Coupling of FEM and BEM for a Nonlinear Interface Problem: The h–p Version

C. Carstensen
Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS, UK

E. P. Stephan
Institut für Angewandte Mathematik, Universität Hannover, Welfengarten 1, D 30167 Hannover, FRG

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This article presents some numerical examples for coupling the finite element method (FEM) and the boundary element method (BEM) as analyzed in [11]. This coupling procedure combines the advantages of boundary elements (problems in unbounded regions) and of finite elements (nonlinear problems with inhomogeneous data). In [28], experimental rates of convergence for the h version are presented, where the accuracy of the Galerkin approximation is achieved by refining the mesh. In this article we treat the h–p version, combining an increase of the degree of the piecewise polynomials with a certain mesh refinement. In our model examples, we obtain theoretically and numerically exponential convergence, which indicates a great efficiency in particular if singularities appear. © 1995 John Wiley & Sons, Inc.

I. INTRODUCTION

The finite element method can be applied to nonlinear or inhomogeneous problems concerning partial differential equations, but is restricted to bounded domains. This is contrary to the boundary element methods, which can be applied to the most important linear and homogeneous partial differential equations with constant coefficients also in unbounded domains (provided that the boundary is bounded).

The coupling of FEM and BEM comes of interest, since it allows a combination of the advantages of both methods. Hence, it is applied for linear transmission problems in scattering problems, elastodynamics, electromagnetism, and elasticity [1–6]; numerical examples may be found in [7, 8]. Recently, a class of nonlinear interface problems is treated in [9–12] using a symmetric coupling method, which allows a variational formulation of a saddle-point problem.

In this article we improve the convergence of the coupling method using the h–p version with a geometric mesh for the first time. Even in the case of singular solutions, we get exponential convergence, which leads to an efficient numerical treatment of the problems.

A motivating interface problem in three-dimensional solid mechanics and a two-dimensional numerical test case are stated in Sections II and III to recall the coupling

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procedure and to describe the error analysis. In particular we contribute an estimate for approximate discrete solutions in Theorem 3. For numerical results in two-dimensional elasticity, we refer to [16]; this article focuses on two-dimensional harmonic examples. In Section IV, the discretization for the finite elements and boundary elements is sketched for the h–p version. Then, we derive exponential convergence of the h–p version of the Galerkin procedure of the coupled problem. The iterative solution and its numerical implementation are described explicitly in Sections V and VI. Numerical experiments are reported in Section VII to underline the exponential convergence and the efficiency of the proposed treatment of such nonlinear interface problems in case of singularities.

II. COUPLING METHOD FOR A MONOTONE PROBLEM FOR HENCKY-ELASTICITY

Let Ω_1 be a three-dimensional bounded Lipschitz domain with $\partial\Omega_1 = \Gamma_u \cup \Gamma$ in which we assume the nonlinear Hencky–von Mises stress–strain relation of the form

$$\sigma = \left(k - \frac{2}{3}\,\mu(\gamma)\right)I \cdot \operatorname{div} u_1 + 2\mu(\gamma)\epsilon\,,$$

where σ and $\epsilon = \frac{1}{2}(\nabla u^T + \nabla u)$ denotes the (Cauchy) stresses and the (linear Green) strain, respectively, see [13–15]. Then, if we define

$$P_1(u_1)_i := \frac{\partial}{\partial x_i}\left(k - \frac{2}{3}\,\mu(\gamma(u_1))\right) \cdot \operatorname{div} u_1 + \sum_{j=1}^{3} 2\,\frac{\partial}{\partial x_j}\,\mu(\gamma(u_1))\epsilon_{ij}(u_1)$$

for $i = 1,2,3$, the equilibrium condition $\operatorname{div}\sigma + F = 0$ gives

$$P_1(u_1) = F \qquad \text{in } \Omega_1\,. \tag{1}$$

Here, the bulk modulus k and the function $\mu(\gamma)$ in P_1 satisfy (cf., e.g., [14])

$$0 < \tilde{\mu}_0 \le \mu(\gamma) \le \frac{3}{2}\,k, \qquad 0 < \tilde{\mu}_1 \le \mu + 2\gamma\,\frac{d\mu}{d\gamma} \le \tilde{\mu}_2 < \infty,$$

where $\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\mu}_2$ are constants and

$$\gamma(u_1) = \sum_{i,j=1}^{3}\left(\epsilon_{ij} - \delta_{ij}\,\frac{1}{3}\cdot\operatorname{div} u_1\right)^2, \qquad \epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_{1i}}{\partial x_j} + \frac{\partial u_{1j}}{\partial x_i}\right).$$

In a surrounding unbounded exterior region Ω_2, we consider the homogeneous Lamé system describing linear isotropic elastic material, with the Lamé constants $\mu_2 > 0$, $3\lambda_2 + 2\mu_2 > 0$,

$$P_2(u_2) = -\mu_2\Delta u_2 - (\lambda_2 + \mu_2)\,grad\;div\;u_2 = 0 \qquad \text{in } \Omega_2\,. \tag{2}$$

The interface problem under consideration [11] reads: *For a given vector field F in Ω_1 find vector fields u_j in $\Omega_j (j = 1,2)$ satisfying $u_1|_{\Gamma_u} = 0$, the differential Eqs.* (1), (2), *the interface conditions*

$$u_1 = u_2, \qquad T_1(u_1) = T_2(u_2) \qquad \text{on } \Gamma\,, \tag{3}$$

and the regularity condition at infinity ($n = 3$)

$$u_2 = \mathcal{O}\left(\frac{1}{|x|}\right) \qquad \text{as } |x| \longrightarrow \infty . \tag{4}$$

Here, with $\mu_1 = \mu(\gamma(u_1))$, $\lambda_1 = k - \frac{2}{3}\mu(\gamma(u_1))$, the tractions are given by

$$T_j(u_j) = 2\mu_j \partial_n u_j + \lambda_j n \, div \, u_j + \mu_j n \times curl \, u_j , \tag{5}$$

and $\partial_n u_j$ is the derivative with respect to the outer normal on Γ.

We are interested in solutions u_j of (1)–(4), which belong to $(H^1_{loc}(\Omega_j))^3$, i.e., which are of finite energy. A variational formulation is obtained as in [11]. An application of the first Green formula to (1) yields

$$\int_{\Omega_1} P_1 u_1 w \, dx = \Phi_1(u_1, w) - \int_\Gamma T_1 u_1 w \, ds \tag{6}$$

for all $w \in H^1(\Omega_1)$, where

$$\Phi_1(u_1, w) := \int_{\Omega_1} \left\{ k - \frac{2}{3} \mu(\gamma(u_1)) \text{ div } u_1 \text{ div } w + \sum_{i,j=1}^{3} 2\mu(\gamma(u_1))\epsilon_{ij}(u_1)\epsilon_{ij}(w) \right\} dx . \tag{7}$$

On the other hand, the solution u_2 of (2) is given by the Somigliana representation formula for $x \in \Omega_2$:

$$u_2(x) = \int_\Gamma \{T_2(x,y)v_2(y) - G_2(x,y)\phi_2(y)\} ds(y) , \tag{8}$$

where $v_2 = u_2$, $\phi_2 = T_2(u_2)$ on Γ, and the fundamental solution $G_2(x,y)$ of $P_2 u_2 = 0$ is the 3×3 matrix function

$$G_2(x,y) = \frac{\lambda_2 + 3\mu_2}{8\pi\mu_2(\lambda_2 + 2\mu_2)} \left\{ \frac{1}{|x - y|} I + \frac{\lambda_2 + \mu_2}{\lambda_2 + 3\mu_2} \frac{(x - y)(x - y)^T}{|x - y|^3} \right\}$$

with the unit matrix I and $T_2(x,y) = T_{2,y}(G_2(x,y))^T$, where T denotes transposition. Taking Cauchy data in (8), i.e., boundary values and tractions on Γ for $x \longrightarrow \Gamma$, we obtain a system of boundary integral equations on Γ,

$$v_2 = \left(\frac{1}{2} + \Lambda_2\right) v_2 - V_2 \phi_2 \quad \text{and} \quad \phi_2 = -W_2 v_2 + \left(\frac{1}{2} - \Lambda_2'\right)\phi_2 \tag{9}$$

with the single layer potential V_2, a weakly singular boundary integral operator, the double layer potential Λ_2 and its dual Λ_2', strongly singular operators, and the hypersingular operator W_2 defined as

$$V_2\phi_2(x) = \int_\Gamma G_2(x,y)\phi_2(y) \, ds(y)$$

$$\Lambda_2 v_2(x) = \int_\Gamma T_2(x,y) v_2(y) \, ds(y)$$

$$\Lambda_2'\phi_2(x) = T_{2,x} \int_\Gamma G_2(x,y)^T \phi_2(y) \, ds(y)$$

$$W_2 v_2(x) = -T_{2,x} \int_\Gamma T_2(x,y) v_2(y) \, ds(y) .$$

As in interface problems for purely linear equations [3], we obtain a variational formulation for the interface problem (1)–(4) by adding a weak form of the boundary integral Eqs. (9) on Γ to the weak form (6). Then we insert it into (6) and make use of the interface conditions (3), i.e., $t_2 = t_1 =: \phi$ and $v_2 = u_1 =: u$.

This yields the following variational problem: *For given* $F \in L^2(\Omega_1)^3$ *find* $u \in H^1(\Omega_1)^3$, $\phi \in H^{-1/2}(\Gamma)^3$ *such that* $u|_{\Gamma_u} = 0$ *and*

$$b(u,\phi;w,\psi) = \int_{\Omega_1} F \cdot w\,dx \quad \text{for all} \quad (w,\psi) \in H^1(\Omega_1)^3 \times H^{-1/2}(\Gamma)^3. \tag{10}$$

Here, with the form $\Phi_1(\cdot,\cdot)$ in (7) and the brackets $\langle\cdot,\cdot\rangle$ denoting the extended L^2-duality duality between the trace space $H^{1/2}(\Gamma)^3$ and its dual $H^{-1/2}(\Gamma)^3$, we define

$$b(u,\phi;w,\psi) := \Phi_1(u,w) + \langle w, W_2 u\rangle - \left\langle w, \left(\frac{1}{2} - \Lambda_2'\right)\phi\right\rangle - \left\langle\left(\frac{1}{2} - \Lambda_2\right)u, \psi\right\rangle - \langle \psi, V_2\phi\rangle. \tag{11}$$

Theorem 1 ([11, 16]). *For* $F \in L^2(\Omega_1)^3$ *there exists exactly one solution* $u \in H^1(\Omega_1)^3$, $\phi \in H^{-1/2}(\Gamma)^3$ *of* (10) *yielding* ($u = u_1$ *in* Ω_1 *and* u_2 *given by* (8) *in* Ω_2) *a solution of the interface problem* (1)–(4).

The proof in [11] is based on the fact that the C^2-functional,

$$\begin{aligned} J_1(u,\phi) :=\ & A(u) + \frac{1}{2}\langle u, W_2 u\rangle \\ & - \int_{\Omega_1} Fu\,dx + \left\langle \phi, \left(\Lambda_2 - \frac{1}{2}\right)u\right\rangle - \frac{1}{2}\langle\phi, V_2\phi\rangle \\ A(u) :=\ & \int_{\Omega_1}\left\{\frac{1}{2}k|\mathrm{div}\, u|^2 + \int_0^{\gamma(u)} \mu(t)\,dt\right\}dx, \end{aligned} \tag{12}$$

$u \in H^1(\Omega_1)^3$, $\phi \in H^{-1/2}(\Gamma)^3$, has a unique saddle-point. The two-dimensional case, treated in [16], requires minor modifications only.

Given finite dimensional subspaces $X_N \times Y_M$ of $H^1(\Omega_1)^3 \times H^{-1/2}(\Gamma)^3$, the Galerkin solution $(u_N, \phi_M) \in X_N \times Y_M$ is the unique saddle-point of the functional J_1 on $X_N \times Y_M$; the *Galerkin scheme* for (10) reads: *Given* $F \in L^2(\Omega_1)^3$ *find* $u_N \in X_N$ *and* $\phi_M \in Y_M$ *such that, for all* $w \in X_N$ *and* $\psi \in Y_M$,

$$b(u_N, \phi_M; w, \psi) = \int_{\Omega_1} f \cdot w\,dx. \tag{13}$$

The Theorem 2 states quasi-optimal convergence in the energy norm for any conforming Galerkin scheme. See [16] for the two-dimensional case.

Theorem 2 ([11, 16]). *There exists exactly one solution* $(u_N, \phi_M) \in X_N \times Y_M$ *of the Galerkin Eqs.* (13). *There exists a constant* C *independent of* X_N *and* Y_M *such that*

$$\|u - u_N\|_{H^1(\Omega_1)^3} + \|\phi - \phi_M\|_{H^{-1/2}(\Gamma)^3} \le C\left\{\inf_{w\in X_N}\|u - w\|_{H^1(\Omega_1)^3} + \inf_{\psi\in Y_M}\|\phi - \psi\|_{H^{-1/2}(\Gamma)^3}\right\} \tag{14}$$

where $(u, \phi) \in H^1(\Omega_1)^3 \times H^{-1/2}(\Gamma)^3$ *is the exact solution of the variational problem* (10).

Within the class of saddle-point problems, the Galerkin solution can, in general, be approximated by an iterative process only. To control the error of an approximation $(\tilde{u}_N, \tilde{\phi}_M)$ to the Galerkin solution (u_N, ϕ_M), we prove the following *a posteriori* estimate.

Theorem 3. *Let* $(u_N, \phi_M) \in X_N \times Y_M$ *be the unique Galerkin solution of* (13) *and let* $(\tilde{u}_N, \tilde{\phi}_M) \in X_N \times Y_M$ *be known such that we can compute*

$$\tilde{r}_N := \|DJ_1 \times (\tilde{u}_N, \tilde{\phi}_M)\|_{H^1(\Omega_1)^* \times H^{1/2}(\Gamma)}.$$

Then,

$$\|(u_N - \tilde{u}_N, \phi_M - \tilde{\phi}_M)\|_{H^1(\Omega_1)\times H^{-1/2}(\Gamma)} \leq C \cdot \tilde{r}_N\,.$$

The constant $C > 0$ *depends on* Ω_1, Γ, *and the constants* k, $\tilde{\mu}_j$, λ_2, μ_2 *only; but not on* $X_N \times Y_M$.

Proof. Since V_2 is positive definite and W_2 is positive semi-definite, and since D^2A is uniformly monotone (see, e.g., [11]) we infer, using the main theorem on calculus,

$$\begin{aligned} C^{-1}\|(u_N - \tilde{u}_N, \phi_M - \tilde{\phi}_M)\|^2_{H^1(\Omega_1)\times H^{-1/2}(\Gamma)} &\leq 2\int_0^1 D^2A(t \cdot u_N + (1-t)\cdot \tilde{u}_N) \\ &\quad \times [u_N - \tilde{u}_N, u_N - \tilde{u}_N]\,dt \\ &\quad + \langle V_2(\phi_M - \tilde{\phi}_M), (\phi_M - \tilde{\phi}_M)\rangle \\ &\quad + \langle W_2(u_N - \tilde{u}_N), (u_N - \tilde{u}_N)\rangle \\ &\leq 2DA(u_N)[u_N - \tilde{u}_N] - DA(\tilde{u}_N)[u_N - \tilde{u}_N] \\ &\quad + \langle V_2(\phi_M - \tilde{\phi}_M), (\phi_M - \tilde{\phi}_M)\rangle \\ &\quad + \langle W_2(u_N - \tilde{u}_N), (u_N - \tilde{u}_N)\rangle. \end{aligned}$$

Noting that $DJ_1(u_N)[u_N - \tilde{u}_N, \tilde{\phi}_M - \phi_M] = 0$ and $DA = \Phi_1$, we derive

$$\begin{aligned} C^{-1}\|(u_N - \tilde{u}_N, \phi_M - \tilde{\phi}_M)\|^2_{H^1(\Omega_1)\times H^{-1/2}(\Gamma)} &\leq -DJ_1(\tilde{u}_N, \tilde{\phi}_N)[(u_N - \tilde{u}_N, \tilde{\phi}_M - \phi_M)] \\ &\leq \tilde{r}_N \cdot \|(u_N - \tilde{u}_N, \tilde{\phi}_M - \phi_M)\|_{H^1(\Omega_1)\times H^{1/2}(\Gamma)}. \end{aligned}$$

From this, we conclude the assertion. ∎

In the numerical examples below, we compute $(\tilde{u}_N, \tilde{\phi}_M)$ such that $\tilde{r}_N$ is of machine precision. Then, by triangle inequality, Theorems 2 and 3 verify that $(\tilde{u}_N, \tilde{\phi}_M)$ is a reasonable approximation of (u, ϕ). This justifies the numerical treatment below and in [8].

III. MODEL PROBLEM

Our numerical experiments with the h–p version are related to the following two-dimensional model problem [8] involving prescribed jumps across the interface Γ: *Given* $F \in L^2(\Omega)$, $f \in H^{1/2}(\Gamma)$, $g \in H^{-1/2}(\Gamma)$, *find* $u_1 \in H^1(\Omega_1)$, $u_2 \in H^1_{\mathrm{loc}}(\Omega_2)$ *satisfying*

$$
\begin{aligned}
P_1 u_1 := -\operatorname{div}(p|\nabla u_1| \cdot \nabla u_1) + u_1 = F \qquad \text{in } \Omega_1 \\
-\Delta u_2 = 0 \qquad \text{in } \Omega_2 \\
u_1 = u_2 + f, p(|\nabla u_1|)\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} + g \qquad \text{on } \Gamma \\
u_2(x) = A \log|x| + o(1) \qquad \text{for } |x| \longrightarrow \infty .
\end{aligned}
\tag{15}
$$

Here, $A \in \mathbb{R}$ is a constant depending on u_2 and $p \in C^1(\mathbb{R})$ satisfies, with some constants $\gamma_1, \gamma_2 > 0$,

$$
\gamma_1 \leq p(r) \leq \gamma_2 \quad \text{and} \quad \gamma_1 \leq p(r) + rp'(r) \leq \gamma_2 \quad (r \geq 0).
$$

As in the previous section, the interface problem (15) allows an equivalent variational formulation:

$$
b(u, \phi; w, \psi) = \int_{\Omega_1} Fw\,dx + l(w, \psi) \tag{16}
$$

for all $(w, \psi) \in H^1(\Omega_1) \times H^{-1/2}(\Gamma)$, where b is given in (11), and

$$
\Phi_1(u, w) := 2 \cdot \int_{\Omega_1} (p|\nabla u| \nabla u \nabla w + uw)\,dx \tag{17}
$$

$$
l(w, \psi) := \langle w, g\rangle + \langle f, \psi\rangle + \langle w, W_2 f\rangle + \langle \Lambda_2' g, w\rangle - \langle \Lambda_2 f, \psi\rangle + \langle V_2 g, \psi\rangle. \tag{18}
$$

Corresponding to the Laplace operator, we have the single-layer potential operator V_2, the double-layer potential operator Λ_2 and its adjoint Λ_2', and the hypersingular operator W_2 as defined above with $-\frac{1}{\pi}\log|x - y|$ replacing $G_2(x, y)$ and $\frac{\partial}{\partial n}$ replacing T (see, e.g., [8] for details). As in [8], we assume $cap(\Gamma) < 1$ so that V_2 is positive definite. Let

$$
J_1(u, \phi) := J(u) + \langle u, (\Lambda_2' - 1)\phi\rangle - \frac{1}{2}\langle \phi, V_2 \phi\rangle \tag{19}
$$

$$
J(u) := 2J_0(u) + \frac{1}{2}\langle u, W_2 u\rangle \tag{20}
$$

$$
J_0(u) := \int_{\Omega_1} \left\{ \int_0^{|\nabla u|} t p(t)\,dt + \frac{1}{2}|u|^2 - fu \right\} dx .
$$

Under the present conditions on p, the second Gateaux derivative of J_0 is uniformly monotone [8], so that the results in [11] are applicable and briefly summarized as follows (see [8, 16]):

a. The weak form of the Euler equation to the variational problem of J_1 coincides with the weak form (16) of the coupling problem (15).
b. The variational problem (16) has exactly one solution (u, ϕ).
c. For any pair of finite dimensional subspaces $X_N \subset H^1(\Omega)$, $Y_M \subset H^{-1/2}(\Gamma)$, there exists exactly one solution (u_N, ϕ_M) of the Galerkin scheme for (16) and a constant C independent of X_N and Y_M such that

$$
\|u - u_N\|_{H^1(\Omega_1)} + \|\phi - \phi_M\|_{H^{-1/2}(\Gamma)} \leq C\left\{ \inf_{w \in X_N} \|u - w\|_{H^1(\Omega_1)} + \inf_{\psi \in Y_M} \|\phi - \psi\|_{H^{-1/2}(\Gamma)} \right\}.
$$

d. Theorem 3 is also valid for the two-dimensional model problem at hand.

IV. DISCRETIZATION

Let the two-dimensional domain Ω_1 have the polygonal boundary Γ, i.e., $\Gamma = \overline{\cup_{j=1}^{m}\Gamma_j}$ is the union of straight lines $\Gamma_1, \ldots, \Gamma_m$ connecting the endpoints $x_0 = x_m, x_1, \ldots, x_m$. Near the corner point x_j we improve the approximation quality of the trial space concerning the corner singularities using a geometric mesh and a particular distribution of the polynomial degrees.

First we define a geometric partition I_σ^n of level n on the interval $I = [0, 1]$ by $x_0 := 0$ and $x_j := \sigma^{n-j}$, $j = 1, \ldots, n$. With a degree vector $q = (q_1, \ldots, q_n)$ the trial space $S^q(I_\sigma^n)$ is the vector space of all continuous functions on I, which are piecewise polynomials with degree q_{j+1} on (x_j, x_{j+1}). Next we introduce the analogous two-dimensional vector space on $Q = [0, 1] \times [0, 1]$ as a space of tensor-products

$$S^{q,r}(Q_\sigma^n) = S^q(I_\sigma^n) \times S^r(I_\sigma^n).$$

In our examples, we use a geometric mesh-refinement towards the origin of Q by using a geometric partition of Ω_1 obtained by affine transformations of $S^{q,r}(Q_\sigma^n)$ as shown in Fig. 1 and Fig. 2. Then we define $X_N := S^{q,r}(Q_\sigma^n)$ with N being the dimension of $S^{q,r}(Q_\sigma^n)$.

The trail space Y_M for the boundary elements is obtained as a trace space of gradients in X_N, i.e.,

$$Y_M := S^s(\Gamma_\sigma^n) := \{\nabla w_N|_\Gamma \colon w_N \in X_N\},$$

where $M := \dim Y_M$ is the number of degrees of freedom. This means we take the partition of the boundary Γ induced by the geometric partition of Ω_1 and take piecewise polynomials there with the degree from the neighboring finite element (along the current side) minus one. Note functions in Y_M are, in general, discontinuous.

By using countable normed spaces $B_\beta^l(\Omega)$ (which are appropriately weighted Sobolev spaces; see Appendix) used by Guo and Babuska in [17], one can prove convergence rates (see [8]) as in the linear case [7]: Denote the internal angle at x_j by $\omega_j(0 < \omega_j < 2\pi, 1 \le j \le m)$ and choose $\beta = (\beta_1, .., \beta_m)$ under the condition $0 < \beta_j < 1/2$, $\beta_j > 1 - \pi/\omega_j$. In the linear case, certain conditions on the data f and g (namely $f \in B_\beta^{3/2}(\Gamma)$ and $g \in B_\beta^{1/2}(\Gamma)$) lead to the regularity of the solution (namely $u \in B_\beta^1(\Omega)$). In the nonlinear case, we have to assume this regularity assumption explicitly and then conclude, as in [7],

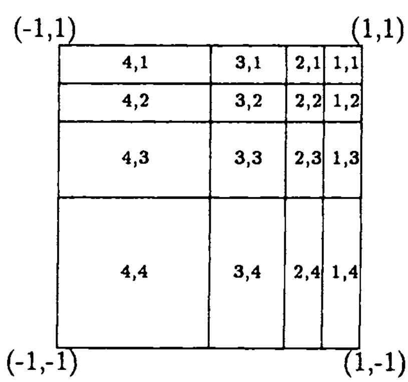

FIG. 1. Geometric mesh with polynomial degrees.

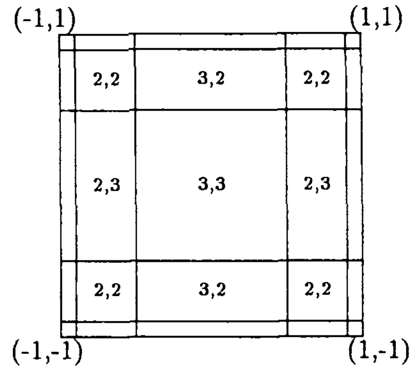

FIG. 2. Symmetric geometric mesh with polynomial degree.

$$\|u - u_N\|_{H^1(\Omega_1)} + \|\phi - \phi_M\|_{H^{1/2}(\Gamma)} \leq C(e^{-b\sqrt[3]{N}} + e^{-b\sqrt{M}}),$$

where the constants b and C are independent of M and N.

V. SOLVING THE DISCRETE PROBLEM

According to the nonlinear function J_1 as in (19), the Galerkin equations

$$DJ_1(u_N, \phi_M)[v, \phi] = 0 \qquad \forall\, (v, \phi) \in X_N \times Y_N, \tag{21}$$

are to be solved within an iterative process. Let $U_N^{(m)}$ and $\Phi_N^{(m)}$ denote the coefficient vectors of the piecewise polynomials $u_N^{(m)}$ and $\phi_M^{(m)}$, respectively, obtained iteratively with Newton–Raphson method or the method of Broyden. One step of Newton's–Raphson's method can be written in a compact form as

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} u_N^{(m)} - u_N^{(m+1)} \\ \phi_N^{(m)} - \phi_N^{(m+1)} \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

with A_{11} being positive definite and A_{22} being negative definite defined by

$$\begin{aligned}
A_{11} &:= D^2 J_0(u_N^{(m)})[v; w] + \langle w, W_2 v \rangle, \\
A_{12} &:= \langle (\Lambda_2' - 1)\xi, v \rangle = A_{21}^T \\
A_{22} &:= -\langle V_2 \xi, \psi \rangle \\
B_1 &:= DJ_0(u_N^{(m)})[v] + \langle u_N^{(m)} - f, W_2 v \rangle + \langle (\Lambda_2' - 1)\phi_N^{(m)}, v \rangle - \langle (\Lambda_2' + 1) g, v \rangle \\
B_2 &:= -\langle V_2 \phi_N^{(m)} - g - (\Lambda_2 - 1) u_N^{(m)} + (\Lambda_2 - 1) f, \psi \rangle.
\end{aligned}$$

One iteration step of the method of Broyden, a quasi-Newton method, reads

$$\begin{pmatrix} U_N^{(m+1)} \\ \Phi_N^{(m+1)} \end{pmatrix} = \begin{pmatrix} U_N^{(m)} \\ \Phi_N^{(m)} \end{pmatrix} + A_m^{-1} \cdot R_m,$$

where $A_0 = (A_{ij})$ is the stiffness matrix evaluated at $(u_N^{(0)}, \phi_M^{(0)})$ and then updated by

$$A_m = A_{m-1} + \frac{1}{d_m^T \cdot d_m} (e_m - A_{m-1} d_m) d_m^T,$$

while $d_m := U_N^{(m)} - U_N^{(m-1)}$, $e_m := R_m - R_{m-1}$, $R_{-1} := U_N^{(-1)} := 0$.

In our numerical examples, the iterations of the Newton- and Broyden-method have been performed until the residual $\tilde{r}_N$ in Theorem 3 was of the order of the machine precision ϵ. Then, Theorem 3 verifies that the computed approximation $(u_N^{(m)}, \phi_N^{(m)})$ might replace the unknown Galerkin-solution (u_N, ϕ_M) in our numerical experiments reported below.

VI. NUMERICAL IMPLEMENTATION

In this subsection, we briefly report on the numerical evaluation of the stiffness matrices involved in the iterative process of Section V.

A. Integrals over the Domain

In the evaluation of the Gateaux-derivatives DJ_0 and D^2J_0 of J_0 [see (20)] we have integrals over Ω_1 to be computed by applying a standard 32×32 point Gaussian quadrature formula on any element.

B. Single-Layer Potential

With ϕ and $\psi \in S^r(\Gamma_\sigma^k)$ we get for the single-layer potential operator V_2:

$$\langle V_2\phi, \psi\rangle = -\frac{1}{\pi}\int_\Gamma \psi(y)\int_\Gamma \phi(x)\log|x-y|\,ds_x\,ds_y,$$

where ψ and ϕ are monomials on $\Gamma_j \in \Gamma_\sigma^k$. To perform the outer integral, we use a 32-point Gauss quadrature formula, whereas the inner integral we compute analytically as follows: An affine transformation mapping Γ_j to $[-1, 1]$ leads to

$$\begin{aligned} I_1(y) &:= \int_{\Gamma_j} \phi(x)\log|x-y|\,ds_x = \frac{ds_x}{2d\xi}\int_{-1}^{1} \xi^r \log(a\xi^2 + b\xi + c)\,d\xi \\ &= \frac{ds_x}{2d\xi}\int_{-1}^{1} \xi^r \log(|a|)\,d\xi + \Re\frac{ds_x}{2d\xi}\int_{-1}^{1} \xi^r\log(\xi - z_1)\,d\xi \\ &\quad + \Re\frac{ds_x}{2d\xi}\int_{-1}^{1} \xi^r \log(\xi - z_2)\,d\xi, \end{aligned}$$

where the constants a, b, c with $b^2 - 4ac \leq 0$, depend on y only, and Γ_j and z_1 and z_2 are complex numbers with $(\xi - z_1)(\xi - z_2) = \xi^2 + \frac{b}{a}\xi + \frac{c}{a}$. The appearing integrals are then evaluated with

$$\int \xi^r \log(\xi - z_0)\,d\xi = \frac{\xi^{1+r} - z_0^{1+r}}{1+r}\log(\xi - z_0) - \frac{1}{1+r}\sum_{k=1}^{1+r}\frac{x^{r-k+2}z_0^{k-1}}{r-k+2}.$$

C. Double-Layer Potential

For $v \in S^{p,q}(\Omega_\sigma)$ and $\psi \in S^r(\Gamma_\sigma^k)$ a typical term involving the double-layer potential operator is

$$\langle \psi, \Lambda_2 v \rangle = \langle \Lambda_2' \psi, v \rangle = -\frac{1}{\pi} \int_\Gamma v(y) \int_\Gamma \psi(x) \frac{\partial}{\partial n_y} \log|x - y| \, ds_x \, ds_y ,$$

where ψ is a monomial on $\Gamma_j \in \Gamma_\sigma^k$. The outer integral we evaluate using a 32-point Gauss quadrature formula, whereas the inner integral we again compute analytically: With an affine transformation and related constants a, b, c, e, f satisfying $b^2 - 4ac \leq 0$,

$$I_2(y) := \int_{\Gamma_j} \phi(x) \frac{\partial}{\partial n_y} \log|x - y| \, ds_x = \frac{ds_x}{2d\xi} \int_{-1}^{1} \xi^r \frac{e\xi + f}{a\xi^2 + b\xi + c} \, d\xi .$$

To evaluate the appearing integrals we let $R := a\xi^2 + b\xi + c$, $\Delta := 4ac - b^2$, and make use of

$$\int \frac{\xi^m}{R} d\xi = \frac{x^{m-1}}{(m-1)A} - \frac{B}{A} \int \frac{\xi^{m-1}}{R} d\xi - \frac{C}{A} \int \frac{x^{m-2}}{R} d\xi$$

$$\int \frac{\xi}{R} d\xi = \frac{1}{2A} \log(R) - \frac{B}{2A} \int \frac{d\xi}{R}$$

$$\int \frac{d\xi}{R} = \frac{2}{\sqrt{\Delta}} \arctan\left(\frac{2A\xi + B}{\sqrt{\Delta}} \right).$$

D. Hypersingular Operator

For $u, v \in S^{p,q}(\Omega_\sigma^n)$, we evaluate the hypersingular operator W_2 with procedures of the single-layer potential operator (see [18]): $\langle u, W_2 v \rangle = -\langle V_2 \frac{d}{ds_x} u, \frac{d}{ds_y} v \rangle$.

VII. NUMERICAL RESULTS

For the computations we consider a couple of examples for the interface problem with Ω_1 being the square $\{(x_1, x_2) \in \mathbf{R}^2 : |x_i| < 1, i = 1, 2\}$. In all examples we have

$$p(r) = 2 + \frac{1}{1 + r} \qquad (r \geq 0),$$

so that $1 \leq p(r) \leq 3$, $1 \leq p(r) + r \cdot p'(r) \leq 3$, $r > 0$. With $G(r) = \int_0^t tp(t)\,dt = r^2 + r - \log(1 + r)$, the functional J_0 on $H^1(\Omega_1)$ becomes

$$J_0 = \int_{\Omega_1} \left\{ |\nabla u|^2 + |\nabla u| - \log(1 + |\nabla u|) + \frac{1}{2} |u|^2 - F \cdot u \right\} dx ,$$

and with (15) we have $P_1 u = -2\Delta u - div\left(\frac{\nabla u}{1 + |\nabla u|}\right) + u$. In Tables I–III and Figs. 3, 4, and 5, we present experimental rates of convergence for the L_2-errors $e := \|u_1 - u_N^{(m)}\|_{L^2(\Omega_1)}$ in Ω_1 and $\epsilon := \|\phi - \phi_M^{(m)}\|_{L^2(\Gamma)}$ on Γ, where $u_1 \in H^1(\Omega_1)$ and $\phi = p(|\nabla u_1|) \frac{\partial u_1}{\partial n} \in H^{-1/2}(\Gamma)$ solve the interface problem (15). In the sequel, m_K denotes the number of iterations of the Newton–method, and N_1 and N_2 denote the dimensions of $S^{p,q}$ and S^r, respectively.

TABLE I. Absolute errors in Example 1.

$\|u - u_N^{(m_K)}\|_{L^2(\Omega_1)}$	N_1	$\|\phi - \phi_M^{(m_K)}\|_{L^2(\Gamma_c)}$	N_2	m_K
		$\sigma = 0.5$:		
0,11358	4	1,5117	4	5
0,03013	17	1,3554	14	5
0,01181	48	1,1910	30	5
0,00518	112	1,0433	52	5
		$\sigma = 0.25$:		
0,11358	4	1,5117	4	5
0,01324	17	1,1679	14	5
0,002815	48	0,9107	30	5
0,000989	112	0,7141	52	5
		$\sigma = 0.1$:		
0,11358	4	1,5117	4	5
0,01201	17	1,02482	14	5
0,00422	48	0,70973	30	5

TABLE II. Absolute errors in Example 2.

$\|u - u_N^{(m_K)}\|_{L^2(\Omega_1)}$	N_1	$\|\phi - \phi_M^{(m_K)}\|_{L^2(\Gamma_c)}$	N_2	m_K
		$\sigma = 0.25$:		
2,47016	4	3,27856	4	2
0,07507	24	2,11075	16	5
0,005175	84	0,7572	36	5
0,000695	217	0,2585	64	5
		$\sigma = 0.2$:		
2,47016	4	3,27856	4	2
0,03830	24	1,86021	16	5
0,01073	84	0,56459	36	5
0,001690	217	0,170889	64	5
		$\sigma = 0.171$:		
2,47016	4	3,27856	4	2
0,02678	24	1,75752	16	5
0,01667	84	0,47349	36	5
0,001246	217	0,130997	64	5

TABLE III. Absolute errors in Example 3.

$\|u - u_N^{(m_K)}\|_{L^2(\Omega_1)}$	N_1	$\|\phi - \phi_M^{(m_K)}\|_{L^2(\Gamma_c)}$	N_2	m_K
		$\sigma = 0.25$:		
1,10564	4	3,27733	4	2
0,07415	24	3,41856	16	5
0,005366	84	2,80817	36	5
0,000620	217	2,23380	64	5
		$\sigma = 0.171$:		
1,10564	4	3,27733	4	2
0,035443	24	3,26434	16	5
0,006374	84	2,50149	36	5
0,000560	217	1,86517	64	5
0,000398	475	1,38318	100	5

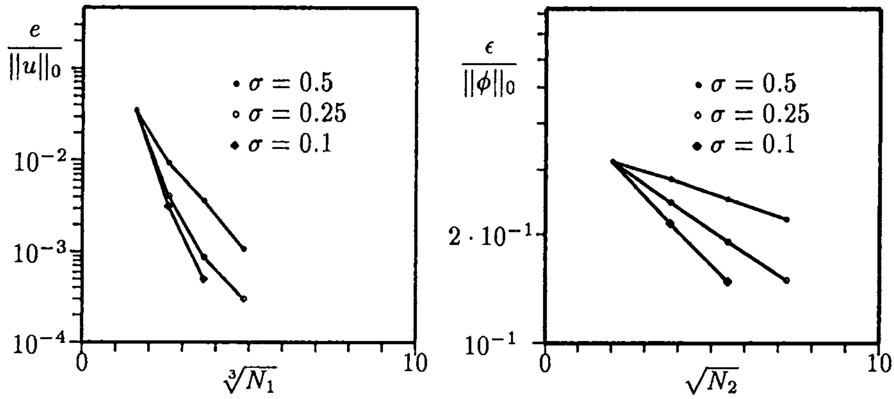

FIG. 3. The relative error of Example 1.

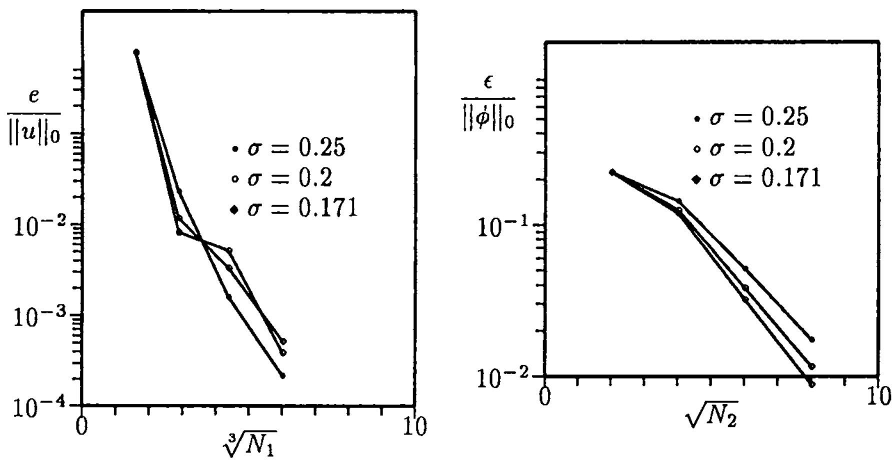

FIG. 4. The relative error of Example 2.

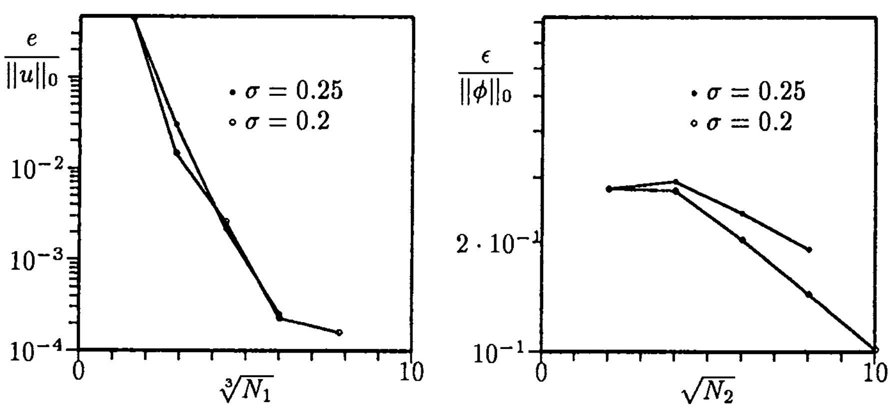

FIG. 5. The relative error of Example 3.

Example 1. Let the data functions be defined by

$$F(x_1,x_2) = \frac{9}{16}A^4\left(2 + \frac{1}{(1+A)^2}\right) + (2 - x_1 - x_2)^{2/3}$$

$$f(x_1,x_2) = \left((2 - x_1 - x_2)^{2/3} - \frac{1}{2}\log(x_1^2 + x_2^2)\right)\Big|_\Gamma ,$$

$$g(x_1,x_2) = -\frac{A}{\sqrt{2}}\left(2 + \frac{1}{1+A}\right)(n_1 + n_2) - \frac{n_1x_1 + n_2x_2}{x_1^2 + x_2^2}$$

$$A := \frac{2\sqrt{2}}{3}(2 - x_1 - x_2)^{-1/3}.$$

For the partition, $\Omega_\sigma^n = \{\Omega_{ij}$ is a rectangle with the corners $(1 - \sigma^{i-1}, 1 - \sigma^{j-1})$, $(1 - \sigma^i, 1 - \sigma^{j-1})$, $(1 - \sigma^{i-1}, 1 - \sigma^j)$, $(1 - \sigma^i, 1 - \sigma^j)$, $1 \le i,j \le n\}$, we use different constants σ and appropriate polynomial degrees. See Fig. 1 for $\sigma = \frac{1}{2}$.

The errors of the Galerkin procedure are shown in Table I and illustrated in Fig. 3. The exact solutions are given by

$$u_1(x_1,x_2) = (2 - x_1 - x_2)^{2/3} \qquad (x \in \Omega_1)$$

$$u_2(x_1,x_2) = \frac{1}{2}\log(x_1^2 + x_2^2) \qquad (x \in \Omega_2).$$

Since $u \sim r^{2/3}$, we have $u \in B_\beta^2(\Omega_1)$ $(1/3 < \beta < 1)$ and $\frac{\partial u}{\partial n} \in B_\beta^{1/2}(\Gamma)$, $(1/3 < \beta < 1)$, and get exponential convergence for $\|u - u_N\|_{H^1(\Omega_1)}$ and $\|\phi - \phi_M\|_{H^{-1/2}(\Gamma)}$ [7], which is confirmed in this example. This is shown in Fig. 2 by the linear dependence of $\log \frac{e}{\|u\|_0}$ and $\sqrt[3]{N_1}$ or $\log \frac{\epsilon}{\|\phi\|_0}$ and $\sqrt{N_2}$. Here $\|u\|_0 := \|u\|_{L^2(\Omega_1)}$ and $\|\phi\|_0 := \|\phi\|_{L^2(\Gamma)}$.

Example 2. Let the data functions be defined by

$$F(x_1,x_2) = -\frac{32r^2}{9A^2} + \frac{32}{3}A + \frac{16}{3(A^{-1} + 8/3r)} - \frac{16r^2}{9(A + 8/3rA^2)^2} - \frac{64r}{9(A^{-1} + 8/3r)^2} + A^4$$

$$f(x_1,x_2) = \left(A^4 - \frac{1}{2}\log(x_1^2 + x_2^2)\right)\Big|_\Gamma ,$$

$$g(x_1,x_2) = -(n_1x_1 + n_2x_2)\left(\frac{16}{3}A + \frac{1}{A^{-1} + \frac{4\sqrt{2}}{3}r} + \frac{1}{2r^2}\right)$$

$$A := (2 - x_1^2 - x_2^2)^{1/3}, \qquad r := \sqrt{x_1^2 + x_2^2}\,.$$

For this and the next example, we use the geometric mesh shown in Fig. 2. The corresponding errors of the Galerkin procedure are given in Table II and illustrated in Fig. 4. The exact solution is given by

$$u_1(x_1,x_2) = (2 - x_1^2 - x_2^2)^{4/3} \qquad (x \in \Omega_1)$$

$$u_2(x_1,x_2) = \frac{1}{2}\log(x_1^2 + x_2^2) \qquad (x \in \Omega_2).$$

Since $u \sim r^{8/3}$, we have $u \in B^3_\beta(\Omega_1)$ $(0 < \beta < 1)$ and $\frac{\partial u}{\partial n} \in B^2_\beta(\Gamma)$ $(0 < \beta < 1)$, and we get exponentially fast convergence in the norms $\|u - u_N\|_{H^1(\Omega_1)}$ and $\|\phi - \phi_M\|_{H^{-1/2}(\Gamma)}$. This is confirmed in the numerical example, where we observe even exponential convergence of (ϕ_M) in $L^2(\Gamma)$.

Example 3. Let the data functions be defined by

$$F(x_1, x_2) = -\frac{16r^2}{9A^4} + \frac{16}{3A} + \frac{8}{3(A + 4/3r)} - \frac{16r}{9(A + 4/3r)^2} + \frac{8r^2}{9(A + 4/3r)^2 \cdot A^2} + A^2,$$

$$f(x_1, x_2) = \left(A^2 - \frac{1}{2}\log(x_1^2 + x_2^2)\right)\Big|_\Gamma,$$

$$g(x_1, x_2) = -(n_1x_1 + n_2x_2)\left(\frac{8}{3}A^{-1} + \frac{1}{A^{-1} + \frac{2\sqrt{2}}{3}r} + \frac{1}{2r^2}\right)$$

$$A := (2 - x_1^2 - x_2^2)^{1/3}, \qquad r := \sqrt{x_1^2 + x_2^2}\,.$$

For this example, we also use the geometric mesh shown in Fig. 2. The corresponding errors of the Galerkin procedure are given in Table III and illustrated in Fig. 5. The exact solution is given by

$$u_1(x_1, x_2) = (2 - x_1^2 - x_2^2)^{2/3} \qquad (x \in \Omega_1)$$

$$u_2(x_1, x_2) = \frac{1}{2}\log(x_1^2 + x_2^2) \qquad (x \in \Omega_2).$$

Since $u \sim r^{4/3}$, we have $u \in B^2_\beta(\Omega_1)$ $(0 < \beta < 1)$ and $\frac{\partial u}{\partial n} \in B^1_\beta(\Gamma)$ $(1/6 < \beta < 1)$ that expect exponentially fast convergence in the energy norms. Numerically, we observe exponential convergence of $\|u - u_N\|_{L^2(\Omega_1)}$ and $\|\phi - \phi_M\|_{L^2(\Gamma)}$.

APPENDIX

Let $\Omega_1 \subset R^2$ be a bounded domain whose curvilinear boundary $\partial\Omega_1$ is a piecewise analytic curve $\Gamma = \cup_{i=1}^M \overline{\Gamma}_i$, where Γ_i is an open arc connecting the vertices A_i and $A_{i+1}(A_{M+1} = A_1)$. Let $\Omega_2 = R^2 \backslash \overline{\Omega}_1$, we denote the internal angle at A_i by ω_i, and assume $0 < \omega_i \leq 2\pi$, $1 \leq i \leq M$. $\partial/\partial n$ denotes the derivative with respect to the normal to Γ pointing from Ω_1 to Ω_2.

Let Ω_1 be a bounded open set in R^2 and let $H^k(\Omega_1)$, $k \geq 0$ integer, denote the usual Sobolev spaces (e.g. [19]):

$$H^k(\Omega_1) = \left\{u\colon \sum_{0\leq|\alpha|\leq k} \|D^\alpha u\|^2_{L^2(\Omega_1)} = \|u\|^2_{H^k(\Omega_1)} < \infty\right\},$$

where $\alpha = (\alpha_1, \alpha_2)$, $\alpha_i \geq 0$ integers, $i = 1,2$, $|\alpha| = \alpha_1 + \alpha_2$, and

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} = u_{x_1^{\alpha_1}/x_2^{\alpha_2}} .$$

$H^{k-1/2}(\Gamma)$ is defined as the restriction of $u \in H^k(\Omega_1)$ to Γ for integer $k \geq 1$ i.e.,

$$H^{k-1/2}(\Gamma) = \{u|_\Gamma : u \in H^k(\Omega_1)\}$$

with

$$\|g\|_{H^{k-1/2}(\Gamma)} = \inf_{u|_\Gamma = g} \|u\|_{H^k(\Omega_1)},$$

and for $k \leq 0$ by duality

$$H^{k-1/2}(\Gamma) = (H^{-(k-1/2)}(\Gamma))'.$$

Let $r_i(x) = dist(x, A_i)$, and let $\beta = (\beta_1, \beta_2, \ldots, \beta_M)$ be an M-tuple of real numbers $0 < \beta_i < 1$. For any integer $k \geq 0$, we shall write $\beta + k = (\beta_1 + k, \beta_2 + k, \ldots, \beta_M + k)$, and $\Phi_{\beta+k}(x) = \Pi_{i=1}^M r_i^{\beta_i+k}(x)$. As in [17], we define the weighted Sobolev space for integers k and $l, k \geq l \geq 0$, by

$$H_\beta^{k,l}(\Omega_1) = \{u: u \in H^{l-1}(\Omega_1) \text{ if } l > 0,\ \|\Phi_{\beta+|\alpha|-l} D^\alpha u\|_{L^2(\Omega_1)} < \infty \text{ for } l \leq |\alpha| \leq k\},$$

and the countably normed space for $l \geq 0$,

$$B_\beta^l(\Omega_1) = \{u: u \in H_\beta^{k,l}(\Omega_1)\ \forall\, k \geq l,\ \|\Phi_{\beta+k-l} D^\alpha u\|_{L^2(\Omega_1)} \leq C d^{k-l}(k-l)! \text{ for } |\alpha| = k = l, l+1, \ldots, \text{ with } C \geq 1, d \geq 1 \text{ independent of } k\}.$$

The space $H_\beta^{k-1/2, l-1/2}(\Gamma)$ [resp. $B_\beta^{l-1/2}(\Gamma)$] k, l integers, $k \geq l \geq 1$, is the trace space of $H_\beta^{k,l}(\Omega_1)$ [resp. $B_\beta^l(\Omega_1)$], i.e., for any $g \in H_\beta^{k-1/2, l-1/2}(\Gamma)$ [resp. $B_\beta^{l-1/2}(\Gamma)$] there exists $G \in H_\beta^{k,l}(\Omega_1)$ [resp. $B_\beta^l(\Omega_1)$] such that $G|_\Gamma = g$, and $\|g\|_{H_\beta^{k-1/2, l-1/2}(\Gamma)} = \inf_{G|_\Gamma = g} \|G\|_{H_\beta^{k,l}(\Omega_1)}$.

In the exterior domain Ω_2, we incorporate the behavior of solutions at infinity. Let $r_i^*(x) = \min(1, r_i(x))$ for $x \in \Omega_2$, then the weight function $\Phi_{\beta+k}(x)$ is modified by

$$\Phi_{\beta+k}(x) = \prod_{i=1}^M (r_i^*(x))^{\beta_i+k}.$$

The weighted Sobolev space, $H_\beta^{k,l}(\Omega_2)$, $k \geq l \geq 2$, is defined by

$$H_\beta^{k,l}(\Omega_2) = \{u: u \in H_{loc}^1(\Omega_2), D^\alpha u \in L^2(\Omega_2) \text{ for } 2 \leq |\alpha| < l, \|\Phi_{\beta+|\alpha|-l} D^\alpha u\|_{L^2(\Omega_2)} < \infty, \text{ for } l \leq |\alpha| \leq k\}.$$

The definition of the space $B_\beta^2(\Omega_2)$ is the same as $B_\beta^2(\Omega_1)$.

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