Part II

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ON QUADRATIC-LIKE CONVERGENCE OF THE MEANS FOR TWO METHODS FOR SIMULTANEOUS ROOTFINDING OF POLYNOMIALS

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Abstract.

Durand-Kerner's method for simultaneous rootfinding of a polynomial is locally second order convergent if all the zeros are simple. If this condition is violated numerical experiences still show linear convergence. For this case of multiple roots, Fraigniaud [4] proves that the means of clustering approximants for a multiple root is a better approximant for the zero and called this Quadratic-Like-Convergence of the Means.
This note gives a new proof and a refinement of this property. The proof is based on the related Grau's method for simultaneous factoring of a polynomial. A similar property of some coefficients of the third order method due to Börsch-Supan, Maehly, Ehrlich, Aberth and others is proved.

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1. Introduction.

One of the most efficient methods for simultaneous approximation of all the zeros $\zeta_1, \ldots, \zeta_m$ of a polynomial

$$(1)\quad P(z) = z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n = (z - \zeta_1)^{\mu_1} (z - \zeta_2)^{\mu_2} \ldots (z - \zeta_m)^{\mu_m}$$

was indicated by Weierstrass in 1891 and much later proposed independently by Durand Kerner and others, see [8] and the related references therein. Provided simple zeros, i.e. $1 = \mu_1 = \ldots = \mu_m$ and $m = n$, *Durand-Kerner's* method with the iteration formula (in total-step mode)

$$(2)\qquad z_i^{(\nu+1)} := z_i^{(\nu)} - W_i^{(\nu)}, \quad W_i^{(\nu)} := P(z_i^{(\nu)}) / \prod_{j=1, j \neq i}^{n} (z_i^{(\nu)} - z_j^{(\nu)}),$$
$$i = 1, \ldots, n;\ \nu = 0, 1, 2, \ldots,$$

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yields locally quadratic convergence.

As noted in [1], [6], numerical experiences show convergence of the method also towards multiple zeros in the correct multiplicity but with slower convergence and lower accuracy. P. Fraigniaud proved in [4] that in this case Durand-Kerner's method has the property of *Quadratic-Like Convergence of the Means*, i.e. for sufficiently good initial values, for a suitable subset $I \subseteq \{1, \ldots, n\}$ of indices and $i \in \{1, \ldots, m\}$ there holds

$$(3) \qquad \left| \zeta_i - \frac{\sum_{j \in I} z_j^{(\nu+1)}}{\operatorname{card} I} \right| = O\left(\max_{k \in \{1, \ldots, n\}} |\omega_k - z_k^{(\nu)}|^2 \right); \quad \nu \to \infty,$$

where $(\omega_1, \ldots, \omega_n) := (\zeta_1, \ldots, \zeta_1, \zeta_2, \ldots \ldots, \zeta_m)$ is the vector of the n zeros of f counting multiplicities. This property, denoted as QLMC, is proved in [4] generalizing a similar result from [7].

Using the related Grau's method [5] this note presents a simple new proof giving more detailed information about QLMC for Durand-Kerner's method. Moreover, the open problem from [4] whether a similar result holds for the method from Börsch-Supan, Maehly, Ehrlich and Aberth is treated; see, e.g., [8] for the related references. This method reads (in total-step mode)

$$(4) \quad z_i^{(\nu+1)} := z_i^{(\nu)} - W_i^{(\nu)} \Big/ \left[1 + \sum_{j=1, j \neq i}^{n} W_j^{(\nu)} / (z_i^{(\nu)} - z_j^{(\nu)}) \right]; \quad i = 1, \ldots, n; \ \nu = 0, 1, 2, \ldots$$

and is locally cubically convergent if all the roots of P are simple. We note that the right hand side of (4) is equal to

$$z_i^{(\nu)} - \left[\frac{P'(z_i^{(\nu)})}{P(z_i^{(\nu)})} - \sum_{j=1, j \neq 1}^{n} (z_i^{(\nu)} - z_j^{(\nu)})^{-1} \right]^{-1}.$$

The proof is left to the reader (use Lagrange-interpolation of P at $z_1^{(\nu)}, \ldots, z_n^{(\nu)}$ to write P in terms of $W_i^{(\nu)}$'s, then one obtains the required expression for $P'(z_i^{(\nu)})/P(z_i^{(\nu)})$).

Using a related third order method for factoring a polynomial from [3], the question of a third-order-like convergence of the means of (4) can be solved. Method (4) does not have the analogous property, but a simple modification of it, does.

The paper is organized as follows: In Section 2, the methods for simultaneous factoring of a polynomial from [2], [3], [5] are described. The connection to the method (2) and (4) is given in Section 3. Using this, the QLMC property is treated for method (2) and (4) in Section 4 and 5, repectively, illustrated by numerical examples. Some remarks in Section 6 conclude this note.

2. Simultaneous factoring of a polynomial.

In this section some results from [5], [2] and [3] are summarized. We refer to these papers for motivations, proofs and details. There, the methods (2) and (4) are generalized for simultaneous factoring of a polynomial.

Consider a factorization of the given polynomial P,

$$P(z) = p_1^*(z) \cdot \ldots \cdot p_m^*(z), \tag{5}$$

where the factors $p_1^*, \ldots, p_m^*$ are pairwise relatively prime monic polynomials. In this factorization we do not care for the $n = \sum_{i=1}^m \mu_i$ zeros of P; we are only interested in the coefficient matrix

$$(a_{i,j}^* \mid i = 1, \ldots, m; j = 0, \ldots, \mu_i - 1) \in \mathbb{C}^n \tag{6}$$

defined via

$$p_i^*(z) = z^{\mu_i} + a_{i,\mu_i-1}^* \cdot z^{\mu_i - 1} + \ldots + a_{i,0}^*. \tag{7}$$

Given sufficiently good approximations

$$(a_{i,j} \mid i = 1, \ldots, m; j = 0, \ldots, \mu_i - 1) \in \mathbb{C}^n \tag{8}$$

of the exact coefficients (6) the approximating factors $p_1, \ldots, p_m$,

$$p_i(z) = z^{\mu_i} + a_{i,\mu_i-1} \cdot z^{\mu_i-1} + \ldots + a_{i,0} =: (z - \xi_{i,1}) \cdot \ldots \cdot (z - \xi_{i,\mu_i}), \tag{9}$$

are also pairwise relatively prime such that the sets of the related zeros $\{\xi_{1,1}, \ldots, \xi_{1,\mu_1}\}, \ldots, \{\xi_{m,1}, \ldots, \xi_{m,\mu_m}\}$ are pairwise disjunct. Here, $\xi_{i,1}, \ldots, \xi_{i,\mu_i}$ are the zeros of p_i counting multiplicities with arbitrarily chosen order.

To improve these approximating factors, we describe two methods theoretically using the zeros of the approximating factors. We stress that there exist other possible descriptions and algorithms to compute the new approximants $p_1^G, \ldots, p_m^G$ for Grau's method [5], [2], and $p_1^H, \ldots, p_m^H$ for the method from [3].

For $i \in \{1, \ldots, m\}$ let $G_i(z)$ be the Lagrange-interpolating polynomial of

$$P(z) \Big/ \prod_{j=1, j\neq i}^{m} p_j(z) \tag{10}$$

with respect to the interpolating points $\xi_{i,1}, \ldots, \xi_{i,\mu_i}$ and let $H_i(z)$ be the Lagrange-interpolating polynomial of

$$P(z) \Big/ \left[\left(1 + \sum_{k=1, k\neq i}^{m} G_k(z)/p_k(z) \right) \cdot \prod_{j=1, j\neq i}^{m} p_j(z) \right] \tag{11}$$

with respect to the interpolating points $\xi_{i,1}, \ldots, \xi_{i,\mu_i}$. Then, define the improved approximating factors by

$$p_i^G := p_i + G_i \quad \text{and} \quad p_i^H := p_i + H_i; \quad i = 1, \ldots, m. \tag{12}$$

The related coefficients converge locally of order two and three, respectively. Moreover, the following is proved in [2, Theorem 4.1] and [3], respectively.

THEOREM 1. *Provided the factors from* (5) *are pairwise relatively prime there exists*

a neighbourhood U of (6) *in* $\mathbb{C}^n$ *and a positive constant K such that for any coefficient vector* (8) *in U and any* $i \in \{1, \ldots, m\}$ *the following holds.*
(a) *The denominators in* (10) *and* (11) *are non-vanishing at* $\xi_{i,1}, \ldots, \xi_{i,\mu_i}$ *such that the Lagrange-interpolating polynomials* G_i *and* H_i *exist uniquely.*
(b) *For the impoved approximating factors*

$$p_i^G(z) = z^{\mu_i} + a_{i,\mu_i-1}^G \cdot z^{\mu_i-1} + \ldots + a_{i,0}^G$$

$$p_i^H(z) = z^{\mu_i} + a_{i,\mu_i-1}^H \cdot z^{\mu_i-1} + \ldots + a_{i,0}^H$$

(*given in* (12)) *there holds*

$$\max_{k=0,\ldots,\mu_i-1} |a_{i,k}^G - a_{i,k}^*| \le K \cdot \varepsilon_i \cdot \max_{j=1,\ldots,m, j \neq i} \varepsilon_j$$

$$\max_{k=0,\ldots,\mu_i-1} |a_{i,k}^H - a_{i,k}^*| \le K \cdot \varepsilon_i^2 \cdot \max_{j=1,\ldots,m, j \neq i} \varepsilon_j$$

where

$$\varepsilon_j := \max_{k=0,\ldots,\mu_j-1} |a_{j,k} - a_{j,k}^*|$$

denotes the max-norm error in the coefficients of the j-th factor from (8), $j \in \{1, \ldots, m\}$.

Note that for linear factors, i.e. $m = n$ and $\mu_1 = \ldots = \mu_m = 1$, the methods reduce to (2) and (4), respectively; (here, the change in some signs is caused by the plus sign in $p_i(z) = z + a_{i,0}$).

3. Connection to Method (2) and (4).

In this section one step of the above methods for simultaneous factoring will be compared with terms of method (2) and (4). We retain the notations of the previous section. Assume that the zeros $\xi_{i,j}$ of the approximating factors (9) are distinct and that they are approximants of the zeros of P. Then, using these initial values, one step of method (2) reads

$$\xi_{i,j}^W := \xi_{i,j} - W_{i,j}, \tag{13}$$

$$W_{i,j} := P(\xi_{i,j}) \Bigg/ \left[\prod_{k=1}^{m} \prod_{\substack{l=1 \\ (k,l) \neq (i,j)}}^{\mu_k} (\xi_{i,j} - \xi_{k,l}) \right]; \quad i = 1, \ldots, m; j = 1, \ldots, \mu_i,$$

and one step of method (4) takes the form

$$\xi_{i,j}^H := \xi_{i,j} - W_{i,j} \Bigg/ \left[1 + \sum_{k=1}^{m} \sum_{\substack{l=1 \\ (k,l) \neq (i,j)}}^{\mu_k} W_{k,l}/(\xi_{i,j} - \xi_{k,l}) \right]. \tag{14}$$

LEMMA 1. *For any $i \in \{1, \ldots, m\}$ there holds*

$$- a^G_{i,\mu_i - 1} = \sum_{k=1}^{\mu_i} \xi^W_{i,k} \tag{15}$$

$$- a^H_{i,\mu_i - 1} = \sum_{k=1}^{\mu_i} \left(\xi_{i,k} - W_{i,k} \middle/ \left[1 + \sum_{j=1, j \neq i}^{m} \sum_{\kappa=1}^{\mu_j} \frac{W_{j,\kappa}}{\xi_{i,k} - \xi_{j,\kappa}} \right] \right). \tag{16}$$

PROOF. Define the polynomials

$$l_{i,k}(z) = \prod_{\kappa=1, \kappa \neq k}^{\mu_i} (z - \xi_{i,\kappa}),$$

$i \in \{1, \ldots, m\}$, $k \in \{1, \ldots, \mu_i\}$. Then, by Lagrange-interpolation and some minor calculations

$$G_i(z) = \sum_{\kappa=1}^{\mu_i} W_{i,\kappa} \cdot l_{i,\kappa}(z)$$

$$H_i(z) = \sum_{k=1}^{\mu_i} W_{i,k} \, l_{i,k}(z) \middle/ \left[1 + \sum_{j=1, j \neq i}^{m} \sum_{\kappa=1}^{\mu_j} \frac{W_{j,\kappa}}{\xi_{i,k} - \xi_{j,\kappa}} \right]$$

which implies the lemma. ∎

Note that the left hand side of (15) is the sum of the related new approximations of Durand-Kerner's method. Note also that (16) is, in general, *not* the sum of the related new approximations (14) of method (4).

4. Higher-order-like convergence of the means in Durand-Kerner's method.

Using the results of the two previous sections, the QLMC property of Durand-Kerner's method (2) is proved and refined in this section.

For the polynomial (1) let method (2) generate approximations $(z_1^{(\nu)}, \ldots, z_n^{(\nu)})$ for $\nu = 1, 2, 3, \ldots$. Therefore, assume that for any ν, $(z_1^{(\nu)}, \ldots, z_n^{(\nu)})$ are distinct. Finally, assume that

$$\lim_{\nu \to \infty} (z_1^{(\nu)}, \ldots, z_n^{(\nu)}) = (\underbrace{\zeta_1, \ldots, \zeta_1}_{\mu_1}, \ldots, \underbrace{\zeta_m, \ldots, \zeta_m}_{\mu_m}). \tag{17}$$

To describe the quadratic-like convergence of the means, identify the following vectors which have distinct indices

$$(z_{1,1}^{(\nu)}, z_{1,2}^{(\nu)}, \ldots, z_{1,\mu_1}^{(\nu)}, z_{2,1}^{(\nu)}, \ldots\ldots, z_{m,\mu_m}^{(\nu)}) := (z_1^{(\nu)}, \ldots, z_n^{(\nu)}), \tag{18}$$

and define the means of all approximants of the zero ζ_i in step ν

(19) $$X_i^{(\nu)} := (1/\mu_i) \sum_{j=1}^{\mu_i} z_{i,j}^{(\nu)}$$

for any $i = 1, \ldots, m$ and $\nu = 1, 2, 3, \ldots$.

The following theorem states that, in the method of Durand-Kerner, the order of the error of a certain mean in step $\nu + 1$ is greater than twice the order of the errors of the approximants in step ν.

THEOREM 2.

$$\zeta_i - X_i^{(\nu+1)} = O\left(\delta_i^{(\nu)} \cdot \max_{j=1,\ldots,m, j \neq i} \delta_j^{(\nu)}\right); \quad i = 1, \ldots, m; \nu \to \infty$$

where

(20) $$\delta_j^{(\nu)} := \max_{k=1,\ldots,\mu_j} |z_{j,k}^{(\nu)} - \zeta_j|.$$

PROOF. Let the approximations $(z_1^{(\nu)}, \ldots, z_n^{(\nu)})$ be generated by Durand-Kerner's method. Fix the iteration index ν and consider Theorem 1 for the polynomials

(21) $$p_i(z) = p_i^{(\nu)}(z) := z^\mu + a_{i,\mu_i-1}^{(\nu)} \cdot z^{\mu_i - 1} + \ldots + a_{i,0}^{(\nu)} := (z - z_{i,1}^{(\nu)}) \cdot \ldots \cdot (z - z_{i,\mu_i}^{(\nu)}).$$

Here, according to (18), $z_{i,j}^{(\nu)}$ is defined with the elements of $(z_1^{(\nu)}, \ldots, z_n^{(\nu)})$. Assume that ν is sufficiently large so that according to (17) we have $(a_{i,j}^{(\nu)} | i = 1, \ldots, m; j = 0, \ldots, \mu_i - 1) \in U$. Since $\varepsilon_j = O(\delta_j^{(\nu)})$, ε_j being defined in Theorem 1, we find

(22) $$\zeta_i + a_{i,\mu_i-1}^G / \mu_i = O\left(\delta_i^{(\nu)} \cdot \max_{j=1,\ldots,m, j \neq 1} \delta_j^{(\nu)}\right)$$

where $a_{i,j}^G$ is a coefficient in p_i^G which is defined in (12) for $p_i = p_i^{(\nu)}$. On the other hand, Lemma 1 shows that for the fixed index ν with $\xi_{i,j} := z_{i,j}^{(\nu)}$ (cf. (9), (13))

(23) $$-a_{i,\mu_i-1}^G / \mu_i = \sum_{j=1}^{\mu_i} \xi_{i,j}^G / \mu_i = X_i^{(\nu+1)}.$$

Using (23) in (22) proves the theorem. ■

REMARK. Theorem 2 improves (3), i.e. the result from [4, Proposition 1].

We conclude this section with a closer look at the QLMC property of Durand-Kerner's method (2) improving the results from Theorem 2 as well as [4, Proposition 1].

Under the assumptions of Theorem 2 let $\delta^{(\nu)}$ denote the max-norm of the error in step ν given in (20), i.e.

(24) $$\delta^{(\nu)} := \max_{i=1,\ldots,m} \delta_i^{(\nu)}.$$

The following theorem states that the means of some clustering approximants for multiple zeros behave in *two* steps of Durand-Kerner's method like having second R-order of convergence.

THEOREM 3. *If* $\delta^{(\nu+1)} = O(\delta^{(\nu)})$ *for* $\nu \to \infty$, *then there holds for* $k = 1$ *and* $k = 2$

$$\zeta_i - X_i^{(\nu+k)} = O((\delta^{(\nu)})^{2k}); \quad i = 1, \ldots, m; \ \nu \to \infty.$$

PROOF. Fix $i \in \{1, \ldots, m\}$ and ν sufficiently large. Then, the assertion for $k = 1$ is implied by Theorem 2.

It is mentioned in [2] that it is not necessary to consider the approximating factors in Grau's method in the form of a finite power series developed at zero as above, cf. (6), (9). Indeed, any other (fixed) basis can be changed to deal with coefficients (8) such that Theorem 1 holds. Therefore, we may consider the approximating factor $p_i^{(\nu)}$ in the form of a power series developed at ζ_i, i.e.

$$\begin{aligned} p_i^{(\nu)}(z) &= b_{i,0}^{(\nu)} + b_{i,1}^{(\nu)}(z - \zeta_i) + \ldots + b_{i,\mu_i-1}^{(\nu)}(z - \zeta_i)^{\mu_i - 1} + (z - \zeta_i)^{\mu_i} \\ &= (z - z_{i,1}^{(\nu)}) \cdot \ldots \cdot (z - z_{i,\mu_i}^{(\nu)}). \end{aligned}$$

Note that the related coefficients $b_{i,j}^*$ of $p_i^*(z) = (z - \zeta_i)^{\mu_i}$ (i.e. the coefficients in the power series developed at ζ_i) vanish, $j \in \{0, \ldots, \mu_i - 1\}$. Therefore, since $|\zeta_i - z_{i,j}^{(\nu)}| \le \delta_i^{(\nu)}$ is small, there holds

$$b_{i,j}^{\nu+1} - b_{i,j}^* = O((\delta_i^{(\nu+1)})^{\mu_i - j}); \quad j = 0, \ldots, \mu_i - 1.$$

Using the assertion for $k = 1$ of the theorem for $j = \mu_i - 1$ and the last estimate (with $K \cdot \delta^{(\nu)} \ge \delta^{(\nu+1)}$) for $j < \mu_i - 1$ we obtain

$$b_{i,j}^{(\nu+1)} - b_{i,j}^* = O((\delta^{(\nu)})^2); \quad j = 0, \ldots, \mu_i - 1. \tag{25}$$

Recall that the polynomials $p_i^{(\nu)}$ are generated by Durand-Kerner's method. The first application of Theorem 1 and Lemma 1 for the iteration step ν proved (25). A second application of Theorem 1 for the iteration index $\nu + 1$ using (25) gives

$$b_{i,\mu_i-1}^{(\nu+2)} - b_{i,\mu_i-1}^* = O((\delta^{(\nu)})^4)$$

and with Lemma 1 the proof is concluded. ■

The next example illustrates the assertion of Theorem 3.

EXAMPLE 1. Consider the polynomial $P(z) = z(z-1)^2(z+1)^3$ of degree $n = 6$ which is discussed in [4]. Letting $z_{1,1}^{(0)} := 0.1$, $z_{2,1}^{(0)} := 1 + 0.1i$, $z_{2,2}^{(0)} := 1.2 - 0.1i$ and $z_{3,1}^{(0)} := -1 + 0.1i$, $z_{3,2}^{(0)} := -1.2$, $z_{3,3}^{(0)} := -1.1 - 0.1i$, $m = 3$, $\mu_1 = 1$, $\mu_2 = 2$, $\mu_3 = 3$, according to (18), Durand-Kerner's method (2) generates a sequence $(z_1^{(\nu)}, \ldots, z_6^{(\nu)})$, $\nu = 1, 2, 3, \ldots$. The errors $\delta_j^{(\nu)}$ of the approximants defined in (20) as well as the errors of the means $X_j^{(\nu)}$ (defined in (19)) are shown in Table 1.

It is mentioned generally in [1] and can be also observed numerically from Table 1 that the errors $\delta_j^{(\nu)}$ (defined in (20)) for the multiple zeros tend linearly

Table 1. *Example for method* (2).
The values of $|X_1^{(v)} - \zeta_1|$ *are equal to* $\delta_1^{(v)}$, $v = 1(1)5$.

v	$\delta_1^{(v)}$	$\delta_2^{(v)}$	$\delta_3^{(v)}$	$\lvert X_2^{(v)} - \zeta_2\rvert$	$\lvert X_3^{(v)} - \zeta_3\rvert$
1	$3.7673\,10^{-2}$	$8.3516\,10^{-2}$	$1.7921\,10^{-1}$	$4.4808\,10^{-3}$	$1.5533\,10^{-2}$
2	$2.7562\,10^{-3}$	$4.0470\,10^{-2}$	$9.9629\,10^{-2}$	$1.2126\,10^{-4}$	$8.6050\,10^{-4}$
3	$3.1126\,10^{-5}$	$2.0228\,10^{-2}$	$5.5576\,10^{-2}$	$1.9292\,10^{-7}$	$1.0330\,10^{-5}$
4	$6.7632\,10^{-8}$	$1.0118\,10^{-2}$	$3.3426\,10^{-2}$	$9.0998\,10^{-8}$	$7.9425\,10^{-8}$
5	$3.2790\,10^{-11}$	$5.0597\,10^{-3}$	$2.1174\,10^{-2}$	$5.3545\,10^{-9}$	$3.5783\,10^{-9}$

towards zero. Consequently, $\delta_j^{(v+1)} = O(\delta^{(v)})$, with $\delta^{(v)}$ defined in (24), and hence Theorem 3 implies

$$|X_j^{(v+2)} - \zeta_j| = O((\delta^{(v)})^4); \quad j = 2, 3; \; v \to \infty.$$

This can be confirmed by the numerical results in Table 1 for large v, such that, in the example, the "QLMC property" looks more like a "fourth-order improvement of the means".

5. Third-order-like convergence of the modified means in method (4).

This section clarifies the open problem from [4] whether a quadratic-like convergence of the means can be proved also for method (4). Using the technique of the previous section this question is solved: A similar result for method (4) is false but modified means can be defined having a *third-order-like convergence.*

For the polynomial (1) let method (4) generate approximations $(z_1^{(v)}, \ldots, z_n^{(v)})$ for $v = 1, 2, 3, \ldots$. As in the previous section we assume that the approximants are distinct and convergent, i.e. assume (17). We also use the notations of (18) and instead of (19) we define the modified means

$$(26) \qquad Y_i^{(v+1)} := (1/\mu_i) \sum_{j=1}^{\mu_i} \left(z_{i,j}^{(v)} - W_{i,j}^{(v)} \middle/ \left[1 + \sum_{\substack{k=1\\k\neq i}}^{m} \sum_{l=1}^{\mu_k} W_{k,l}^{(v)} / (z_{i,j}^{(v)} - z_{k,l}^{(v)}) \right] \right).$$

$Y_i^{(v+1)}$ is related with the right hand side of (16) and, even in the present case of method (4), it is *not* the means of all approximants of step v or $v + 1$ for the zero ζ_i.

The following theorem states that, in the method (4) due to Börsch-Supan, Maehly, Ehrlich, Aberth and others the order of the error of $Y_i^{(v+1)}$ is greater than three times the order of the errors of the approximants in step v.

THEOREM 4. *For any* $i \in \{1, \ldots, m\}$ *there holds*

$$\zeta_i - Y_i^{(v+1)} = O\left(\delta_i^{(v)2} \cdot \max_{j=1,\ldots,m, j\neq i} \delta_j^{(v)} \right); \quad v \to \infty.$$

PROOF. The proof is similar to that of Theorem 2 so we only sketch it. Fix the iteration index ν sufficiently large. Apply Theorem 1 using the polynomials (21). Then, similarly as in (22) we have

$$\zeta_i + a^H_{i,\mu_i - 1}/\mu_i = O\left(\delta_i^{(\nu)2} \cdot \max_{j=1,\ldots,m, j\neq i} \delta_j^{(\nu)} \right).$$

Since Lemma 1 gives $-a^H_{i,\mu_i - 1}/\mu_i = Y_i^{(\nu+1)}$, this concludes the proof. ∎

The following example illustrates the *third-order-like-convergence of the modified means* of method (4) in Theorem 4 in comparison with the behaviour of the means.

EXAMPLE 2. Using the data from Example 1 we applied method (4) and obtained a sequence of approximants for the zeros of (1). Table 2 shows the errors $\delta_j^{(\nu)}$ of the approximants (given in (20)) as well as the errors of the means $X_j^{(\nu)}$ (defined in (19)) and, in addition, the errors of the modified means $Y_j^{(\nu+1)}$ (defined in (26)).

From Table 2, the "third-order-improvement of the modified means" $Y_j^{(\nu)}$ can be observed whereas the means $X_j^{(\nu)}$ have minor accuracy.

REMARK. For good approximations (18) we have $W_i^{(\nu)} = O(\delta^{(\nu)})$ and therefore $Y_i^{(\nu+1)} = X_i^{(\nu+1)} + O(\delta^{(\nu)2})$. Thus, according to Theorem 4 we have only quadratic-like convergence of the means for method (4).

Table 2. *Examples for Method* (4). *The values of* $|X_1^{(\nu)} - \zeta_1|$ *and* $|Y_1^{(\nu)} = \zeta_1|$ *are equal to* $\delta_1^{(\nu)}$.

ν	$\delta_1^{(\nu)}$	$\delta_2^{(\nu)}$	$\delta_3^{(\nu)}$
1	$5.3270\ 10^{-4}$	$6.1769\ 10^{-2}$	$9.7469\ 10^{-2}$
2	$2.7287\ 10^{-8}$	$1.8526\ 10^{-2}$	$5.0364\ 10^{-2}$
3	$2.7869\ 10^{-17}$	$5.9509\ 10^{-3}$	$2.5125\ 10^{-2}$
4	$1.3458\ 10^{-35}$	$1.9592\ 10^{-3}$	$1.2255\ 10^{-2}$
5	$2.9182\ 10^{-54}$	$6.5039\ 10^{-4}$	$6.0102\ 10^{-3}$

ν	$\lvert X_2^{(\nu)} - \zeta_2\rvert$	$\lvert X_3^{(\nu)} - \zeta_3\rvert$	$\lvert Y_2^{(\nu)} - \zeta_2\rvert$	$\lvert Y_3^{(\nu)} - \zeta_3\rvert$
1	$1.2221\ 10^{-2}$	$3.8746\ 10^{-2}$	$4.4769\ 10^{-4}$	$3.0075\ 10^{-3}$
2	$1.3413\ 10^{-3}$	$1.2636\ 10^{-2}$	$9.1029\ 10^{-6}$	$2.9012\ 10^{-5}$
3	$1.4824\ 10^{-4}$	$3.1409\ 10^{-3}$	$3.7013\ 10^{-8}$	$3.4303\ 10^{-7}$
4	$1.6448\ 10^{-5}$	$7.2152\ 10^{-4}$	$1.1221\ 10^{-10}$	$2.3707\ 10^{-9}$
5	$1.8270\ 10^{-6}$	$1.6773\ 10^{-4}$	$3.1421\ 10^{-13}$	$1.3793\ 10^{-11}$

6. Conclusion.

In this note the QLMC property from [4] of Durand-Kerner's method (2) was proved by a connection to Grau's method for simultaneous factorization of a polynomial. A similar idea can be applied to method (4) due to Börsch-Supan, Maehly, Ehrlich, Alberth and others. It proves a third-order-improvement not for the means themselves but for some modified means (given in Theorem 4). A more detailed analysis of Durand-Kerner's method in the case of multiple roots suggests numerically a "fourth-order improvement of the means".

These properties facilitate a modification of the considered methods in the presence of multiple roots as given in [4]. However, in the case of multiple roots, the author proposes the application of the method for simultaneous factorization. The advantage of the preferred factoring methods [5] and [3] is that the same convergence behaviour can be expected also in the presence of clusters of zeros and not only in the particular case of an exact multiple zero.

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