Numerical Algorithms 3 (1992) 29–44

A reliable modification of the cross rule for rational Hermite interpolation

Bernhard Beckermann and Carsten Carstensen

Institut für Angewandte Mathematik, Universität Hannover, Welfengarten 1, D-3000 Hannover 1, Germany

Dedicated to Professor G. Mühlbach on the occasion of his 50th birthday

Claessens' cross rule [8] enables simple computation of the values of the rational interpolation table if the table is normal, i.e. if the denominators in the cross rule are non-zero. In the exceptional case of a vanishing denominator a singular block is detected having certain structural properties so that some values are known without further computations. Nevertheless there remain entries which cannot be determined using only the cross rule.

In this note we introduce a simple recursive algorithm for computation of the values of neighbours of the singular block. This allows to compute entries in the rational interpolation table along antidiagonals even in the presence of singular blocks. Moreover, in the case of non-square singular blocks, we discuss a facility to monitor the stability.

Subject classification: AMS(MOS) 65D05, 30E05; CR G1.2.

Keywords: Rational interpolation; Newton–Padé approximation; cross rule; reliability; singular rules.

1. Introduction

Let $(z_i)_{i \in \mathbb{N}_0}$ be a sequence of (not necessarily distinct) knots in the complex plane. Let f be a function which is sufficiently smooth in a neighbourhood of these knots. Then, the *rational interpolation problem* consists in computing polynomials p and q with given maximal degree m and n such that $f - p/q$ has the zeros $z_0, z_1, \dots, z_{m+n}$ counting multiplicities.

It is well-known (cf. [15]) that this problem is closely connected with the *linearized rational interpolation problem* (or *Newton–Padé approximation problem*) which is also under consideration here. As was pointed out by Claessens [5, theorem 1], for any $m, n \in \mathbb{N}_0$ there exist unique polynomials $p^*_{m,n}$ and $q^*_{m,n}$ of "minimal degree"

$$deg\, p^*_{m,n} \leqslant m \quad \text{and} \quad deg\, q^*_{m,n} \leqslant n,$$

$q^*_{m,n}$ being a monic polynomial and $a_{m,n}$ being the leading coefficient of $p^*_{m,n}$, which satisfy the *interpolation conditions*

$$(p^*_{m,n} - f \cdot q^*_{m,n}) \text{ has the zeros } z_0, \ldots, z_{m+n} \text{ counting multiplicities,}$$

such that other solutions of the linearized rational (m, n)-interpolation problem are of the form $s \cdot p^*_{m,n}$, $s \cdot q^*_{m,n}$, where s is a polynomial of degree less than or equal to $\min\{m - deg\, p^*_{m,n}, n - deg\, q^*_{m,n}\}$. $p^*_{m,n}$, $q^*_{m,n}$ is the *minimal solution* of the linearized rational (m, n)-interpolation problem.

Throughout this note we are interested in a certain value of the rational interpolant, i.e. in the value of the meromorphic function

$$r_{m,n} := \frac{p^*_{m,n}}{q^*_{m,n}}$$

at a point $z \in \mathbb{C} \backslash \{z_0, \ldots, z_{m+n}\}$. Note that two solutions of the linearized rational (m, n)-interpolation problem have the same reduced form. Therefore, $r_{m,n}$ is called *the solution* of the (m, n)-rational interpolation problem.

The polynomials of the *minimal solution* $p^*_{m,n}$ and $q^*_{m,n}$ are not necessarily irreducible. But their greatest common divisor $d_{m,n}$ only has zeros from $z_0, \ldots, z_{m+n}$ which are called *unattainable points* of the rational interpolant; cf. [5, theorem 2].

Note also that $z \notin \{z_0, \ldots, z_{m+n}\}$ excludes the exceptional case $p^*_{m,n}(z) = 0 = q^*_{m,n}(z)$.

So far we have fixed the maximal degrees m and n. Actually, as in most applications, we wish to compute *the table of rational interpolants*, i.e. to compute some elements of $(r_{m,n}(z) \mid m, n \in \mathbb{N}_0)$. This can be done recursively by various algorithms, like, e.g., Claessens' cross rule [8]:

$$\frac{1}{z - z_{m+n}} \left\{ \frac{1}{W(z) - C(z)} - \frac{1}{N(z) - C(z)} \right\}$$
$$= \frac{1}{z - z_{m+n+1}} \left\{ \frac{1}{S(z) - C(z)} - \frac{1}{E(z) - C(z)} \right\},$$

where neighbouring interpolants are identified with compass points

$$\begin{array}{ccccc} & N(z) & & & r_{m-1,n}(z) & \\ W(z) & C(z) & E(z) := r_{m,n-1}(z) & r_{m,n}(z) & r_{m,n+1}(z). \\ & S(z) & & & r_{m+1,n}(z) & \end{array}$$

Here, the initializations $r_{-1,n}(z) := 0$ and $r_{n,-1}(z) := \infty$ $(n \in \mathbb{N}_0)$ are used and the quantities $r_{n,0}(z)$ are determined by, e.g., the Newton interpolation formula.

Unfortunately, Claessens' cross rule fails if certain related values of the table of rational interpolants are equal to each other; such neighbouring entries will be combined in so-called *singular blocks*.

The aim of this note is the introduction of an algorithm for computing the remaining entries in the rational interpolation table, i.e. we will generalize Cordellier's identity [9]. This reliable modification of the cross rule can be seen complementary to Werner's and Arndt's modification of the continued fraction of Thiele [18,1]. It was shown independently by Gutknecht [14], Van Barel and Bultheel [16] and Beckermann [4, pp. 266–272] that such a modified continued fraction can be given also without reordering of interpolation knots. In this context, let us also mention a modification of a class of rhombus-type algorithms due to Cordellier [10;11, Annexe 6] including Claessens' ϵ-algorithm which yields a quite different approach for computing the table of values of rational interpolants in the presence of singular blocks.

The paper is organized as follows: In section 2 some preliminaries, partly known, are stated which give a brief review of the structure of a singular block as it was essentially pointed out by Claessens [6]. We describe the block structure in detail and fix the notation. Then, in section 3 – as our main result – we prove dependencies for certain polynomials related to neighbours of singular blocks and construct the modification of Claessens' cross rule. The Padé case of confluent knots is considered more explicitly and Cordellier's identity is obtained for the Padé approximation problem as a particular case. We illustrate this note with a numerical example in section 4 where a quite complicated singular table is discussed.

2. Singular blocks

In this section we summarize some facts for later use which are partly well-known. For $j < k$ we define monic polynomials $\omega_{j,k}$ of degree $k-j$ by

$$\omega_{j,k}(z) := (z - z_j) \cdot (z - z_{j+1}) \cdot \cdots \cdot (z - z_{k-1}),$$

$z \in \mathbb{C}$, $j, k \in \mathbb{N}_0$, while by convention empty products equal one.

The first result concerns relations between neighbours and introduces the polynomials $\alpha^k_{m,n}$ which will be essential in the sequel.

THEOREM 1

(a) For any $m, n \in \mathbb{N}_0$

$$r_{m+1,n} - r_{m,n} = \begin{cases} 0 & \text{iff } deg\, p^*_{m+1,n} < m+1 \text{ or } deg\, q^*_{m,n} < n, \\ a_{m+1,n} \dfrac{\omega_{0,m+n+1}}{q^*_{m,n} \cdot q^*_{m+1,n}} & \text{otherwise}; \end{cases} \tag{1}$$

$$r_{m,n} - r_{m,n+1} = \begin{cases} 0 & \text{iff } deg\, p^*_{m,n} < m \text{ or } deg\, q^*_{m,n+1} < n+1, \\ a_{m,n} \dfrac{\omega_{0,m+n+1}}{q^*_{m,n} \cdot q^*_{m,n+1}} & \text{otherwise}; \end{cases} \tag{2}$$

$$r_{m+1,n} - r_{m,n+1} = \begin{cases} 0 & \text{iff } deg\, p^*_{m+1,n} < m+1 \text{ or } deg\, q^*_{m,n+1} < n+1, \\ a_{m+1,n} \dfrac{\omega_{0,m+n+2}}{q^*_{m+1,n} \cdot q^*_{m,n+1}} & \text{otherwise.} \end{cases} \tag{3}$$

(b) For any $m, n, k \in \mathbb{N}_0$ there exists a unique polynomial $\alpha^k_{m,n}$ which is identical zero if $deg\, p^*_{m+k,n-1} + deg\, q^*_{m-1,n+k} < m+n+k$ and monic and of degree

$$deg\, p^*_{m+k,n-1} + deg\, q^*_{m-1,n+k} - (m+n+k)$$

otherwise, such that

$$r_{m+k,n-1} - r_{m-1,n+k} = a_{m+k,n-1} \cdot \alpha^k_{m,n} \frac{\omega_{0,m+n+k}}{q^*_{m+k,n-1} \cdot q^*_{m-1,n+k}}. \tag{4}$$

Proof

Assertions (1), (2) have been given in [7, lemma 2; 17] supposing that there are no singular blocks. We prove only (b) since (3) is a particular case of (4) and (1), (2) can be shown by applying similar techniques.

Let $h := q^*_{m-1,n+k} \cdot p^*_{m+k,n-1} - p^*_{m-1,n+k} \cdot q^*_{m+k,n-1}$. Then h has the zeros $z_0, \dots, z_{m+n+k-1}$ because $f \cdot q^*_{m-1,n+k} - p^*_{m-1,n+k}$ and $f \cdot q^*_{m+k,n-1} - p^*_{m+k,n-1}$ have these zeros by construction and $h = q^*_{m-1,n+k} \cdot (p^*_{m+k,n-1} - f \cdot q^*_{m+k,n-1}) + q^*_{m+k,n-1} \cdot (f \cdot q^*_{m-1,n+k} - p^*_{m-1,n+k})$. Note that this remains valid if some or all knots coincide. The leading coefficient of h is $a_{m+k,n-1}$ and hence $\alpha^k_{m,n} := h/(a_{m+k,n-1} \cdot \omega_{0,m+n+k})$ has the claimed properties. □

Remark

Theorem 1 states that two neighbours in the rational interpolation table (like $r_{m,n}$, $r_{m+1,n}$ or $r_{m,n}$, $r_{m,n+1}$) are identical iff they are equal at only one arbitrary point $z \in \mathbb{C} \setminus \{z_0, z_1, z_2, \dots\}$. Note that this remains valid if z is a common pole.

In the second theorem we describe the form of singular blocks in the non-normal rational Hermite interpolation table. We will not give a proof of this theorem which is an immediate consequence of the local result of theorem 1. Instead we refer to the original proof of [6] and the more compact results of Gutknecht [14]. Also, the structure of the Newton–Padé table is covered by [3] where a more general interpolation problem has been studied. The notation chosen in this theorem is extended in view of the new algorithm.

THEOREM 2

Suppose that $m, n \in \mathbb{N}_0$ with $r_{m,n}(z) \neq r_{m-1,n}(z)$ and $r_{m,n}(z) \neq r_{m,n-1}(z)$. There exists a (minimal) $p \in \mathbb{N} \cup \{+\infty\}$ such that $r_{m,n}$ interpolates f at exactly p of the knots $z_{m+n}, \dots, z_{m+n+2p-1}$ counting multiplicities, namely z_{m+n+j} for $j \in A \subset \{0, 1, \dots, 2p-1\}$. Let $U := \{0, 1, \dots, 2p-1\} \setminus A$ denote the set of unattainable knots.

$r_{m,n}$ and A, U induce a singular block consisting of squares with a common main diagonal as follows. For $0 \leqslant j \leqslant 2p$ let

$$l_j := card\{i < j \mid i \in U\},$$
$$k_j := card\{i < j \mid i \in A\} - l_j = j - 2l_j.$$

Then $k_0 = k_{2p} = 0$ and (since p is minimal) $k_1, k_2, \ldots, k_{2p-1} > 0$. For $0 < j < 2p$, $0 \leqslant i < k_j$ there holds

$$r_{m+l_j+k_j-1-i,n+l_j+i} = C, \tag{5}$$
$$(p^*_{m+l_j+k_j-1-i,n+l_j+i};\ q^*_{m+l_j+k_j-1-i,n+l_j+i}) = d_j(p^*_{m,n};\ q^*_{m,n}),$$
$$d_j(z) = \prod_{i<j, i\in U}(z - z_{m+n+i}), \tag{6}$$

while for $0 \leqslant j \leqslant 2p$ there holds

$$SW_j := r_{m+l_j+k_j,n+l_j-1} \neq C \neq r_{m+l_j-1,n+l_j+k_j} =: NE_j. \tag{7}$$

Hence k_j describes the number of occurrences of C, $l_j = deg\ d_j$ the number of unattainable knots with respect to antidiagonal no.$(m+n+j-1)$. Locally, for the growth of the block we have to distinguish between two cases [14, p. 555]:
(a) $j \in A$ (the block becomes "wider"): here $k_{j+1} = k_j + 1$ and $l_{j+1} = l_j$.
(b) $j \in U$ (the block becomes "narrower"): here $k_{j+1} = k_j - 1$ and $l_{j+1} = l_j + 1$.
□

EXAMPLE 1

As in [6, example 2], let $z_{0+4i} := -3$, $z_{1+4i} := 0$, $z_{2+4i} := 1$, $z_{3+4i} := 2$, $z_{12+i} := 3$ for $i = 0, 1, 2$ and $f(-3) = 1/2$, $f(0) = 2$, $f(1) = 3/2$, $f(2) = 4/3$, $f'(-3) = -1/4$, $f'(0) = 1$, $f'(1) = 1$, $f'(2) = -1/9$, $f''(-3) = 1$, $f''(0) = 2$, $f''(1) = 1$, $f''(2) = 2/27$, $f(3) = 5/4$, $f'(3) = 1$, $f''(3) = 1$. For $z = -2$, applying Claessens' cross rule and proceeding by antidiagonals from south west to north east, we obtain the incomplete table of values of rational interpolants as shown in table 1.

The entries marked by • are singular since Claessens' cross rule fails and no information from theorem 2 is available. It is stated in theorem 2 that the singular entries build a symmetric union of squares which are overlapping along the diagonal. Using the initializations and the fact that all entries in the singular block are equal to 0, we can determine the western, southern and some of the northern neighbours such that the size of the singular block is known. With the notation of theorem 2, the singular block appears as follows:

	NE_0	NE_1	NE_2	NE_3	
SW_0	C	C	C	NE_4	
SW_1	C	C	C	NE_5	NE_6
SW_2	C	C	C	C	NE_7
SW_3	SW_4	SW_5	C	C	NE_8
		SW_6	SW_7	SW_8	

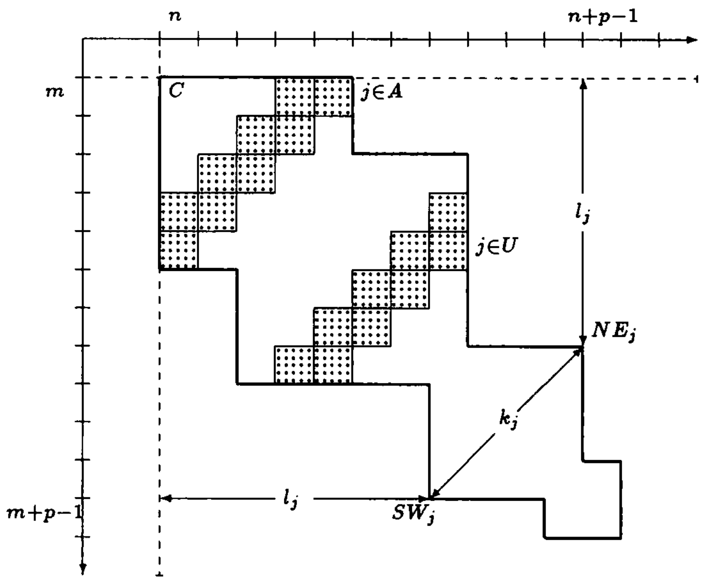

Fig. 1. An example of a singular block.

The first entry C has the proposition $(m, n) := (1, 1)$ and the boundary is counted starting with 0 and ending with $2p$ once along the top of the structure and once along the bottom of the structure writing NE_j or SW_j for $j = 0, \dots, 2p$; $p := 4$. Moreover, we have $(k_0, \dots, k_8) = (0, 1, 2, 3, 2, 1, 2, 1, 0)$, $(l_0, \dots, l_8) = (0, 0, 0, 0, 1, 2, 2, 3, 4)$, $A = \{0, 1, 2, 5\}$ and $U = \{3, 4, 6, 7\}$. In example 2 we will show how to compute $\mathrm{NE}_4, \dots, \mathrm{NE}_8$ (and hence the rest of the table), using our new modification as a singular rule. □

The next theorem provides an explicit representation of the quantities involved in Claessens' cross rule and Cordellier's singular rule.

Table 1

$r_{m,n}(-2)$	$n=-1$	0	1	2	3	4	5	6
$m=-1$		0	0	0	0	0	0	0
0	∞	1/2	2/3	6/7	12/11	12/23	•	
1	∞	1	0	0	0	•		
2	∞	3/2	0	0	0	•		
3	∞	2	0	0	0	0	•	
4	∞	1	8/13	120/193	0	0	•	
5	∞	−2/3	156/251	240/389	240/371	80/121	600/1013	•
6	∞	−35/4	64/195	102/157	816/1247	34224/52439	4800/7471	0

THEOREM 3

With the notation of theorem 2, let the functions a_j be defined by

$$\frac{1}{SW_j(z)-C(z)}-\frac{1}{NE_j(z)-C(z)}=a_j(z)\frac{d_j(z)^2\cdot q^*_{m,n}(z)^2}{\omega_{0,m+n+j}(z)\cdot a_{m,n}}, \tag{8}$$

$0\leqslant j\leqslant 2p$. Then a_j is a monic polynomial of degree k_j.

Proof

Let $0<j<2p$. By assumption the left hand side of (8) is equal to

$$\frac{r_{m+l_j-1,n+l_j+k_j}-r_{m+l_j+k_j,n+l_j-1}}{\left(r_{m+l_j+k_j,n+l_j-1}-r_{m+l_j+k_j-1,n+l_j}\right)\cdot\left(r_{m+l_j-1,n+l_j+k_j}-r_{m+l_j,n+l_j+k_j-1}\right)}.$$

By theorem 1 any of the occurring differences can be expressed through certain right hand sides of (3), (4), where $\alpha^{k_j}_{m+l_j,n+l_j}=:a_j$ with $deg\ a_j = deg\ p^*_{m+l_j+k_j,n+l_j-1}+deg\ q^*_{m+l_j-1,n+l_j+k_j}-(m+l_j+n+l_j+k_j)=k_j$. Using (6) we obtain (8). For $j=0$, $j=2p$, assertion (8) follows directly from (1), (2) and (3), in this case a_j equals one. □

Equation (8) for $j=0$, $j=2p$ yields the following singular rule including Claessens' cross rule [8] for $p=1$.

COROLLARY 1

$$\prod_{i\in A}(z-z_{m+n+i})^{-1}\cdot\left\{\frac{1}{\mathrm{SW}_0-\mathrm{C}}-\frac{1}{\mathrm{NE}_0-\mathrm{C}}\right\}$$
$$=\prod_{i\in U}(z-z_{m+n+i})^{-1}\cdot\left\{\frac{1}{\mathrm{SW}_{2p}-\mathrm{C}}+\frac{1}{\mathrm{NE}_{2p}-\mathrm{C}}\right\}. \quad \square$$

The Kronecker algorithm is a coefficient algorithm for computing rational interpolants on antidiagonals. There exists a modification based on the Euclidean algorithm which can be applied in the presence of singularities (cf. [2, p. 7]). It can be shown that the auxiliary polynomial required for the basic relation of the reliable Kronecker algorithm on antidiagonal no.$(m+n+j-1)$ coincides with our polynomial a_j, $j=1,2,\dots,2p-1$.

3. The reliable modification

For $p\geqslant 2$, theorem 3 leads us to consider relations between $a_0,\dots,a_{2p}$. In the sequel, let

$$R(t):=\frac{a_{m,n}\cdot\omega_{0,m+n}(t)}{q^*_{m,n}(t)\cdot q^*_{m,n}(t)}, \tag{9}$$

$$b_{\mu,\nu} := \lim_{t\to\infty} t^{\nu-\mu} \cdot r_{\mu,\nu}(t) \in \{0, \infty, a_{\mu,\nu}\}, \quad \mu, \nu \in \mathbb{N}_0, \tag{10}$$

$$c_j := \frac{b_{m,n}}{b_{m+l_j+k_j,n+l_j-1}} + \frac{b_{m+l_j-1,n+l_j+k_j}}{b_{m,n}} \in \mathbb{C}. \tag{11}$$

THEOREM 4

For $0 \leqslant j \leqslant 2p$, there holds

$$a_j(t) \frac{\prod_{\nu<j,\nu\in U} (t - z_{m+n+\nu})}{\prod_{\nu<j,\nu\in A} (t - z_{m+n+\nu})} \frac{\mathrm{C}(t)}{R(t)} = 1 + c_j t^{-1-k_j} + O(t^{-2-k_j})_{t\to\infty}.$$

Proof

From $\mathrm{NE}_j \neq \mathrm{C}$ and (3) we conclude

$$\lim_{t\to\infty} t^{n-m+k_j+1} \cdot \mathrm{NE}_j(t) = \lim_{t\to\infty} t^{n-m+k_j+1} \cdot r_{m+l_j-1,n+l_j+k_j}(t)$$
$$= b_{m+l_j-1,n+l_j+k_j} \neq \infty.$$

Thus

$$\frac{\mathrm{NE}_j(t)}{\mathrm{C}(t)} = \frac{b_{m+l_j-1,n+l_j+k_j}}{b_{m,n}} t^{-1-k_j} + O(t^{-2-k_j})_{t\to\infty},$$

$$b_{m,n} \neq 0,\ b_{m+l_j-1,n+l_j+k_j} \neq \infty,$$

and similarly

$$\frac{\mathrm{C}(t)}{\mathrm{SW}_j(t)} = \frac{b_{m,n}}{b_{m+l_j+k_j,n+l_j-1}} t^{-1-k_j} + O(t^{-2-k_j})_{t\to\infty},$$

$$b_{m,n} \neq \infty,\ b_{m+l_j+k_j,n+l_j-1} \neq 0.$$

Because of $k_j \geqslant 0$ we obtain

$$\frac{\mathrm{C}(t)}{\mathrm{SW}_j(t) - \mathrm{C}(t)} - \frac{\mathrm{C}(t)}{\mathrm{NE}_j(t) - \mathrm{C}(t)}$$

$$= \frac{\mathrm{C}(t)/\mathrm{SW}_j(t)}{1 - \mathrm{C}(t)/\mathrm{SW}_j(t)} + 1 + \frac{\mathrm{NE}_j(t)/\mathrm{C}(t)}{1 - \mathrm{NE}_j(t)/\mathrm{C}(t)}$$

$$= 1 + c_j t^{-1-k_j} + O(t^{-2-k_j})_{t\to\infty}.$$

Hence the assertion follows from (8). □

COROLLARY 2

For $0 \leqslant i,\ j \leqslant 2p$ with $k_j \leqslant k_i$

$$a_j(t)\frac{\prod_{\nu<j,\nu\in U}(t-z_{m+n+\nu})}{\prod_{\nu<j,\nu\in A}(t-z_{m+n+\nu})} = a_i(t)\frac{\prod_{\nu<i,\nu\in U}(t-z_{m+n+\nu})}{\prod_{\nu<i,\nu\in A}(t-z_{m+n+\nu})} + O(t^{-1-k_j})_{t\to\infty}.$$

□

Remark

Cordellier's singular rule (see (12) below) for Padé approximation is an immediate consequence of corollary 2. To prove it let us have a closer look at the Padé case which is included for $0=z_0=z_1=z_2=\cdots$. A knot which has once become unattainable cannot be an attainable point any longer (cf., e.g. [6,14]). Since we have exactly one knot 0, singular blocks must be squares, finite or infinite (except perhaps those lying at the border of the table). Hence $A=\{0, 1,\dots,p-1\}$, $U=\{p, p+1,\dots,2p-1\}$ and we obtain $k_j=\min\{j, 2p-j\}$. For this special case, corollary 2 with $i=2p-j$ takes the simple form $a_j(t)=a_{2p-j}(t)+O(t^{-1})_{t\to\infty}$. a_j and a_{2p-j} are both polynomials, consequently $a_j=a_{2p-j}$ and with (8)

$$\frac{1}{\mathrm{SW}_j-\mathrm{C}}-\frac{1}{\mathrm{NE}_j-\mathrm{C}}=\frac{1}{\mathrm{SW}_{2p-j}-\mathrm{C}}-\frac{1}{\mathrm{NE}_{2p-j}-\mathrm{C}} \tag{12}$$

$(j=p+1, p+2,\dots,2p)$. Note that for $j=2p$ (12) coincides with corollary 1.

□

Let us return to the more general case of arbitrary knots. A more careful discussion of the asymptotic expansion of theorem 4 as in the proof of corollary 2 yields

COROLLARY 3

For $j\in\{0, 1,\dots,2p-1\}$, there holds

$$a_{j+1}=a_j\cdot\omega_{m+n+j,m+n+j+1}-c_j \qquad \text{if } j\in A,$$
$$a_j=a_{j+1}\cdot\omega_{m+n+j,m+n+j+1}-c_{j+1} \qquad \text{if } j\in U.$$

□

Remark

Corollary 3 enables the computation of the polynomials a_j provided that A and the values $a_j(z)$ are known for $j-1\in A$. Beside the initialization $a_0(t)=1$, we apply the formulas

$$a_{j+1}(t)=a_{j+1}(z)+(t-z_{m+n+j})\cdot a_j(t)-(z-z_{m+n+j})\cdot a_j(z) \quad \text{if } j\in A, \tag{13}$$

$$a_{j+1}(t)=\frac{a_j(t)-a_j(z_{m+n+j})}{t-z_{m+n+j}}, \qquad \text{if } j\in U. \tag{14}$$

Moreover, the unknown values $(a_j(z) \mid j-1 \in U)$ (corresponding to "S-" and "E-neighbours") are uniquely determined by $(a_j(z) \mid j-1 \notin U,\ j \in A)$ (corresponding to "W-" and "N-neighbours") which can be computed using Claessens' cross rule for SW_j and NE_j. □

Of course, (13) and (14) can be used as a singular rule of Claessens' cross rule. Instead, we want to propose a different reliable modification which, for non-square singular blocks, requires less input data and, in addition, provides a facility to monitor the stability of our method if exact arithmetic is not available. This will be done by exploiting the information of corollary 2. In the next definition, we give a representation of the auxiliary quantities involved in our reliable efficient modification of Claessens' cross rule.

DEFINITION

Let the monic polynomial a of degree p be defined by

$$a(t) := \frac{R(t)}{C(t)} \prod_{j \in A} (t - z_{m+n+j}) + O(t^{-1})_{t \to \infty}$$
$$= \frac{a_{m,n} \cdot \omega_{0,m+n}(t)}{p^*_{m,n}(t) \cdot q^*_{m,n}(t)} \prod_{j \in A} (t - z_{m+n+j}) + O(t^{-1})_{t \to \infty}.$$

We denote the attainable and unattainable knots in $(z_{m+n}, \ldots, z_{m+n+2p-1})$, $(z_1^A, \ldots, z_p^A)$ and $(z_1^U, \ldots, z_p^U)$ more explicitly: for $1 \leqslant j \leqslant 2p$

$$z_{m+n+j-1} = \begin{cases} z^A_{l_j+k_j} & \text{if } j-1 \in A, \\ z^U_{l_j} & \text{if } j-1 \in U. \end{cases}$$

Finally, for $0 \leqslant j \leqslant 2p$ let ($[\cdot]$ denotes divided differences)

$$A^j_\nu := \frac{1}{R(z)} \cdot \begin{cases} \left[z,\, z_1^U,\, z_2^U, \ldots, z^U_{l_j},\, z^A_{j+1-\nu}, \ldots, z_p^A\right] a & \text{if } l_j \leqslant \nu \leqslant l_j + k_j, \\ \left[z,\, z_1^U,\, z_2^U, \ldots, z^U_\nu,\, z^A_{l_j+k_j+1}, \ldots, z_p^A\right] a & \text{if } 0 \leqslant \nu \leqslant l_j. \end{cases} \quad \square$$

COROLLARY 4

For $0 \leqslant j \leqslant 2p$ we have

$$A^j_{l_j} = \frac{a_j(z)}{R(z)}$$
$$= \frac{\prod_{i<j, i \in A} (z - z_{m+n+i})}{\prod_{i<j, i \in U} (z - z_{m+n+i})} \left(\frac{1}{\mathrm{SW}_j(z) - \mathrm{C}(z)} - \frac{1}{\mathrm{NE}_j(z) - \mathrm{C}(z)} \right).$$

Proof

Apply theorem 4 to verify that $a_j(t)\Pi_{i<j,i\in U}(t-z_{m+n+i})\Pi_{i\geqslant j,i\in A}(t-z_{m+n+i})-a(t)$ is a polynomial in t of degree at most $p-k_j-1$. □

By definition of the divided differences, the following recurrence relations for the quantities A_ν^j are immediate

$$A_\nu^j=\begin{cases} A_{\nu-1}^{j-1} & \text{if } j-1\in A \text{ and } l_j<\nu\leqslant l_j+k_j,\\ A_{\nu-1}^{j-1}+\left(z_\nu^U-z_{l_j+k_j}^A\right)A_\nu^{j-1} & \text{if } j-1\in A \text{ and } 0<\nu\leqslant l_j,\\ A_\nu^{j-1} & \text{if } j-1\in U \text{ and}\\ & \quad\left(0\leqslant\nu<l_j \text{ or } \nu=l_j+k_j\right),\\ A_\nu^{j-1}+\left(z_{l_j}^U-z_{j-\nu}^A\right)A_{\nu+1}^j & \text{if } j-1\in U \text{ and } l_j\leqslant\nu<l_j+k_j. \end{cases} \tag{15}$$

Define for $0\leqslant j\leqslant 2p$

$$\kappa_j:=\max\{k_0,k_1,\dots,k_j\},$$

$$M:=\left\{j\in\{0,\dots,2p\}\mid j=0 \text{ or } k_j>\kappa_{j-1}\right\}$$

$$\subseteq\{j\mid j=0 \text{ or } j-1\in A\}.$$

It is easy to prove with the recurrence relations (by induction on j) that, given $R(z)$ and $(a_j(z)\mid j\in M)$, we can compute all quantities $(A_\nu^j\mid l_j+k_j-\kappa_j\leqslant\nu\leqslant l_j+k_j)$ and therefore the unknown values $(a_j(z)\mid j-1\in U)$. Moreover, if the singular block is not square, the quantities $(a_j(z)\mid j-1\in A,\ j\notin M)$ can be computed by two different rules, which allows us to monitor the stability of our algorithm (cf. the step "CHECK STABILITY" in the algorithm below). These preliminary remarks prove our reliable modification of Claessens' cross rule which is stated below. For an implementation of the following procedure, we can drop all indices j.

THE RELIABLE MODIFICATION

Initialize for $j=0\in A$: $k_0=l_0=\kappa_0=0$,

$$A_0^0=\frac{1}{\mathrm{SW}_0(z)-C(z)}-\frac{1}{\mathrm{NE}_0(z)-C(z)};$$

for $j=1$: $k_1=1$, $l_1=0$, $z_1^A=z_{m+n}$, $\Omega_1=z-z_{m+n}$

FOR $j=1,2,\dots$ until EXIT ($k_j<0$) do

  COMPUTE A_ν^j for $l_j+k_j-\kappa_{j-1}\leqslant\nu\leqslant k_j+l_j$ by (15)

  FILL jth antidiagonal of singular block with $C(z)$ by (5)

  IF $k_{j-1}<k_j$

THEN $(j-1 \in A)$ COMPUTE $\mathrm{NE}_j(z)$ directly (by cross rule or by reliable modification with respect to another singular block)

$$H := \Omega_j \left(\frac{1}{\mathrm{SW}_j(z) - C(z)} - \frac{1}{\mathrm{NE}_j(z) - C(z)} \right)$$

IF $\kappa_{j-1} < k_j$
THEN $(j \in M)$ $\kappa_j = \kappa_{j-1} + 1$, $A^j_{l_j} := H$
ELSE $(j \notin M)$ $\kappa_j = \kappa_{j-1}$, CHECK STABILITY $A^j_{l_j} \stackrel{?}{=} H$

ELSE $(j-1 \in U)$ Compute $\mathrm{NE}_j(z)$ by

$$A^j_{l_j} = \Omega_j \left(\frac{1}{\mathrm{SW}_j(z) - C(z)} - \frac{1}{\mathrm{NE}_j(z) - C(z)} \right)$$

IF $r_{m+l_j+k_j, n+l_j}(z) = C(z)$ AND $k_j \neq 0$
THEN $(j \in A)$ $k_{j+1} = k_j + 1$, $l_{j+1} = l_j$, $z^A_{l_{j+1}+k_{j+1}} = z_{m+n+j}$, $\Omega_{j+1} = \Omega_j(z - z_{m+n+j})$
ELSE $(j \in U$ OR $j = 2p)$ $k_{j+1} = k_j - 1$, $l_{j+1} = l_j + 1$, $z^U_{l_{j+1}} = z_{m+n+j}$, $\Omega_{j+1} = \Omega_j/(z - z_{m+n+j})$

EXIT IF $k_{j+1} < 0$

EXAMPLE 2

We continue example 1 from the last section. The reliable modification proposed above allows us to fill the singular table of values of rational interpolants at $z = -2$ (cf. table 1) as shown in table 2. Since with $m = n = 1$, $r_{m,n}(t) = C(t) = (t+2)/(t+1)$, we have $a(t) = t^4 - 2t^3 - 9t^2 + 30t - 24$, our results can be compared with those obtained directly from the definition. □

Remarks

(i) Provided exact arithmetic the proposed strategy is reliable. It allows recursive computation of the values of the non-normal rational Hermite interpolation table and detects the size of a singular block. Its complexity has the same order as if one computes an equivalent part of a normal rational interpolation table applying Claessens' cross rule.

(ii) If exact arithmetic is not available, then, of course, it is numerically delicate to detect singular blocks and to determine their size. For our strategy, we have to give a modified criterion for the decision $j \in A$ or $j \in U$, taking into account the "global" numerical error accumulated during our computations. But once a singular block is detected, from the step "CHECK STABILITY" we obtain an estimate for the additional "local" numerical error occurring during the computations connected with this singular block. Since the step "CHECK STABILITY" is executed only for singular blocks consisting of several squares, we obtain a (heuristic) tool for monitoring the size of the tail of squares.

(iii) Let us briefly discuss the problem of reordering interpolation knots as proposed by Werner's modification of the Thiele interpolating continued frac-

Table 2

$j=$	0	1	2	3	4	5	6	7	8
$j \in$	A, M	A, M	A, M	U, M	A	U	U	U	
$z_{m+n+j}=$	$z_1^A=1$	$z_2^A=2$	$z_3^A=-3$	$z_1^U=0$	$z_4^A=1$	$z_2^U=2$	$z_3^U=-3$	$z_4^U=0$	
Ω_j		-3	12	12	-6	2	-8	-8	4
$SW_j(-2)$ In	1	3/2	2	1	8/13	120/193	240/389	240/371	80/121
$NE_j(-2)$ In	2/3	6/7	12/11	12/23		120/163			
$A_0^j=$	$-1/2$	3/2	-5	-11	-11	-11			
$A_1^j=$		$-1/2$	3/2	-5	5/2	5/2	-16	-16	-16
$A_2^j=$			$-1/2$	3/2	5/2	1/2	2	2	2
$A_3^j=$				$-1/2$	$-1/2$	$-1/2$	1/2	3	3
$A_4^j=$							$-1/2$	$-1/2$	$-1/2$
$NE_j(-2)$ Out					24/49	120/163	240/449	240/461	80/131

tion. An advantage of this proceeding is that we only have to discuss the simpler case of square blocks. More important, such an algorithm might be numerically more stable since we recognize all zeros during the computation and react immediately, hence this information will not be distroyed by non-exact arithmetic (under certain restrictions, the stability of such a procedure has been proved by Graves-Morris [12]). On the other hand, as pointed out, e.g., by [14, p. 578], after a reordering, the entries on certain antidiagonals will not have any connection to the original problem which is to compute a part of the table $(r_{m,n}(z) \mid m, n \in \mathbb{N}_0)$. In particular, a reordering is not compatible with a progressive form of the algorithm where we add the data z_{K+1}, $f(z_{K+1})$ after having determined the triangular part $(r_{m,n}(z) \mid m, n \in \mathbb{N}_0,\ m+n \leqslant K)$. Our reliable modification does not have this important disadvantage. But it seems to be a new approach to give singular rules for rational interpolation where the technique of reordering interpolation knots is applied implicitly such that a monitoring of the stability is still available.

4. Numerical example

We conclude this note with a numerical example. Based on the proposed algorithm, a recursive strategy allows the calculation of the values in the non-normal rational Hermite-interpolation table along antidiagonals and overcomes singular blocks, thus proving reliability. We present a more delicate example. Note that there remain some entries which cannot be calculated by the cross rule even if we use the initializations for $r_{0,j}(z)$ and $r_{j,0}(z)$, $j = 0, 1, 2, \ldots$.

EXAMPLE 3

Using the knots $z_{5i+j} := j - 2$, $j = 0, 1, 2, 3, 4$, $i = 0, 1, 2$ we take the data from a polynomial of degree 8, namely $f(x) = 10 - 20x - 12x^2 + 17x^3 + 11.5x^4 - 8.25x^5 - 5.25x^6 + 1.25x^7 + 0.75x^8$, but set the value $f'(0) := -10$ (instead of -20). For $z := 3/7$ the algorithm gives the entries of table 3.

Note that, in the beginning, we have no information about the number or size of the singular blocks which are rather complicated. In contrast to our previous example we used a double precision floating point arithmetic in this example. Thus we cannot hope that the decision whether a denominator is zero or whether case A or U is true, is exact; in the present example, instead of equality, we observe at most a difference in the last four digits (six digits for $m \geqslant 15$). In our implementation we consider two values of neighbouring rational interpolants to be equal iff they are equal using simple precision. The step "CHECK STABILITY" at the positions (4; 4), (11; 2), (15; 3) of the center indeed verifies that for large m we obtain a "local" relative error of four digits. □

Table 3

$r_{m,n}(3/7)$	$n=0$	1	2	3	4	5	6
$m=0$	10.0000000000	52.5000000000	7.4242424242	28.114754098	−455.88607594	46.419628061	18.951259848
1	22.142857142	10.0000000000	1.2068965517	1.2068965517	1.2068965517	6.2661740062	4.8241577257
2	4.7959183673	4.7959183673	1.2068965517	1.2068965517	1.2068965517	4.9508030087	4.3398886559
3	4.7959183673	4.7959183673	1.2068965517	1.2068965517	1.2068965517	1.2068965517	4.3770799232
4	3.9462723865	3.9462723865	3.9462723865	5.3461296513	1.2068965517	1.2068965517	6.0848202855
5	3.9462723865	3.9462723865	3.9462723865	7.9183673469	5.9936213023	9.7784535187	0.6845902668
6	3.9462723865	3.9462723865	3.9462723865	0.9141388289	4.8329758872	4.6960412823	3.5338765567
7	5.1043175645	12.052588632	−0.1068857365	−28.611883026	4.7073426628	5.4826151114	2.0001424261
8	4.1117074119	4.8469741916	4.4569064807	3.9335122755	3.8040313447	1.6722505643	55.670530091
9	3.4027001601	0.8030069034	0.8030069034	3.7934708720	3.6963952909	3.3763950697	3.5633868257
10	3.9597772865	0.8030069034	0.8030069034	0.8030069034	0.8030069034	0.8030069034	0.8030069034
11	5.3126788793	3.5480246279	0.8030069034	0.8030069034	0.8030069034	0.8030069034	0.8030069034
12	2.4136040377	2.9252054803	0.8030069034	0.8030069034	0.8030069034	0.8030069034	0.8030069034
13	2.8277575865	2.8277575865	0.8030069034	0.8030069034	0.8030069034	0.8030069034	0.8030069034
14	2.8277575865	2.8277575865	0.8030069034	0.8030069034	0.8030069034	0.8030069034	0.8030069034
15	2.7347843408	2.9409424073	4.1611299875	0.8030069034	0.8030069034	0.8030069034	0.8030069034
16	2.0574078368	3.4372488635	6.6743521677	0.8030069034	0.8030069034	0.8030069034	0.8030069034
17	5.6055704769	2.7286818498	20.120781277	0.8030069034	0.8030069034	0.8030069034	0.8030069034
18	4.2231694483	2.3799680768	7.2739855114	0.8030069034	0.8030069034	0.8030069034	0.8030069034

Remark

Since the original submission of this paper, we have become aware that, for the special case of square singular blocks ($k_j = \min\{j, 2p - j\}$), the assertions of theorems 3 and 4 and their connections to the reliable Kronecker algorithm where independently discovered by Graves-Morris [13]. Since in general the functions $p^*_{m,n}$ and $q^*_{m,n}$ and therefore the rational function R of theorem 4 are not known during the computations, it seems that these results are not sufficient to obtain a reliable modification of Claessens' cross rule, even if we suppose that the table of rational interpolants only contains square singular blocks.

References

[1] H. Arndt, Ein verallgemeinerter Kettenbruch zur rationalen Hermite-Interpolation, Numer. Math. 36 (1980) 99–107.

[2] G.A. Baker and P. Graves-Morris, *Padé Approximants*, Part II (Addison–Wesley, 1981).

[3] B. Beckermann, The structure of the singular solution table of the M-Padé approximation problem, J. Comput. Appl. Math. 32 (1990) 3–15.

[4] B. Beckermann, Zur Interpolation mit polynomialen Linearkombinationen beliebiger Funktionen, Thesis, Univ. Hannover (1990).

[5] G. Claessens, On the Newton–Padé approximation problem, J. Approx. Th. 22 (1978) 150–160.

[6] G. Claessens, On the structure of the Newton–Padé table, J. Approx. Th. 22 (1978) 304–319.

[7] G. Claessens, A new algorithm for osculatory rational interpolation, Numer. Math. 27 (1976) 77–83.

[8] G. Claessens, A useful identity for the rational Hermite interpolation table, Numer. Math. 29 (1978) 227–231.

[9] F. Cordellier, Démonstration algébrique de l'extension de l'identité de Wynn aux table Padé non normal, in: *Padé Approximation and its Applications*, Antwerp 1979, Lecture Notes in Mathematics 765 (Springer, Berlin, 1979) pp. 257–277.

[10] F. Cordellier, Utilisation de l'invariance homographique dans les algorithmes de losange, in: *Padé Approximation and its Applications*, Bad Honnef 1983, Lecture Notes in Mathematics 1071 (Springer, Berlin, 1983) pp. 62–94.

[11] F. Cordellier, Interpolation rationelle et autres questions: aspects algorithmes et numériques, Thèse d'Etat, Lille (1989).

[12] P.R. Graves-Morris, Efficient reliable rational approximation, in: *Padé Approximation and its Applications*, Amsterdam 1980, Lecture Notes in Mathematics 888 (Springer, Berlin, 1980) pp. 28–63.

[13] P.R. Graves-Morris, Talk at the JCAM Conf., Leuven (1984), unpublished.

[14] M.H. Gutknecht, Continued fraction associated with the Newton–Padé table, Numer. Math. 56 (1989) 547–589.

[15] H.E. Salzer, Note on osculatory rational interpolation, Math. Comp. 16 (1962) 486–491.

[16] M. Van Barel and A. Bultheel, A new approach to the rational interpolation problem, J. Comput. Appl. Math. 32 (1990) 281–289.

[17] D.D. Warner, Hermite interpolation with rational functions, Ph.D. Thesis, Univ. of California, San Diego (1974).

[18] H. Werner, A reliable method for rational interpolation, in: *Padé Approximation and its Applications*, Antwerp 1979, Lecture Notes in Mathematics 765 (Springer, Berlin, 1979) pp. 257–277.