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On Some Interval Methods for Algebraic, Exponential and Trigonometric Polynomials

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Abstract — Zusammenfassung

On Some Interval Methods for Algebraic, Exponential and Trigonometric Polynomials. New inclusion methods for the simultaneous determination of the zeros of algebraic, exponential and trigonometric polynomials are presented. These methods are realized in real interval arithmetic and do not use any derivatives. Using Weierstrass' correction some modified methods with the increased convergence rate are constructed. Convergence analysis and numerical example are included.

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Key words: Zeros of generalized polynomials, simultaneous methods, convergence rate, interval arithmetic.

Über Intervallmethoden für algebraische, exponentielle und trigonometrische Polynome. Die Arbeit behandelt neue Einschliessungsmethoden zur simultanen Berechnung aller Nullstellen von algebraischen, exponentiellen und trigonometrischen Polynomen. Die Verfahren sind für reelle Intervallarithmetik formuliert und benötigen keine Auswertungen von Ableitungen des gegebenen verallgemeinerten Polynomes. Unter Verwendung der sog. Weierstrass-Korrektoren werden verbesserte modifizierte Verfahren konstruiert. Hierzu enthält die Arbeit Konvergenzuntersuchungen und numerische Beispiele.

1. Introduction

In the last two decades a lot of methods for finding *a posteriori* error bounds of an approximation z_m, say, to a zero ξ of a given function f were developed. One of the often applied methods for finding, from the data provided by the algorithm, a bound α_m for the error $|z_m - \xi|$ of the last approximation is based on the combination of an iterative method implemented in ordinary floating-point arithmetic and a suitably chosen disk of the form $|z - z_m| \leq \alpha_m$ which includes at least one zero of f. A quite different approach to error estimate is based on the use of interval arithmetic (see [2] and references cited there). In this manner, not only very close zero approximations (given by the midpoints of intervals) but also upper error bounds for the zeros (expressed by the semi-width of intervals) are obtained which means the automatic verification of results and a control of errors in each iteration. For a long time the computational cost of interval methods was rather great, until the development of very efficient programming languages for scientific computation (SC) as PASCAL-SC and ACRITH-SC, and very recently PASCAL-XSC [11] and

ACRITH-XSC [10]. These languages possess the maximum accuracy (there is no other floating-point number between the rounded result and the exact result) including the directed roundings which enable the implementation of a maximum accurate interval arithmetic. Besides, the CPU time of basic interval operations is considerably decreased which caused a reasonably higher computational efficiency of interval methods. This improvement, together with the possibility of rounding an production of self-verifying results, made these methods to be competitive and so often applied in practice.

Many problems of applied mathematics and mathematical models in various scientific disciplines reduce to the problem of finding real zeros of algebraic, exponential and trigonometric polynomials (generalized polynomials, for brevity). In the last decade several algorithms for the determination of zeros of this type of polynomials were proposed (see, e.g. [3], [6], [8], [9], [12], [19]). In the recent paper [6] Carstensen presented a new approach for the simultaneous computation of all zeros of generalized polynomials. Using some results derived in [6] we propose in this paper some new methods for the simultaneous inclusion of all real zeros of algebraic, exponential and trigonometric polynomials. The employed real interval arithmetic provides the resulting intervals that contain the wanted zeros. As far as we know, interval methods for exponential and trigonometric polynomials appear for the first time in this paper.

We note that the determination of zeros of trigonometric and exponential polynomials can also be done using suitable transformations as it was proposed by Weidner in [19]. In this way the considered problem reduces to solving complex algebraic polynomial. Because of roundoff errors the transformations lead to falsified coefficients and hence to perturbed approximations (even if they are computed as the exact zeros of the transformed polynomial). Hence a direct method is of principal interest; cf. also the comparison of interval methods based on Weidner's transformation and the presented direct interval methods given in Section 4.

2. Real Interval Arithmetic

Before deriving new interval algorithms we give the basic real interval operations, introduced by R. E. Moore [14].

A subset of the set of real numbers $\mathbb{R}$ of the form

$$A := [a_1, a_2] = \{x \colon a_1 \leq x \leq a_2, a_1, a_2 \in \mathbb{R}\}$$

is called a closed *real interval*. The set of all closed real intervals is denoted by $I(\mathbb{R})$. If $*$ is one of the symbols $+, -, \cdot, :$, the arithmetic operations on $I(\mathbb{R})$ are defined by

$$A * B = \{x = a * b \colon a \in A, b \in B\} \qquad (A, B \in I(\mathbb{R})).$$

The basic operations on intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$ are calculated explicitly as

$$
\begin{aligned}
A + B &= [a_1 + b_1, a_2 + b_2] \\
A - B &= [a_1 - b_2, a_2 - b_1] \\
A \cdot B &= [\min(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2), \max(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2)] \\
B^{-1} &= \left[\frac{1}{b_2}, \frac{1}{b_1}\right] \qquad (0 \notin B) \\
A : B &= [a_1, a_2] \cdot \left[\frac{1}{b_2}, \frac{1}{b_1}\right] \qquad (0 \notin B).
\end{aligned}
\tag{1}
$$

To control rounding errors, we can apply the rounding real arithmetic (see e.g. [1], [10] and [11]).

To simplify our analysis it is preferable to deal with a modified form of a real interval $A = [a_1, a_2]$ which reads in a parametric notation as $A := \{a, r_a\}$, where $a := mid(A) = \frac{1}{2}(a_1 + a_2)$ is the *midpoint* of A and $r_a := rad(A)$ is the *semi-width* or *radius* of A. If $A = \{a, r_a\}$ and $B = \{b, r_b\}$ then the basic interval operations may be expressed as

$$
\begin{aligned}
A \pm B &\div \{a \pm b, r_a + r_b\} \\
B^{-1} &= \left\{\frac{b}{b^2 - r_b^2}, \frac{r_b}{b^2 - r_b^2}\right\} \qquad (0 \notin B, |b| > r_b).
\end{aligned}
\tag{2}
$$

We will also use the *centered form* of the inverse of a real non-zero interval $B = [b_1, b_2]$ given by

$$
\begin{aligned}
B^I &:= \{[mid(B)]^{-1}, max(1/b_1 - [mid(B)]^{-1}, [mid(B)]^{-1} - 1/b_2)\} \\
&= \left\{\frac{2}{b_1 + b_2}, \frac{b_2 - b_1}{b_1(b_1 + b_2)}\right\} = \left[\frac{4}{b_1 + b_2} - \frac{1}{b_1}, \frac{1}{b_1}\right]
\end{aligned}
$$

with

$$B^I \supset B^{-1} = [1/b_2, 1/b_1].$$

The inverse interval B^I is wider than B^{-1} but in the case of small intervals appearing (for example) in iterative interval processes the difference in size is negligible. This simply follows from the fact that

$$rad(B^I) = \frac{\delta}{b_1(b_1 + b_2)}, \qquad rad(B^{-1}) = \frac{\delta}{2b_1 b_2}$$

for reasonably small δ.

In addition, if $\varphi: \mathbb{R} \to \mathbb{R}$ is a strictly monotonic function on a real interval $D \subseteq \mathbb{R}$, then the *interval function*

$$\Phi(X) = \{y = \varphi(x): x \in X = [x_1, x_2] \subseteq D\}$$

is defined as

$$\Phi(X) = [\min(\varphi(x_1), \varphi(x_2)), \max(\varphi(x_1), \varphi(x_2))]. \tag{3}$$

3. Iterative Inclusion Method

Let $f\colon G \to \mathbb{R}$ be a real function having n simple real zeros $\xi_1, \ldots, \xi_n$ in the open set $G \subseteq [-\pi, \pi]$ in the case of trigonometric polynomials. Let $q\colon \mathbb{R} \to \mathbb{R}$ be the real function

$$q(t) = t, \qquad q(t) = \sinh(t/2), \qquad q(t) = \sin(t/2) \tag{4}$$

if and only if f is an algebraic, exponential or trigonometric polynomial with (exact) degree n, $n/2$ or $n/2$ respectively.

Assume that we have found disjoint real intervals $X_1, \ldots, X_n$ belonging G with $\xi_j \in X_j \subset G$ $(j = 1, \ldots, n)$. Let $x_j = min(X_j)$ and $r_j = ran(X_j)$ be the midpoint and the semi-width (radius) of the real interval X_j, that is, $X_j = \{x_j, r_j\}$. Let $x_0 \in G \backslash (X_1 \cup \cdots \cup X_n)$ be fixed but chosen so that $q(x_j - x_k) \neq 0$ if $j \neq k, j, k = 0, 1, \ldots, n$.

Lemma 1. *For* $(x_1, \ldots, x_n) \in X = X_1 \times \cdots \times X_n$ *and* $x_0 \in G\backslash(X_1 \cup \cdots \cup X_n)$ *define*

$$c_j := \frac{f(x_j)}{\prod\limits_{\substack{k=0 \\ k \neq j}}^{n} q(x_j - x_k)} \qquad (j = 0, 1, \ldots, n) \tag{5}$$

and

$$c^* := \frac{f(x_0)}{\prod\limits_{k=1}^{n} q(x_0 - \xi_k)} \neq 0. \tag{6}$$

Then for all $t \in G$

$$f(t) = c^* \prod_{k=1}^{n} q(t - \xi_k) = \sum_{j=0}^{n} c_j \prod_{\substack{k=0 \\ k \neq j}}^{n} q(t - x_k). \tag{7}$$

The proof of Lemma 1 is simple and can be found in [6]. Namely, by simple calculation it is easy to see that all expressions in (7) are polynomials of degree n. In view of (5) and (6) the identity (7) is valid for $t = x_0, \xi_1, \ldots, \xi_n$ and $t = x_0, \ldots, x_n$ and therefore for all t.

Remark 1. The choice $x_0 = \infty$ in the algebraic case yields the constant $c_0 = c^*$ which becomes the leading coefficient of f. Then the terms $q(x_0 - x_k)$ $(k = 1, \ldots, n)$ can be substituted by c_0 in (5) and (7). In this case (7) reduces to the Lagrangean interpolation of f at the points $x_1, \ldots, x_n, \infty$ (see Braess, Hadeler [5]).

Lemma 1 is the base for the derivation of a suitable fixed-point relation necessary for the construction of new inclusion methods. Let $\xi_j \neq x_k$ $(j \in \{1, \ldots, n\}, k = 0, 1, \ldots, n)$ be a zero of f. Substituting $t = \xi_j$ in (7) we find

$$\frac{c_j}{q(\xi_j - x_j)} = -\sum_{\substack{k=0 \\ k \neq j}}^{n} \frac{c_k}{q(\xi_j - x_k)},$$

whence

$$q(\xi_j - x_j) = \frac{-c_j}{\sum\limits_{\substack{k=0\\k\neq j}}^{n} \frac{c_k}{q(\xi_j - x_k)}} \qquad (j = 1,\ldots,n). \tag{8}$$

The functions q defined by (4) are strictly monotonic on G so that they have their inverse functions. This implies *the fixed-point relation*

$$\xi_j = x_j + q^{-1}\left(\frac{-c_j}{\sum\limits_{\substack{k=0\\k\neq j}}^{n} \frac{c_k}{q(\xi_j - x_k)}}\right) \qquad (j = 1,\ldots,n) \tag{9}$$

from (8). Replacing the zero ξ_j by its inclusion interval X_j on the right hand side of (9), according to the inclusion property we get

$$\xi_j \in x_j + q^{-1}\left(\frac{-c_j}{\sum\limits_{\substack{k=0\\k\neq j}}^{n} \frac{c_k}{q(X_j - x_k)}}\right) = x_j + q^{-1}\left(\frac{-c_j}{A_j}\right) = \hat{X}_j \qquad (j = 1,\ldots,n), \tag{10}$$

where we put

$$A_j = \sum_{\substack{k=0\\k\neq j}}^{n} \frac{c_k}{q(X_j - x_k)}.$$

If $0 \notin A_j$ then $-c_j/A_j$ is a closed real interval so that we have the implication

$$\xi_j \in X_j \Rightarrow \xi_j \in \hat{X}_j = x_j + q^{-1}(-c_j/A_j)$$

in the case of algebraic and exponential polynomials. For trigonometric polynomials one requires the additional condition

$$\frac{-c_j}{A_j} \subseteq [-1, 1]$$

since the interval function $\arcsin(X)$ is defined for $X \subseteq [-1, 1]$.

The relation (10) suggests the following **iterative method** for the simultaneous inclusion of all zeros of generalized polynomials defined in the beginning of this section.

Let $X_1^{(0)} = \{x_1^{(0)}, r_1^{(0)}\}$, ..., $X_n^{(0)} = \{x_n^{(0)}, r_n^{(0)}\}$ be the initial disjoint real intervals containing the real zeros $\xi_1, \ldots, \xi_n$ of f, and let $X_j^{(m)} = \{x_j^{(m)}, r_j^{(m)}\}$ for $m = 0, 1, \ldots$. Then the successive interval approximations to these zeros are calculated by

$$X_j^{(m+1)} = x_j^{(m)} + q^{-1}\left(\frac{-c_j^{(m)}}{\sum\limits_{\substack{k=0\\k\neq j}}^{n} \frac{c_k^{(m)}}{q(X_j^{(m)} - x_k^{(m)})}}\right) \qquad (j = 1,\ldots,n; m = 0, 1, \ldots), \tag{11}$$

where $x_0^{(m)} = x_0$ *and*

$$c_j^{(m)} = \frac{f(x_j^{(m)})}{\prod_{\substack{k=0\\k\neq j}}^{n} q(x_j^{(m)} - x_k^{(m)})} \qquad (j = 0, 1, \ldots, n). \tag{12}$$

Remark 2. If the initial intervals $X_1^{(0)}, \ldots, X_n^{(0)}$ are small enough, then the conditions $0 \notin A_j^{(m)}$ $(j = 1, \ldots, n)$ and $-c_j^{(m)}/A_j^{(m)} \subseteq [-1, 1]$ (for trigonometric polynomials), mentioned above, will be satisfied and the iterative method (11) is defined in each iteration. Besides, such choice of initial intervals provides the convergence of the interval sequences $(X_j^{(m)})$ $(j = 1, \ldots, n)$ in the sense that the sequences of radii $(r_j^{(m)})$ tend to 0 when $m \to \infty$.

Remark 3. If we choose $x_0 = \infty$ in the case of an algebraic polynomial, then formula (11) becomes

$$X_j^{(m+1)} = x_j^{(m)} - \frac{-c_j^{(m)}}{1 + \sum_{\substack{k=0\\k\neq j}}^{n} \frac{c_k^{(m)}}{X_j^{(m)} - x_k^{(m)}}} \qquad (j = 1, \ldots, n; m = 0, 1, \ldots), \tag{13}$$

which is the third order method proposed by Petković in [17]. We note that complex intervals can be used in (13).

Now we will derive some estimations which are necessary for the convergence analysis of the presented methods. The real functions given by (4) and their inverse functions are strictly monotonic increasing ($q(t) = \sin(t/2)$ on $[-\pi, \pi]$ and its inverse $q^{-1}(t) = 2 \arcsin t$ on $[-1, 1]$). Therefore, for an interval $X = [x_1, x_2]$ which belongs to the domain of monotonicity, we have (according to (3))

$$q(X) = [q(x_1), q(x_2)], \qquad q^{-1}(X) = [q^{-1}(x_1), q^{-1}(x_2)].$$

According to Theorem 5 from [2, Ch. 3] we obtain

$$rad(q(X)) = O(rad(X)) \tag{14}$$

and

$$rad(q^{-1}(X)) = O(rad(X)), \tag{15}$$

where "O" is Landau's symbol. Besides, from (2) we observe that

$$rad\left(\frac{1}{B}\right) = O(rad(B)) \qquad (0 \notin B). \tag{16}$$

The determination of the convergence speed of the interval methods presented in this paper reduces to the convergence analysis of positive null-sequences where the corresponding asymptotic error constants are positive and finite. As it is known (see [16, Exercise 9.3–4]), in this case the Q-, R- and C-orders are identical and we will use the unified notion "order of convergence".

Theorem 1. *Let $X_1^{(0)}, \ldots, X_n^{(0)}$ be initial real intervals containing the simple real zeros $\xi_1, \ldots, \xi_n$ of a generalized polynomial f and let q satisfy* (4). *Then for the interval method* (11) *we have for each $j = 1, \ldots, n$ and $m = 0, 1, \ldots$*

1^o $\xi_j \in X_j^{(m)}$;
2^o $r_j^{(m+1)} = O(r_j^{(m)2})$.

Proof. The choice $x_0 = \infty$ in the case of the algebraic polynomials, that provides a cubic convergence of the method (11), is already discussed in Remark 3 and will be omitted in the proof.

Assume that $\xi_j \in X_j^{(0)}$. Then, using (13) and the mathematical induction, the proof of 1^o follows with (10).

In the sequel, we will neglect iteration indices for simplicity and write X_j, $\hat{X}_j$, x_j, r_j, $\hat{r}_j$ instead of $X_j^{(m)}$, $X_j^{(m+1)}$, $x_j^{(m)}$, $r_j^{(m)}$, $r_j^{(m+1)}$.

Since the intervals $X_1, \ldots, X_n$ are disjoint we have $0 \notin X_j - x_k$ for $j \neq k$. This implies $0 \notin q(X_j - x_k)$ due to the monotonicity of q and (3). Therefore, the inverse of the interval $q(X_j - x_k)$ exists. According to (14) and (16) we estimate using Landau's symbol

$$rad\left(\frac{1}{q(X_j - x_k)}\right) = O(rad(q(X_j - x_k))) = O(rad(X_j)). \tag{17}$$

Assuming that the interval X_j is reasonably small, the midpoint x_j will be close to the zero ξ_j so that

$$|f(x_j)| = O(|x_j - \xi_j|) = O(|\varepsilon_j|).$$

Hence, in regard to the definition of c_j, there follows

$$|c_j| = O(|\varepsilon_j|) \qquad (j = 1, \ldots, n), \tag{18}$$

but only

$$|c_0| = O(1). \tag{19}$$

By (17), (18) and (19) we find

$$rad(A_j) = O\left(|c_0| r_j + \sum_{\substack{k=1 \\ k \neq j}}^{n} |c_k| r_j\right) = O(r_j). \tag{20}$$

Finally, by (15), (16), (18) and (20) we get

$$\hat{r}_j = rad(\hat{X}_j) = rad\left(q^{-1}\left(\frac{-c_j}{A_j}\right)\right) = O\left(rad\left(\frac{-c_j}{A_j}\right)\right),$$

that is

$$\hat{r}_j = |c_j| O\left(rad\left(\frac{1}{A_j}\right)\right) = O(|\varepsilon_j| r_j). \tag{21}$$

Hence

$$\hat{r}_j = O(r_j^2)$$

because of $|\varepsilon_j| \leq r_j$, which completes the proof. □

Remark 4. The estimation (19) yields the explanation for the quadratic convergence. Namely, (19) does not provide a better estimation for $rad(A_j)$ in (20). In the previously commented case $x_0 = \infty$ the term $|c_0| r_j$ does not appear in (20) so that a cubic convergence is feasible.

4. Improved Methods

Let

$$w_j := \frac{c_j q(x_j - x_0)}{c_0 q'(0)} \qquad (j = 1, \ldots, n), \tag{22}$$

where c_j is defined by (5). The iterative method

$$\hat{x}_j = x_j - w_j \qquad (j = 1, \ldots, n) \tag{23}$$

of the second order was considered in the papers [3], [6], [9], [12]. If $x_0 = \infty$ in the algebraic case, then (23) is the Durand-Kerner method

$$\hat{x}_j = x_j - \frac{f(x_j)}{\prod\limits_{k=1, k \neq j}^{n} (x_j - x_k)} \qquad (j = 1, \ldots, n),$$

known also as Weierstrass' method. Therefore, w_j is often called the *Weierstrass correction.*

Using Nourein's approach [15] Petković and Carstensen modified formula (13) in the recent paper [18] for algebraic polynomials incorporating the Weierstrass correction $w_j = c_j = f(x_j)/\prod\limits_{k \neq j} (x_j - x_k)$. It was proved that the new interval method

$$\hat{X}_j = x_j - \frac{c_j}{1 + \sum\limits_{k=1, k \neq j}^{n} (X_j - w_j - x_k)^{-1}} \qquad (j = 1, \ldots, n)$$

has the order of convergence equals to $\dfrac{3 + \sqrt{17}}{2} \cong 3.562$. The further improvements were achieved applying the centered form of inversion of intervals introduced in Section 2. The presented procedure with correction will be applied in this section to generalized polynomials starting from the iterative formula (11).

Before constructing new interval formulas we will present the iterative method with Weierstrass' correction in floating-point arithmetic. Taking Weierstrass' approximation $x_j^* := x_j - w_j$ instead of ξ_j in the fixed-point relation (9) we get the iterative formula

$$\hat{x}_j = x_j + q^{-1}\left(\frac{-c_j}{\displaystyle\sum_{k=0,k\neq j}^{n} \frac{c_k}{q(x_j - w_j - x_k)}}\right) \qquad (j = 1,\dots,n) \tag{24}$$

for the simultaneous approximation of zeros of generalized polynomials defined above.

For the convergence analysis of interval methods developed in this section the following assertion concerning convergence rate of the algorithm (24) is necessary:

Theorem 2. *If the initial approximations are sufficiently close to the zeros* $\xi_1, \dots, \xi_n$ *then the order of convergence of the iterative method* (24) *is three.*

Proof. Let us introduce the errors

$$\varepsilon_j := x_j - \xi_j, \qquad \varepsilon_j^* := x_j^* - \xi_j = x_j - w_j - \xi_j, \qquad \hat{\varepsilon}_j := \hat{x}_j - \xi_j$$

and let $co\{a,b\} := [min(a,b), max(a,b)]$ denote a convex hull of two real numbers a and b. We will assume that all errors are of the same order, that is $|\varepsilon_i| = O(|\varepsilon_j|) = O(|\varepsilon|)\,(i,j = 1,\dots,n)$, where $|\varepsilon| := max(|\varepsilon_k|: k \in \{1,\dots,n\})$. Using the identity (8) and $q(-\hat{x}_j + x_j) = q(\varepsilon_j - \hat{\varepsilon}_j)$ we obtain from (24)

$$c_j\left\{\frac{1}{q(\varepsilon_j)} - \frac{1}{q(\varepsilon_j - \hat{\varepsilon}_j)}\right\} = \sum_{k=0,k\neq j}^{n} c_k\left\{\frac{1}{q(\xi_j - x_k)} - \frac{1}{q(\xi_j - x_k + \varepsilon_j^*)}\right\}. \tag{25}$$

Applying the mean value theorem to the both sides of (25) we obtain

$$c_j\frac{\hat{\varepsilon}_j}{q^2(\eta_j)}q'(\eta_j) = -\sum_{k=0,k\neq j}^{n} c_k\frac{\varepsilon_j^*}{q^2(\theta_k)}q'(\theta_k), \tag{26}$$

where $\eta_j \in co\{\varepsilon_j, \varepsilon_j - \hat{\varepsilon}_j\}$ and $\theta_k \in co\{\xi_j - x_k, \xi_j - x_k + \varepsilon_j^*\}$.

For the iterative method (23) we have $\varepsilon_j^* = O(\varepsilon\varepsilon_j)$ (see [6]), and hence, $\varepsilon_j^* = O(\varepsilon_j^2)$ because of the above assumption. Furthermore, $c_0 = O(1)$, $c_j = O(\varepsilon_j)$ $(j = 1,\dots,n)$, $q(\eta_j) = O(\varepsilon_j)$, while $|q(\theta_k)|$ is lower bounded. According to these facts from (26) there follows

$$|\hat{\varepsilon}_j| = O\left(\frac{|\varepsilon_j^*||q^2(\eta_j)|}{|c_j|}\right) = O(|\varepsilon_j^*||\varepsilon_j|) = O(|\varepsilon|^3),$$

which proves the theorem. □

We will consider the interval method (11) in a general form. As in the case of the iterative method (24), the basic point in the construction of improved interval methods consists of the substitution of the inclusion interval X_j by the interval $X_j - w_j$. Conditions under which this substitution saves the inclusion property are considered in Lemma 2. As it can see *the underlying idea* consists of *the improvement of the midpoint to improve the radii.* Applying the inverse $(\cdot)^{-1}$ and $(\cdot)^I$ and Weierstrass' correction (22) in (11) we can construct the following *interval methods*:

$$\hat{X}_j = x_j + q^{-1}\left(\frac{-c_j}{\sum_{k=0,k\neq j}^{n} c_k[q(X_j - w_j - x_k)]^{-1}}\right) \qquad (j = 1,\dots,n), \tag{27}$$

$$\hat{X}_j = x_j + q^{-1}\left(\frac{-c_j}{\sum_{k=0,k\neq j}^{n} c_k[q(X_j - w_j - x_k)]^{I}}\right) \qquad (j = 1,\dots,n), \tag{28}$$

$$\hat{X}_j = x_j + q^{-1}\left(-c_j\left\{\sum_{k=0,k\neq j}^{n} c_k[q(X_j - w_j - x_k)]^{I}\right\}^{I}\right) \qquad (j = 1,\dots,n). \tag{29}$$

Let $O_R(IM)$ denote the order of convergence of an interval iterative method (IM) defined as in [2]. The order of convergence of the interval methods (27), (28) and (29) is given in the following theorem.

Theorem 3. *Let $X_1^{(0)}, \dots, X_n^{(0)}$ be initial intervals containing the zeros $\xi_1, \dots, \xi_n$ of the generalized polynomial f. If these intervals are sufficiently small then the iterative methods* (27)–(29) *converge and there holds*

$$O_R(27), \qquad O_R(28) \geq 1 + \sqrt{2} \cong 2.414, \qquad O_R(29) \geq 3.$$

The proof of Theorem 3 will be divided into several lemmas. As it was noted the improved method is constructed using the substitution of the inclusion interval X_j by the interval $X_j - w_j$. The following lemma yields the conditions under which this substitution is fruitful in the sense that provides the enclosure of zeros.

Lemma 2. *There exists a sufficiently small real number $\delta > 0$ such that for any real intervals $X_1, \dots, X_n$ with a length smaller than δ there holds for any $j = 1, \dots, n$*

$$\xi_j \in X_j \Rightarrow \xi_j \in X_j - w_j. \tag{30}$$

Proof. The implication (30) is equivalent to

$$|x_j - \xi_j| \leq r_j \Rightarrow |x_j - w_j - \xi_j| \leq r_j,$$

where $x_j := mid(X_j)$ and $r_j := rad(X_j)$. For the Weierstrass method (23) we have $|x_j - W_j - \xi_j| = O(|\varepsilon\varepsilon_j|)$ (see [6]), which is smaller than r_j provided when $|\varepsilon| := max_{j=1,\dots,n}|\varepsilon_j| \leq \delta$ is sufficiently small. □

Lemma 3. *Let X be a real interval, $y \in \mathbb{R}$ and let Φ: $G \to \mathbb{R}$ be analytic and monotone on the interval $y + X \subseteq G$. Then*

$$mid(\Phi(y + X)) = \Phi(mid(X)) + O(|y| + (rad(X))^2).$$

The proof of Lemma 3 is very simple: it merely uses Taylor series and the mean value theorem.

Lemma 4. *Let $r_j = rad(X_j)$ and $\hat{r}_j = rad(\hat{X}_j)$. Then we have the estimation*

$$|\hat{\varepsilon}_j| = O(|\varepsilon_j| r_j^2) \tag{31}$$

for the interval methods (27) *and* (28), *and*

$$|\hat{\varepsilon}_j| = O(|\varepsilon_j^3|) \tag{32}$$

for (29). *The radii of the new intervals* $\hat{X}_j$ *produced by all three methods* (27)–(29) *are given by*

$$\hat{r}_j = O(|\varepsilon_j| r_j). \tag{33}$$

Proof. We use a similar procedure as in [7] and [18] so that only the sketch of the proof will be given. Since

$$rad(q(X_j - x_k)) = O(rad(q(X_j - w_j - x_k)))$$

the proof of (33) is quite analogous to that of Theorem 1 (see the estimation (21)). The estimation (32) immediately follows from Theorem 2; namely, applying twice the centered form of the inverse of real intervals in (29) we obtain that the center of the interval $\hat{X}_j$ produced by (29) coincides with the "point" approximation $\hat{x}_j$ given by (24).

To prove (31) we estimate $mid(\hat{X}_j) - \hat{x}_j$ by elementary calculations using Lemma 3 and the relation

$$mid\left(\frac{1}{c + [a,b]}\right) = \frac{1}{mid([a,b])} + O(|c| + (b-a)^2),$$

where $\hat{X}_j$ is given by (27) or (28) and $\hat{x}_j$ by (24). Then the assertion follows with the triangle inequality and Theorem 2,

$$|\hat{\varepsilon}_j| = |x_j - \xi_j| + |x_j - mid(\hat{X}_j)| = O(\varepsilon\varepsilon_j^2) + O(\varepsilon_j r_j^2). \quad \square$$

To determine the order of convergence of the interval methods (27), (28) and (29) we use the following lemma which can be easily proved in the similar way as in [4]. We will say that C_m is a convergence factor if the sequence (C_m) of positive numbers C_m is bounded.

Lemma 5. *Let* (s_m) *be a positive null-sequence satisfying* $s_{m+2} \leq C_m s_{m+1}^h s_m^p$. *Then the order of convergence of* (s_m) *is at least* $(h + \sqrt{h^2 + 4p})/2$.

Proof of Theorem 3. First of all, by the mathematical induction we see that $\xi_j \in X_j^{(m)}$ for each $j = 1, \ldots, n$ and $m = 0, 1, \ldots$. Assuming that the initial intervals are narrow enough so that the interval methods (27)–(29) are convergent, that is, $r_j^{(m+1)} < r_j^{(m)}$ for each $j = 1, \ldots, n$ and $m = 0, 1, \ldots$, we get from Lemma 2 the implications

$$\xi_j \in X_j^{(m)} \Rightarrow \xi_j \in X_j^{(m)} - w_j^{(m)} \qquad (j = 1, \ldots, n; m = 0, 1 \ldots).$$

Therefore, the improved methods (27)–(29) are feasible. We note that conditions which enable the safe convergence of the applied interval method are most frequently sufficient for the fulfillment of the last implications (see, e.g., [7], [18]).

In regard to (31) and (33) we have $|\varepsilon_j^{(m+1)}|/r_j^{(m+1)} = O(r_j^{(m)})$ so that

$$r_j^{(m+2)} = O(|\varepsilon_j^{(m+1)}| r_j^{(m+1)}) = O((r_j^{(m+1)})^2 r_j^{(m)}).$$

According to Lemma 5 and the last relation we conclude that $O_R(27)$, $O_R(28) \geq 1 + \sqrt{2}$. Similarly, by (32) and (33) we have the order 3 for the midpoints and hence for the radii as well. $\square$

The accelerated interval methods presented in this section use the already calculated values incorporated in (22). Thus, the increase of the convergence rate of the iterative methods (27), (28) and (29) is attained without additional calculations which enables a high computational efficiency of these methods.

The presented algorithms (11), (27), (28) and (29) can also be applied in the case when the coefficients of f are intervals. This fact is of interest in practice since these coefficients can appear naturally as uncertain quantities (for example, if f is involved in mathematical models of some engineering disciplines or in simulation of processes). In this case the use of Weidner's transformation and suitable complex rectangular arithmetic method is not convenient. For illustration, let us consider the trigonometric polynomial

$$T_\nu(x) = A_0 + \sum_{k=1}^{\nu} (A_k \cos kx + B_k \sin kx),$$

where $A_0, A_1, \ldots, A_\nu, B_1, \ldots, B_\nu$ are real intervals. In order to reduce this trigonometric polynomial to an algebraic polynomial it is necessary to introduce the substitution $e^{ix} = w$ and construct the resulting algebraic polynomial

$$T_\nu^*(w) = w^{2\nu} + D_{2\nu-1} w^{2\nu-1} + \cdots + D_1 w + D_0$$

by the transformation

$$D_j = \frac{A_{\nu-j} + iB_{\nu-j}}{A_\nu - iB_\nu} \quad (j = 0, \ldots, \nu - 1), \qquad D_\nu = \frac{2A_0}{A_\nu - iB_\nu},$$

$$D_{\nu+j} = \frac{A_j + iB_j}{A_\nu - iB_\nu} \quad (j = 1, \ldots, \nu).$$

Since the coefficients of $T_\nu(x)$ are real intervals, the coefficients $D_0, D_1, \ldots, D_{2\nu-1}$ must be calculated in *rectangular interval arithmetic* ([2, Ch. 5]). But, the operations of inversion and multiplicaton (and, therefore, division) in rectangular arithmetic are *not exact* ones which can produce reasonable large rectangles (coefficients of $T_\nu^*(w)$). Consequently, the resulting inclusion intervals will be larger compared with those obtained directly by the interval methods proposed in this paper. This is an important advantage of the presented methods.

Example. The interval methods (27), (28) and (29) have been tested on the example of the exponential polynomial

$$E_2(x) = a_0 + a_1 e^{-x} + b_1 e^{x} + a_2 e^{-2x} + b_2 e^{2x},$$

where

$$a_0 = e^3 + e^{-3} + ps, \qquad a_1 = -(e^{7/2}p + e^{1/2}s), \qquad b_1 = -(e^{-7/2}p + e^{-1/2}s)$$

$$a_2 = e^4, \qquad b_2 = e^{-4}, \qquad p = 2\cosh\tfrac{3}{2}, \qquad s = 2\cosh\tfrac{1}{2}$$

(example taken from [13]). The exact zeros of this exponential polynomial are $\xi_1 = -1$, $\xi_2 = 2$, $\xi_3 = 3$ and $\xi_4 = 4$. As the initial interval containing these zeros we have taken (using a grafic presentation in the programming package *Mathematica*)

$$X_1^{(0)} = [-1.5, -0.6], \quad X_2^{(0)} = [1.75, 2.2], \quad X_3^{(0)} = [2.8, 3.25], \quad X_4^{(0)} = [3.7, 4.4]$$

and $X_0 = [1, 1]$. The corresponding routines have been realized in the programming language PASCAL-XSC.

After the third iterative step the interval method (11) produced the following inclusion intervals.

$$X_1^{(3)} = [-1.000000000112175, -0.9999999998313206], \qquad r_1^{(3)} = 1.4 \times 10^{-10},$$

$$X_2^{(3)} = [1.999999999999167, 2.000000000000719], \qquad r_2^{(3)} = 7.8 \times 10^{-13},$$

$$X_3^{(3)} = [2.999999999998647, 3.000000000000898], \qquad r_3^{(3)} = 1.1 \times 10^{-12},$$

$$X_4^{(3)} = [3.999999999959944, 4.000000000025437], \qquad r_4^{(3)} = 3.3 \times 10^{-11}.$$

For the simultaneous inclusion of all zeros of the above polynomial the modified interval methods (27), (28) and (29) with Weierstrass's corrections were applied with the same initial inclusion intervals. Let $r^{(m)} = max_{1 \le i \le 4} r_i^{(m)}$ $(m = 0, 1, \ldots)$ be the maximal semi-width (radius) in the m-th iterative step. These values are given in Table 1 for all three modified methods and for the Weierstrass method (11) too. The improvements by the methods (27), (28) and (29) can be seen in the later iterations when we are interested in more than 15 accurate digits.

Table 1. The maximal semi-widths

	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$
Method (11)	3.12×10^{-2}	6.25×10^{-5}	2.81×10^{-10}
Method (27)	3.05×10^{-2}	2.89×10^{-5}	7.74×10^{-13}
Method (28)	3.01×10^{-2}	8.23×10^{-5}	2.20×10^{-11}
Method (29)	3.84×10^{-2}	1.55×10^{-5}	2.82×10^{-13}

For comparison, we have applied the method based on Weidner's transformation and the interval Durand-Kerner method [2, Ch. 8]. The obtained inclusion intervals were of the same size as in the case of the inerval method (11), while the CPU time of (11) was slightly greater compared with the interval Weidner-Durand-Kerner method. Other tested examples have shown the similar results.

We finish our consideration with some remarks concerning exponential and trigonometric polynomials particularly. Makrelov and Semerdziev [13] and Frommer [9] have always given several practical observations about the domain of convergence of methods of the same type in ordinary floating-point arithmetic, their convergence properties, and an implementation on a vector or parallel computer. In the case of multiple zeros all these metods converge only linear. The situation is even worse for the presented interval methods (and, more generally, for the most of interval methods when the orders of multiplicities are not known) since the division by zero-intervals of the form $q(X_j - x_k)$ appears. Finally, we note that inclusion methods for the complex zeros of trigonometric and exponential polynomials are generally rather complicated since they deal with circular or rectangular complex functions and, thus, the same is valid for the considered methods.

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