Math. Nachr. 163 (1993) 269–279

An Interface Problem in Solid Mechanics with a Linear Elastic and a Hyperelastic Material

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(Received December 4, 1992)
(Revised Version January 12, 1993)

Abstract. The three dimensional interface problem is considered with the homogeneous Lamé system in an unbounded exterior domain and some quasistatic nonlinear elastic material behavior in a bounded interior Lipschitz domain. The nonlinear material is of the Mooney-Rivlin type of polyconvex materials. We give a weak formulation of the interface problem based on minimizing the energy, and rewrite it in terms of boundary integral operators. Then, we prove existence of solutions.

1. Introduction

This paper is concerned with interface (or transmission) problems in three dimensional solid mechanics which consist of a nonlinear elastic problem in a bounded (non-empty) Lipschitz domain $\Omega = \Omega_1$ and the homogeneous linear elasticity problem–subject to Sommerfeld's radiation condition–in the unbounded exterior domain $\Omega_2 := \mathbb{R}^3 \setminus \bar{\Omega}_1$. On the interface $\Gamma := \bar{\Omega}_1 \cap \bar{\Omega}_2$ we have continuity for the displacements and tractions defined as traces of Ω_j for $j = 1, 2$.

We start giving some notations concerning the interior and exterior problem in § 2 and § 3, respectively. In § 4 we give a weak energetic formulation of the interface problem incorporating ideas of [6] for the linear exterior part and [1] for the nonlinear interior part. Using the Calderon projections we rewrite the exterior problem in terms of boundary integral operators related to the Poincare-Steklov operator. This yields a non-local boundary condition for the interior part which can be included in the polyconvex stored energy framework and results in a nice additive term. Due to the properties of this perturbation we can modify Ball's arguments and prove existence for the interface problem at hand in § 5.

Although we only study the Mooney-Rivlin material we remark that the proofs also work for the other polyconvex materials considered in [1, 4, 10, 12].

2. The interior problem

In this paper we consider the Mooney-Rivlin material behavior in Ω as an example of the class of polyconvex materials [1, 4, 10, 11, 12] which became important in applications since Ball's existence theorem in [1].

We need some notations concerning 3×3 matrices. For $A \in \mathbb{R}^{3 \times 3}$ let A_{ij}, $\operatorname{adj} A$ and $\det A$ denote its component in the i-th row and j-the column, its adjugate, and its determinant, respectively. Let I be the 3×3 unit matrix. $\mathbb{R}^{3 \times 3}$ is a Hilbert space with respect to the product ":" defined by $A:B := \sum_{i,j=1,2,3} A_{ij} \cdot B_{ij}$; write $|A|^2 := A:A$.

Let $\varphi: (0, \infty) \to \mathbb{R}$ be a continuous and convex function with

(1) $$\lim_{x \to 0^+} \varphi(x) = +\infty$$

and such that there exist $a > 0$ and $1 < s < \infty$ with

(2) $$\varphi(x) \geq a \cdot x^s$$

for all $x \in (0, \infty)$.

Define the stored energy function $e(F)$ with $F := I + \operatorname{grad} u$ for the Mooney-Rivlin material through

$$e(F) := P(F, \operatorname{adj} F, \det F)$$

where there exist a constant c_0 and positive constants c_1, c_2 with

(3) $$P(F, H, d) := c_0 + c_1 \cdot |F|^2 + c_2 \cdot |H|^2 + \varphi(d).$$

The nonlinear material behavior in Ω and the equilibrium condition with the body forces $f \in (H^1(\Omega; \mathbb{R}^3))^*$ and surface tractions $t \in H^{-1/2}(\Gamma; \mathbb{R}^3)$ are given by minimizing the energy functional

$$E: \begin{cases} \mathbb{H} \times H^{-1/2}(\Gamma; \mathbb{R}^3) \to \mathbb{R} \cup \{\infty\} \\ (u, t) \mapsto \int_\Omega e(I + \operatorname{grad} u)\, d\Omega - \int_\Omega f u \, d\Omega - \langle t, \gamma u \rangle . \end{cases}$$

Here, $\gamma: H^1(\Omega; \mathbb{R}^3) \to H^{1/2}(\Gamma; \mathbb{R}^3)$ is the trace of Γ, $\langle , \rangle$ is the (extended) $L^2(\Gamma; \mathbb{R}^3)$-duality between $H^{1/2}(\Gamma; \mathbb{R}^3)$ and its dual $H^{-1/2}(\Gamma; \mathbb{R}^3)$, and

$$\mathbb{H} := \{u \in H^1(\Omega; \mathbb{R}^3) \mid \operatorname{adj}(I + \operatorname{grad} u) \in L^2(\Omega; \mathbb{R}^{3 \times 3}), \det(I + \operatorname{grad} u) \in L^s(\Omega; \mathbb{R}), \det(I + \operatorname{grad} u) > 0 \text{ a.e. in } \Omega\},$$

$1 < s < \infty$ (cf. (2)).

Definition 1. Given $f \in (H^1(\Omega; \mathbb{R}^3))^*$ and $t \in H^{-1/2}(\Gamma; \mathbb{R}^3)$, the *interior problem* consists in finding $u \in \mathbb{H}$ with

(4) $$E(u, t) = \min\{E(v, t) : v \in \mathbb{H}\}.$$

Remark 1. Other polyconvex materials can also be included in the considerations of the paper; we restrict ourselves to the Mooney-Rivlin material in order to be explicit and to simplify notations.

Remark 2. If the solution of the minimization problem was smooth its Frechét derivative would vanish giving a weak form of equilibrium, namely the Euler-Lagrange equations, cf. [4, Theorem 4.1–1]. From this we see that f is the applied body force and t is the surface force.

Remark 3. We remark that (instead of $\Omega_0 = \emptyset$) we may allows that $\Omega = \Omega_1 \setminus \bar{\Omega}_0$ for some Lipschitz domain Ω_0 lying compactly in Ω_1. Then, we may have Dirichlet, Neumann, or mixed boundary conditions on $\partial\Omega_0$. This causes only obvious modifications of the present analysis.

3. The exterior problem

The exterior problem is the homogeneous Lamé system of linear elasticity [6, 7]

$$\Delta^* u := -\mu_2 \Delta u - (\lambda_2 + \mu_2)\,\mathrm{grad}\,\mathrm{div}\, u = 0 \quad \text{in} \quad \Omega_2$$

with $\Delta = \mathrm{div}\,\mathrm{grad}$ denoting the Laplace operator and μ_2, λ_2 being the positive Lamé constants [4].

Due to the trace lemma $u_2|_\Gamma \in H^{1/2}(\Gamma;\mathbb{R}^3)$ whenever $u_2 \in H^1_{\mathrm{loc}}(\Omega_2;\mathbb{R}^3)$, $H^1_{\mathrm{loc}}(\Omega_2;\mathbb{R}^3)$ denoting the displacements of locally finite energy.

The traction $T_2(u_2)|_\Gamma$ is the conormal derivative defined (for smooth u_2) by

$$T_2(u_2) := 2\mu_2 \partial_n u_2 + \lambda_2 n\,\mathrm{div}\, u_2 + \mu_2 n \times \mathrm{curl}\, u_2 .$$

∂_n denotes the normal derivative, n being the unit normal pointing into Ω_2. In Sobolev spaces the traction can also be defined via the First Green formula [6, 7]. In order to do this, we introduce the following notation

$$a_{ijkl} := \lambda_2 \delta_{ij}\delta_{kl} + \mu_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

$\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. The strain tensor $\varepsilon(u)$ is defined by

$$\varepsilon_{ij}(u) := \frac{1}{2}\,(u_{i,j} + u_{j,i}),$$

$(u_{i,j}) := (u_{i,j})_{i,j=1,2,3} := \mathrm{grad}\, u$. Let the brackets $\langle \cdot, \cdot \rangle$ denote duality between $H^{1/2}(\Gamma;\mathbb{R}^3)$ and its dual $H^{-1/2}(\Gamma;\mathbb{R}^3)$. Then, for $u_2 \in H^1_{\mathrm{loc}}(\Omega_2;\mathbb{R}^3)$ with $\Delta^* u_2 \in L^2_{\mathrm{loc}}(\Omega_2, \mathbb{R}^3)$, $T_2(u_2)|_\Gamma \in H^{-1/2}(\Gamma;\mathbb{R}^3)$ is defined by

$$(5) \qquad \int_{\Omega_2} \Delta^* u_2 v \, d\Omega_2 = \langle T_2(u_2)|_\Gamma, v|_\Gamma \rangle + \Phi_2(u_2, v)$$

for any $v \in H^1(\Omega_2;\mathbb{R}^3)$ with compact support and

$$\Phi_2(u_2, v) = \int_{\Omega_2} \sum_{ijkl=1}^{3} a_{ijkl}\varepsilon_{kl}(u_2)\,\varepsilon_{ij}(v)\, d\Omega_2 .$$

Thus, for any $u_2 \in H^1_{\mathrm{loc}}(\Omega_2, \mathbb{R}^n)$ with $\Delta^* u_2 = 0$ its Cauchy data are

$$(u_2|_\Gamma, T_2(u_2)|_\Gamma) \in H^{1/2}(\Gamma;\mathbb{R}^3) \times H^{-1/2}(\Gamma;\mathbb{R}^3).$$

Following e.g. [2, 6, 7, 8, 9] we consider solutions which are regular at infinity, i.e. (in three dimensions) u_2 satisfies the Sommerfeld's radiation condition

$$(6) \qquad u_2(x) = O\left(\frac{1}{|x|}\right) \quad \text{as} \quad |x| \to \infty .$$

Definition 2. The *exterior problem* consists in finding $u_2 \in \mathscr{L}_2$,

(7) $$\mathscr{L}_2 := \{u_2 \in H^1_{\mathrm{loc}}(\Omega_2; \mathbb{R}^3) : u_2 \textit{ satisfies } (6) \textit{ and } \Delta^* u_2 = 0\},$$

subject to some interface conditions concerning the Cauchy data $(u_2|_\Gamma, T_2(u_2))$ of u_2.

In order to rewrite the exterior problem we follow [2, 6, 7]. The fundamental solution G_2 for the Lamé operator Δ^* has the kernel $G_2(x, y)$, the Kelvin-matrix,

$$G_2(x, y) = \frac{\lambda_2 + 3\mu_2}{8\pi\mu_2(\lambda_2 + 2\mu_2)} \left\{ \frac{1}{|x-y|} I + \frac{\lambda_2 + \mu_2}{\lambda_2 + 3\mu_2} \frac{(x-y)(x-y)^T}{|x-y|^3} \right\}.$$

I is the unit matrix and ^{T} denotes the transposed matrix. Since G is analytic in $\mathbb{R}^3 \times \mathbb{R}^3$ without the diagonal we may define its traction

$$T_2(x, y) := T_{2,y}(G_2(x, y))^T, \quad x \neq y.$$

Due to Green's formula we have the following Somigliana representation formula for $x \in \mathbb{R}^3 \setminus \Gamma$

(8) $$u_2(x) = \langle T_2(x, \cdot), v \rangle - \langle G_2(x, \cdot), \phi \rangle$$

which is proved for Lipschitz domains in [5]. Differentiation of (8) gives a representation formula for the stresses $T_2(u_2)$. By using the classical jump relations for $x \to \Gamma$ and inserting the Cauchy data into these formulas one obtains on Γ

(9) $$\begin{pmatrix} v \\ \phi \end{pmatrix} = \mathscr{C}_2 \cdot \begin{pmatrix} v \\ \phi \end{pmatrix}, \quad \mathscr{C}_2 = \begin{bmatrix} \frac{1}{2} + \Lambda_2 & -V_2 \\ -D_2 & \frac{1}{2} - \Lambda_2' \end{bmatrix},$$

with the Calderón projector $\mathscr{C}_2$ being defined via

$$(V_2\phi)(x) = \langle G_2(x, \cdot), \phi \rangle, \quad (D_2 v)(x) = -T_{2,x}(\langle T_2(x, \cdot), v \rangle),$$
$$(\Lambda_2 v)(x) = \langle T_2(x, \cdot), v \rangle, \quad (\Lambda_2'\phi)(x) = -T_{2,x}(\langle G_2(x, \cdot), \phi \rangle),$$

$(x \in \Gamma)$. $V_2: H^{-1/2}(\Gamma; \mathbb{R}^3) \to H^{1/2}(\Gamma; \mathbb{R}^3)$ is the single layer potential, $\Lambda_2: H^{1/2}(\Gamma; \mathbb{R}^3) \to H^{1/2}(\Gamma; \mathbb{R}^3)$ is the double layer potential with its dual $\Lambda_2': H^{-1/2}(\Gamma; \mathbb{R}^3) \to H^{-1/2}(\Gamma; \mathbb{R}^3)$, and $D_2: H^{1/2}(\Gamma; \mathbb{R}^3) \to H^{-1/2}(\Gamma; \mathbb{R}^3)$ is the hypersingular operator. It is known from [5, 7] that these operators are linear and bounded and that D_2 is symmetric and positive semi-definite and V_2 is symmetric and positive definite.

Lemma 1 ([6, 7]). (i) *If* $u_2 \in \mathscr{L}_2$, *then* (8) *holds for* $v := u_2|_\Gamma \in H^{1/2}(\Gamma; \mathbb{R}^3)$ *and* $\phi := T_2(u_2)|_\Gamma \in H^{-1/2}(\Gamma; \mathbb{R}^3)$.

(ii) *For any* $v \in H^{1/2}(\Gamma; \mathbb{R}^3)$ *and* $\phi \in H^{-1/2}(\Gamma; \mathbb{R}^3)$ *the vector field* u_2 *defined via* (8) *belongs to* $\mathscr{L}_2$ *and its Cauchy data are given by* $\mathscr{C}_2 \begin{pmatrix} v \\ \phi \end{pmatrix}$.

(iii) *For* $(v, \phi) \in H^{1/2}(\Gamma; \mathbb{R}^3) \times H^{-1/2}(\Gamma; \mathbb{R}^3)$ *the following statements (a) and (b) are equivalent:*

(a) (v, ϕ) *are Cauchy data of some* $u_2 \in \mathscr{L}_2$, *i.e.* $v = u_2|_\Gamma$, $\phi = T_2(u_2)|_\Gamma$ *for some* $u_2 \in \mathscr{L}_2$;

(b) (v, ϕ) *satisfies* (9).

Since V_2 is positive definite, whence invertible, we may define the Poincaré-Steklov operator (sometimes called Dirichlet-Neumann map)

$$S_2 := D_2 + (1/2 - \Lambda_2')\, V_2^{-1}(1/2 - \Lambda_2) : H^{1/2}(\Gamma;\mathbb{R}^3) \to H^{-1/2}(\Gamma;\mathbb{R}^3)$$

which is linear, bounded, symmetric, and positive semi-definite. It is proved in [2] that S_2 is also positive definite.

Lemma 2 ([3]). *u_2 solves the exterior problem (i.e. $u_2 \in \mathscr{L}_2$) if and only if its Cauchy data $(v, \phi) := (u_2|_\Gamma, T_2(u_2)|_\Gamma)$ satisfy $\phi = -S_2 v$ and* (8) *is valid.*

Proof. Using Lemma 1, short calculations show the assertion, cf. [2, Proof of Theorem 1], [3]. □

In order to define the energy of displacements in the unbounded exterior domain Ω_2, let B_R denote the ball in $\mathbb{R}^3$ with center 0 and assume the radius $R > 0$ sufficiently large such that $\Omega_1 \subseteq B_R$. Then, define

$$\Phi_{2R}(u_2, v_2) := \int_{\Omega_2 \cap B_R} \sum_{ijkl=1}^{3} a_{ijkl}\varepsilon_{kl}(u_2)\,\varepsilon_{ij}(v_2)\,\mathrm{d}\Omega_2 .$$

Lemma 3. *Given $u_2 \in \mathscr{L}_2$ and $v_2 \in H^1_{\mathrm{loc}}(\Omega_2;\mathbb{R}^3)$ with $v_2(x) = O\left(\frac{1}{|x|}\right)(|x| \to \infty)$ there holds*

$$\lim_{R\to\infty} \phi_{2R}(u_2, v_2) = \langle S_2 u_2|_\Gamma, v_2|_\Gamma\rangle .$$

Proof. Since $u_2 \in \mathscr{L}_2$ and from Lemma 2 we have (8) which additionally leads to $\operatorname{grad} u_2(x) = O\left(\frac{1}{|x|^2}\right)(|x| \to \infty)$. Hence, using Greens formula (compare (5)) in $\Omega_2 \cap B_R$, we have from $\Delta^* u_2 = 0$,

$$\Phi_{2R}(u_2, v_2) + \langle T_2(u_2)|_\Gamma, v_2|_\Gamma\rangle = o(1)$$

because the boundary integrals on ∂B_R are $O(1/R)\cdot O(1/R^2)\cdot O(R^2) = o(1)$. Letting $R \to \infty$ we obtain existence of the limit and the claimed equality. □

4. The interface problem

We start with an energetic description of the interface problem and prove an equivalent formulation.

Definition 3. The *interface problem* consists in finding $(u_1, u_2) \in \mathbb{L}$,

$$\mathbb{L} := \{(v_1, v_2) \in \mathbb{H} \times H^1_{\mathrm{loc}}(\Omega_2;\mathbb{R}^3) : v_1|_\Gamma = v_2|_\Gamma,\ v_2 \textit{ satisfies } (6)\},$$

with

$$E(u_1, 0) + \frac{1}{2}\Phi_2(u_2, u_2) = \min\{E(v_1, 0) + \frac{1}{2}\Phi_2(v_2, v_2) : (v_1, v_2) \in \mathbb{L}\}.$$

Remark 4. $E(v_1, 0) + \frac{1}{2}\Phi_2(v_2, v_2)$ is the sum of the interior and exterior part of the energy in the interface problem where we only prescribe continuity of the displacements in $\mathbb{L}$, i.e. only the essential interface conditions, at the interface Γ. In particular t, as introduced in the definition of E, is an applied surface force in the interior problem but not in the interface problem and consequently t does not occur explicitly here. We consider homogeneous interface conditions on Γ for simplicity and because of physical relevance. Inhomogeneties can be simply included as in the linear case [6].

The natural interface condition, namely the equilibrium of the stresses, i.e. $t = T_2(u_2)|_\Gamma$, will be a property of any solution of the interface problem, compare Remark 7 below.

Remark 5. Note that, in view of Lemma 3, the exterior energy $\frac{1}{2}\phi(v_2, v_2) \in [0, \infty]$ is finite for v_2 with compact support as well as for $v_2 \in \mathscr{L}_2$.

Remark 6. An alternative approach could be to minimize $E(u|_{\Omega_1}, 0) + \frac{1}{2}\Phi(u|_{\Omega_2}, u|_{\Omega_2})$ with respect to $u \in H^1(\mathbb{R}^3; \mathbb{R}^3)$, where (6) does not appear. If we did not assume (6) we would need the boundedness of $\|u|_{\Omega_2}\|^2_{H^1(\Omega_2;\mathbb{R}^3)}/\Phi(u|_{\Omega_2}, u|_{\Omega_2})$ to guarantee coercivity. Since $\|u|_{\Omega_2}\|^2_{H^1(\Omega_2;\mathbb{R}^3)}/\Phi(u|_{\Omega_2}, u|_{\Omega_2})$ is in general not bounded we have to work in other spaces.

We have the following equivalent minimization problem.

Theorem 1. *$(u_1, u_2) \in \mathbb{L}$ solves the interface problem if and only if $u_1 \in \mathbb{H}$ satisfies*

$$E\left(u_1, -\frac{1}{2}S_2 u_1|_\Gamma\right) = \min\left\{E\left(v_1, -\frac{1}{2}S_2 v_1|_\Gamma\right) : v_1 \in \mathbb{H}\right\} \tag{10}$$

and $u_2 \in \mathscr{L}_2$ is defined by (8) *with $(u_1|_\Gamma, -S_2 u_1|_\Gamma)$ replacing (v, ϕ).*

Proof. Assume that $(u_1, u_2) \in \mathbb{L}$ solves the interface problem. Then, we have for any $\eta \in C_c^\infty(\Omega_2; \mathbb{R}^3)$

$$E(u_1, 0) + \frac{1}{2}\Phi(u_2, u_2) \le E(u_1, 0) + \frac{1}{2}\Phi(u_2 + \eta, u_2 + \eta)$$

and hence,

$$\Phi(u_2, u_2) \le \Phi(u_2, u_2) + 2\Phi(\eta, u_2) + \Phi(\eta, \eta).$$

Since η is arbitrary, this inequality shows $\phi_2(u_2, \eta) = 0$ for all $\eta \in C_c^\infty(\Omega_2; \mathbb{R}^3)$ which is $\Delta^* u_2 = 0$ in the distributional sense (compare (5)). Thus $u_2 \in \mathscr{L}_2$ and Lemma 3 gives

$$\begin{aligned} E(u_1, 0) + \frac{1}{2}\Phi(u_2, u_2) &= E(u_1, 0) + \frac{1}{2}\langle S_2 u_1|_\Gamma, u_1|_\Gamma\rangle \\ &\ge \inf\left\{E\left(v_1, -\frac{1}{2}S_2 v_1|_\Gamma\right) : v_1 \in \mathbb{H}\right\}. \end{aligned} \tag{11}$$

Given $v_1 \in \mathbb{H}$ define v_2 by (8) with $(v_2, v_1|_\Gamma, -S_2 v_1|_\Gamma)$ replacing (u_2, v, ϕ). Then, $v_2 \in \mathscr{L}_2$

(cf. Lemma 1) and $(v_1, v_2) \in \mathbb{L}$ by Lemma 2. Lemma 3 gives

$$E\left(v_1, -\frac{1}{2}S_2 v_1|_\Gamma\right) = E(v_1, 0) + \frac{1}{2}\Phi_2(v_2, v_2)$$
$$\geq \min\left\{E(w_1, 0) + \frac{1}{2}\Phi_2(w_2, w_2) : (w_1, w_2) \in \mathbb{L}\right\}$$
$$= E(u_1, 0) + \frac{1}{2}\Phi_2(u_2, u_2).$$

This and (11) prove (10). The claimed properties of u_2 are proved below at the end of this proof.

For the moment assume conversely that $u_1 \in \mathbb{H}$ satisfies (10). Then, define $\tilde{u}_2 \in \mathscr{L}_2$ by (8) with $(\tilde{u}_2, u_1|_\Gamma, -S_2 u_1|_\Gamma)$ replacing (u_2, v, ϕ) so that $(u_1, \tilde{u}_2) \in \mathbb{L}$. Lemma 3 gives

$$\text{(12)} \qquad \min\left\{E\left(v_1, -\frac{1}{2}S_2 v_1|_\Gamma\right) : v_1 \in \mathbb{H}\right\} = E\left(u_1, -\frac{1}{2}S_2 u_1|_\Gamma\right)$$
$$= E(u_1, 0) + \frac{1}{2}\Phi_2(\tilde{u}_2, \tilde{u}_2)$$
$$\geq \inf\left\{E(v_1, 0) + \frac{1}{2}\Phi_2(v_2, v_2) : (v_1, v_2) \in \mathbb{L}\right\}.$$

Given $(v_1, v_2) \in \mathbb{L}$ define $\tilde{v}_2 \in \mathscr{L}_2$ by (8) with $(\tilde{v}_2, v_1|_\Gamma, -S_2 v_1|_\Gamma)$ replacing (u_2, v, ϕ). Thus $\tilde{v}_2 \in \mathscr{L}_2$ and $(v_1, \tilde{v}_2) \in \mathbb{L}$ (cf. Lemma 1 and Lemma 2). Note that $(v_2 - \tilde{v}_2)|_\Gamma = 0$ so that Lemma 3 yields $\Phi_2(\tilde{v}_2, v_2 - \tilde{v}_2) = 0$ which gives

$$0 \leq \Phi_2(v_2 - \tilde{v}_2, v_2 - \tilde{v}_2) = \Phi_2(v_2, v_2 - \tilde{v}_2) = \Phi_2(v_2, v_2) - \Phi_2(\tilde{v}_2, \tilde{v}_2)$$

and hence

$$\text{(13)} \qquad \Phi_2(\tilde{v}_2, \tilde{v}_2) \leq \Phi_2(v_2, v_2) \in [0, \infty].$$

By (10), (13), and Lemma 3, we have

$$E(v_1, 0) + \frac{1}{2}\Phi_2(v_2, v_2) \geq E(v_1, 0) + \frac{1}{2}\Phi_2(\tilde{v}_2, \tilde{v}_2) = E\left(v_1, -\frac{1}{2}S_2 v_1|_\Gamma\right)$$
$$\geq \min\left\{E\left(w_1, -\frac{1}{2}S_2 w_1|_\Gamma\right) : w_1 \in \mathbb{H}\right\}$$
$$= E\left(u_1, -\frac{1}{2}S_2 u_1|_\Gamma\right) = E(u_1, 0) + \frac{1}{2}\Phi_2(\tilde{u}_2, \tilde{u}_2).$$

This and (12) show that $(u_1, \tilde{u}_2)$ solves the interface problem.

Finally it remains to prove that for any solution (u_1, u_2) of the interface problem there holds $u_2 = \tilde{u}_2$ where $\tilde{u}_2$ is defined as above. If (u_1, u_2) solves the interface problem then, as we have already proved, $(u_1, \tilde{u}_2)$ solves the interface problem as well where $\tilde{u}_2 \in \mathscr{L}_2$ is defined by (8) with $(\tilde{u}_2, u_1|_\Gamma, -S_2 u_1|_\Gamma)$ replacing (u_2, v, ϕ). Therefore, we have

$$\text{(14)} \qquad \Phi_2(\tilde{u}_2, \tilde{u}_2) = \Phi_2(u_2, u_2).$$

Arguing as above (with $(u_1, u_2, \tilde{u}_2)$ replacing $(v_1, v_2, \tilde{v}_2)$ in the proof of (13)) we see that (14) implies $0 = \Phi_2(u_2 - \tilde{u}_2, u_2 - \tilde{u}_2)$, i.e. $u_2 - \tilde{u}_2$ is a rigid body motion. Since, by construction, $(u_2 - \tilde{u}_2)|_\Gamma = 0$ this rigid body motion is zero, i.e. $u_2 = \tilde{u}_2$. □

Remark 7. Arguing as in the proof of Theorem 1 one shows that the interface problem is equivalent to the following problem: Find $(u_1, u_2) \in \mathbb{L}$ such that $u_2 \in \mathscr{L}_2$ and $u_1 \in \mathbb{H}$ satisfies (4) with t replaced by $T_2(u_2)|_\Gamma$.

Hence, any solution (u_1, u_2) of the interface problem satisfies $u_1|_\Gamma = u_2|_\Gamma$ (by definition of $\mathbb{L}$) and $t = T_2(u_2)|_\Gamma$ which is equilibrium of the stresses on Γ; compare Remark 2.

Remark 8. Theorem 1 holds without any particular properties of the function e in the definition of E. We have only used that the interior problem is written as a minimization problem where an applied surface load t leads to the additive term $-\langle t, \gamma u_1\rangle$ in the energy functional. Hence the nonlinear interface problems under consideration in [7, 8] dealing with the nonlinear Hencky material are included in the framework of this note. Since S_2 is positive definite, the Dirichlet boundary conditions (cf. Remark 3 with, e.g., $u|_{\partial\Omega_0} = 0$) needed there can be omitted.

5. Existence of solutions

According to Theorem 1, the following result shows existence of solutions of the interface problem. Define $\mathscr{E}: \mathbb{H} \to \mathbb{R} \cup \{\infty\}$ by

$$\mathscr{E}(u) := E\left(u, -\frac{1}{2} S_2 \gamma u\right) = \int_\Omega (e(I + \operatorname{grad} u) - fu)\, d\Omega + \frac{1}{2}\langle S_2 \gamma u, \gamma u\rangle,$$

$\gamma u = u|_\Gamma, u \in \mathbb{H}$.

Theorem 2. *The interface problem has solutions, i.e. the functional $\mathscr{E}$ attains its minimum in $\mathbb{H}$ for some $u \in \mathbb{H}$.*

The proof below follows Ball's notions [1] as described in, e.g., [4, 10, 12]. Unfortunately, the formulations in [4, 10, 12] do not allow an explicit application to $\mathscr{E}$. Therefore, we give a simple sketch of the main idea first in Lemma 4 and then we verify the technical hypothesis in Lemma 5, 6, 7 and 8.

Lemma 4. *Let X be a real reflexive Banach space and let $I: D \to \mathbb{R} \cup \{\infty\}$ be a weakly sequential lower semicontinuous functional defined on the nonempty subset D of X. Assume (i), (ii), (iii):*

(i) $\infty > \inf_{x \in D} I(x) > -\infty$;

(ii) *For any sequence (x_n) in D with $\lim_{n\to\infty} \|x_n\|_X = \infty$ there holds $\lim_{n\to\infty} I(x_n) = \infty$;*

(iii) *For any sequence (x_n) in D which is weakly convergent to $x \in X$, $(x_n) \rightharpoonup x$, there holds $x \in D$ provided $\lim_{n\to\infty} I(x_n)$ exists as a real number.*

Then I attains its minimum in D, i.e. there exists at least one $x^* \in D$ such that

$$I(x^*) = \inf_{x \in D} I(x).$$

Proof. The simple proof is given only for completeness. Since D is not empty and by (i) there exists a sequence (x_n) in D with

$$\lim_{n\to\infty} I(x_n) = I_0 := \inf_{x \in D} I(x). \tag{15}$$

Because of (ii), the sequence (x_n) is bounded in the reflexive Banach space X. According to the Banach-Alaoglu theorem there exists a subsequence which converges weakly in X. Thus, without loss of generality, we may assume that (x_n) converges weakly towards $x \in X, (x_n) \rightharpoonup x$. According to (15) (holding also for a subsequence (x_n)) and (iii) we have $x \in D$, whence $I(x) \geq I_0$. On the other hand I is weakly sequential lower semicontinuous which implies

$$I(x) \leq \liminf_{n \to \infty} I(x_n) = I_0 .$$

Altogether, $I_0 = I(x)$, i.e. x is a minimizer of I. □

Remark 9. The abstract conditions of Lemma 4 have the following interpretations. $D \neq \emptyset$ and (i) are natural conditions for minimization problems. The weakly sequential lower semicontinuity of I as well as the coercitivity condition (ii) are usually required in convex analysis. The last condition (iii) seems to be technical and generalizes the weak closedness of D. Indeed (iii) is just needed to ensure that bounded minimizing sequences in D having a weak limit have a weak limit in D.

We will apply Lemma 4 in the following situation.

Definition 4. Consider the reflexive real Banach space

$$X := H^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^{3\times 3}) \times L^2(\Omega; \mathbb{R}^{3\times 3}) \times L^s(\Omega; \mathbb{R}),$$

$1 < s < \infty$, and its subset

$$D := \{(u, F, H, d) \in X : I + \operatorname{grad} u = F,\ H = \operatorname{adj} F,\ d = \det F,\ d > 0 \textit{ a.e. in } \Omega\}.$$

D is nonempty since, *e.g.*, $(0, I, I, 1) \in D$.

Consider $I: D \to \mathbb{R}$ defined for any $(u, F, H, d) \in D$ by

$$I(u, F, H, d) := \int_\Omega P(F, H, d)\, d\Omega + \frac{1}{2}\langle S_2 \gamma u, \gamma u\rangle - \int_\Omega f u \, d\Omega . \tag{16}$$

Note that $\mathscr{E}(u) = I(u, F, H, d)$ for any $(u, F, H, d) \in D$. Thus, in order to prove Theorem 2, it remains to verify the conditions in Lemma 4. This can be done modifying some proofs given in the literature [1, 4, 12, 10]. The proof are listed here for completeness.

Lemma 5. *I is weakly sequential lower semicontinuous.*

Proof. $P: \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \times (0, \infty) \to \mathbb{R}$, as defined in (3), is continuous and convex (cf. [1, 4, 12, 10]). Hence,

$$\begin{cases} L^2(\Omega; \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3}) \times L^s(\Omega; (0, \infty)) \to \mathbb{R} \\ (F, H, d) \mapsto \int_\Omega P(F, H, d)\, d\Omega \end{cases}$$

is weakly sequential lower semicontinuous (cf. e.g. [4, Theorem 7.3–1]). Since $\langle S_2 \gamma, \gamma\rangle : H^1(\Omega; \mathbb{R}^3) \to \mathbb{R}$ is convex and continuous we have that

$$I: H^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3}) \times L^s(\Omega; (0, \infty)) \to \mathbb{R} \cup \{\infty\},$$

as defined in (16), is lower semicontinuous. □

Lemma 6. $\infty > \inf_{x \in D} I(x) > -\infty$.

Proof. Let $c_3, \ldots, c_{10}$ denote constants. Note that, by (2), $\inf \varphi \geq 0$ so that

$$I(u, F, H, d) \geq c_3 \cdot \left(\int_\Omega F : F \, \mathrm{d}\Omega - 1 \right) + \frac{1}{2} \langle S\gamma u, \gamma u \rangle - \int_\Omega f u \, \mathrm{d}\Omega . \tag{17}$$

Since S_2 is positive definite, we conclude for any $x = (u, F, H, d) \in D$

$$I(u, F, H, d)$$
$$\geq c_4 \cdot (\|I + \operatorname{grad} u\|^2_{L^2(\Omega; \mathbb{R}^{3 \times 3})} + \|\gamma u\|^2_{H^{1/2}(\Gamma; \mathbb{R}^3)} - 1) - c_5 \|\gamma u\|_{H^1(\Omega; \mathbb{R}^3)} .$$

By the generalized Poincaré's inequality (cf. e.g. [4]) and since S_2 is positive definite

$$\|u\|^2_{L^2(\Omega; \mathbb{R}^3)} \leq c_6 \left(\|\operatorname{grad} u\|^2_{L^2(\Omega; \mathbb{R}^{3 \times 3})} + \frac{1}{2} \langle S\gamma u, \gamma u \rangle \right).$$

This yields $c_7 > 0$ with

$$I(u, F, H, d) \geq c_7 \|u\|^2_{H^1(\Omega; \mathbb{R})} - c_8 \|u\|_{H^1(\Omega; \mathbb{R})} - c_9 \geq c_{10} \tag{18}$$

which tends to infinity whenever $\|u\|_{H^1(\Omega; \mathbb{R})} \to \infty$. This and $\inf_{x \in D} I(x) \leq I(0, I, I, 1) < \infty$ gives the lemma. □

Lemma 7. *For any sequence (x_n) in D with $\lim_{n \to \infty} \|x_n\|_X = \infty$ there holds $\lim_{n \to \infty} I(x_n) = \infty$.*

Proof. We conclude from (18) and (17) that $I(u, F, H, d)$ tends to infinity whenever $\|u\|_{H^1(\Omega; \mathbb{R}^3)} \to \infty$ or $\|F\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})} \to \infty$. From

$$I(u, F, H, d) \geq c_0 + \int_\Omega H : H \, \mathrm{d}\Omega + \int_\Omega \varphi(d) \, \mathrm{d}\Omega \geq c_0 + \|H\|^2_{L^2(\Omega; \mathbb{R}^{3 \times 3})} + c_{11} \|d\|^s_{L^s(\Omega; \mathbb{R})}$$

(which is obtained using (2)) we conclude that $I(u, F, H, d)$ tends to infinity whenever $\|H\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})} \to \infty$ or $\|d\|_{L^s(\Omega)} \to \infty$ as well. This proves the lemma. □

Lemma 8. *For any sequence (x_n) in D which is weakly convergent to $x \in X$, $(x_n) \rightharpoonup x$, there holds $x \in D$ provided $\lim_{n \to \infty} I(x_n)$ exists as a real number.*

Proof. Let $(x_n) =: (u_n, F_n, H_n, d_n)$ be a sequence in D converging weakly towards $x = (u, F, H, d)$ in X such that $I(x_n)$ is bounded. It remains to prove that $x \in D$.

Let $id: \Omega \to \mathbb{R}^3$, $x \mapsto x$ be the identity in Ω. Note that $u_n \rightharpoonup u$ in $H^1(\Omega; \mathbb{R}^3)$ implies $F_n - I = \operatorname{grad} u_n \rightharpoonup \operatorname{grad} u$ in $L^2(\Omega; \mathbb{R}^{3 \times 3})$. Since

$$F_n \rightharpoonup F \quad in \quad L^2(\Omega; \mathbb{R}^{3 \times 3})$$

we have $F = I + \operatorname{grad} u$. From [4, Theorem 7.6–1] and

$$\begin{aligned} id + u_n &\rightharpoonup id + u && \text{in} \quad H^1(\Omega; \mathbb{R}^3) \\ \operatorname{adj}(I + \operatorname{grad} u_n) &\rightharpoonup H && \text{in} \quad L^2(\Omega; \mathbb{R}^{3 \times 3}) \\ \det(I + \operatorname{grad} u_n) &\rightharpoonup d && \text{in} \quad L^s(\Omega; \mathbb{R}) \end{aligned}$$

we obtain $H = \operatorname{adj}(I + \operatorname{grad} u)$ and $d = \det(I + \operatorname{grad} u)$.

It remains to prove that $d > 0$ a.e. in Ω which is based on (1) and $I(x) < \infty$ and can be proved as in [4, p. 374f.].

Proof of Theorem 2. Recall that $I(u, F, H, d) = \mathscr{E}(u)$ whenever $(u, F, H, d) \in D$, and $(u, F, H, d) \in D$ corresponds bijectivly to $u \in \mathbb{H}$. Hence any minimizer of I in D is a minimizer of $\mathscr{E}$ in $\mathbb{H}$ and vice versa. Since the hypotheses of Lemma 4 are satisfied, I has a minimizer. This implies the existence result of Theorem 2. □

Remark 10. Note that, for any solution (u_1, u_2) of the interface problem, $u_2 \in \mathscr{L}_2$ is uniquely determined by Theorem 1. But, in general, we cannot expect uniqueness or regularity of solutions, cf., e.g. [4, Section 7.10] and the references cited there.

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