\pi\mathrm{i}\hat{\varpi}_x(y)}\,.$$

Call a function $\hat{\mathcal{L}}$-invariant if $f(x+l) = \hat{e}_l(x)f(x)$ for all $x\in\mathbb{R}^n$ and $l\in\hat{\mathcal{L}}$. The distinguished connection on $\hat{\omega}$ is $\mathrm{d}+\hat{A}^{\mathrm{D}}$ with $\hat{A}^{\mathrm{D}}_x := -2\pi\mathrm{i}\hat{\varpi}_x$, and for any harmonic imaginary-valued 1-form a on M set $\hat{a} := -a$ and let $\hat{\Delta}^{\mathrm{D}}_{\hat{a}}$ denote the Laplacian corresponding to the connection $\mathrm{d}+\hat{A}^{\mathrm{D}}+\hat{a}$. Also, write $\hat{\mu}(v) := -\nu(v)$ and $\hat{\nu}(u) := -\mu(u)$.

Lemma 6.15. Define a map $\gamma\colon \mathrm{L}^2(\omega)\to\mathrm{L}^2(\hat{\omega})$ by

$$f \mapsto \overline{f}\cdot\hat{e} \qquad \text{for all } f\in\mathrm{L}^2(\mathbb{R}^n,\mathbb{C})^{\mathcal{L}},\ \text{where} \qquad \hat{e}(x) := \hat{e}_x(x)\,.$$

More precisely, $\gamma(s) := \mathfrak{s}_{\hat{\omega}}(\overline{\mathfrak{f}_\omega s}\cdot\hat{e})$. If Q is an arbitrary smooth real-valued potential on M, then γ is a transplantation

$$\gamma\colon (\mathrm{L}^2(\omega),\Delta_a^{\mathrm{D}}+Q) \to (\mathrm{L}^2(\hat{\omega}),\hat{\Delta}^{\mathrm{D}}_{\hat{a}}+Q)\,.$$

Proof. The map γ is well-defined because $\gamma(f)$ is $\hat{\mathcal{L}}$-invariant if f is $\mathcal{L}$-invariant: With $e_x(y) = \hat{e}_y(x)$ one has for all $l \in \mathcal{L}$ that

$$\gamma(f)(x+l) = (\overline{f} \cdot \hat{e})(x+l) = \overline{f}(x+l)\hat{e}_{x+l}(x+l) = \\ e_{-l}(x)\overline{f}(x)\hat{e}_x(x)\hat{e}_x(l)\hat{e}_l(x) = (\overline{f} \cdot \hat{e})(x) \cdot \hat{e}_l(x) = \gamma(f)(x) \cdot \hat{e}_l(x)$$

by Remark 3.15. Thus, $\gamma(f)$ yields a section of $\hat{\omega}$.

The map is also linear and unitary. Its inverse is again the map $g \mapsto \overline{g} \cdot e$ and therefore it is a unitary isomorphism. Also, γ intertwines the two Schrödinger operators $\Delta_a^{\mathrm{D}} + Q$ and $\hat{\Delta}_{\hat{a}}^{\mathrm{D}} + Q$: The first Laplacian is given by $-\sum_i (\nabla_{U_i}^{A^{\mathrm{D}}+a} \circ \nabla_{U_i}^{A^{\mathrm{D}}+a}/a_i{}^2 + \nabla_{V_i}^{A^{\mathrm{D}}+a} \circ \nabla_{V_i}^{A^{\mathrm{D}}+a}/b_i{}^2)$. The second Laplacian is given by the same formula except that the role of A_x^{D} is taken by $\hat{A}_x^{\mathrm{D}} = -2\pi \mathrm{i} \hat{w}_x$ and a is replaced by $\hat{a} = -a$. Note that $A^{\mathrm{D}}(U_i) = 0$, $\hat{A}^{\mathrm{D}}(U_i) = -2\pi \mathrm{i} r_i v^i$ and $U_i(\hat{e}) = 2\pi \mathrm{i} r_i v^i \hat{e}$. Thus,

$$\nabla_{U_i}^{\hat{A}^{\mathrm{D}}+\hat{a}}(\gamma f) = U_i(\gamma f) + (\hat{A}^{\mathrm{D}}(U_i) + \hat{a}(U_i))\gamma f = \\ \hat{e} U_i(\overline{f}) + \overline{f} U_i(\hat{e}) + \big(-2\pi \mathrm{i} r_i v^i - a(U_i)\big)\hat{e}\overline{f} = \hat{e} \cdot \big(U_i(\overline{f}) - \overline{f} \cdot a(U_i)\big) = \\ \gamma\big(U_i(f) + (A^{\mathrm{D}}(U_i) + a(U_i))f\big) = \gamma \nabla_{U_i}^{A^{\mathrm{D}}+a} f$$

and analogously $\nabla_{V_i}^{\hat{A}^{\mathrm{D}}+\hat{a}}(\gamma f) = \gamma \nabla_{V_i}^{A^{\mathrm{D}}+a} f$. Hence, γ intertwines the connections given by $\hat{A}^{\mathrm{D}} + \hat{a}$ and $A^{\mathrm{D}} + a$ and therefore γ intertwines the corresponding Laplacians. It also intertwines the multiplications by a real-valued potential on the two function spaces:

$$\gamma(Qf) = \hat{e} \cdot \overline{Qf} = Q \cdot \hat{e}\overline{f} = Q \cdot \gamma(f)\,.$$

Since both γ and its inverse map smooth sections to smooth sections, γ is a transplantation. □

Theorem 6.13 and Lemma 6.15 yield the following Corollary. But first an analogue of Definition 6.6 is needed:

Definition 6.16. Define a norm on $C^\infty(\mathbb{R}\mathfrak{V}, \mathbb{C})$ by

$$\|f\|^2_{\mathrm{L}^2(\mathbb{R}\mathfrak{V},\mathbb{C})} := \int_{\mathbb{R}\mathfrak{V}} |f(v)|^2 \,\mathrm{d}v \qquad \text{with} \qquad \mathrm{d}v := \mathrm{d}v^1 \cdots \mathrm{d}v^m$$

and let $\mathrm{L}^2(\mathbb{R}\mathfrak{V}, \mathbb{C})$ be the completion of $C^\infty(\mathbb{R}\mathfrak{V}, \mathbb{C})$ with respect to this norm. Define an operator $D_{\mathfrak{V}}$ on the dense subspace of smooth functions in $\mathrm{L}^2(\mathbb{R}\mathfrak{V}, \mathbb{C})$ by

$$D_{\mathfrak{V}} := \sum_{i=1}^{m} \Big((2\pi r_i v^i / a_i)^2 - \frac{1}{b_i{}^2} \frac{\partial^2}{\partial v^{i2}} \Big).$$

For a system of representatives R of the finite quotient space $F(\mathcal{L}_{\mathfrak{V}}) \backslash \mathcal{L}_{\mathfrak{U}}{}'$ define a Hilbert space

$$\mathcal{H}_{\mathfrak{V}} := \bigoplus\nolimits_{d \in R} \mathrm{L}^2(\mathbb{R}\mathfrak{V}, \mathbb{C})\,,$$

where the inner product is induced by the product on $L^2(\mathbb{R}\mathfrak{V},\mathbb{C})$. With an $\mathcal{L}_{\mathfrak{V}}$-periodic function $q \in C^\infty(\mathbb{R}\mathfrak{V})$ and $\hat{a}(u,v) = 2\pi\mathrm{i}(\hat{\mu}(v) + \hat{\nu}(u))$ let

$$D_{\mathfrak{V},\hat{q}} := (D_{\mathfrak{V}} + \hat{q}^d)_{d\in R} \qquad \text{with} \qquad \hat{q}^d := q\Big(\,.\, + \hat{G}(\hat{\nu} - d)\Big)$$

act componentwise on $\mathcal{H}_{\mathfrak{V}}$, where $\hat{G}$ is defined with respect to $\hat{\Omega} = -\Omega$ in the same way as G is defined with respect to Ω. Thus, $\hat{G} = -G$ and $\hat{\nu}(u) = -\mu(u)$.

This Definition and the following Corollary 6.17 imply a small error in signs in [GGKW08, Theorem 3.4 (2)]: The spectrum $\mathrm{Spec}_\alpha(Q, L_r)$ of the potential $Q(u,v) = q(v)$ does not coincide with $S(q,r,b,a,\mu)$ but with $S(q,r,b,a,-\mu)$, provided one uses the definition $q_{r,c,\mu}(s) := q(s_1 + (\mu_1 - c_1)/r_1, \ldots, s_m + (\mu_m - c_m)/r_m)$. (Note that $\hat{G}(\mathrm{d}u^i) = V_i/r_i$.)

Corollary 6.17. If ω is a nondegenerate rectangular line bundle and $q \in C^\infty(\mathbb{R}\mathfrak{V})$ an arbitrary $\mathcal{L}_{\mathfrak{V}}$-periodic function, then one has

$$\mathrm{Spec}_a(q,\omega) = \mathrm{Spec}(D_{\mathfrak{V},\hat{q}})\,.$$

A unitary transplantation is given for any chosen set of representatives R of $\hat{F}(\mathcal{L}_{\hat{\mathfrak{U}}})\backslash\mathcal{L}_{\hat{\mathfrak{V}}}{}' = F(\mathcal{L}_{\mathfrak{V}})\backslash\mathcal{L}_{\mathfrak{U}}{}'$ by

$$\hat{\Psi}_R \circ \gamma = \hat{\mathcal{T}}_R \circ \hat{\mathcal{F}}_R \circ \gamma \colon (L^2(\omega), \Delta_a^{\mathrm{D}} + q) \to (\mathcal{H}_{\mathfrak{V}}, D_{\mathfrak{V},\hat{q}})\,,$$

where $\hat{\mathcal{T}}_R$ and $\hat{\mathcal{F}}_R$ are defined for $\hat{\omega}$ analogously as $\mathcal{T}_S$ and $\mathcal{F}_S$ are for ω:

$$\hat{\mathcal{F}}_R(f) = (f_d)_{d\in R} \qquad \text{and} \qquad \hat{\mathcal{T}}_R(f_d)_{d\in R} = \Big(E_{-\hat{\mu}} \cdot f_d(\,.\, + \hat{G}(\hat{\nu} - d))\Big)_{d\in R}$$

Proof. By Lemma 6.15 γ is a transplantation from $(L^2(\omega), \Delta_a^{\mathrm{D}} + q)$ to $(L^2(\hat{\omega}), \hat{\Delta}_{\hat{a}}^{\mathrm{D}} + q)$ and one can apply Theorem 6.13 to the line bundle $\hat{\omega}$. Thus, the variables and parameters $(u,v,U,V,\mu,\nu,a_i,b_i,G)$ are replaced by $(v,u,V,U,\hat{\mu},\hat{\nu},b_i, a_i,\hat{G})$, where $\hat{\mu}(v) = -\nu(v)$, $\hat{\nu}(u) = -\mu(u)$ and $\hat{G} = -G$. Therefore, $D_{\mathfrak{V},\hat{q}}$ and the transplantation

$$\hat{\Psi}_R \colon (L^2(\hat{\omega}), \hat{\Delta}_{\hat{a}}^{\mathrm{D}} + q) \to (\mathcal{H}_{\mathfrak{V}}, D_{\mathfrak{V},\hat{q}})$$

have the given form and together with γ this yields the desired transplantation

$$\hat{\Psi}_R \circ \gamma \colon (L^2(\omega), \Delta_a^{\mathrm{D}} + q) \to (\mathcal{H}_{\mathfrak{V}}, D_{\mathfrak{V},\hat{q}})\,. \qquad \square$$

Finally, the transplantation $\Psi_S \colon L^2(\omega) \to \mathcal{H}$ shall be used to construct a transplantation from one torus to another. In the next section this transplantation will be used to find various isospectral but "different" tori and potentials.

Definition 6.18. For two flat tori with line bundles (M,ω) and $(\tilde{M},\tilde{\omega})$ and Chern bases $\mathfrak{U} \cup \mathfrak{V}$ and $\tilde{\mathfrak{U}} \cup \tilde{\mathfrak{V}}$ a linear bijection $\theta \colon \tilde{\mathcal{L}} \to \mathcal{L}$ between the lattices mapping $\tilde{\mathfrak{U}}$ onto $\mathfrak{U}$ and $\tilde{\mathfrak{V}}$ onto $\mathfrak{V}$ shall be called *lattice transplantation* if it respects the line bundle structure: $\tilde{\Omega}(\tilde{U}_i, \tilde{V}_j) = \Omega(\theta\tilde{U}_i, \theta\tilde{V}_j)$. For brevity denote the dual map $\mathcal{L}' \to \tilde{\mathcal{L}}'$ of θ also by θ.

Lemma 6.19. Assume we are given two rectangular tori M and $\tilde{M}$ of the same dimension with two nondegenerate rectangular line bundles ω and $\tilde{\omega}$ and a lattice transplantation θ with respect to the Chern bases $\tilde{\mathfrak{U}} \cup \tilde{\mathfrak{V}}$ and $\mathfrak{U} \cup \mathfrak{V}$. Specify connections on the bundles via two harmonic imaginary-valued 1-forms a and $\tilde{a}$.

If two periodic potentials p on $\mathbb{R}\mathfrak{U}$ and $\tilde{p}$ on $\mathbb{R}\tilde{\mathfrak{U}}$ satisfy $p^c = P$ for all $c \in \mathcal{L}_{\mathfrak{V}}'$ and $\tilde{p}^{\tilde{c}} = \tilde{P}$ for all $\tilde{c} \in \mathcal{L}_{\tilde{\mathfrak{V}}}'$ and if there is a unitary transplantation

$$\psi\colon (\mathrm{L}^2(\mathbb{R}\mathfrak{U}, \mathbb{C}), D_{\mathfrak{U}} + P) \to (\mathrm{L}^2(\mathbb{R}\tilde{\mathfrak{U}}, \mathbb{C}), \tilde{D}_{\tilde{\mathfrak{U}}} + \tilde{P}),$$

then a transplantation $\Psi\colon (\mathrm{L}^2(\omega), \Delta_a^{\mathrm{D}} + p) \to (\mathrm{L}^2(\tilde{\omega}), \tilde{\Delta}_{\tilde{a}}^{\mathrm{D}} + \tilde{p})$ is given by

$$\Psi := \tilde{\Psi}_{\theta S}^{-1} \circ \psi \circ \Psi_S$$

with ψ acting componentwise on $\mathcal{H} = \bigoplus_{c \in S} \mathrm{L}^2(\mathbb{R}\mathfrak{U}, \mathbb{C})$. This map is independent of the choice of representatives S of $F(\mathcal{L}_{\mathfrak{U}}) \backslash \mathcal{L}_{\mathfrak{V}}'$. If one abbreviates the notation of the transplantation by omitting the isomorphisms $\mathfrak{s}_{\omega}$ and $\mathfrak{s}_{\tilde{\omega}}$, then the transplantation is explicitly given by

$$\big(\Psi f\big)(\tilde{u}, \tilde{v}) = \sum_{c \in S} \sum_{\tilde{k} \in \mathcal{L}_{\tilde{\mathfrak{U}}}} E_{\theta c + \tilde{F}\tilde{k}}(\tilde{v}) \cdot \\ \Big(E_{\tilde{\mu}} \cdot \psi\Big(E_{-\mu} \cdot f_c(\,.\, + G(\nu - c))\Big)\Big)\Big(\tilde{u} + \tilde{k} - \tilde{G}(\tilde{v} - \theta c)\Big).$$

Proof. To begin with, a note on lattice transplantations: Since $\theta\colon \tilde{\mathcal{L}} \to \mathcal{L}$ is a lattice transplantation, $F(\theta\tilde{k}) = \Omega(\theta\tilde{k}, .) = \tilde{\Omega}(\tilde{k}, \theta^{-1}.) = \theta^{-1}\tilde{F}(\tilde{k})$ for $\tilde{k} \in \mathcal{L}_{\tilde{\mathfrak{U}}}$. In particular,

$$\tilde{G}\theta F = \theta^{-1} \qquad \text{and} \qquad \theta c + \tilde{F}(\tilde{k}) = \theta d \Leftrightarrow c + F(\theta\tilde{k}) = d\,.$$

This equivalence states that θ descends to an isomorphism $F(\mathcal{L}_{\mathfrak{U}}) \backslash \mathcal{L}_{\mathfrak{V}}' \to \tilde{F}(\tilde{\mathcal{L}}_{\tilde{\mathfrak{U}}}) \backslash \tilde{\mathcal{L}}_{\tilde{\mathfrak{V}}}'$. Thus, $\theta S = \{\theta c\}_{c \in S}$ is a system of representatives of the quotient space $\tilde{F}(\tilde{\mathcal{L}}_{\tilde{\mathfrak{U}}}) \backslash \tilde{\mathcal{L}}_{\tilde{\mathfrak{V}}}'$.

It follows that the three maps

$$(\mathrm{L}^2(M), \Delta_a^{\mathrm{D}} + p) \xrightarrow{\Psi_S} (\mathcal{H}, D_{\mathfrak{U},p}) \xrightarrow{\psi} (\mathcal{H}, D_{\tilde{\mathfrak{U}},\tilde{p}}) \xrightarrow{\tilde{\Psi}_{\theta S}^{-1}} (\mathrm{L}^2(\tilde{M}), \tilde{\Delta}_{\tilde{a}}^{\mathrm{D}} + \tilde{p})$$

are unitary transplantations and consequently $\tilde{\Psi}_{\theta S}^{-1} \circ \psi \circ \Psi_S$ is a unitary transplantation. Explicitly, (with the abbreviation $f := \mathfrak{f}_{\omega} s$)

$$\begin{aligned} \tilde{\Psi}_{\theta S}^{-1} \circ \psi \circ \Psi_S(s) &= \mathfrak{s}_{\tilde{\omega}} \tilde{\mathcal{F}}_{\theta S}^{-1} \circ \tilde{\mathcal{T}}_{\theta S}^{-1} \circ \psi\Big(E_{-\mu} f_c(\,.\, + G(\nu - c))\Big)_{c \in S} \\ &= \mathfrak{s}_{\tilde{\omega}} \tilde{\mathcal{F}}_{\theta S}^{-1}\Big(\Big(E_{\tilde{\mu}} \cdot \psi\Big(E_{-\mu} \cdot f_c(\,.\, + G(\nu - c))\Big)\Big)(\,.\, - \tilde{G}(\tilde{\nu} - \theta c))\Big)_{c \in S}. \end{aligned}$$

Evaluating this function at the point $(\tilde{u}, \tilde{v}) \in \mathbb{R}\tilde{\mathfrak{U}} \oplus \mathbb{R}\tilde{\mathfrak{V}}$ gives the desired formula.

It remains to show that Ψ is independent of the choice of representatives S of $F(\mathcal{L}_{\mathfrak{U}}) \backslash \mathcal{L}_{\mathfrak{V}}'$. Define Ψ_c by setting $\Psi(s) = \sum_{c \in S} \Psi_c(s)$. In fact, not only Ψ but all the Ψ_c in the explicit formula are independent of the choice of representatives: If two representatives of the same class, $d = c + k'$, are given, then

$$\mathfrak{f}_{\tilde{\omega}}\Psi_d(s)(\tilde{u},\tilde{v}) = \sum_{\tilde{k}\in\mathcal{L}_{\tilde{\mathfrak{U}}}} E_{\theta c+\theta Fk+\tilde{F}\tilde{k}}(\tilde{v})\cdot$$
$$\Big(E_{\tilde{\mu}}\cdot\psi\Big(E_{-\mu}\cdot f_{c+Fk}(\,.\,+G\nu-Gc-k)\Big)\Big)\Big(\tilde{u}+\tilde{k}+\tilde{G}\theta Fk-\tilde{G}(\tilde{\nu}-\theta c)\Big).$$

Since $\tilde{G}\theta Fk = \theta^{-1}k \in \mathcal{L}_{\tilde{\mathfrak{U}}}$, one can substitute $\tilde{l}$ for $\tilde{k}+\tilde{G}\theta Fk$ and this gives, together with $f_{c+k'}(\,.-k) = f_c$, that $\Psi_c = \Psi_d$. □

7 Negative Results

In this section some isospectrality statements following from Theorem 6.13 and Lemma 6.19 are provided, together with examples and transplantations. To simplify the explicit formulae a little bit the rather uninteresting isomorphisms $\mathfrak{f}_\omega$ and $\mathfrak{s}_\omega$ will now be omitted and thus the formulae will not be defined on $\mathrm{L}^2(\omega)$ but $\mathrm{L}^2(\mathbb{R}^n,\mathbb{C})^{\mathcal{L}}$.

Corollary 7.1. For every even-dimensional rectangular flat torus M and each nondegenerate rectangular line bundle ω over this torus $\mathrm{Spec}(\omega) := \mathrm{Spec}_a(0,\omega)$ is independent of a and is called the *spectrum of* ω. If $\tilde{a}$ is another harmonic imaginary-valued 1-form on M, then a transplantation

$$\Psi\colon (\mathrm{L}^2(\omega),\Delta_a^{\mathrm{D}}) \to (\mathrm{L}^2(\omega),\Delta_{\tilde{a}}^{\mathrm{D}}) \qquad \text{is given by}$$
$$\Big(\Psi f\Big)(u,v) = E_{\tilde{\mu}-\mu}(u-G\tilde{v})\cdot f\Big(u+G(\nu-\tilde{\nu}),v+G(\mu-\tilde{\mu})\Big).$$

Proof. Choose as a lattice transplantation $\theta = \mathrm{id}$ and as a transplantation of $(\mathrm{L}^2(\mathbb{R}\mathfrak{U},\mathbb{C}),D_{\mathfrak{U}})$ to itself $\psi = \mathrm{id}$. With those choices one obtains a transplantation from the line bundle ω with the Laplacian given by a to the bundle with the Laplacian given by $\tilde{a}$. By Lemma 6.19 this transplantation is explicitly given by

$$\Big(\Psi f\Big)(u,v) = \sum_{c\in S}\sum_{k\in\mathcal{L}_{\mathfrak{U}}} E_{c+Fk}(v)\cdot\Big(E_{\tilde{\mu}-\mu}\cdot f_c(\,.\,+G(\nu-c))\Big)(u+k-G(\tilde{\nu}-c))$$

because $\omega = \tilde{\omega}$. Recall that S is an arbitrary system of representatives of $F(\mathcal{L}_{\mathfrak{U}})\backslash\mathcal{L}_{\mathfrak{V}}'$. Thus, one can simplify this expression to

$$\sum_{c\in\mathcal{L}_{\mathfrak{V}}'} E_c(v)E_{\tilde{\mu}-\mu}\Big(u-G(\tilde{\nu}-c)\Big)\cdot f_c\Big(u+G(\nu-\tilde{\nu})\Big).$$

Since $\mu(Gc) = -c(G\mu)$, one has $E_\mu(Gc) = E_c(-G\mu)$ and therefore

$$\Big(\Psi f\Big)(u,v) = E_{\tilde{\mu}-\mu}(u-G\tilde{v})\sum_{c\in\mathcal{L}_{\mathfrak{V}}'} E_c(v+G(\mu-\tilde{\mu}))\cdot f_c(u+G(\nu-\tilde{\nu})) =$$
$$E_{\tilde{\mu}-\mu}(u-G\tilde{v})\cdot f\Big(u+G(\nu-\tilde{\nu}),v+G(\mu-\tilde{\mu})\Big). \quad □$$

Definition 7.2. Two nondegenerate rectangular line bundles are called *isospectral* if their spectra—defined in the sense just mentioned—are equal including multiplicities.

Corollary 7.3. For every even integer $n \geq 4$ there is an n-dimensional flat torus M and two topologically distinct but isospectral nondegenerate rectangular line bundles over M.

Proof. Choose any lattice with an orthogonal basis $\mathfrak{U} \cup \mathfrak{V}$ and lengths such that $a_1 = a_2$, $b_1 = 1$ and $b_2 = 2$. Further, choose two tuples of Chern invariant factors $r, \tilde{r} \in \mathbb{N}^m$ such that $r = (1, 4, r_3, \ldots, r_m)$ and $\tilde{r} = (2, 2, r_3, \ldots, r_m)$. Then the two line bundles ω_r and $\omega_{\tilde{r}}$ defined by those Chern invariant factors have different Chern classes and are thus topologically distinct, but the operators $D_{\mathfrak{U}}$ corresponding to ω_r and $\tilde{D}_{\mathfrak{U}}$ corresponding to $\omega_{\tilde{r}}$ are intertwined by the map $\psi \colon \mathrm{L}^2(\mathbb{R}\mathfrak{U}, \mathbb{C}) \to \mathrm{L}^2(\mathbb{R}\mathfrak{U}, \mathbb{C})$ interchanging the two coordinates u^1 and u^2: $(\psi f)(u^1, u^2, u^3, \ldots) := f(u^2, u^1, u^3, \ldots)$. Thus, $D_{\mathfrak{U}}$ and $\tilde{D}_{\mathfrak{U}}$ are isospectral and it follows that ω_r and $\omega_{\tilde{r}}$ are isospectral line bundles.

Note that $\theta = \mathrm{id}$ is not a lattice transplantation from (M, ω) to $(M, \tilde{\omega})$ in this example: $\Omega(U_1, V_1) = 1 \neq 2 = \tilde{\Omega}(U_1, V_1)$. In fact there exists *no* lattice transplantation in this case because the Chern invariant factors are different. □

Corollary 7.4. For every even-dimensional rectangular flat torus M and every nondegenerate rectangular line bundle ω over M there exists another nonisometric rectangular flat torus $\tilde{M}$ and a rectangular line bundle $\tilde{\omega}$ over this torus with the same Chern invariant factors such that (M, ω) and $(\tilde{M}, \tilde{\omega})$ are isospectral.

Example 7.5. Let r be the Chern invariant factors with respect to the lattice basis $\{U_1, \ldots, U_m, V_1, \ldots, V_m\}$ of the lattice of M. Given any tuple $\alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbb{R}^+)^m$ one can rescale this basis by setting $\tilde{U}_i := \alpha_i \cdot U_i$ and $\tilde{V}_i := V_i / \alpha_i$. One can choose the factors α such that the new basis defines a nonisometric torus $\tilde{M}$. One can use the Chern invariant factors r_i of ω to define a line bundle $\tilde{\omega}$ over this new torus $\tilde{M}$.

Those two tori and line bundles are isospectral: Since the coordinates with respect to the new lattice basis $\tilde{\mathfrak{U}} \cup \tilde{\mathfrak{V}} := \{\tilde{U}_1, \ldots, \tilde{U}_m, \tilde{V}_1, \ldots, \tilde{V}_m\}$ are given by $\tilde{u}^i = u^i / \alpha_i$ and since $\tilde{a}_i = \alpha_i a_i$, $\tilde{b}_i = b_i / \alpha_i$ and $\partial / \partial \tilde{u}^i = \tilde{U}_i = \alpha_i U_i = \alpha_i \partial / \partial u^i$, one has

$$\tilde{D}_{\tilde{\mathfrak{U}}} = \sum_{i=1}^{m} \Big((2\pi r_i \tilde{u}^i / \tilde{b}_i)^2 - \frac{1}{\tilde{a}_i{}^2} \frac{\partial^2}{\partial \tilde{u}^{i2}} \Big) = D_{\mathfrak{U}} \qquad \text{and in particular}$$

$$\big(\psi f\big)(\tilde{u}) := \sqrt{\alpha_1 \cdots \alpha_m} \cdot f(\tilde{u})$$

is a unitary transplantation from $(\mathrm{L}^2(\mathbb{R}\mathfrak{U}, \mathbb{C}), D_{\mathfrak{U}})$ to $(\mathrm{L}^2(\mathbb{R}\tilde{\mathfrak{U}}, \mathbb{C}), D_{\tilde{\mathfrak{U}}})$:

$$\|\psi f\|^2_{\mathrm{L}^2(\mathbb{R}\tilde{\mathfrak{U}}, \mathbb{C})} = \alpha_1 \cdots \alpha_m \cdot \int_{\mathbb{R}\tilde{\mathfrak{U}}} |f(\tilde{u})|^2 \, \mathrm{d}\tilde{u} = \int_{\mathbb{R}\mathfrak{U}} |f(u)|^2 \, \mathrm{d}u = \|f\|^2_{\mathrm{L}^2(\mathbb{R}\mathfrak{U}, \mathbb{C})} \, .$$

Setting $\theta(\tilde{U}_i) := U_i$ and $\theta(\tilde{V}_i) := V_i$ defines a lattice transplantation. By Corollary 7.1 the spectrum of a line bundle is independent of the chosen harmonic imaginary-valued 1-form a used to construct the Laplacian Δ_a^{D}. Choosing $a = 0$ on M and $\tilde{a} = 0$ on $\tilde{M}$ one can apply Lemma 6.19 to see that a transplantation is provided by (with $C := \sqrt{\alpha_1 \cdots \alpha_m}$)

$$\begin{aligned}
\big(\Psi f\big)(\tilde{u},\tilde{v}) &= C \cdot \sum_{c\in S} \sum_{\tilde{k}\in\mathcal{L}_{\tilde{\mathfrak{U}}}} E_{\theta c+\tilde{F}\tilde{k}}(\tilde{v}) \cdot f_c(\tilde{u} + \tilde{k} + \tilde{G}\theta c - Gc) \\
&= C \cdot \sum_{\tilde{c}\in\theta S} \sum_{\tilde{k}\in\mathcal{L}_{\tilde{\mathfrak{U}}}} E_{\tilde{c}+\tilde{F}\tilde{k}}(\tilde{v}) \cdot f_{\theta^{-1}\tilde{c}}(\tilde{u} + \tilde{k} + \tilde{G}\tilde{c} - G\theta^{-1}\tilde{c}) \\
&= C \cdot \sum_{\tilde{c}\in\mathcal{L}_{\tilde{\mathfrak{V}}}'} E_{\tilde{c}}(\tilde{v}) \cdot f_{\theta^{-1}\tilde{c}}(\tilde{u} + \tilde{G}\tilde{c} - G\theta^{-1}\tilde{c}) \\
&= \sqrt{\alpha_1 \cdots \alpha_m} \cdot \sum_{c\in\mathcal{L}_{\mathfrak{V}}'} E_{\theta c}(\tilde{v}) \cdot f_c(\tilde{u} + \tilde{G}\theta c - Gc)\,.
\end{aligned}$$

Definition 7.6. Two potentials Q_1 and Q_2 on a flat torus M are called *congruent* if there is an isometry $\sigma\colon M \to M$ such that $Q_1 = Q_2 \circ \sigma$. Assume there is a nondegenerate line bundle ω given over M. Two potentials are called *isospectral* with respect to a translation-invariant connection if the spectra of the Schrödinger operators given by the connection and the respective potentials are equal including multiplicities.

Corollary 7.7. Let ω be a nondegenerate rectangular line bundle over any flat torus M and let $\mathfrak{U} \cup \mathfrak{V}$ be an orthogonal Chern basis. If there is a $j \in \{1, \ldots, m\}$ such that b_j/a_j is rational but not equal to 1, then for every translation-invariant connection there exist two isospectral but noncongruent potentials Q_1 and Q_2 on M. Those potentials can be chosen to be analytic and $\mathbb{Z}_2$-invariant.

Example 7.8. This example will be constructed with the help of Lemma 6.19, which uses Theorem 6.13. It will be shown that Lemma 6.19 yields a transplantation $(\mathrm{L}^2(\omega), \Delta_a^{\mathrm{D}} + Q_1) \to (\mathrm{L}^2(\hat{\omega}), \hat{\Delta}_{\hat{a}}^{\mathrm{D}} + Q_2)$ and then Lemma 6.15 can be applied to give a transplantation to $(\mathrm{L}^2(\omega), \Delta_a^{\mathrm{D}} + Q_2)$.

But before one can construct transplantations, the potentials need to be defined. Write $b_j/a_j = \beta/\alpha \in \mathbb{Q}$ with $\alpha, \beta \in \mathbb{N}$. Let $\tilde{P} \in C^\infty(\mathbb{R}U_j)$ be any nonconstant $\mathcal{L}_{\{U_j/\alpha r_j\}}$-periodic function and with the projection $\pi\colon \mathbb{R}\mathfrak{U} \to \mathbb{R}U_j$ sending all $U_i \neq U_j$ to 0 set $P := \tilde{P} \circ \pi$. Define $p := P(\,.\, - Gv)$ and $q := P \circ A(\,.\, - \hat{G}\hat{v})$, where $A\colon \mathbb{R}\mathfrak{V} \to \mathbb{R}\mathfrak{U}$ is the linear isomorphism sending each V_i/b_i to $\hat{V}_i/a_i = U_i/a_i$ and $a(u,v) = 2\pi\mathrm{i}(\mu(u) + \nu(v)) = -2\pi\mathrm{i}(\hat{\mu}(v) + \hat{\nu}(u))$, see Definition 6.14.

Since α and r_j are integers and since $G(\mathrm{d}v^i) = U_i/r_i$ and $G(\mathrm{d}u^i) = -V_i/r_i$, p is not only $\mathcal{L}_{\mathfrak{U}}$-periodic but it also satisfies

$$p^c = P(\,.\, + Gv - Gc - Gv) = \tilde{P}(\pi - c(V_j)/r_j \cdot U_j) = \tilde{P} \circ \pi = P$$

for every $c \in \mathcal{L}_{\mathfrak{V}}'$. Analogously, q is $\mathcal{L}_{\mathfrak{V}}$-periodic and satisfies

$$\begin{aligned}
\hat{q}^d = P \circ A(\,.\, + \hat{G}\hat{v} - \hat{G}d - \hat{G}\hat{v}) = \tilde{P}(\pi \circ A - d(U_j)b_j \cdot U_j/(r_j a_j)) = \\
\tilde{P}(\pi \circ A - d(U_j)\beta \cdot U_j/(r_j\alpha)) = P \circ A
\end{aligned}$$

for all $d \in \mathcal{L}_{\mathfrak{U}}'$. With the projections $\pi_{\mathfrak{U}} \colon \mathbb{R}^n \to \mathbb{R}\mathfrak{U}$ and $\pi_{\mathfrak{V}} \colon \mathbb{R}^n \to \mathbb{R}\mathfrak{V}$ the two functions p and q define two potentials $Q_1 := p \circ \pi_{\mathfrak{U}}$ and $Q_2 := q \circ \pi_{\mathfrak{V}}$, which are noncongruent. To see this consider the connected subsets of M on which Q_1 is minimal. Let $N \in \mathbb{N}$ denote their number. Since Q_1 is nonconstant, the number of such sets of Q_2 is

$$N \cdot b_j / a_j = N \cdot \beta / \alpha \neq N\,, \quad (\alpha \text{ divides } N \text{ since } Q_1 \text{ is } \mathcal{L}_{\{U_1/\alpha r_j\}}\text{-periodic.})$$

since $b_j / a_j \neq 1$. Thus, Q_1 and Q_2 cannot be congruent. They are isospectral though: To apply Lemma 6.19 one needs the map

$$\psi \colon \mathrm{L}^2(\mathbb{R}\mathfrak{U}, \mathbb{C}) \to \mathrm{L}^2(\mathbb{R}\mathfrak{V}, \mathbb{C}) \qquad \text{with} \qquad \psi f := \sqrt{\tfrac{b_1 \cdots b_m}{a_1 \cdots a_m}} \cdot f \circ A\,,$$

which is a transplantation from the operator $D_{\mathfrak{U},p} = D_{\mathfrak{U}} + P$ to the operator $D_{\mathfrak{V}} + P \circ A = D_{\mathfrak{V},\hat{q}}$ because $\partial^2/\partial v^{i2}(f \circ A) = \partial^2 f/\partial u^{i2} \circ A \cdot b_i{}^2/a_i{}^2$ and $(2\pi r_i v^i/a_i)^2(f \circ A) = ((2\pi r_i u^i/b_i)^2 f) \circ A$ give

$$D_{\mathfrak{V},\hat{q}}(f \circ A) = \Big(\sum_{i=1}^{m}\Big((2\pi r_i u^i/b_i)^2 - \frac{1}{a_i{}^2}\frac{\partial^2}{\partial u^{i2}}\Big) f + Pf\Big) \circ A = \Big(D_{\mathfrak{U}} f + Pf\Big) \circ A$$

for every smooth function f in $\mathrm{L}^2(\mathbb{R}\mathfrak{U}, \mathbb{C})$. ψ maps smooth functions to smooth functions and so does $\psi^{-1} \colon g \mapsto \sqrt{\tfrac{a_1 \cdots a_m}{b_1 \cdots b_m}} \cdot g \circ A^{-1}$, which implies that ψ is a transplantation. Also, ψ is unitary:

$$\begin{aligned}\|\psi f\|^2_{\mathrm{L}^2(\mathbb{R}\mathfrak{V}, \mathbb{C})} = \tfrac{b_1 \cdots b_m}{a_1 \cdots a_m} \cdot \int_{\mathbb{R}\mathfrak{V}} f \circ A(v)\, \mathrm{d}v &= \tfrac{b_1 \cdots b_m}{a_1 \cdots a_m} \cdot \int_{\mathbb{R}\mathfrak{V}} f\Big(\tfrac{b_1}{a_1} v^1, \ldots, \tfrac{b_m}{a_m} v^m\Big)\, \mathrm{d}v \\ &= \int_{\mathbb{R}\mathfrak{U}} f(u^1, \ldots, u^m)\, \mathrm{d}u = \|f\|^2_{\mathrm{L}^2(\mathbb{R}\mathfrak{U}, \mathbb{C})}\,.\end{aligned}$$

Now, as in Definition 6.14 let $\hat{\omega}$ be the line bundle over M defined by the Chern basis $\hat{\mathfrak{U}} := \mathfrak{V}$ and $\hat{\mathfrak{V}} := \mathfrak{U}$ and the Chern invariant factors r_i. As a lattice transplantation from $\hat{\omega}$ to ω choose $\theta \colon \hat{\mathcal{L}} \to \mathcal{L}$ with $\theta \hat{U}_i := U_i$ and $\theta \hat{V}_i := V_i$, which satisfies $\hat{\Omega}(\hat{U}_i, \hat{V}_i) = r_i = \Omega(U_i, V_i) = \Omega(\theta \hat{U}_i, \theta \hat{V}_i)$. With the transplantation ψ and the lattice transplantation θ one can use Lemma 6.19 to obtain a transplantation

$$\Psi \colon (\mathrm{L}^2(\omega), \Delta^{\mathrm{D}}_a + Q_1) \to (\mathrm{L}^2(\hat{\omega}), \hat{\Delta}^{\mathrm{D}}_{\hat{a}} + Q_2)\,,$$

which is explicitly given by

$$\begin{aligned}\Big(\Psi f\Big)(\hat{u}, \hat{v}) = \sqrt{\tfrac{b_1 \cdots b_m}{a_1 \cdots a_m}} \cdot \sum_{c \in S} \sum_{\hat{k} \in \mathcal{L}_{\hat{\mathfrak{U}}}} & E_{\theta c + \hat{F}\hat{k}}(\hat{v}) \cdot \\ & \Big(E_{\hat{\mu}} \cdot E_{-\mu} \circ A \cdot f_c(A\,. + G(\nu - c))\Big)(\hat{u} + \hat{k} - \hat{G}(\hat{v} - \theta c))\end{aligned}$$

with an arbitrary system of representatives $S \subset F(\mathcal{L}_{\mathfrak{U}}) \backslash \mathcal{L}_{\mathfrak{V}}'$. Using Lemma 6.15 one obtains the desired transplantation

$$\Phi := \gamma^{-1} \circ \Psi \colon (\mathrm{L}^2(\omega), \Delta^{\mathrm{D}}_a + Q_1) \to (\mathrm{L}^2(\omega), \Delta^{\mathrm{D}}_a + Q_2)$$

and using $(\hat{F},\hat{G},\hat{u},\hat{v},\hat{\mu},\hat{\nu}) = (-F,-G,v,u,-\nu,-\mu)$ one obtains

$$\big(\Phi f\big)(u,v) = \sqrt{\tfrac{b_1\cdots b_m}{a_1\cdots a_m}}\cdot e_u(v)\,\mathrm{conj}\sum_{c\in S}\sum_{k\in\mathcal{L}_{\mathfrak{V}}} E_{\theta c-Fk}(u)\cdot \\ \Big(E_{-\nu}\cdot E_{-\mu}\circ A\cdot f_c(A\,.+G(\nu-c))\Big)(v+k-G(\mu+\theta c))$$

In particular, the two potentials are isospectral.

Corollary 7.9. For *any* nondegenerate line bundle over any two-dimensional nonsquare rectangular torus with rational ratio of side lengths and any translation-invariant connection on this line bundle there are two noncongruent but isospectral potentials. Those potentials can be chosen to be analytic and $\mathbb{Z}_2$-invariant.

Proof. By assumption, the torus is given by a two-dimensional lattice with an orthogonal basis $\{U_1,V_1\}$. The line bundle is represented by an antisymmetric and nondegenerate map Ω having integer values on the lattice. Let $0 < r_1 := \Omega(U_1,V_1)$. As the lattice is two-dimensional, the tuple $(r_1)\in\mathbb{N}^1$ satisfies the conditions of Lemma 3.8 and therefore r_1 is the only Chern invariant factor and $\{U_1,V_1\}$ is an orthogonal Chern basis. This means that—in this case—any such line bundle is already rectangular and that Corollary 7.7 can be applied. □

Remark 7.10. It is possible to simplify the transplantation Φ in Example 7.8 under additional assumptions: If the connection is translation- and $\mathbb{Z}_2$-invariant, if the Chern invariant factors are all equal to one, $r=(1,\dots,1)$, and if $b_i/a_i\in\mathbb{Z}$ for all $i\in\{1,\dots,m\}$, then the transplantation is given by

$$\Phi(f)(u,v) = \sqrt{\tfrac{b_1\cdots b_m}{a_1\cdots a_m}}\cdot e_u(v)\frac{1}{\#S}\sum_{l\in S}\overline{f}\Big(Av,A^{-1}(u+l)\Big),$$

where $S\subset\mathcal{L}_{\mathfrak{U}}$ is any (finite) system of representatives of the quotient space $\mathcal{L}_{\mathfrak{U}}/A\mathcal{L}_{\mathfrak{V}}$.

Proof. Since all Chern invariant factors are one, F is bijective as a map from $\mathcal{L}_{\mathfrak{U}}$ to $\mathcal{L}_{\mathfrak{V}}'$. Choose $c=0$ as a representative for the only class in $F(\mathcal{L}_{\mathfrak{U}})\backslash\mathcal{L}_{\mathfrak{V}}'$. Also, the only translation- and $\mathbb{Z}_2$-invariant connection is the distinguished connection and therefore $\mu=0$ and $\nu=0$, confer Lemma 4.11. This yields

$$\big(\Phi f\big)(u,v) = \sqrt{\tfrac{b_1\cdots b_m}{a_1\cdots a_m}}\cdot e_u(v)\,\mathrm{conj}\sum_{k\in\mathcal{L}_{\mathfrak{V}}} E_{-Fk}(u)\cdot f_0\circ A(v+k)\,.$$

The assumption that $b_i/a_i\in\mathbb{Z}$ for all $i\in\{1,\dots,m\}$ is equivalent to the demand that $Ak\in\mathcal{L}_{\mathfrak{U}}$ for all $k\in\mathcal{L}_{\mathfrak{V}}$. In particular, the quotient space $\mathcal{L}_{\mathfrak{U}}/A\mathcal{L}_{\mathfrak{V}}$ is well-defined and finite. Also, one can use that $f_0(Av+Ak) = f_{FAk}(Av)$. Together with $E_{-Fk}(u) = E_{FAk}(A^{-1}u)$ this gives

$$\big(\Phi f\big)(u,v) = \sqrt{\tfrac{b_1\cdots b_m}{a_1\cdots a_m}}\cdot e_u(v)\,\mathrm{conj}\sum_{k\in\mathcal{L}_{\mathfrak{V}}} E_{FAk}(A^{-1}u)\cdot f_{FAk}(Av) = \\ \sqrt{\tfrac{b_1\cdots b_m}{a_1\cdots a_m}}\cdot e_u(v)\frac{1}{\#S}\sum_{l\in S}\overline{f}\Big(Av,A^{-1}(u+l)\Big).$$

To see the latter equation consider the map

$$\pi\colon \mathrm{L}^2(\mathbb{R}\mathfrak{V},\mathbb{C})^{\mathcal{L}_{\mathfrak{V}}} \to \mathrm{L}^2(\mathbb{R}\mathfrak{V},\mathbb{C})^{A^{-1}\mathcal{L}_{\mathfrak{U}}} \quad \text{with} \quad g \mapsto \frac{1}{\#S}\sum_{l\in S} g(\,.+A^{-1}l)\,,$$

which is the projection of the $\mathcal{L}_{\mathfrak{V}}$-periodic functions onto the $A^{-1}\mathcal{L}_{\mathfrak{U}}$-periodic functions. In particular, every trigonometric monomial E_c corresponding to an element $c \in \mathcal{L}_{\mathfrak{V}}{}'$ *not* in the coarser dual lattice $(A^{-1}\mathcal{L}_{\mathfrak{U}})' = \{FAk \mid k \in \mathcal{L}_{\mathfrak{V}}\}$ of the finer lattice $A^{-1}\mathcal{L}_{\mathfrak{U}}$ is mapped to zero. On the other hand, $\pi(g_{FAk}E_{FAk}) = g_{FAk}E_{FAk}$ for all $k \in \mathcal{L}_{\mathfrak{V}}$. □

Remark 7.11. One can also prove Remark 7.10 directly instead of calculating the Fourier coefficients of the map

$$\Phi(f)(u,v) = \sqrt{\tfrac{b_1\cdots b_m}{a_1\cdots a_m}} \cdot e_u(v)\frac{1}{\#S}\sum_{l\in S}\overline{f}\big(Av, A^{-1}(u+l)\big).$$

It is easy to check that one has $\nabla_{V_i}\circ\nabla_{V_i}\Phi(f)/{b_i}^2 = \Phi(\nabla_{U_i}\circ\nabla_{U_i}f/{a_i}^2)$ and $\nabla_{U_i}\circ\nabla_{U_i}\Phi(f)/{a_i}^2 = \Phi(\nabla_{V_i}\circ\nabla_{V_i}f/{b_i}^2)$. In fact, this holds true for every summand of Φ individually. Also, $\Phi(pf)(u,v) = p\circ A\cdot\Phi(f) = q\cdot\Phi(f)$ and hence Φ intertwines the Schrödinger operators of Q_1 and Q_2.

However, only $\Phi(f)$ but not each summand is $\mathcal{L}$-invariant and descends to a section of the line bundle ω. Note that Φ is independent of the choice of representatives S of $\mathcal{L}_{\mathfrak{U}}/A\mathcal{L}_{\mathfrak{V}}$ because f is $\mathcal{L}$-invariant and therefore $\mathcal{L}_{\mathfrak{V}}$-periodic. For any $h \in \mathcal{L}_{\mathfrak{U}}$ and $k \in \mathcal{L}_{\mathfrak{V}}$ one has with $C := \sqrt{(b_1\cdots b_m)/(a_1\cdots a_m)}$ that

$$\begin{aligned}\Phi(f)(u+h,v+k) = e_u(v)e_u(k)e_h(v)\frac{C}{\#S}\sum_{l\in S}\overline{f}\big(A(v+k),A^{-1}(u+l+h)\big) =\\ e_h(v)e_u(v)e_u(k)\frac{C}{\#S}\sum_{l\in S}e_{-Ak}\big(A^{-1}(u+l+h)\big)\overline{f}\big(Av,A^{-1}(u+l+h)\big) =\\ e_h(v)e_u(v)\frac{C}{\#S}\sum_{l\in R}\overline{f}\big(Av,A^{-1}(u+l)\big) = e_h(v)\Phi(f)(u,v)\end{aligned}$$

since $R := S+h$ is also a system of representatives of $\mathcal{L}_{\mathfrak{U}}/A\mathcal{L}_{\mathfrak{V}}$.

Hence, Φ is a well-defined map $\mathrm{L}^2(\omega) \to \mathrm{L}^2(\omega)$ intertwining the Schrödinger operators of the two potentials Q_1 and Q_2. This means that Φ is a transplantation.

Example 7.12. Finally, an illustration of Remark 7.10 shall be given. Let M be the two-dimensional torus defined by the lattice $\mathcal{L} := \{U_1, V_1\}$, where $U_1 := e_1$ and $V_1 := 2e_2$ are given by the standard unit vectors. This means $a_1 = 1$ and $b_1 = 2$. Let ω be the line bundle with the Chern basis $\{U_1\}\cup\{V_1\}$ and the Chern invariant factor $r_1 = 1$. If one chooses the only translation- and $\mathbb{Z}_2$-invariant connection, i. e. the distinguished connection ∇^{D}, one can apply the previous simplification to obtain the transplantation (with $u := u^1$ and $v := v^1$)

$$\Phi(f)(u,v) = \frac{1}{\sqrt{2}}\mathrm{e}^{-2\pi\mathrm{i}uv}\cdot\Big(\overline{f}\big(2v,\frac{u}{2}\big)+\overline{f}\big(2v,\frac{u+1}{2}\big)\Big).$$

This map is a transplantation from the Schrödinger operator $\Delta_0^{\mathrm{D}} + p$ of an arbitrary $\mathcal{L}_{\{U_1\}}$-periodic potential $p \in C^\infty(\mathbb{R}U_1)$ to the Schrödinger operator $\Delta_0^{\mathrm{D}} + q$ of the corresponding potential $q(v) := p(2v)$. For instance, one can choose $p(u) := \cos(2\pi u)$ and correspondingly $q(v) = \cos(4\pi v)$, which are shown in Figure 1. Note that both potentials are analytic and $\mathbb{Z}_2$-invariant.

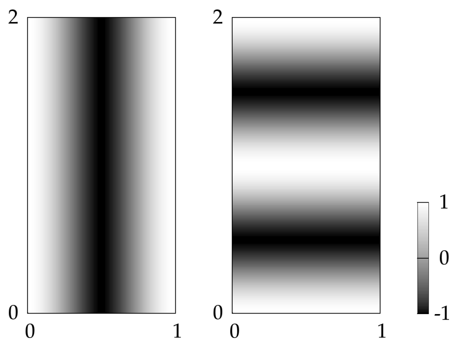

Figure 1: Two noncongruent even potentials on a nonsquare torus, which are isospectral with respect to the distinguished connection on a line bundle with Chern invariant factor $r = 1$.

8 Positive Results

To contrast the negative results obtained in this work I will give some positive results by constructing some spectral invariants in this final section.

Definition 8.1. An *invariant* f of an equivalence relation $\sim$ on a set X is a function $f\colon X \to Y$ which descends to a function on the quotient space $X/\sim$. Provided that every element of X has a spectrum, two elements $x, y \in X$ are said to be equivalent if they have the same spectrum. An invariant on X of this equivalence relation is called a *spectral invariant*.

It was shown in the previous section in Corollary 7.7 that a potential is in general not determined by its spectrum with respect to a given translation-invariant connection. However, under one additional condition a potential is determined by its entire ω-Bloch spectrum. To prove this some invariants are needed. Let $|.|$ denote the Euclidean norm on $\mathbb{R}^n$. Denote by $\mathrm{d}V$ the Riemannian volume form of the flat torus M or on $\mathbb{R}^n$ and set $\operatorname{Vol} M := \int_M 1 \,\mathrm{d}V$. Abbreviate $\mathrm{d}x := \mathrm{d}V(x)$.

Proposition 8.2. Let $M = \mathcal{L}\backslash\mathbb{R}^n$ be a torus defined by a lattice $\mathcal{L}$ and let ω be a line bundle over this torus such that all Chern invariant factors are equal to one, $r = (1, \ldots, 1)$. With $\alpha \in (\mathbb{R}^n)'$ and $l \in \mathcal{L}$ set

$$W_l^\alpha(Q) := E_\alpha(l) \frac{1}{\operatorname{Vol} W} \int_W \int_0^1 E_{Fl}(x) \cdot Q(x - l \cdot t) \,\mathrm{d}t \,\mathrm{d}x\,,$$

where W is a unit cell of the lattice, $Fl := \Omega(l, .)$ and Q a smooth potential on M. Then for every length d occurring in $|\mathcal{L}|$,

$$C^\infty(M) \ni Q \mapsto \sum_{l \in \mathcal{L},\, |l| = d} W_l^\alpha(Q) \in \mathbb{C}$$

is an invariant of $\operatorname{Spec}_{2\pi i\alpha}(Q,\omega)$ on the set of smooth potentials.

A torus $M = \mathcal{L}\backslash\mathbb{R}^n$ is said to have a *nondegenerate length spectrum* if the only lattice vectors of the same length $|l| \in |\mathcal{L}|$ are l and $-l$. If $\mathcal{L}$ has a nondegenerate length spectrum, then the above invariants are equal to $W_l^\alpha + W_{-l}^\alpha$ for each $l \in \mathcal{L}\backslash\{0\}$.

Proof. A proof for the case $d = 0$ will be given later in Proposition 8.6. For the general case see [GGKW08]. □

Corollary 8.3. If $Q_c := \langle Q, E_c\rangle_{L^2(M)} := \frac{1}{\operatorname{Vol} M}\int_M Q\cdot\overline{E_c}\,\mathrm{d}V$ denotes the Fourier coefficient of Q for $c \in \mathcal{L}'$, then for every $d \in |\mathcal{L}|$

$$V_d^\alpha(Q) := \sum_{|l|=d} Q_{-Fl}\cdot E_\alpha(l) \qquad \text{is equal to } \sum_{|l|=d} W_l^\alpha(Q) \text{ and hence}$$

an invariant of $\operatorname{Spec}_{2\pi i\alpha}(Q,\omega)$.

Proof. One has for every $l \in \mathcal{L}$ with $|l| = d$ that

$$\begin{aligned} W_l^\alpha(Q) &= E_\alpha(l)\frac{1}{\operatorname{Vol} W}\int_W\int_0^1 E_{Fl}(x)\sum_{c\in\mathcal{L}'} Q_c E_c(x - l\cdot t)\,\mathrm{d}t\,\mathrm{d}x \\ &= E_\alpha(l)\sum_{c\in\mathcal{L}'} Q_c\cdot\langle E_{Fl}, E_{-c}\rangle_{L^2(M)}\cdot\int_0^1 E_c(-l\cdot t)\,\mathrm{d}t\,. \end{aligned}$$

Since all Chern invariant factors are one, F is an isomorphism from $\mathcal{L}$ to $\mathcal{L}'$ and there is one and only one $c \in \mathcal{L}'$ with $Fl = -c$. For this c one has $c(-l\cdot t) = tFl(l) = t\Omega(l,l) = 0$ and thus $E_c(-l\cdot t) \equiv 1$. Hence, $W_l^\alpha(Q) = Q_{-Fl}E_\alpha(l)$, which gives

$$\sum_{|l|=d} W_l^\alpha(Q) = \sum_{|l|=d} Q_{-Fl}\cdot E_\alpha(l) = V_d^\alpha(Q)\,.$$

Since the left hand side is a spectral invariant by Proposition 8.2, the same holds for the right hand side. □

Theorem 8.4. Every smooth potential on a fixed even-dimensional torus with line bundle ω whose Chern invariant factors are $r_1 = \cdots = r_m = 1$ is uniquely determined by its ω-Bloch spectrum.

Proof. Let $l_1,\ldots,l_k \in \mathcal{L}$ be the lattice vectors with length $d \in |\mathcal{L}|$ and assume that some $\alpha_1,\ldots,\alpha_k \in (\mathbb{R}^n)'$ are given. The vector $V_d := (V_d^{\alpha_1}(Q),\ldots,V_d^{\alpha_k}(Q))$ is a spectral invariant and $V_d = Aq$, where $q := (Q_{-Fl_1},\ldots,Q_{-Fl_k})$ and

$$A := \begin{pmatrix} E_{\alpha_1}(l_1) & \cdots & E_{\alpha_1}(l_k) \\ \vdots & \ddots & \vdots \\ E_{\alpha_k}(l_1) & \cdots & E_{\alpha_k}(l_k) \end{pmatrix}.$$

Thus, if one chooses $\alpha_1 = 0$ and α_2 such that all $E_{\alpha_2}(l_i)$ are pairwise distinct, and if one further sets $\alpha_j = (j-1)\cdot\alpha_2$, then A is a Vandermonde matrix with nonvanishing determinant. This means A is invertible and the Fourier coefficients q are determined by the invariants V_d. Since $F\colon \mathcal{L} \to \mathcal{L}'$ is bijective, all Fourier coefficients are determined this way and are thus invariants. □

This Theorem shows that the entire ω-Bloch spectrum contains a lot of information. Under stronger assumptions, however, some parts and properties of a potential are already determined by the spectra associated with certain given connections. This shall be shown in the rest of this section.

Remark 8.5. If, additionally to the assumptions of the previous Theorem, the length spectrum of the torus is nondegenerate, then Q is already determined by $\operatorname{Spec}_{2\pi i\alpha}(Q,\omega)$ and $\operatorname{Spec}_{2\pi i\beta}(Q,\omega)$ for two suitable elements $\alpha, \beta \in (\mathbb{R}^n)'$. The connections must be chosen subject to the weak condition that $(\alpha-\beta)(l)$ is irrational for all nonzero lattice vectors l. Indeed, in that case the matrix

$$\begin{pmatrix} E_\alpha(-l) & E_\alpha(l) \\ E_\beta(-l) & E_\beta(l) \end{pmatrix}$$

has nonvanishing determinant for each $l \in \mathcal{L}\backslash\{0\}$ and therefore the spectral invariants $W_l^\alpha + W_{-l}^\alpha = V_{|l|}^\alpha$ and $W_l^\beta + W_{-l}^\beta = V_{|l|}^\beta$ determine the Fourier coefficients $Q_{\pm Fl}$ for all $l \in \mathcal{L}\backslash\{0\}$. Finally, Q_0 is determined since it equals $V_0^\alpha = V_0^\beta$.

For later use and as promised, I will now prove Proposition 8.2 for $d = 0$.

Proposition 8.6. Given any line bundle ω over a torus M and a Laplacian Δ associated to any (fixed) connection ∇ on the line bundle ω

$$Q_0 = \frac{1}{\operatorname{Vol} M}\int_M Q \,\mathrm{d}V \qquad \text{and} \qquad \|Q\|^2_{L^2(M)} = \frac{1}{\operatorname{Vol} M}\int_M Q^2 \,\mathrm{d}V$$

are spectral invariants of the Schrödinger operator $\Delta + Q$.

Proof. In [Gil95, Chapter 1.6] it is shown that for every symmetric elliptic differential operator P of positive order d acting on the smooth sections of any vector bundle over a closed Riemannian manifold M (like Schrödinger operators over flat tori) there is a well-defined operator e^{-tP} for every $t > 0$ such that

$$\mathrm{e}^{-tP} f(x) = \int_M K(t,x,y) f(y) \,\mathrm{d}y\,,$$

where $K(t,x,y)$ is a homomorphism from the fibre over y to the fibre over x and $\mathrm{d}x := \mathrm{d}V(x)$. It is also shown that the trace of this heat kernel is a spectral invariant since

$$\operatorname{trace}_{L^2(M)} \mathrm{e}^{-tP} = \sum \mathrm{e}^{-t\lambda_n} = \int_M \operatorname{trace}_{\omega_x} K(t,x,x) \,\mathrm{d}x\,,$$

where $(\lambda_n)_{n\in\mathbb{N}}$ is the spectrum of P (with multiplicities). In [Gil95, Lemma 1.8.2] it is shown that there is an asymptotic expansion

$$K(t,x,x) \sim \sum_{i=0}^{\infty} t^{\frac{i-n}{d}} e_i(x) \qquad \text{as } t \searrow 0,$$

where $n = \dim M$ and (only in this proof) $e_i(x)$ is an endomorphism of the fibre over x of the vector bundle under consideration. From those two equations one can deduce that the dimension n and for each $i \in \mathbb{N}$ the number $\int_M \operatorname{trace} e_i(x)\,\mathrm{d}x$ are spectral invariants. Note that $e_i(x) = 0$ if i is odd. Applying [Gil95, Theorem 4.1.6] to the operator $\Delta + Q$ on a line bundle over the flat manifold M yields $e_0(x) = \mathrm{id}_x/B$, $e_2 = -Q/B$ and $e_4 = (\frac{1}{2}Q^2 - \frac{1}{6}\Delta Q + \frac{1}{12}\Omega_{ij}\Omega_{ij})/B$, where the Ω_{ij} are given by the connection and $B := (4\pi)^{n/2}$. Thus,

$$B \cdot \int_M \operatorname{trace} e_0(x)\,\mathrm{d}x = \operatorname{Vol} M \quad \text{and} \quad B \cdot \int_M \operatorname{trace} e_2(x)\,\mathrm{d}x = -\int_M Q(x)\,\mathrm{d}x$$

are spectral invariants. Therefore, Q_0 is a spectral invariant. Since M is closed, the Divergence Theorem gives $\int_M \Delta Q(x)\,\mathrm{d}x = 0$ and

$$2(4\pi)^{n/2} \int_M \operatorname{trace} e_4(x)\,\mathrm{d}x = \int_M Q^2(x)\,\mathrm{d}x + C\,,$$

where $C = 2(4\pi)^{n/2} \int_M \operatorname{trace} \frac{1}{12}\Omega_{ij}\Omega_{ij}\,\mathrm{d}x$ does not depend on Q. Thus, the norm $\frac{1}{\operatorname{Vol} M}\int_M Q^2\,\mathrm{d}V$ is a spectral invariant. Since Q is a real function, this integral is equal to $\|Q\|_{\mathrm{L}^2(M)}$.

A simpler introduction to the heat kernel of the Laplacian acting on functions can be found in [Ros97]. □

The notation needed for the following Remark 8.7 is introduced in Definition 4.10.

Remark 8.7. If Q is a potential on a torus M, then the even and the odd part of Q are defined by

$$Q^+ := \frac{1}{2}(Q + \check{Q}) \qquad \text{and} \qquad Q^- := \frac{1}{2}(Q - \check{Q})\,,$$

respectively, and $Q = Q^+ + Q^-$. By substitution one obtains $\check{Q}_c = \langle \check{Q}, E_c\rangle_{\mathrm{L}^2(M)} = \langle Q, \check{E}_c\rangle_{\mathrm{L}^2(M)} = Q_{-c}$ for each $c \in \mathcal{L}'$. Thus, the Fourier coefficients of the even and odd part of the potential are $(Q_c + Q_{-c})/2$ and $(Q_c - Q_{-c})/2$, respectively. Also note that Q^+ and Q^- are $\mathrm{L}^2(M)$-orthogonal and by Pythagoras' Theorem

$$\|Q\|^2_{\mathrm{L}^2(M)} = \|Q^+\|^2_{\mathrm{L}^2(M)} + \|Q^-\|^2_{\mathrm{L}^2(M)}\,.$$

Theorem 8.8. *Let M be an even-dimensional torus with nondegenerate length spectrum and let ω be a line bundle with Chern invariant factors $r_1 = \cdots = r_m = 1$. If one is given an arbitrary but fixed translation- and weakly $\mathbb{Z}_2$-invariant connection, then the even part of a potential and the norm of the odd part are spectrally determined.*

Proof. The lattice having nondegenerate length spectrum means that there are only two lattice vectors of a given length $d \in |\mathcal{L}|\backslash\{0\}$. Proposition 4.9 states that every translation-invariant connection on the line bundle ω is gauge equivalent to a connection $\nabla^{\mathrm{D}} + a$, where a is a harmonic imaginary-valued 1-form on M and ∇^{D} the distinguished connection on the line bundle ω. This notation is introduced in Remark 4.1. If one denotes the given connection by $\nabla^{\mathrm{D}} + 2\pi \mathrm{i}\alpha$, then one obtains for all $c \in \mathcal{L}' \backslash \{0\}$ that

$$V_d^\alpha(Q) = Q_{-c} E_\alpha(Gc) + Q_c E_\alpha(-Gc)$$

is a spectral invariant by Corollary 8.3. Recall that G is the inverse of the map $F\colon \mathbb{R}^n \to (\mathbb{R}^n)'$ with $F(x) = \Omega(x, .)$. The restriction of F to $\mathcal{L}$ is bijective map $\mathcal{L} \to \mathcal{L}'$ because all Chern invariant factors are equal to 1. Thus, $G\colon \mathcal{L} \to \mathcal{L}'$, see Definition 6.5.

Since the connection is assumed to be weakly $\mathbb{Z}_2$-invariant, one has $\alpha(\mathcal{L}) \subset \frac{1}{2}\mathbb{Z}$ by Lemma 4.11. Hence, $E_\alpha(Gc) = E_\alpha(-Gc) = \pm 1$ and thus $(Q_c + Q_{-c})/2$ is also an invariant. Additionally, Q_0 is spectrally determined by the previous Proposition. Hence, by Remark 8.7 all Fourier coefficients of the even part of the potential are spectral invariants.

According to Proposition 8.6 and Remark 8.7 the L^2-norm of Q is a spectral invariant and $\|Q\|^2_{\mathrm{L}^2(M)} = \|Q^+\|^2_{\mathrm{L}^2(M)} + \|Q^-\|^2_{\mathrm{L}^2(M)}$. Therefore, if the even part is known, one can calculate the norm of the odd part. □

Corollary 8.9. Let $\mathrm{Iso}(Q, \omega, \nabla)$ be the set of potentials on ω having the same spectrum as Q with respect to the connection ∇. Under the assumptions of the Theorem 8.8 one has that if Q is even then $\mathrm{Iso}(Q, \omega, \nabla) = \{Q\}$ for every translation- and weakly $\mathbb{Z}_2$-invariant connection ∇.

Proof. Let $P \in \mathrm{Iso}(Q, \omega, \nabla)$. By Theorem 8.8 one has $Q^+ = P^+$ and $0 = \|Q^-\|_{\mathrm{L}^2(M)} = \|P^-\|_{\mathrm{L}^2(M)}$. Thus, the odd part of P vanishes and $P = P^+ = Q^+ = Q$. □

Finally, a variant of Theorem 8.8 for two-dimensional tori will be shown.

Definition 8.10. A dual lattice vector $a \in \mathcal{L}'$ shall be called *minimal* if $a(\mathcal{L}) = \mathbb{Z}$. Let $\mathcal{L}'_+$ be a set of minimal dual lattice vectors such that for any minimal $a \in \mathcal{L}'$ either $a \in \mathcal{L}'_+$ or $-a \in \mathcal{L}'_+$ but not both. For $a \in \mathcal{L}'_+$ and a given potential Q on M define a *one-dimensional potential* on the one-dimensional torus $\mathbb{Z}\backslash\mathbb{R}$ by setting

$$Q^a(s) := \sum_{k \in \mathbb{Z}\backslash\{0\}} Q_{ka} \mathrm{e}^{-2\pi \mathrm{i} k s} .$$

This sum converges with respect to the $\mathrm{L}^2(\mathbb{Z}\backslash\mathbb{R})$-norm because the maps $\{s \mapsto \mathrm{e}^{-2\pi \mathrm{i} k s}\}_{k \in \mathbb{Z}\backslash\{0\}}$ form an orthonormal basis of $\mathrm{L}^2(\mathbb{Z}\backslash\mathbb{R})$ and $\{Q_{ka}\}_{k \in \mathbb{Z}\backslash\{0\}}$ is a subset of the set of Fourier coefficients of Q.

Lemma 8.11. Any smooth potential on M can be written as

$$Q = Q_0 + \sum_{a \in \mathcal{L}'_+} Q^a \circ a$$

and for any minimal a one has $(Q^a)^\pm = (Q^\pm)^a$.

Proof. First note that $Q^a \circ a$ is a well-defined function on M because for $x, y \in \mathbb{R}^n$ with $x = y + l$ for an $l \in \mathcal{L}$ one has $a(l) \in \mathbb{Z}$ and thus $Q^a \circ a(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} Q_{ka} \mathrm{e}^{-2\pi \mathrm{i} k(a(y)+a(l))} = Q^a \circ a(y)$. The last note is a simple calculation:

$$(Q^a)^{\pm}(s) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \tfrac{1}{2}(Q_{ka}\mathrm{e}^{-2\pi \mathrm{i} ks} \pm Q_{ka}\mathrm{e}^{2\pi \mathrm{i} ks}) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \tfrac{1}{2}(Q_{ka} \pm Q_{-ka})\mathrm{e}^{-2\pi \mathrm{i} ks} = (Q^{\pm})^a$$

by Remark 8.7. To show the decomposition into one-dimensional potentials it is sufficient to prove that $\mathcal{L}'$ is the disjoint union of $\{0\}$ and all $\mathbb{Z}\setminus\{0\} \cdot a$ with $a \in \mathcal{L}'_+$ since

$$Q^a \circ a = \sum_{k\in\mathbb{Z}\setminus\{0\}} Q_{ka} E_{ka} \,.$$

First, assume there is a $c \in \mathbb{Z}\setminus\{0\} \cdot a \cap \mathbb{Z}\setminus\{0\} \cdot b$ for $a \neq b \in \mathcal{L}'_+$, i. e. assume that $c = k_a a = k_b b$ with $k_a, k_b \in \mathbb{Z}\setminus\{0\}$. By the minimality of a and b there are two lattice vectors l_a and l_b such that $a(l_a) = 1 = b(l_b)$ and thus

$$k_a = k_b b(l_a) \quad \wedge \quad k_b = k_a a(l_b) \qquad \text{giving} \qquad k_b | k_a \quad \wedge \quad k_a | k_b \,.$$

However, this means that $k_a = \pm k_b$ and $a = \pm b$, which is not possible by the definition of $\mathcal{L}'_+$. Hence, the considered union is disjoint.

Second, choose any $a \in \mathcal{L}'$ and some basis $\{W_1, \dots, W_n\}$ of $\mathcal{L}$. Set $a_i := a(W_i)$ and let k be the greatest common divisor of the a_i. Define a new dual lattice vector b by setting $b(W_i) := a_i/k$. Since the greatest common divisor of the $b(W_i)$ is 1 by construction, there are $k_1, \dots, k_n \in \mathbb{Z}$ with $\sum_{i=1}^n k_i b(W_i) = b(\sum_{i=1}^n k_i W_i) = 1$. Therefore, $1 \in b(\mathcal{L})$ and b is minimal. This shows that $b \in \mathcal{L}'_+$ or $-b \in \mathcal{L}'_+$ and, since $a \in \mathbb{Z}\setminus\{0\} \cdot (\pm b)$, the union contains all of $\mathcal{L}'$. □

The previous Lemma holds for arbitrary tori. In the next Lemma, however, it is necessary to assume that M is two-dimensional.

Lemma 8.12. For any line bundle ω with Chern invariant factor $r_1 = 1$ over an arbitrary two-dimensional torus M and any smooth potential Q with vanishing mean on this torus one has the following equation of functions on M:

$$\int_0^1 Q(.-tGka)\,\mathrm{d}t = Q^a \circ a \qquad \text{for all } k \in \mathbb{Z}\setminus\{0\} \text{ and all } a \in \mathcal{L}'_+,$$

where G is the inverse of the map $F\colon \mathcal{L} \to \mathcal{L}'$ with $F(l) = \Omega(l, .)$. In particular, Q^a is a real-valued function.

Proof. By the minimality of a there is an $l \in \mathcal{L}$ with $1 = a(l) = \Omega(Ga, l)$. Since Ω is skew-symmetric, Ga and l form a basis of $\mathbb{R}^2$. They even form a basis of $\mathcal{L}$: Let $k \in \mathcal{L}$ be arbitrary. There are two real numbers $\alpha, \beta \in \mathbb{R}$ such that $k = \alpha Ga + \beta l$. But

$$-\beta = \Omega(k, Ga) \in \mathbb{Z} \qquad \text{and} \qquad \alpha = \Omega(k, l) \in \mathbb{Z}\,.$$

As $F\colon \mathcal{L} \to \mathcal{L}'$ is an isomorphism ($r_1 = 1$), a and Fl form a basis of $\mathcal{L}'$ and every $b \in \mathcal{L}'$ satisfies: $b(Ga) = 0$ if and only if $b = \alpha \cdot a$ with $\alpha \in \mathbb{Z}$. Therefore,

$$\int_0^1 Q(.-tGka)\,\mathrm{d}t = \int_0^1 \sum_{b\in\mathcal{L}'} Q_b E_b(.-tGka)\,\mathrm{d}t = \sum_{b\in\mathcal{L}'} Q_b E_b \int_0^1 E_b(-tGka)\,\mathrm{d}t = \sum_{i\in\mathbb{Z}\setminus\{0\}} Q_{ia}E_{ia} = Q^a \circ a\,.$$

The left hand side of this equation is real-valued and it follows that $Q^a \circ a$ is real-valued. From $a(l) = 1$ one concludes that $Q^a(s) = Q^a \circ a(s\cdot l) \in \mathbb{R}$ for all $s \in \mathbb{R}$. □

Theorem 8.13. Let ω be a line bundle over a two-dimensional torus M with Chern invariant factor $r_1 = 1$. Let Q be any smooth potential on M. If the torus M has a nondegenerate length spectrum, then one has for all $P \in \mathrm{Iso}(Q,\omega,\nabla^{\mathrm{D}})$ and $a \in \mathcal{L}'_+$ that $|(P^-)^a| = |(Q^-)^a|$, where the distinguished connection ∇^{D} is the unique translation- and $\mathbb{Z}_2$-invariant connection on ω.

Proof. Assume for any $k \in \mathbb{Z}$ and $a \in \mathcal{L}'_+$ that

$$\frac{1}{\mathrm{Vol}\,M}\int_M E_{ka}(x)\Big(\int_0^1 Q^-(x-tGka)\,\mathrm{d}t\Big)^2\,\mathrm{d}V(x)$$

is a spectral invariant of $\mathrm{Spec}_0(Q,\omega)$, see [GGKW08] for a proof. The previous Lemma shows that those invariants are equal to

$$\Big\langle\big((Q^-)^a \circ a\big)^2, E_{-ka}\Big\rangle_{\mathrm{L}^2(M)}.$$

Per definitionem $((Q^-)^a \circ a)^2$ lies in the subspace of $\mathrm{L}^2(M)$ spanned by $\{E_{-ka}\}_{k\in\mathbb{Z}}$, which proves that all $((Q^-)^a \circ a)^2 = ((Q^-)^a)^2 \circ a$ are determined by those invariants. Hence, the $|(Q^-)^a|$ are spectral invariants of $\mathrm{Spec}_0(Q,\omega)$. □

Corollary 8.14. If M is a two-dimensional torus with nondegenerate length spectrum, ω a line bundle with Chern invariant factor $r_1 = 1$ over this torus and if one considers only the translation- and $\mathbb{Z}_2$-invariant connection, then there are no nontrivial continuous isospectral deformations within the space of smooth potentials.

Proof. By Lemma 4.11 the only translation- and $\mathbb{Z}_2$-invariant connection is the distinguished connection ∇^{D}. If $\{Q_t\}_{t\in[0,\varepsilon]}$ is an $\mathrm{L}^2(M)$-continuous family of smooth and mutually ∇^{D}-isospectral potentials, it easily follows that $\{\check{Q}_t\}$ and $\{Q_t^-\}$ are also continuous. For $t, s \in [0,\varepsilon]$

$$\|(Q_t^-)^a \circ a - (Q_s^-)^a \circ a\|^2_{\mathrm{L}^2(M)} = \sum_{k\in\mathbb{Z}\setminus\{0\}} |(Q_t^-)_{ka} - (Q_s^-)_{ka}|^2 \le \|Q_t^- - Q_s^-\|^2_{\mathrm{L}^2(M)}$$

shows that $\{(Q_t^-)^a \circ a\}$ is also a continuous family.

By Theorem 8.13 $|(Q_t^-)^a| = |(Q_0^-)^a|$. Thus, all one-dimensional potentials $(Q_t^-)^a$ in this family have the same zeros Z and are equal up to sign on the connected subsets of $M\backslash Z$. The continuity of the deformation implies that $(Q_t^-)^a \circ a = (Q_0^-)^a \circ a$. Thus, Lemma 8.11 and $(Q_t^-)_0 = 0$ yield

$$Q_t^- = \sum_{a \in \mathcal{L}'_+} (Q_t^-)^a \circ a = \sum_{a \in \mathcal{L}'_+} (Q_0^-)^a \circ a = Q_0^- \,.$$

Since $Q_t^+ = Q_0^+$ by Theorem 8.8, one concludes $Q_t = Q_0$. □

Corollary 8.15. Assume that M is a two-dimensional torus with nondegenerate length spectrum and ω a line bundle with Chern invariant factor $r_1 = 1$ over this torus. Let Q be a potential on this torus such that Q^- is one-dimensional in the sense that $Q^- = (Q^-)^a \circ a$ for an $a \in \mathcal{L}'_+$. If $P \in \mathrm{Iso}(Q, \omega, \nabla^{\mathrm{D}})$, then $P^- = (P^-)^a \circ a$ and $|P^-| = |Q^-|$. If P^- and Q^- are real analytic, then $P = Q$ or $P = \check{Q}$.

Proof. By Theorem 8.13 $|(P^-)^b| = |(Q^-)^b|$ for any $b \in \mathcal{L}'_+$. In particular, $(P^-)^b = 0$ for any $b \in \mathcal{L}'_+\backslash\{a\}$, which shows with $(P^-)_0 = 0$ that both $P^- = (P^-)^a \circ a$ and $|P^-| = |Q^-|$. This means that $P^-(x) = \pm Q^-(x)$ for all $x \in M$. Since both functions are analytic, only one of those two cases can occur. Thus, $P^- = Q^-$ or $P^- = -Q^-$, which gives together with $P^+ = Q^+$ by Theorem 8.8 that $P = Q$ or $P = Q^+ - Q^- = \check{Q}$. □

Example 7.12 shows that Corollary 8.15 does not hold if the length spectrum is degenerate.

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