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On the adaptive coupling of FEM and BEM in 2–d–elasticity *

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Summary. This paper concerns the combination of the finite element method (FEM) and the boundary element method (BEM) using the symmetric coupling. As a model problem in two dimensions we consider the Hencky material (a certain nonlinear elastic material) in a bounded domain with Navier–Lamé differential equation in the unbounded complementary domain. Using some boundary integral operators the problem is rewritten such that the Galerkin procedure leads to a FEM/BEM coupling and quasi–optimally convergent discrete solutions. Beside this a priori information we derive an a posteriori error estimate which allows (up to a constant factor) the error control in the energy norm. Since information about the singularities of the solution is not available a priori in many situation and having in mind the goal of an automatic mesh–refinement we state adaptive algorithms for the h–version of the FEM/BEM–coupling. Illustrating numerical results are included.

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1. Introduction

The mathematical justification of the ”mariage á la mode” proposed by engineers started in the later seventies by Brezzi, Johnson, Nedelec, Bielak, MacCamy and others. Further progress in the analysis of the coupling of finite elements (FE) and boundary elements (BE) concerns Lipschitz boundaries, systems of equations, and nonlinear problems cf. e.g. [5, 8, 9, 12, 13, 18, 27] and the literature quoted therein.

In order to get asymptotically a good convergence but also when dealing with a few degrees of freedom, we need a good mesh in particular when singularities

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appear. If the nature and the position of a singularity are known a priori, the mesh refinement can reflect on this. Otherwise one requires the information we may achieve from an analysis of the discrete solution. Nowadays the main topics in the adaptive feedback steering of mesh refinements, usually based on the residuals, are mathematically understood for the finite element methods — we refer only to the pioneering works [1, 11], to [19, 25] for nonlinear problems, and to [26] for a recent review. Comparably little is known for the boundary element method — cf. e.g. [2, 23, 24, 29, 30].

In this paper adaptive h–versions of the symmetric FEM/BEM—coupling are presented for linear and nonlinear interface problems. They are based on an a posteriori error estimate which gives a computable error estimate up to a multiplicative constant. Then, following the approach of Eriksson and Johnson (elaborated for the finite element method) we present an adaptive feedback algorithm for the mesh refinement of the coupling procedure and report on numerical experiments.

We consider a model problem for the FE and BE coupling in two dimensional elasticity described in the sequel. Let Ω be a bounded Lipschitz domain in the plane with boundary Γ and complement $\Omega_c := \mathbb{R}^2 \setminus \overline{\Omega}$. Neglecting the functional analytic framework (outlined in Sect. 2) we have in Ω a displacement field u, a strain field ϵu, and a stress field σ satisfying the elasticity material behaviour

$$\sigma = A(\epsilon u) \qquad \text{in } \Omega \tag{1}$$

which reads for the Hencky material (in components $i, j = 1, 2$)

$$\sigma_{ij} = \Big(\kappa - \overline{\mu}(\gamma(u))\Big)\delta_{ij} \operatorname{div} u + 2\overline{\mu}(\gamma(u))\epsilon_{ij}(u), \tag{2}$$

with $\delta_{ij} = 1$ iff $i = j$ and $\delta_{ij} = 0$ iff $i \neq j$,

$$\epsilon_{ij}(u) := \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \tag{3}$$

and $\gamma(u) := \operatorname{dev} \epsilon u : \operatorname{dev} \epsilon u$ with

$$\operatorname{dev} \zeta := \zeta - \frac{1}{2}\operatorname{tr} \zeta, \quad \operatorname{tr} \zeta := \zeta_{11} + \zeta_{22}, \quad \zeta \in \mathbb{R}^{2\times 2}_{\text{sym}}.$$

In (2), κ is a constant bulk modulus and $\overline{\mu}$ is a certain function (cf. Example 1 below).

Given a volume force f the equilibrium equations read $\sigma^{\mathrm{T}} = \sigma$ and

$$\operatorname{div} \sigma + f = 0 \qquad \text{in } \Omega. \tag{4}$$

The exterior problem consists of the Navier–Lamé equations

$$0 = -\Delta^* u := -\mu_2 \Delta u - (\lambda_2 + \mu_2)\operatorname{grad} \operatorname{div} u \qquad \text{in } \Omega_c \tag{5}$$

and a radiation condition of the form [15, 16]

(6) $$D^{\alpha}(u-a)(x) = O(|x|^{-1-\alpha}), \qquad \alpha = 0, 1, \quad (|x| \to \infty)$$

where $D = \partial/\partial x_j$ and $a \in \mathbb{R}^2$ is a constant vector.

The two problems are coupled on the interface Γ where we have in the simplest case continuity of the displacements and equilibrium of the tractions, i.e.

(7) $$u|_{\Omega} = u|_{\Omega_c} \quad \text{and} \quad \sigma n = T_2(u|_{\Omega_c}) \qquad \text{on } \Gamma$$

where σ is the stress field in Ω, n is the unit normal vector on Γ pointing from Ω into Ω_c, and T_2 is the conormal derivative related to the Lamé operator Δ^*,

$$T_2(u) := 2\mu_2 \partial_n u + \lambda_2 n \operatorname{div} u + \mu_2 n \times \operatorname{curl} u$$

with the normal derivative ∂_n.

In this paper we consider the transmission problem (1)—(7) and extend results in [3] in three aspects at least: regarding the Lamé system instead of the scalar Laplacian; allowing more general nonlinearities; using a different and more general coupling (zero means for the discrete tractions need no assumptions on the size of the domain).

This paper is organized as follows: In Sect. 2 we give a functional analytic framework and rewrite the exterior part equivalently using boundary integral operators (as in [28]). Then, we discuss the resulting weak form of the transmission problem and prove existence and uniqueness of solutions. The numerical approximation of the problem is given in Sect. 3 via the coupling of boundary and finite elements. As an a priori result we prove quasi-optimal convergence estimates while we prove a posteriori error estimates in Sect. 4. These error estimates can be used to derive an adaptive algorithm for an automatic mesh–refinement as performed in Sect. 5. In order to give numerical examples we explain computational details and study a class of examples with singular solutions as well as a more practical example in Sect. 6. Thereby we prove efficiency of our adaptive algorithms and illustrate that the h–method yields efficient solutions.

We finally emphasize that this model problem combines the advantages of the two methods (FEM for nonlinear problems, BEM for simple problems in unbounded domains) but can be used also as a model for generally combining FE and BE where many subdomains are discretized via FEM or BEM also for bounded domains.

2. A nonlinear transmission problem

We use the following notations. $H^s(\Omega)$ denotes the usual Sobolev spaces [20] with the trace spaces $H^{s-1/2}(\Gamma)$ ($s \in \mathbb{R}$) for a bounded Lipschitz domain Ω with boundary Γ. $\|\cdot\|_{H^k(\omega)}$ and $|\cdot|_{H^k(\omega)}$ denote the norm and semi-norm in $H^k(\omega)$ for $\omega \subseteq \Omega$ and an integer k.

In Ω we have the non–linear elastic Hencky material [22, 32] determined by a stress strain relation (1) where

$$\epsilon : H^1(\Omega; \mathbb{R}^2) \to L^2(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}}), \quad u \mapsto \frac{1}{2}(\mathrm{grad}\, u + \mathrm{grad}^{\,\mathrm{T}} u)$$

maps the displacements u to the (linear) Green's strains ϵu, $\sigma \in L^2(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ denotes the stress field and

$$A : L^2(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}}) \to L^2(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$$

describes the elastic material behavior, $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ is the 3–dimensional real vector space of real 2×2 symmetric matrices.

Example 1. For example let A be the derivative of a mapping

$$M : L^2(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}}) \to \mathbb{R}$$

with

$$M(\zeta) := \frac{1}{2}(\lambda + \mu)\mathrm{tr}(\zeta)^2 + \mu\varphi(\mathrm{dev}\,\zeta : \mathrm{dev}\,\zeta), \quad \zeta \in L^2(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}}),$$

where λ, μ are the positive Lamé constants and the function $\varphi : [0, \infty) \to \mathbb{R}$ is a C^2 map with $\varphi(0) = \varphi''(0) = 0$ and

$$a \le \varphi'(t) \le 1, \quad -b \le \varphi''(t) \le 0, \quad \frac{1}{n} \le \varphi'(t) + 2\varphi''(t) \le n \tag{8}$$

for all $t \in [0, \infty)$ and constants $a, b > 0$ and a natural number n. Thus,

$$\begin{aligned}\int_\Omega A(\xi)\cdot\zeta\, d\Omega \;=\; &\int_\Omega \{(\lambda + \mu)\, tr(\xi) tr(\zeta) \\ &+ 2\mu\varphi'(\mathrm{dev}\,\xi : \mathrm{dev}\,\xi)\cdot \mathrm{dev}\,\xi : \mathrm{dev}\,\zeta\}\, d\Omega\end{aligned}$$

$(\xi, \zeta \in L^2(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}}))$.

It is proved in [32, Sect. 62] that A is uniformly monotone and Lipschitz continuous.

In the sequel we provide the weaker assumptions that $A : L \to L^*$, $L := L^2(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ is strongly monotone and Lipschitz continuous for bounded arguments, i.e. there exists a convex function

$$\alpha : [0, \infty) \to [0, \infty) \tag{9}$$

with

$$0 = \lim_{t\to 0^+} \alpha(t)/t, \quad \infty = \lim_{t\to\infty} \alpha(t)/t, \quad \alpha(t)/t \;\text{ strongly monotone in t },$$

such that for any $\xi, \zeta \in L := L^2(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$

$$\int_{\Omega}(A(\xi)-A(\zeta))\cdot(\xi-\zeta)\,d\Omega \geq \alpha(\|\xi-\zeta\|_L) \tag{10}$$

and there exists a function $a : [0,\infty) \to [0,\infty)$ such that for any $r > 0$ and $\xi, \zeta, \rho \in L$ we have

$$\|\,\xi\,\|_L, \|\,\zeta\,\|_L \leq r \quad \Rightarrow \quad \int_{\Omega}(A(\xi)-A(\zeta))\cdot\rho\,d\Omega \leq a(r)\|\xi-\zeta\|_L\|\,\rho\,\|_L.$$

The quasi–static equilibrium condition is (4) in the sense of distributions where $f \in L^2(\Omega;\mathbb{R}^2)$ is a given body force. Due to Green's formula and the symmetry of $\sigma = A(\epsilon u)$ the equilibrium yields the weak form

$$\int_{\Omega} A(\epsilon u)\cdot\epsilon v\,d\Omega = \int_{\Omega} f\cdot v\,d\Omega + \int_{\partial\Omega}\sigma n\cdot v\,d\partial\Omega \tag{11}$$

for any test function $v \in H^1(\Omega;\mathbb{R}^2)$.

Remark 1. For Lipschitz domains we may use Green's formula (used in (11)) to define $\sigma\cdot n \in H^{-1/2}(\Gamma;\mathbb{R}^2)$ as well as the conormal derivative T_2 (cf. [9] in 3 dimension or e.g. [6] for the general case).

Definition 1. The *transmission problem* has the data $f \in L^2(\Omega;\mathbb{R}^2)$, $u_0 \in H^{1/2}(\Gamma;\mathbb{R}^2)$, and $t_0 \in H^{-1/2}(\Gamma;\mathbb{R}^2)$ and consists in finding $(u_1,u_2) \in H^1(\Omega;\mathbb{R}^2)\times H^1_{\mathrm{loc}}(\Omega_{\mathrm{c}};\mathbb{R}^2)$ satisfying

$$\operatorname{div} A(\epsilon u) + f = 0 \qquad \text{in } \Omega \tag{12}$$

with $u = u_1$ and (5), (6) with replacing u by u_2 and the interface conditions

$$u_1 = u_2 + u_0 \quad \text{and} \quad A(\epsilon u_1)n = T_2(u_2) + t_0 \qquad \text{on } \Gamma. \tag{13}$$

Remark 2. Although the physical interpretation of inhomogeneous transmission data u_0 and t_0 is not already clear, the present formulation is more general than (7) which is included by $u_0 = 0 = t_0$.

In the sequel we recall definitions and some properties of boundary integral operators yielding the rewritten form of the transmission problem of Definition 1 following a particular case of the general description in [17, 28].

Definition 2. For any u in

$$\begin{aligned}\mathscr{L}_2 \quad := \quad \{u \in H^1_{\mathrm{loc}}(\Omega_{\mathrm{c}};\mathbb{R}^2) :\ & \text{there exists a constant vector } a \text{ such that} \\ & u \text{ satisfies (6) and (5)}\}\end{aligned}$$

let $(u|_\Gamma, T_2(u)|_\Gamma)$ denote its *Cauchy data.*

Remark 3. Due to the trace lemma $u_2|_\Gamma \in H^{1/2}(\Gamma;\mathbb{R}^2)$ whenever $u_2 \in H^1_{\text{loc}}(\Omega_c;\mathbb{R}^2)$, $H^1_{\text{loc}}(\Omega_c;\mathbb{R}^2)$ denoting the displacements of locally finite energy. The brackets $< \cdot,\cdot >$ always denote the duality between $H^{1/2}(\Gamma;\mathbb{R}^2)$ and $H^{-1/2}(\Gamma;\mathbb{R}^2) = (H^{1/2}(\Gamma;\mathbb{R}^2))^*$ such that for $v \in H^{1/2}(\Gamma;\mathbb{R}^2)$ and $w \in L^2(\Gamma;\mathbb{R}^2)$

$$< w, v >= \int_\Gamma w \cdot v \, d\Gamma.$$

Then, the Cauchy data of $u_2 \in H^1_{\text{loc}}(\Omega_c,\mathbb{R}^2)$ with $\Delta^* u_2 = 0$ satisfy (cf. e.g. [9])

$$(u_2|_\Gamma, T_2(u_2)|_\Gamma) \in H^{1/2}(\Gamma;\mathbb{R}^2) \times H^{-1/2}(\Gamma;\mathbb{R}^2).$$

For the Lamé operator the fundamental solution E with the kernel $E(x,y)$ –called Kelvin–matrix– is well-known,

$$E(x,y) = \frac{\lambda_2 + 3\mu_2}{4\pi\mu_2(\lambda_2 + 2\mu_2)} \left\{ \log \frac{1}{|x-y|} \cdot I + \frac{\lambda_2 + \mu_2}{\lambda_2 + 3\mu_2} \frac{(x-y)(x-y)^{\mathrm{T}}}{|x-y|^2} \right\}.$$

I is the 2×2 unit matrix and $^{\mathrm{T}}$ denotes the transposed matrix. Since E is analytic in $\mathbb{R}^2 \times \mathbb{R}^2$ without the diagonal we may define its traction

$$T(x,y) := T_{2,y}(E(x,y))^{\mathrm{T}}, \quad x \neq y.$$

As it is derived e.g. in [15, 16, 17] we have the following Betti representation formula for $x \in \Omega_c$

$$u_2(x) =< T(x,\cdot), v > - < E(x,\cdot), \phi > +a \tag{14}$$

for all $u_2 \in \mathscr{L}_2$ with $v = u_2|_\Gamma$, $\phi = T_2(u_2)|_\Gamma$.

Remark 4. Note that $< T(x,\cdot), v > - < E(x,\cdot), \phi > +a$ satisfies (6) if and only if $< \phi^1, 1 >= 0 =< \phi^2, 1 >$ [16], i.e. $\phi \in H_0^{-1/2}(\Gamma) := \{\psi \in H^{-1/2}(\Gamma;\mathbb{R}^2) : < \psi^1, 1 >= 0 =< \psi^2, 1 >\}$.

For any $x \in \Omega_c$, (14) can be differentiated giving a representation formula for the stresses $T_2(u_2)$. By using the classical jump relations for $x \to \Gamma$ and inserting the Cauchy data into these formulas one obtains on Γ

$$\begin{pmatrix} v \\ \phi \end{pmatrix} = \mathscr{C}_2 \cdot \begin{pmatrix} v \\ \phi \end{pmatrix} \tag{15}$$

where the Calderón projector

$$\mathscr{C}_2 = \begin{pmatrix} \frac{1}{2} + K & -V \\ -W & \frac{1}{2} - K' \end{pmatrix}$$

is defined by

$$\begin{aligned} (V\phi)(x) &= < E(x,\cdot), \phi > \\ (Kv)(x) &= < T(x,\cdot), v > \\ (Wv)(x) &= -T_{2,x}(< T(x,\cdot), v >) \\ (K'\phi)(x) &= -T_{2,x}(< E(x,\cdot), \phi >) \quad (x \in \Gamma). \end{aligned}$$

V is the single layer potential, K is the double layer potential with its dual K', and W is the hypersingular operator.

It is known (cf. e.g. [5, 6, 9]) that, $\mathscr{L}(X;Y)$ denotes the real Banach space of bounded linear operators mapping X into Y,

$$\begin{aligned} V &\in \mathscr{L}(H^{-1/2}(\Gamma;\mathbb{R}^2); H^{1/2}(\Gamma;\mathbb{R}^2)) \\ K &\in \mathscr{L}(H^{1/2}(\Gamma;\mathbb{R}^2); H^{1/2}(\Gamma;\mathbb{R}^2)) \\ K' &\in \mathscr{L}(H^{-1/2}(\Gamma;\mathbb{R}^2); H^{-1/2}(\Gamma;\mathbb{R}^2)) \\ W &\in \mathscr{L}(H^{1/2}(\Gamma;\mathbb{R}^2); H^{-1/2}(\Gamma;\mathbb{R}^2)). \end{aligned}$$

W and V are symmetric, K' is the dual of K, W is positive semi–definite and V is positive definite on $H_0^{-1/2}(\Gamma)$, i.e. there exists a constant $\gamma_2 > 0$ such that for all $v \in H^{1/2}(\Gamma;\mathbb{R}^2)$ and all $\phi \in H^{-1/2}(\Gamma;\mathbb{R}^2)$ with $< \phi^1, 1 >= 0 =< \phi^2, 1 >$ there holds

$$< Wv, v > \geq 0 \quad \text{and} \quad < \phi, V\phi > \geq \gamma_2 \|\phi\|^2_{H^{-1/2}(\Gamma;\mathbb{R}^2)}. \tag{16}$$

This may be proved as in the three dimensional case in [9] since we assume the radiation condition (6).

As it is already proved for transmission problems concerning the Laplacian or the Navier–Lamé equations in three dimensions (cf. e.g. [7, 9]) the Calderón projector is a projection in $H^{1/2}(\Gamma;\mathbb{R}^2) \times H^{-1/2}(\Gamma;\mathbb{R}^2)$ onto its subspace of Cauchy data of weak solutions.

We summarize this briefly reviewing descriptions of the exterior problem.

Theorem 1. *For any* $(v,t) \in H^{1/2}(\Gamma;\mathbb{R}^2) \times H_0^{-1/2}(\Gamma)$ *there exists* $u \in H^1_{\text{loc}}(\Omega_c)$ *solving* (5) *and* (6) *and having Cauchy data* (v,t) *if and only if* (15) *holds. In this case the solution* u_2 *of the exterior problem is unique and given by the right hand side of the representation formula* (14).

Remark 5. Note $T_2(u_2) \in H_0^{1/2}(\Gamma)$ and (13) lead to the further assumption

$$\int_\Omega f\,dx + \int_\Gamma t_0\,ds = 0 \tag{17}$$

(cf. (11)) which will be used in the sequel.

Remark 6. We note that for any rigid body motion with r we have

$$Wr = 0 \quad \text{and} \quad Kr = -\frac{1}{2}r. \tag{18}$$

Remark 7. It should be emphasized that in related works (e.g. [8, 9, 13, 18]) the rigid body motions in elasticity in the interface problem are prevented by an additional Dirichlet boundary inside of the interior domain. It is shown in this paper that this technical restriction is not necessary. Instead with one solution u_1 in Ω, u_2 in Ω_c any $u_1 + c$, $u_2 + c$ with a constant vector c is a solution as well.

We need some subspaces in order to treat the constant displacements.

Definition 3. Let

$$
\begin{aligned}
H_0^{-1/2}(\Gamma) &:= \{\phi \in H^{-1/2}(\Gamma;\mathbb{R}^2) :< \phi^1, 1 >= 0 =< \phi^2, 1 >\}\\
H_0^{1/2}(\Gamma) &:= \{v \in H^{1/2}(\Gamma;\mathbb{R}^2) :< v^1, 1 >= 0 =< v^2, 1 >\}\\
H_0^{1}(\Omega) &:= \{u \in H^{1}(\Omega;\mathbb{R}^2) : u|_\Gamma \in H_0^{1/2}(\Gamma)\}
\end{aligned}
$$

and define $P : H^{1/2}(\Gamma;\mathbb{R}^2) \to H_0^{1/2}(\Gamma)$ by $Pv := v - v_0$ where $v \in H^{1/2}(\Gamma;\mathbb{R}^2)$ and $v_0 \in \mathbb{R}^2$ is defined by $v_0^j =< v^j, 1 > / < 1, 1 >$. With the integral operator V, W we have the continuous mappings

$$
V_0 := PV|_{H_0^{-1/2}(\Gamma)} : H_0^{-1/2}(\Gamma) \to H_0^{1/2}(\Gamma)
$$

and

$$
S_0 := W + \left(\frac{1}{2} - K'\right) V_0^{-1} P \left(\frac{1}{2} - K\right) : H_0^{1/2}(\Gamma) \to H_0^{-1/2}(\Gamma).
$$

Lemma 1. *The operators V_0 and S_0 are well defined, linear, bounded, symmetric and positive definite.*

Proof. Since $P : H^{1/2}(\Gamma;\mathbb{R}^2) \to H_0^{1/2}(\Gamma)$ is linear and bounded V_0 is well defined, linear, and bounded as a composition of linear and bounded operators. According to (16), V_0 is positive definite, hence invertible, and V_0^{-1} is bounded and positive definite as well; (the symmetry of V_0 follows from that of V). By (18) we have for $\mathrm{e}^1 = (1,0)$ and $\mathrm{e}^2 = (0,1)$, $v \in H^{1/2}(\Gamma;\mathbb{R}^2)$, $\psi = (\psi^1,\psi^2) \in H^{-1/2}(\Gamma;\mathbb{R}^2)$, and $j = 1,2$

$$
< Wv, e_j >=< v, We_j >= 0
$$

and

$$
- < \psi^j, 1 >=< \psi, \left(K - \frac{1}{2}\right) \mathrm{e}^j >=< \left(K' - \frac{1}{2}\right) \psi, \mathrm{e}^j > .
$$

Thus $(K' - \frac{1}{2})$ maps $H_0^{-1/2}(\Gamma)$ into itself and W maps $H_0^{1/2}(\Gamma)$ into $H_0^{-1/2}(\Gamma)$. Thus, S_0 is well defined, linear, and bounded.

The symmetry of S_0 follows from

$$
< \left(\frac{1}{2} - K'\right) V_0^{-1} P \left(\frac{1}{2} - K\right) v, w >=< V_0^{-1} P \left(\frac{1}{2} - K\right) v, P \left(\frac{1}{2} - K\right) w >
$$

since $< V_0^{-1} P(\frac{1}{2} - K)v, \mathrm{e}^j >= 0$ for any $v, w \in H_0^{1/2}(\Gamma)$.

Note S_0 is positive semi–definite (cf. (16)). In order to prove positive definiteness of S_0 let us assume that this is false, i.e. there exists a sequence $(u_n)_n$ in $H_0^1(\Omega)$ with

$$
\| \gamma u_n \|_{H_0^{1/2}(\Gamma)} = 1 \quad \text{and} \quad 0 \le < S_0 u_n, u_n > \le 1/n, \qquad n = 1, 2, 3, \ldots .
$$

It is known form the analysis in [9] (performed for three dimensions which works also in this case according to (6)) that W is positive definite on

$H^{1/2}(\Gamma;\mathbb{R}^2)/\mathrm{Ker}\,W$ and Ker W are the rigid body motions. Therefore, we may write $u_n = v_n + r_n$ where $v_n \in H^{1/2}(\Gamma;\mathbb{R}^2)/$ Ker W and r_n is a rigid body motion. Because of $< W v_n, v_n > \leq < S_0 v_n, v_n > \leq \frac{1}{n}$ we obtain that $(v_n)_n$ tends in $H^{1/2}(\Gamma;\mathbb{R}^2)/$ Ker W towards 0. Since (r_n) is a bounded sequence in a finite dimensional normed space we may and will choose a subsequence, also denoted by (r_n), which converges strongly in $H^{1/2}(\Gamma;\mathbb{R}^2)$ towards some rigid body motion r. By assumption and the strong convergence we have

$$1 = \lim_{n\to\infty} \| \gamma u_n \|_{H_0^{1/2}(\Gamma)} = \| r \|_{H_0^{1/2}(\Gamma)} \tag{19}$$

and $< S_0 r, r >= 0$. By (18) this implies $< V_0^{-1} r, r >= 0$, i.e. $r = 0$. This contradicts (19). □

We are now in the position to reformulate the transmission problem of Definition 1.

Definition 4 (Problem (P)).

$$\int_\Omega A(\epsilon u)\cdot \epsilon\eta\, d\Omega + < Wu|_\Gamma + (K' - \frac{1}{2})\phi, \eta|_\Gamma > \tag{20}$$

$$= \int_\Omega f \cdot \eta\, d\Omega + < t_0 + Wu_0, \eta|_\Gamma > \quad (\eta \in H_0^1(\Omega))$$

$$< \psi, V\phi + \left(\frac{1}{2} - K\right) u|_\Gamma > = < \psi, \left(\frac{1}{2} - K\right) u_0 > \quad (\psi \in H_0^{-1/2}(\Gamma)). \tag{21}$$

The transmission problem of Definition 1 and problem (P) are equivalent; compare also [8, 9, 13, 18] for related results.

Theorem 2. *The transmission problem and problem (P) are equivalent in the following sense.*
(i) If $(u_1, u_2) \in H^1(\Omega;\mathbb{R}^2)\times H^1_{\mathrm{loc}}(\Omega_c;\mathbb{R}^2)$ is a solution of the transmission problem stated in Definition 1 then let $c \in \mathbb{R}^2$ be a constant vector with $u = u_1 + c \in H_0^1(\Omega)$ and let $\phi := T_2(u_2)$. Then $(u,\phi) \in H_0^1(\Omega)\times H_0^{-1/2}(\Gamma)$ solves problem (P).
(ii) If (u,ϕ) is a solution of problem (P) then for any $a \in \mathbb{R}^2$ let $u_1 = u + a$ and define $u_2 \in H^1_{\mathrm{loc}}(\Omega_c;\mathbb{R}^2)$ by (14) *with replacing v by $u_1|_\Gamma - u_0$ on the right hand side of* (14)*. Then (u_1, u_2) solves the transmission problem.*

Proof. The proof is based on arguments concerning (14) and (15) and quite similar to the proof in [3]. Hence we omit the details. □

We rewrite the problem (P) using some forms B and L.

Definition 5. Define the continuous mapping $B : (H_0^1(\Omega)\times H_0^{-1/2}(\Gamma))^2 \to \mathbb{R}$ and the linear form $L : H_0^1(\Omega)\times H_0^{-1/2}(\Gamma) \to \mathbb{R}$ by

$$\begin{aligned} B\left(\binom{u}{\phi}, \binom{v}{\psi}\right) &:= \int_\Omega A(\epsilon u)\cdot\epsilon v\, d\Omega \\ &+ < Wu|_\Gamma + (K' - 1/2)\phi, v|_\Gamma > \end{aligned}$$

$$
\begin{aligned}
& + \quad < \psi, V\phi + (1/2 - K)u|_\Gamma > \\
L\binom{v}{\psi} \quad := \quad & \int_\Omega f \cdot v \, d\Omega + < \psi, (1/2 - K)u_0 > \\
& + \quad < t_0 + Wu_0, v|_\Gamma >
\end{aligned}
$$

for any $(u, \phi), (v, \psi) \in H_0^1(\Omega) \times H_0^{-1/2}(\Gamma)$.

Remark 8. Problem (P) is equivalent to finding $(u, \phi) \in H_0^1(\Omega) \times H_0^{-1/2}(\Gamma)$ with

(22) $$B(\binom{u}{\phi}, (\cdot)) = L,$$

i.e. for any $(v, \psi) \in H_0^1(\Omega) \times H_0^{-1/2}(\Gamma)$ there holds $B(\binom{u}{\phi}, \binom{v}{\psi}) = L\binom{v}{\psi}$.

In the case that A is a linear mapping, the following result proves that the bilinear form B satisfies the Babuška–Brezzi condition.

Theorem 3. *There exists a constant* $\beta > 0$ *such that for all* $(u, \phi), (v, \psi) \in H_0^1(\Omega) \times H_0^{-1/2}(\Gamma)$ *we have with* α *from* (9)

(23)
$$
\begin{aligned}
& \alpha(\| \epsilon u - \epsilon v \|_{L^2(\Omega; \mathbb{R}^{2\times 2}_{\text{sym}})}) + \beta \| \binom{\gamma u - \gamma v}{\phi - \psi} \|^2_{H^{1/2}(\Gamma; \mathbb{R}^2) \times H^{-1/2}(\Gamma; \mathbb{R}^2)} \\
& \quad \le B(\binom{u}{\phi}, \binom{u-v}{\eta-\delta}) - B(\binom{v}{\psi}, \binom{u-v}{\eta-\delta})
\end{aligned}
$$

with $2\eta := \phi + V_0^{-1} P(\frac{1}{2} - K)u|_\Gamma$, $2\delta := \psi + V_0^{-1} P(\frac{1}{2} - K)v|_\Gamma \in H_0^{-1/2}(\Gamma)$.

Proof. The proof is similar to that in [3] and given here for completeness. Some calculations show

$$
\begin{aligned}
& B(\binom{u}{\phi}, \binom{u-v}{\eta-\delta}) - B(\binom{v}{\psi}, \binom{u-v}{\eta-\delta}) \\
& = \int_\Omega \Big(A(\epsilon u) - A(\epsilon v) \Big) \cdot \epsilon(u - v) \, d\Omega \\
& + \frac{1}{2} < W(u - v), u - v > + \frac{1}{2} < S_0(u - v), u - v > \\
& + \frac{1}{2} < V_0(\phi - \psi), \phi - \psi > .
\end{aligned}
$$

Since A is strongly monotone and by the definiteness of S_0 and V_0 we have that the right hand side is bounded below by

$$
\begin{aligned}
& \alpha(\| \epsilon u - \epsilon v \|_{L^2(\Omega; \mathbb{R}^{2\times 2}_{\text{sym}})}) + \frac{c_1}{4} \| \gamma u - \gamma v \|^2_{H^{1/2}(\Gamma; \mathbb{R}^2)} \\
& + \frac{c_2}{4} \| \phi - \psi \|^2_{H^{-1/2}(\Gamma; \mathbb{R}^2)}
\end{aligned}
$$

with constants $c_1, c_2 > 0$. This proves (23). □

In case A is linear, Theorem 3 and continuity of the forms $B(\cdot, \cdot)$ and $L(\cdot)$ give with the Lax–Milgram lemma give existence and uniqueness of solutions of the transmission problem (unique up to constant displacements) as well as of the rewritten problem (P) (due to Theorem 1 and 2).

Theorem 4. *The problem (P) has a unique solution.*

Proof. Note (21) is equivalent to

$$\phi = -V_0^{-1}P(1/2 - K)(u|_\Gamma - u_0) \tag{24}$$

which may be used to eliminate ϕ in (20). This leads to the problem of finding $u \in H_0^1(\Omega)$ with

$$\widehat{A}(u)(\eta) = L'(\eta) \quad (\eta \in H_0^1(\Omega)). \tag{25}$$

Here, L' is some bounded linear functional. The operator

$$\widehat{A}(u)(\eta) := \int_\Omega A(\epsilon u)\cdot \epsilon\eta \, d\Omega + < S_0 u|_\Gamma, \eta|_\Gamma > \qquad (u, \eta \in H_0^1(\Omega))$$

maps $H^1(\Omega;\mathbb{R}^2)$ into its dual, it is continuous, bounded and strongly monotone. From the main theorem on monotone operators [31] we obtain that $\widehat{A}$ is bijective. This yields the existence of u satisfying (25). Letting ϕ as in (24) we have that (u, ϕ) solves Problem (P). □

Remark 9. We emphasize the different meanings of $< u^j, 1 >= 0$, $(j = 1, 2)$ and $< \phi^j, 1 >= 0$, $(j = 1, 2)$ for $\phi = (\phi^1, \phi^2)$. $\phi \in H_0^{-1/2}(\Gamma)$ guarantees that we consider solutions of the exterior domain having a correct physical relevance, namely finite energy, whereas the constrains $< u^1, 1 >= 0 =< u^2, 1 >$ for $u \in H^1(\Omega;\mathbb{R}^2)$ just fix an (otherwise undetermined) additive constant which may also be chosen in another way (compare Remark 7).

3. The discrete problem (P_h)

In this section we treat the discretization of problem (P) in the form (22).

Let $(H_h \times H_h^{-1/2} : h \in I)$ be a family of finite dimensional subspaces of $H_0^1(\Omega) \times H_0^{-1/2}(\Gamma)$. Then, the coupling of finite elements and boundary elements consists in the following Galerkin procedure.

Definition 6 (Problem (P_h)). For $h \in I$ find $(u_h, \phi_h) \in H_h \times H_h^{-1/2}$ such that

$$B\left(\binom{u_h}{\phi_h}, \binom{v_h}{\psi_h}\right) = L\binom{v_h}{\psi_h} \tag{26}$$

for all $(v_h, \psi_h) \in H_h \times H_h^{-1/2}$.

In order to prove a discrete Babuška–Brezzi condition if A is linear, we need some notations and a discrete analogue of the positive definite operator S_0.

Assumption 1. *Let* $I \subseteq (0, 1)$ *with* $0 \in \overline{I}$ *and for any* $h \in I$ *let* $H_h \times H_h^{-1/2} \subseteq H_0^1(\Omega) \times H_0^{-1/2}(\Gamma)$. *Let us assume that for any* $(v, \psi) \in H_0^1(\Omega) \times H_0^{-1/2}(\Gamma)$ *and* $h \in I$ *there exists* $(v_h, \psi_h) \in H_h \times H_h^{-1/2}$ *with*

$$\lim_{I \ni h \to 0} \| (v - v_h, \psi - \psi_h) \|_{H_0^1(\Omega)\times H_0^{-1/2}(\Gamma)} = 0.$$

Definition 7 (Notations). Let $i_h : H_h \hookrightarrow H_0^1(\Omega)$ and $j_h : H_h^{-1/2} \hookrightarrow H_0^{-1/2}(\Gamma)$ denote the canonical injections with their duals $i_h^* : H_0^1(\Omega)^* \to H_h^*$ and $j_h^* : H_0^{1/2}(\Gamma) \to (H_h^{-1/2})^*$ being projections. Let $\gamma : H_0^1(\Omega) \to H_0^{1/2}(\Gamma)$ denote the trace operator, $\gamma u = u|_\Gamma$ for all $u \in H_0^1(\Omega)$, with the dual γ^*. Then, define

$$(27)\quad V_h := j_h^* V j_h, \quad K_h := j_h^* K \gamma i_h, \quad W_h := i_h^* \gamma^* W \gamma i_h, \quad K_h' := i_h^* \gamma^* K^* j_h$$

and, since V_h is positive definite as well,

$$(28)\qquad S_h := W_h + \left(\frac{1}{2}1_h^* - K_h'\right) V_h^{-1} \left(\frac{1}{2}1_h - K_h\right) : H_h \to H_h^*$$

with $1_h := j_h^* \gamma i_h$ and its dual 1_h^*.

Lemma 2. *There exist constants $c_0 > 0$ and $h_0 > 0$ such that for any $h \in I$ with $h < h_0$ we have*

$$< S_h u_h, u_h > \geq c_0 \cdot \| \gamma u_h \|^2_{H^{1/2}(\Gamma;\mathbb{R}^2)} \quad \textit{for all } u_h \in H_h.$$

Proof. The proof is quite analogue to that in [3] and is included for completeness. Assume that the assertion is false. Then one can construct a sequence of functions $(u_{h_n})_{n=1,2,3,\ldots}$ in $H_0^1(\Omega)$ with

$$u_{h_n} \in H_{h_n}, \ \| \gamma u_{h_n} \|_{H^{1/2}(\Gamma;\mathbb{R}^2)} = 1, \quad < S_{h_n} u_{h_n}, u_{h_n} > \leq \frac{1}{n},$$

$(n = 1, 2, 3, \ldots)$, and $\lim_{n\to\infty} h_n = 0$. Due to the Banach–Alaoglu theorem we may assume that a subsequence of $(u_{h_n}|_\Gamma)_{n=1,2,3,\ldots}$ (also denoted as $(u_{h_n}|_\Gamma)_{n=1,2,3,\ldots}$) converges towards some $w \in H_0^{1/2}(\Gamma)$ weakly in $H_0^{1/2}(\Gamma)$. Then, by definition of S_h, we firstly conclude that $< W u_{h_n}|_\Gamma, u_{h_n}|_\Gamma >$ tends towards zero so that (by weak convexity of $< W\cdot,\cdot >$) $< Ww, w >= 0$, i.e. $w|_\Gamma$ is a rigid body motion. A decomposition of $u_{h_n}|_\Gamma = v_n + w_n$ with $v_n \in H_0^{1/2}(\Gamma)$ and w_n a rigid body motion shows additionally that $(v_n)_{n=1,2,3,\ldots}$ tends towards zero strongly in $H_0^{1/2}(\Gamma)$ so that we have also strong convergence of $(u_{h_n}|_\Gamma)_{n=1,2,3,\ldots}$ towards w in $H_0^{1/2}(\Gamma)$.

On the other hand we have $0 = \lim_{n\to\infty} < V z_n, z_n >$ with $z_n := V_{h_n}^{-1}(\phi_n) \in H_{h_n}^{-1/2} \subseteq H_0^{-1/2}(\Gamma)$, $\phi_n := j_{h_n}^* y_n \in (H_{h_n}^{-1/2})^*$, $y_n := \frac{1}{2}u_{h_n} - K u_{h_n} \in H_0^{1/2}(\Gamma)$. Thus, $0 = \lim_{n\to\infty} \| z_n \|_{H_0^{-1/2}(\Gamma)}$ whence $0 = \lim_{n\to\infty} \| \phi_n \|_{(H_{h_n}^{-1/2})^*}$. Because of $(u_{h_n}|_\Gamma)_{n=1,2,3,\ldots} \to w$ we get $(y_n)_{n=1,2,3,\ldots} \to w$ strongly in $H_0^{1/2}(\Gamma)$ (compare (18)). Hence, since $\| w \|_{H^{1/2}(\Gamma;\mathbb{R}^2)} = \lim_{n\to\infty} \| \gamma u_{h_n} \|_{H^{1/2}(\Gamma;\mathbb{R}^2)} = 1$ we find $r_n \in H_{h_n}^{-1/2}$ with $\lim_{n\to\infty} \| r_n - w \|_{H_0^{-1/2}(\Gamma)} = 0$. (cf. Assumption 1). Then, since $\| r_n \|_{H_0^{-1/2}(\Gamma)}$ is bounded, we obtain

$$0 = \lim_{n\to\infty} < \phi_n, r_n > = \lim_{n\to\infty} < y_n, r_n > = < w, w >,$$

a contradiction. □

Theorem 5. *There exist constants* $\beta_0 > 0$ *and* $h_0 > 0$ *such that for any* $h \in I$ *with* $h < h_0$ *we have that for any* $(u_h, \phi_h), (v_h, \psi_h) \in H_h \times H_h^{-1/2}$

$$\alpha\Big(\| \epsilon u_h - \epsilon v_h \|_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\rm sym})}\Big) + \beta_0 \cdot \| \big(\begin{smallmatrix} \gamma u_h - \gamma v_h \\ \phi_h - \psi_h \end{smallmatrix}\big) \|^2_{H^{1/2}(\Gamma;\mathbb{R}^2)\times H^{-1/2}(\Gamma;\mathbb{R}^2)}$$
$$\leq B\big(\big(\begin{smallmatrix} u_h \\ \phi_h \end{smallmatrix}\big), \big(\begin{smallmatrix} u_h - v_h \\ \eta_h - \delta_h \end{smallmatrix}\big)\big) - B\big(\big(\begin{smallmatrix} v_h \\ \psi_h \end{smallmatrix}\big), \big(\begin{smallmatrix} u_h - v_h \\ \eta_h - \delta_h \end{smallmatrix}\big)\big)$$

with $2\eta_h := \phi_h + V_h^{-1}(\frac{1}{2}1_h - K_h)u_h$, $2\delta_h := \psi_h + V_h^{-1}(\frac{1}{2}1_h - K_h)v_h \in H_h^{-1/2}$.

Proof. The proof is quite analogue to that of Theorem 3 dealing now with the discrete operators (27) and (28). All calculations in the proof of Theorem 3 can be repeated with obvious modifications. Due to Lemma 2 the corresponding constants are independent of h so that β_0 does not depend on $h < h_0$; we omit the details. □

Theorem 6. *There exist constants* $c_0 > 0$ *and* $h_0 > 0$ *such that for any* $h \in I$ *with* $h < h_0$ *the problem* (P_h) *has a unique solution* (u_h, ϕ_h) *and, if* (u, ϕ) *denotes the solution of (P), there holds with constants* $\alpha > 0$ *in* (9) *and* $\beta > 0$

$$\alpha\Big(\| \epsilon u - \epsilon u_h \|_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\rm sym})}\Big) + \beta \cdot \| \big(\begin{smallmatrix} \gamma u - \gamma u_h \\ \phi - \phi_h \end{smallmatrix}\big) \|^2_{H^{1/2}(\Gamma;\mathbb{R}^2)\times H^{-1/2}(\Gamma;\mathbb{R}^2)}$$
$$\leq \quad c_0 \cdot \inf_{(v_h,\psi_h)\in H_h \times H_h^{-1/2}} \Big\{ \alpha^*\Big(c_0 \| \epsilon u - \epsilon v_h \|_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\rm sym})}\Big)$$
$$+ \| \big(\begin{smallmatrix} \gamma u - \gamma v_h \\ \phi - \psi_h \end{smallmatrix}\big) \|^2_{H^{1/2}(\Gamma;\mathbb{R}^2)\times H_0^{-1/2}(\Gamma)} \Big\}$$

letting $\alpha^*(s) := \sup_{t>0}(s \cdot t - \alpha(t))$ *for* $s > 0$, *the dual of* α.

Proof. The existence and uniqueness of the discrete solutions follow as in the continuous case.

Let $(v_h, \psi_h) \in H^h \times H_h^{-1/2}$ be an approximation of (u, ϕ), the solution of Problem (P), (cf. Assumption 1) such that we may assume that $\| (v_h, \psi_h) \|_{H_0^1(\Omega)\times H_0^{-1/2}(\Gamma)}$ is bounded. Let (u_h, ϕ_h) solve problem (P_h) such that, from Theorem 5, we get

$$\alpha\Big(\| \epsilon u_h \|_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\rm sym})}\Big) + \beta \cdot \| \big(\begin{smallmatrix} \gamma u_h \\ \phi_h \end{smallmatrix}\big) \|^2_{H^{1/2}(\Gamma;\mathbb{R}^2)\times H^{-1/2}(\Gamma;\mathbb{R}^2)} \leq L\big(\big(\begin{smallmatrix} u_h \\ \eta_h \end{smallmatrix}\big)\big).$$

Using equivalence of $\| \epsilon \cdot \|_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\rm sym})} + \| \gamma \cdot \|_{H^{1/2}(\Gamma;\mathbb{R}^2)}$ and $\| \cdot \|_{H^1(\Omega;\mathbb{R}^2)}$ one concludes that (u_h, ϕ_h), (u, ϕ), and (v_h, ψ_h) are bounded in $H_0^1(\Omega) \times H_0^{-1/2}(\Gamma)$ by $r_0 > 0$, say, where r_0 is independent of h. Therefore, we may and will assume in the sequel that A is Lipschitz continuous with Lipschitz constant $a(r_0)$ (see below).

From Theorem 5 we conclude with appropriate η_h, $\delta_h \in H_h^{-1/2}$ that

$$\alpha\Big(\| \epsilon v_h - \epsilon u_h \|_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\rm sym})}\Big) + \beta_0 \cdot \| \big(\begin{smallmatrix} \gamma v_h - \gamma u_h \\ \psi_h - \phi_h \end{smallmatrix}\big) \|^2_{H^{1/2}(\Gamma;\mathbb{R}^2)\times H^{-1/2}(\Gamma;\mathbb{R}^2)}$$
$$\leq B\big(\big(\begin{smallmatrix} v_h \\ \psi_h \end{smallmatrix}\big), \big(\begin{smallmatrix} v_h - u_h \\ \eta_h - \delta_h \end{smallmatrix}\big)\big) - B\big(\big(\begin{smallmatrix} u_h \\ \phi_h \end{smallmatrix}\big), \big(\begin{smallmatrix} v_h - u_h \\ \eta_h - \delta_h \end{smallmatrix}\big)\big) .$$

Using the Galerkin condition,

$$B((\begin{smallmatrix} u \\ \phi \end{smallmatrix}),(\begin{smallmatrix} v_h \\ \psi_h \end{smallmatrix})) - B((\begin{smallmatrix} u_h \\ \phi_h \end{smallmatrix}),(\begin{smallmatrix} v_h \\ \psi_h \end{smallmatrix})) = 0 \quad \text{for all } (v_h,\psi_h) \in H_h \times H_h^{-1/2},$$

and the continuous dependence of $\eta_h - \delta_h$ from $(\gamma(v_h - u_h), \psi_h - \phi_h)$, i.e.

$$\begin{aligned} &\| \eta_h - \delta_h \|_{H^{-1/2}(\Gamma;\mathbb{R}^2)} \\ &\quad \le \quad C \left(\| \gamma v_h - \gamma u_h \|_{H^{1/2}(\Gamma;\mathbb{R}^2)} + \| \psi_h - \phi_h \|_{H^{-1/2}(\Gamma;\mathbb{R}^2)} \right) , \end{aligned}$$

we get a constant c_1 such that the right hand side of the last inequality is bounded by

$$\begin{aligned} &\int_\Omega (A(\epsilon v_h) - A(\epsilon u)) \cdot (\epsilon v_h - \epsilon u_h)\, d\Omega \\ &\quad + \quad c_1 \| (\begin{smallmatrix} \gamma v_h - \gamma u_h \\ \psi_h - \phi_h \end{smallmatrix}) \|_{H^{1/2}(\Gamma;\mathbb{R}^2)\times H^{-1/2}(\Gamma;\mathbb{R}^2)} \\ &\qquad \cdot \| (\begin{smallmatrix} \gamma v_h - \gamma u \\ \psi_h - \phi \end{smallmatrix}) \|_{H^{1/2}(\Gamma;\mathbb{R}^2)\times H^{-1/2}(\Gamma;\mathbb{R}^2)} . \end{aligned}$$

Using that A is Lipschitz continuous (see above in this proof) we obtain

$$\alpha(a) + \beta_0 \cdot c^2 \le c_2 \cdot a \cdot b + c_1 \cdot c \cdot d$$

with some constant $c_2 > 0$ and real numbers

$$\begin{aligned} a &:= \| \epsilon v_h - \epsilon u_h \|_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}})} \\ b &:= \| \epsilon v_h - \epsilon u \|_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}}} \\ c &:= \| (\begin{smallmatrix} \gamma v_h - \gamma u_h \\ \psi_h - \phi_h \end{smallmatrix}) \|_{H^{1/2}(\Gamma;\mathbb{R}^2)\times H^{-1/2}(\Gamma;\mathbb{R}^2)} \\ d &:= \| (\begin{smallmatrix} \gamma v_h - \gamma u \\ \psi_h - \phi \end{smallmatrix}) \|_{H^{1/2}(\Gamma;\mathbb{R}^2)\times H^{-1/2}(\Gamma;\mathbb{R}^2)}. \end{aligned}$$

Next we use $t \cdot s \le \frac{1}{2}\alpha(t) + \frac{1}{2}\alpha^*(2s)$ for $t = a$ and $s = c_2 b$ and a similar standard argument for $c \cdot d$ to obtain with some constant $c_4 > 0$

$$\alpha(a) + \beta_0 \cdot c^2 \le \alpha^*(2c_2 b) + c_4 \cdot d^2 .$$

Thus $c^2 = \| (\begin{smallmatrix} \gamma v_h - \gamma u_h \\ \psi_h - \phi_h \end{smallmatrix}) \|^2_{H^{1/2}(\Gamma;\mathbb{R}^2)\times H^{-1/2}(\Gamma;\mathbb{R}^2)}$ is bounded by the right hand side of the claimed inequality of the theorem.

With $2\eta := \phi + V_0^{-1} P(\frac{1}{2} - K)u|_\Gamma \in H_0^{-1/2}(\Gamma)$ and $2\delta := \phi_h + V_0^{-1} P(\frac{1}{2} - K)u_h|_\Gamma \in H_0^{-1/2}(\Gamma)$ we obtain from (23)

$$\begin{aligned} &\alpha\Big(\| \epsilon u - \epsilon u_h \|_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}})} \Big) + \beta \| (\begin{smallmatrix} \gamma u - \gamma u_h \\ \phi - \phi_h \end{smallmatrix}) \|^2_{H^{1/2}(\Gamma;\mathbb{R}^2)\times H^{-1/2}(\Gamma;\mathbb{R}^2)} \\ &\quad \le B((\begin{smallmatrix} u \\ \phi \end{smallmatrix}),(\begin{smallmatrix} u - u_h \\ \eta - \delta \end{smallmatrix})) - B((\begin{smallmatrix} u_h \\ \phi_h \end{smallmatrix}),(\begin{smallmatrix} u - u_h \\ \eta - \delta \end{smallmatrix})) \\ &\quad = B((\begin{smallmatrix} u \\ \phi \end{smallmatrix}),(\begin{smallmatrix} u - v_h \\ \eta - \delta \end{smallmatrix})) - B((\begin{smallmatrix} u_h \\ \phi_h \end{smallmatrix}),(\begin{smallmatrix} u - v_h \\ \eta - \delta \end{smallmatrix})) \end{aligned} \tag{29}$$

using the Galerkin property for $v_h \in H_h$. Since A is Lipschitz continuous (see above in this proof) and since $\eta - \delta$ depends continuously on $(\gamma(u - u_h), \phi - \phi_h)$ we get

$$\alpha(e) + \beta \cdot f^2 \le c_5 \cdot e \cdot b + c_6 \cdot f \cdot (f + d)$$

with a, b, c, d defined above and

$$\begin{aligned} e &:= \| \epsilon u - \epsilon u_h \|_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}})} \\ f &:= \| \left({\gamma(u-u_h) \atop \phi-\phi_h} \right) \|_{H^{1/2}(\Gamma;\mathbb{R}^2)\times H^{-1/2}(\Gamma;\mathbb{R}^2)}. \end{aligned}$$

According to triangle inequality, $f^2 \leq 2(c^2 + d^2)$ which gives

$$\alpha(e) + \beta \cdot f^2 \leq c_5 \cdot e \cdot b + c_7 \cdot (c^2 + d^2).$$

Again using $t \cdot s \leq \frac{1}{2}\alpha(t) + \frac{1}{2}\alpha^*(2s)$ for $t = e$ and $s = c_5 b$ yields

$$\alpha(e) + \beta \cdot f^2 \leq \alpha^*(2c_5 b) + c_8 \cdot (c^2 + d^2).$$

Finally, combining this with the above bound for c^2 concludes the proof. □

4. A posteriori error estimate

In this section we present an a posteriori error estimate, which is the base of our adaptive feedback procedure. For simplicity, we restrict ourselves to piecewise linear functions on triangles as finite elements in H_h and to piecewise constant functions on Γ as boundary elements in $H_h^{-1/2}$ assuming the following.

Assumption 2. *Let Ω be a two-dimensional domain with polygonal boundary Γ on which we consider a family $\mathcal{T} := (\mathcal{T}_h : h \in I)$ of decompositions $\mathcal{T}_h = \{\Delta_1, \ldots, \Delta_N\}$ of Ω in closed triangles $\Delta_1, \ldots, \Delta_N$ such that $\overline{\Omega} = \cup_{i=1}^N \Delta_i$ and two different triangles are disjoint or have a side in common or have a vertex in common. Let $\mathcal{S}_h$ denote the sides, i.e.*

$$\mathcal{S}_h = \{\partial\Delta_i \cap \partial\Delta_j : i \neq j \ \text{ with } \ \partial\Delta_i \cap \partial\Delta_j \text{ is a common side}\},$$

$\partial\Delta_j$ being the boundary of Δ_j. Let

$$\mathcal{G}_h = \{E : E \in \mathcal{S}_h \text{ with } E \subseteq \Gamma\}$$

be the set of "boundary sides" and let

$$\mathcal{S}_h^0 = \mathcal{S}_h \setminus \mathcal{G}_h$$

be the set of "interior sides".

We assume that all the angles of some $\Delta \in \mathcal{T}_h \in \mathcal{T}$ are $\geq \Theta$ for some fixed $\Theta > 0$ which does not depend on Δ or $\mathcal{T}_h$.

Then, define

$$\begin{aligned} H_h &:= \{\eta_h \in H_0^1(\Omega) : \eta_{hj}|_\Delta \in P_1 \text{ for any } \Delta \in \mathcal{T}_h \ (j = 1, 2)\} \\ H_h^{-1/2} &:= \{\psi_h \in H_0^{-1/2}(\Gamma) : \psi_{hj}|_E \in P_0 \text{ for any } E \in \mathcal{G}_h \ (j = 1, 2)\} \end{aligned}$$

where P_k denotes the polynomials with degree $\leq k$.

For fixed $\mathcal{T}_h$ let h be the piecewise constant function defined such that the constants $h|_\Delta$ and $h|_E$ equal the sizes $\operatorname{diam}(\Delta)$ *of* $\Delta \in \mathcal{T}_h$ *and* $\operatorname{diam}(E)$ *of* $E \in \mathcal{S}_h$.

We assume that A is of the form $(A\zeta)(x) = (a_{ij}(x, \zeta(x)))_{i,j=1,2}$ for some coefficients a_{ij} which are piecewise smooth with respect to both variables such that $A(\epsilon v_h) \in C^1(\overline{\Delta})$ for any $\Delta \in \mathcal{T}_h \in \mathcal{T}$ and any trial function $v_h \in H_h$. Finally, let $u_0 \in H^1(\Gamma; \mathbb{R}^2)$ and $f \in L^2(\Omega; \mathbb{R}^2)$, , $t_0 \in L^2(\Gamma; \mathbb{R}^2)$ satisfy (17).

Remark 10. We emphasize that a standard basis for $H_h^{-1/2}$ is given by the derivatives with respect to the arc–length of the standard piecewise linear hat functions on the polygon Γ (piecewise with respect to $\mathcal{G}_h$). As noted above (cf. Remark 9) the restriction of H_h to be a subspace of $H_0^1(\Omega)$ is not really necessary, it determines just the constant in the (discrete) solution.

Instead of constructing a basis of piecewise linear trial functions in $H_0^1(\Omega)$ we may use the standard basis of piecewise linear trial functions $\hat{H}_h^1$ (neglecting the above condition $< \eta^1, 1 >= 0 =< \eta^2, 1 >$ for $\eta = (\eta^1, \eta^2)$) but adding one artificial boundary condition on the discrete problem determining a constant like, e.g. $\eta_h(x_0) = 0$ for some fixed node in the mesh (or on the boundary) giving $\tilde{H}_h$. Then, the modified discrete problem yields a unique solution $(\tilde{u}_h, \phi_h) \in \tilde{H}_h^1 \times H_h^{-1/2}$ of

$$B\left(\binom{\tilde{u}_h}{\phi_h}, \binom{v_h}{\psi_h}\right) = L\binom{v_h}{\psi_h} \qquad \left(\binom{v_h}{\psi_h} \in \tilde{H}_h^1 \times H_h^{-1/2}\right).$$

Letting $u_h := P\tilde{u}_h$ we obtain a solution $(u_h, \phi_h) \in H_h \times H_h^{-1/2}$ of Problem (P_h).

Definition 8 (Notations). Let n be the exterior normal on Γ and on any element boundary $\partial\Delta$, let n have a fixed orientation so that $[(A\epsilon u_h)\cdot n]|_E \in L^2(E)$ denotes the jump of the discrete tractions $(A\epsilon u_h)\cdot n$ over the side $E \in \mathcal{S}_h^0$. Define

$$\begin{aligned}
R_1^2 &:= \sum_{\Delta \in \mathcal{T}_h} \operatorname{diam}(\Delta)^2 \cdot \int_\Delta |f + \operatorname{div}(A\epsilon u_h)|^2 \, d\Omega \\
R_2^2 &:= \sum_{E \in \mathcal{S}_h^0} \operatorname{diam}(E) \cdot \int_E |[A(\epsilon u_h)\cdot n]|^2 \, ds \\
R_3 &:= \| \sqrt{h} \cdot \Big(t_0 - A(\epsilon u_h)\cdot n + W(u_0 - \gamma u_h) \\
&\qquad -(K' - 1/2)\phi_h \Big) \|_{L^2(\Gamma;\mathbb{R}^2)} \\
R_4 &:= \sum_{E \in \mathcal{G}_h} \operatorname{diam}(E)^{1/2} \cdot \| \frac{\partial}{\partial s}\{(1/2 - K)(u_0 - \gamma u_h) \\
&\qquad - V\phi_h\} \|_{L^2(E;\mathbb{R}^2)}.
\end{aligned}$$

Remark 11. Note that $R_1, \ldots, R_4$ can be computed (at least numerically) as far as the solution (u_h, ϕ_h) of problem (P_h) is known (see also Sect. 6 below for the computational details).

Under the above assumptions and notations there holds the following a posteriori estimate where (u, ϕ) and (u_h, ϕ_h) solve problem (P) and (P_h), respectively.

Theorem 7. *There exists some constant $c > 0$ such that for any $h \in I$ with $h < h_0$ (h_0 from Lemma 2) we have*

$$\alpha\Big(\|\,\epsilon u - \epsilon u_h\,\|_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\rm sym})}\Big) + \beta\|\,\binom{\gamma u-\gamma u_h}{\phi-\phi_h}\,\|^2_{H_0^{1/2}(\Gamma)\times H_0^{-1/2}(\Gamma)} \tag{30}$$
$$\leq \quad \alpha^*\Big(c\cdot(R_1+R_2+R_3)\Big) + c\cdot(R_1^2+R_2^2+R_3^2+R_4^2)$$

Proof. The proof of Theorem 7 is of some length but analogous to corresponding results in [3] so that we give only a sketch of the proof. We adopt the notation and the assumptions of this section. $c > 0$ is a generic constant and depends only on $\mathcal{T}$ but not on h, Δ, N, u, etc.

We start as in the proof of Theorem 6 and use (29) to see that the left hand side in (30) is bounded by

$$L\binom{e-e_h}{\rho-\rho_h} - B\Big(\binom{u_h}{\phi_h},\binom{e-e_h}{\rho-\rho_h}\Big)$$

where $e := u - u_h$, $\rho := \frac{1}{2}(\phi - \phi_h) + \frac{1}{2}V_0^{-1}P(1/2 - K)(\gamma u - \gamma u_h)$ and $(e_h, \rho_h) \in H_h \times H_h^{-1/2}$ will be chosen later on.

Elementwise integration by parts of the terms $\int_\Delta A(\epsilon u_h)\cdot\epsilon(e - e_h)\,d\Omega$ and direct calculations yield that the left hand side in (30) is bounded by

$$T_1 + T_2 + T_3 + T_4$$

where

$$\begin{aligned}
T_1 &:= \sum_{\Delta\in\mathcal{T}_h}\int_\Delta (f + \operatorname{div} A(\epsilon u_h))(e - e_h)\,d\Omega\\
T_2 &:= -\sum_{E\in\mathcal{S}_h^0}\int_E [A(\epsilon u_h)\cdot n](e - e_h)|_E\,ds\\
T_3 &:= < t_0 - A(\epsilon u_h)\cdot n + W(u_0 - \gamma u_h)\\
&\qquad -(K' - 1/2)\phi_h, (\gamma e - \gamma e_h) >\\
T_4 &:= < \rho - \rho_h, (1/2 - K)(u_0 - \gamma u_h) - V\phi_h > .
\end{aligned}$$

It remains to estimate $T_1, \ldots, T_4$ corresponding to $R_1, \ldots, R_4$.

Under the Assumption 2 the results of [4] apply here which are recalled in the present notations. Firstly, there exists a family of interpolation operators $(I_h : H^1(\Omega;\mathbb{R}^2) \to H_h : h \in I)$ — obtained by local L^2–projection — such that for any $\Delta \in \mathcal{T}_h \in \mathcal{T}$ and integers k, q with $0 \leq k \leq q \leq 2$ and with $N_\Delta := \cup\{\Delta' \in \mathcal{T}_h : \Delta' \cap \Delta \neq \emptyset\}$, the union of all neighbor elements of Δ, and for all $u \in H^q(N_\Delta)$,

$$|\,I_h u - u\,|^2_{H^k(\Delta;\mathbb{R}^2)} \leq c\cdot\operatorname{diam}(\Delta)^{2(q-k)}\cdot|\,u\,|^2_{H^q(N_\Delta;\mathbb{R}^2)}. \tag{31}$$

Secondly, (cf. [4, Lemma 4]) for any side E side of $\Delta \in \mathcal{T}_h \in \mathcal{T}$, and any $u \in H^1(\Delta;\mathbb{R}^2)$ there holds

(32) $\operatorname{diam}(\Delta)\| u \|^2_{L^2(E;\mathbb{R}^2)} \leq c \cdot \left(\| u \|^2_{L^2(\Delta;\mathbb{R}^2)} + \operatorname{diam}(\Delta)^2 \cdot | u |^2_{H^1(\Delta;\mathbb{R}^2)} \right).$

We choose $e_h := I_h e$ and let ρ_h be arbitrary.

Using Cauchy's inequality, (31) ($k = 0$, $q = 1$) and since the number of neighbors is bounded, one concludes

$$T_1 \leq c \cdot | e |_{H^1(\Omega;\mathbb{R}^2)} \cdot R_1.$$

Combining (31) (with $e - I_h e$ replacing u) and (32) (with e replacing u, $k = 0$, $q = 1$ and $k = 1 = q$) we obtain for any $E \in \mathcal{S}_h^0$, $E \subseteq \Delta$, $\Delta \in \mathcal{T}_h \in \mathcal{T}$,

$$\| e - I_h e \|^2_{L^2(E;\mathbb{R}^2)} \leq c \cdot \operatorname{diam}(\Delta) | e |^2_{H^1(N_\Delta;\mathbb{R}^2)}.$$

Therefore,

$$\begin{aligned} T_2 &\leq \sum_{E \in \mathcal{S}_h^0} \| [A(\epsilon u_h) \cdot n] \|_{L^2(E;\mathbb{R}^2)} \cdot \| e - I_h e \|_{L^2(E;\mathbb{R}^2)} \\ &\leq c \sum_{E \in \mathcal{S}_h^0} \sqrt{\operatorname{diam}(E)} \| [A(\epsilon u_h) \cdot n] \|_{L^2(E;\mathbb{R}^2)} \cdot | e |_{H^1(N_\Delta;\mathbb{R}^2)} \\ &\leq c R_2 \cdot | e |_{H^1(\Omega;\mathbb{R}^2)}. \end{aligned}$$

Note that $t_0 \in L^2(\Gamma;\mathbb{R}^2)$, $W(u_0 - \gamma u_h) \in L^2(\Gamma;\mathbb{R}^2)$ since $u_0 - \gamma u_h \in H^1(\Gamma;\mathbb{R}^2)$, $(K' - 1/2)\phi_h \in L^2(\Gamma;\mathbb{R}^2)$ since $\phi_h \in L^2(\Gamma;\mathbb{R}^2)$, and $A(\epsilon u_h) n|_\Gamma \in L^2(\Gamma;\mathbb{R}^2)$ since ϵu_h is piecewise constant and a_{ij} is piecewise smooth. Thus, we may repeat the above arguments to see

$$T_3 \leq c \cdot | e |_{H^1(\Omega;\mathbb{R}^2)} \cdot R_3.$$

Define $\psi := (1/2 - K)(u_0 - \gamma u_h) - V\phi_h$ and note

$$T_4 = < \rho - \rho_h, P\psi >$$

for any $\rho_h \in H_h^{-1/2}$. Since $\eta := P\psi \in H^{-1/2}(\Gamma;\mathbb{R}^2)$ we obtain that $\eta \in H^1(\Gamma;\mathbb{R}^2)$ is orthogonal to any piecewise constant function (not only these from $H_h^{-1/2}$). As it is proved in [2, Proposition 1] this properties include

$$\| \eta \|_{H^{1/2}(\Gamma;\mathbb{R}^2)} \leq c \cdot \sum_{j=1}^{M} \| \sqrt{h} \cdot \eta' \|_{L^2(\Gamma_j;\mathbb{R}^2)},\ c = \tilde{c}\sqrt{k},\ \tilde{\in}\mathbb{R}$$

when $\{\Gamma_1, \ldots, \Gamma_M\} = \mathcal{G}_h$. We remark that [2, Proposition 1] the factor $\sqrt{k}$ appears with $k := \max\{\operatorname{diam}(\Gamma_i)/\operatorname{diam}(\Gamma_j) : \Gamma_i$ and Γ_j have a common node$\}$ which is bounded because of the angle property in Assumption 2. Thus, choosing $\rho_h = 0$ and noting $\| \rho \|_{H^{-1/2}(\Gamma;\mathbb{R}^2)} \leq c \cdot \| \binom{\gamma e}{\phi - \phi_h} \|_{H_0^{1/2}(\Gamma) \times H_0^{-1/2}(\Gamma)}$, we get

$$T_4 \leq c \cdot \| \binom{\gamma e}{\phi - \phi_h} \|_{H_0^{1/2}(\Gamma) \times H_0^{-1/2}(\Gamma)} \cdot \sum_{j=1}^{M} \| \sqrt{h} \cdot \psi' \|_{L^2(\Gamma_j;\mathbb{R}^2)}.$$

By the above estimates he have that the left hand side in (30) is bounded by

$$c(R_1+R_2+R_3)\cdot|\,e\,|_{H^1(\Omega;\mathbb{R}^2)} + cR_4\cdot\|\begin{pmatrix}\gamma e\\ \phi-\phi_h\end{pmatrix}\|_{H_0^{1/2}(\Gamma)\times H_0^{-1/2}(\Gamma)}.$$

Since $\|\,\epsilon\cdot\,\|_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\rm sym})}+\|\,\gamma\cdot\,\|_{H_0^{1/2}(\Gamma)}$ is an equivalent norm in $H_0^1(\Omega)$ we obtain

$$\alpha(a)+\beta\cdot b^2 \le c(R_1+R_2+R_3+R_4)\cdot b + c(R_1+R_2+R_3)\cdot a$$

where

$$\begin{aligned} a &:= \|\,\epsilon e\,\|_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\rm sym})}\\ b &:= \|\begin{pmatrix}\gamma e\\ \phi-\phi_h\end{pmatrix}\|_{H_0^{1/2}(\Gamma)\times H_0^{-1/2}(\Gamma)}\end{aligned}$$

Using $t\cdot s\le \frac{1}{2}\alpha(t)+\frac{1}{2}\alpha^*(2s)$ for $t=a$ and $s=c(R_1+R_2+R_3)$ and a similar standard argument for $c(R_1+R_2+R_3+R_4)\cdot b$ we obtain

$$\alpha(a)+\beta\cdot b^2\le \alpha^*\Big(c(R_1+R_2+R_3)\Big)+c(R_1+R_2+R_3+R_4)^2$$

concluding the proof. □

Example 2. In the case of Example 1 A is uniformly monotone, i.e. we have a global positive constant α_0 such that

$$\alpha_0(\xi-\zeta):(\xi-\zeta)\le(A(\xi)-A(\zeta))(\xi-\zeta)$$

for any $\xi,\zeta\in\mathbb{R}^{2\times 2}_{\rm sym}$. Hence, in the above notations, $\alpha(t)=\alpha_0\cdot t^2$. Then $\alpha^*(s)=s^2/(4\alpha_0)$. Therefore, in this example, Theorem 7 gives

$$\|\begin{pmatrix}u-u_h\\ \phi-\phi_h\end{pmatrix}\|_{H_0^1(\Omega)\times H_0^{-1/2}(\Gamma)}\le c\cdot(R_1+R_2+R_3+R_4). \tag{33}$$

5. Adaptive feedback procedure

For a given triangulation $\mathcal{T}_h=\{\Delta_1,\dots,\Delta_N\}$ of Ω and the related partition $\{\Gamma_1,\dots,\Gamma_M\}=\mathcal{G}_h$ of the boundary Γ we can consider one element $\Delta_j\in\mathcal{T}_h$ and compute its contributions a_j, b_k to the right hand side of the a posteriori error estimate in Theorem 7

$$\begin{aligned} a_j^2 &:= \operatorname{diam}(\Delta_j)^2\cdot\int_{\Delta_j}|f+\operatorname{div}A(\epsilon u_h)|^2\,d\Omega\\ &\quad+\sum_{E\in\mathcal{S}_h^0,E\subseteq\partial\Delta_j}\operatorname{diam}(E)\cdot\int_E|[A(\epsilon u_h)\cdot n]|^2\,ds\\ &\quad+\operatorname{diam}(\Gamma\cap\partial\Delta_j)\cdot\|\,t_0-A(\epsilon u_h)\cdot n+W(u_0-\gamma u_h)\\ &\qquad -(K'-1/2)\phi_h\,\|^2_{L^2(\Gamma\cap\partial\Delta_j;\mathbb{R}^2)}\\ b_k &:= \operatorname{diam}(\Gamma_k)^{1/2}\cdot\|\,\frac{\partial}{\partial s}\{(1/2-K)(u_0-\gamma u_h)-V\phi_h\}\,\|_{L^2(\Gamma_k;\mathbb{R}^2)}.\end{aligned}$$

The computational details for the computation of a_j, b_k are given in the next section.

According to Theorem 7, the error in the energy norm is bounded by

$$\alpha^* \left(c \sum_{j=1}^{N} a_j^2 \right) + c \sum_{j=1}^{N} a_j^2 + c \left(\sum_{k=1}^{M} b_k \right)^2 . \tag{34}$$

This a posteriori error estimate is almost useless for absolute error control without the computation of an upper bound for the constant $c > 0$. — But it may be used to compare the contributions to the local error.

In order to simplify notations and to stress the physical importance of the Hencky material we consider the particular case of Example 1 and 2 and obtain (cf. (33))

$$\| \left(\begin{smallmatrix} u-u_h \\ \phi-\phi_h \end{smallmatrix} \right) \|_{H_0^1(\Omega)\times H_0^{-1/2}(\Gamma)} \le c \sqrt{\sum_{j=1}^{N} a_j^2 + c \sum_{k=1}^{M} b_k} \, . \tag{35}$$

For any element Δ_j let

$$c_j := a_j + \sum_{k=1, \Gamma_k \subseteq \overline{\Delta_j}}^{N} b_k \tag{36}$$

where the sum may be zero or consists of a finite number of summands. Note that c_j describes the contribution of the element Δ_j to the right hand side of (35). The notion for the construction of an automatic mesh–refinement is to refine these elements having a larger ”contribution” c_j. The meshes in our numerical examples are steered by the following algorithm where $0 \le \theta \le 1$ is a global parameter.

Algorithm (A) *Given some coarse e.g. uniform mesh refine it successively by halving some of the elements due to the following rule. For any triangulation define $a_1, \dots, a_N$ as above and divide some element Δ_j by halving the largest side if*

$$c_j \ge \theta \cdot \max_{k=1,\dots,N} c_k \, .$$

In a subsequent step all hanging nodes are avoided by further refinement in order to obtain a regular mesh.

Remark 12. (i) Note that in Algorithm (A) $\theta = 0$ gives a uniform triangulation and with increasing θ the number of refined elements in the present step decreases.
(ii) By observing (35) we have some error control which, in some sense, yields a *reliable* algorithm. In particular, the relative improvement of (35) may be used as a reasonable termination criterion.
(iii) If in some step of Algorithm (A), (35) does not become smaller then we may add some uniform refinement steps ($\theta = 0$). It can be proved that in this case (35) decreases and tends towards zero. If we allow this modification we get *convergence* of the adaptive algorithm.

6. Numerical experiments

We consider two numerical examples for the solution of nonlinear interface problems related to Example 1.

First, we describe the numerical implementation of the Algorithm (A).

6.1. Implementation of the Galerkin procedure

We treat the case $\varphi(t) = \frac{1}{2}(t - (1+t)^{-1})$, $t \geq 0$ yielding a nonlinear operator A as explained in Example 1 with $a = \frac{1}{2}$, $b = 1$ and $n = 3$ in (8). In the sequel we explain the computation of the form in (26) where it is sufficient to describe the approximation of

$$B(\binom{\eta_j}{\psi_k}, \binom{\eta_n}{\psi_m}) \quad \text{and} \quad L\binom{\eta_n}{\psi_m}$$

used in the numerical examples. Here $\eta_j, \eta_k \in H_h^1$ are linear basis functions on triangular or quadrilateral elements and $\psi_m, \psi_n \in H_h^{-1/2}$ are piecewise constants on boundary elements Γ_m, Γ_n and vanish on the remaining part of Γ partitioned by $\Gamma_1, \ldots, \Gamma_M$.

The integral

$$\int_\Delta \{(\lambda+\mu)\, tr(\epsilon\eta_j) tr(\epsilon\eta_k) + 2\mu\varphi'(\text{dev}\ \epsilon\eta_j : \text{dev}\ \epsilon\eta_j) \cdot \text{dev}\ \epsilon\eta_j : \text{dev}\ \epsilon\eta_k\}\, d\Omega$$

is computed by a symmetric quadrature rule of order 19 on any triangle Δ as presented in [10]. The dualities $< V\phi, \varphi >$ and $< K\phi, \varphi >$ where ϕ, φ are polynomial functions can be calculated almost analytically [21]. The remaining integrals which appear are performed by a 32 point Gaussian quadrature rule. By using the relation $< K\eta_j, \psi_m > = < K'\psi_m, \eta_j >$ and $-W\eta_j = \frac{d}{ds} V^\star \frac{d}{ds} \eta_j$ [14] the computation of the Galerkin matrix is performed. The operator $V^\star$ is defined by

$$(V^\star\phi)(x) = < E^\star(x,\cdot), \phi >$$

where

$$E^\star(x,y) = \frac{\mu_2(\lambda_2+\mu_2)}{\pi(\lambda_2+2\mu_2)} \left\{ \log \frac{1}{|x-y|} \cdot I + \frac{(x-y)(x-y)^{\mathrm{T}}}{|x-y|^2} \right\}.$$

In order to approximate the right hand side for given functions $f \in L^2(\Gamma, I\!R^2)$, $u_0 \in H^{1/2}(\Gamma, I\!R^2)$, and $t_0 \in H^{-1/2}(\Gamma, I\!R^2)$ we compute $\int_\Delta f \cdot \eta_j \, d\Omega$ via a quadrature rule with order 19 on any triangle Δ.

The integrals $< \psi_k, (\frac{1}{2} - K)u_0 >$ and $< t_0 + Wu_0, \gamma\eta_j >$ are calculated in the following way. The terms are rewritten such that the integral operator acts on the polynomial function. These integrals can be computed analytically. The outer integral is approximated by a 32 point Gaussian quadrature formula.

Altogether the above descriptions determine the (approximate) computation of the mappings B and L when applied to discrete functions. Since A is a nonlinear operator we get a nonlinear system of equations which is solved via a Newton method until the termination error is of the magnitude of the machine precession.

6.2. Calculation of norms and residuals

In the examples below the error of the displacements u and hence their gradient grad u and traction $\phi = T_2 v$ (cf. Theorem 1) are known explicitly. Hence the $L^2(\Omega)$ norm of $u - u_h$ and grad $(u - u_h)$ can be calculated via the quadrature rule of order 19 [10] on any triangle. This yields an approximation of the error $u - u_h$ in the $H^1(\Omega; \mathbb{R}^2)$–norm.

The calculation of the integrals for the residuals $R_1, \ldots, R_4$ over the finite element Δ and the boundary element Γ_k is performed as follows: The integral

$$\int_\Delta |\text{div}\, A(\epsilon u_h) + f|^2 \, d\Omega$$

is approximated via the above mentioned quadrature rule of order 19 [10]. Here, f is given explicitly and $A(\epsilon u_h)$ is performed by the central difference replacement of ϵu_h which is known for each point. The jumps on the interior element boundaries in R_2 are polynomial functions and their L^2–norm is determined analytically. The $L^2(\Gamma_k)$–norm of

$$t_0 - A(\epsilon u_h) \cdot n + W(u_0 - \gamma u_h) - (K' - \frac{1}{2})\phi_h$$

is approximated by a 32 point Gaussian quadrature formula. Here, $t_0(x)$ is known, $A(\epsilon u_h) \cdot n$ is determined for each point on Γ_k, while the analytic function u_0 is replaced by its best approximation $\overline{u}_0$ in $\mathbb{P}_8$ and the terms $((K' - 1)\phi_h)(x)$ and $W(\overline{u}_0 - \gamma u_h)(x) = -(\frac{\partial}{\partial s} V^* \frac{\partial}{\partial s}(\overline{u}_0 - \gamma v_h))(x)$ are calculated analytically.

For any $x \in \Gamma_j$ the first summand of

$$\psi(x) := \frac{1}{2}(u_0 - \gamma u_h)(x) - (K(u_0 - \gamma u_h))(x) - (V\phi_h)(y)$$

is given explicitly, the third can be calculated analytically and by using the best approximation $\overline{u}_0$ instead of $u_0|_{\Gamma_k}$ the term $K(u_0|_{\Gamma_k} - \gamma u_h)(x)$ is calculated analytically. Then, $\| \psi' \|_{L^2(\Gamma_k;\mathbb{R}^2)}$ is approximated by a 32 point Gaussian quadrature rule on Γ_k where the value $\psi'(x_i)$ is determined for any Gaussian knot x_i as follows. For $1 < i < 32$, the values of $\psi(x_{i-1})$, $\psi(x_i)$, and $\psi(x_{i+1})$ are interpolated by a second order polynomial p_i and its derivative $p_i'(x_i)$ replaces $\psi'(x_i)$. For $i = 1$ we take $\psi(x_1)$, $\psi(x_2)$, and $\psi(x_3)$ and for $i = 32$ we take $\psi(x_{30})$, $\psi(x_{31})$, and $\psi(x_{32})$ to determine p_1 and p_{32}, respectively.

6.3. Numerical experiments

Let us consider the interface problem (1)-(7), i.e. Problem (P) for Example 1 on the L-shaped domain in Fig. 1 with exact solution

$$u_1(x,y) = \begin{pmatrix} r^{\frac{2}{3}} \sin(\frac{2}{3}\theta) - C \\ r^{\frac{2}{3}} \sin(\frac{2}{3}\theta) - C \end{pmatrix} \qquad \text{where } C = \frac{\int_\Gamma r^{\frac{2}{3}} \sin(2/3\theta) ds}{\int_\Gamma ds}$$

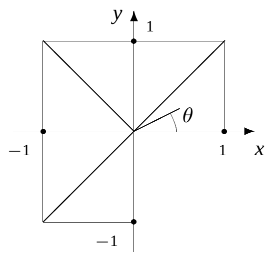

Fig. 1. L-shape

and

$$u_2(r,\theta) = \begin{pmatrix} \bar{r}^{-1} \\ 0 \end{pmatrix} \qquad \text{where } \bar{r}^2 = (x+0.5)^2 + (y-0.5)^2$$

in polar and Cartesian coordinates (r,θ) and (x,y), respectively. Young's modulus $E = 2 \cdot 10^7 \mathrm{Ncm}^{-2}$ and Poisson's coefficient $\nu = 0.3$ in both domains. We have $\varphi(t) = \frac{1}{2}(1-(1+t)^{-1})$, $t \geq 0$ in (8), compare Example 1. The solution has a typical corner singularity such that the convergence rate of the h–version with a uniform mesh does not lead to the optimal convergence rate even if the right hand side is smooth. The right hand side f and the jumps u_0 and t_0 are computed by (12) and (13) from u_1 and u_2 above. The Lamé coefficients are given by the relations

$$\lambda = \frac{E\,\nu}{(1-2\nu)(1+\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}\,.$$

As initial mesh we use a partition of Ω in six similar triangles with vertex at the origin (see Fig. 1).

Numerical results for the h–versions are shown (see Tables 1-6). The algorithm (A) generates meshes as shown in Fig. 1 for $\theta = 0.4$. The meshes automatically refine towards the origin where we have the expected singularity of the solution.

In Table 1 we have the numerical results for the uniform mesh ($\theta = 0$) and in Table 2-6 for the meshes generated by Algorithm (A) for *theta* $= \theta = 0.2, 0.4, 0.6, 0.8, 1.0$. We show the number of degree of freedom N and the corresponding relative error of the displacements e_N in the $H^1(\Omega;\mathbb{R}^2)$–norm. From Table 1-6 we may compute experimental convergence rates. For the uniform mesh we get experimentally a convergence of the form $O(h^\alpha)$ with a mesh size $h = O(1/N^2)$ and $\alpha \approx \frac{2}{3}$. We compare the degrees of freedom N needed to make the relative error smaller than 6.5%. For θ = 0, 0.2, 0.4, 0.6 and 0.8 we have N = 450,236,168,166,166 and the number of adaptive steps are 4,8,10,12,15. This shows that, in this example, the adapted meshes yield better results than a uniform triangulation, but it is not clear which θ leads to the most efficient procedure.

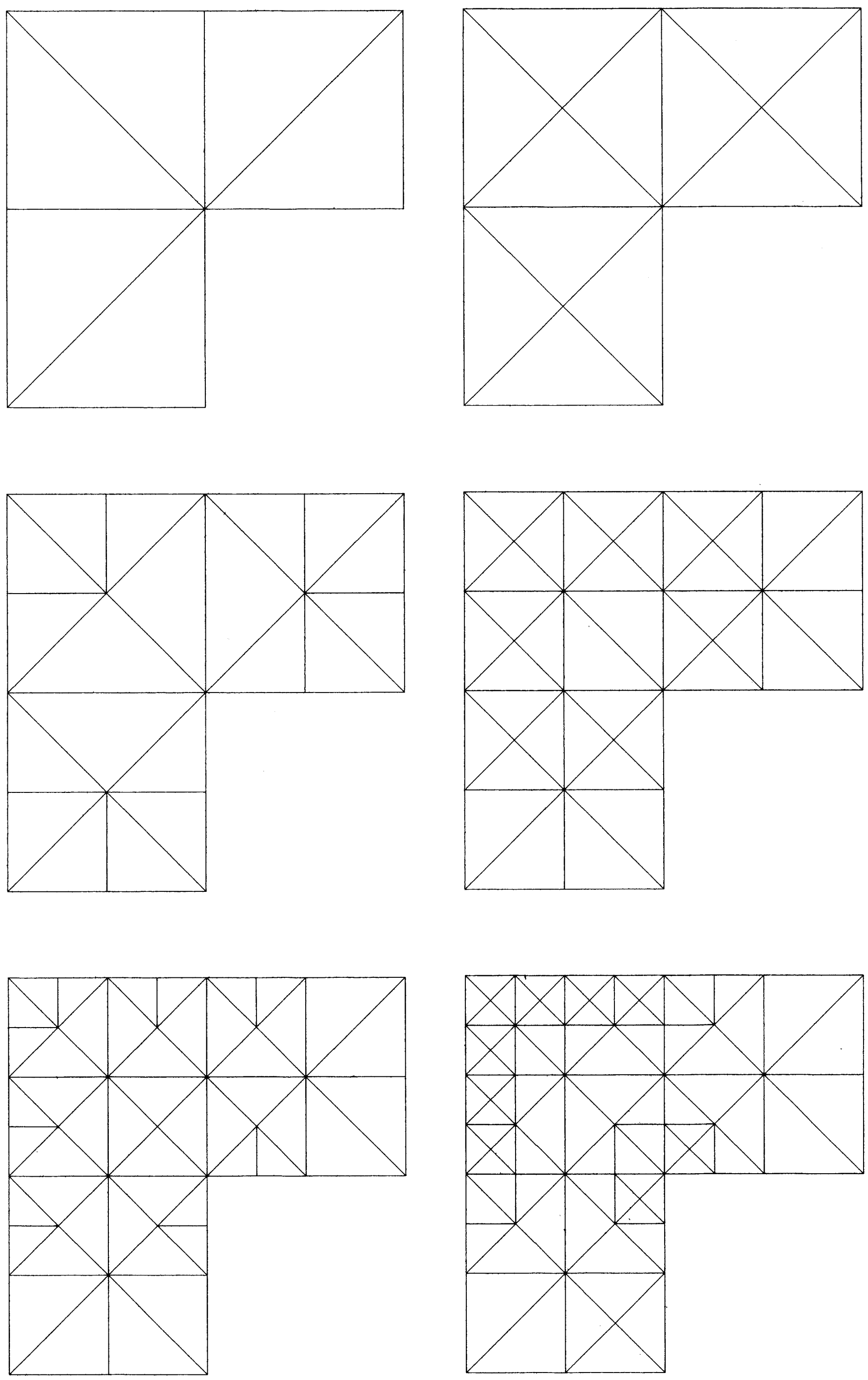

Fig. 2. Adapted meshes for the nonlinear transmission problem

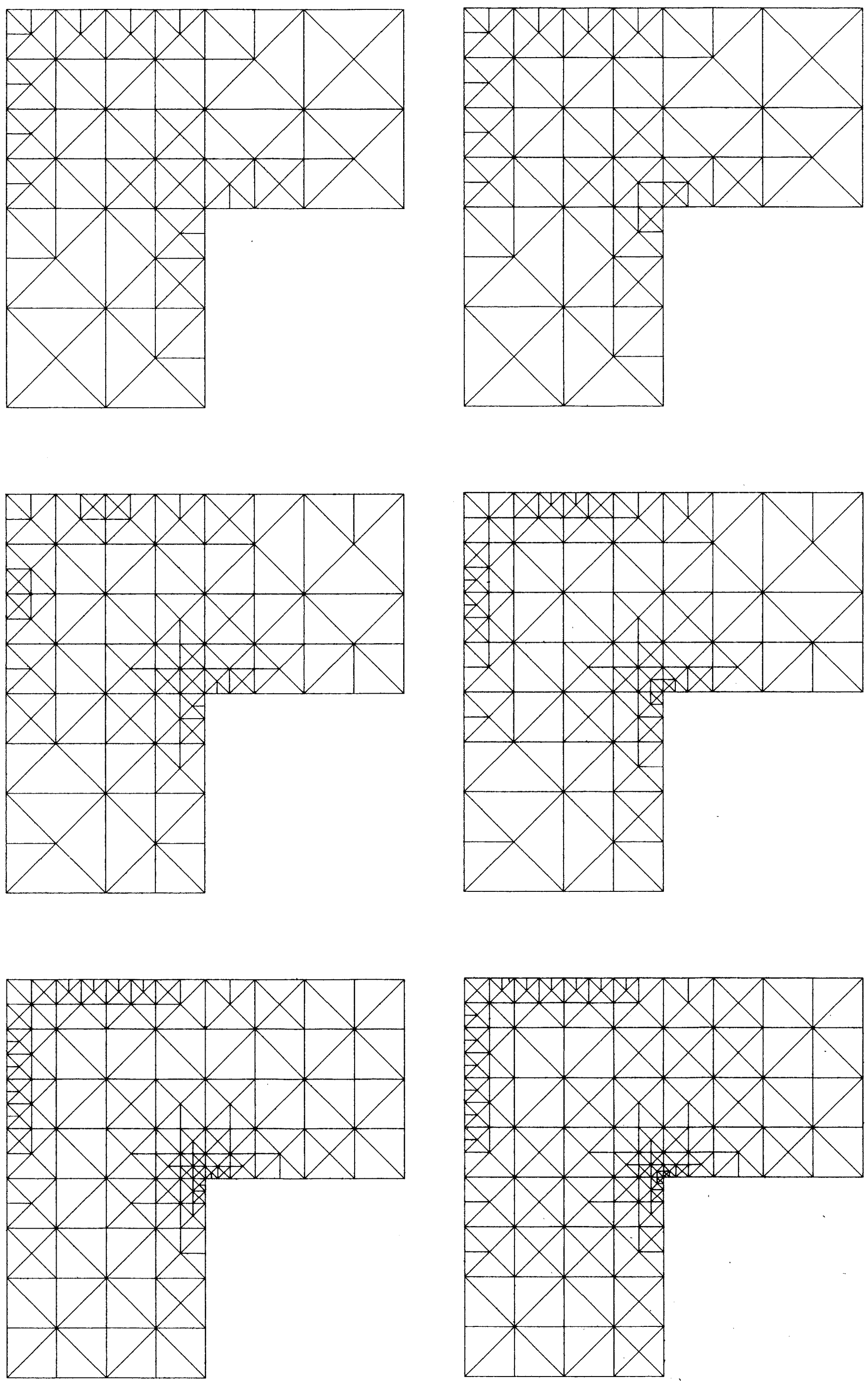

Fig. 2. (continued)

Table 1. Numerical results for the nonlinear transmission problem

$\theta = 0$		
N	$\frac{\Vert u_1 - u_N \Vert_{H^1(\Omega)}}{\Vert u_1 \Vert_{H^1(\Omega)}}$	$\frac{\log\left(\frac{\Vert u_1 - u_{N_1}\Vert}{\Vert u_1 - u_{N_2}\Vert}\right)}{\log \frac{N_2}{N_1}}$
16	0.25281	
		0.41376
42	0.16958	
		0.47234
130	0.09945	
		0.35313
450	0.06414	
	expected:	0.33

Table 2. Numerical results for the nonlinear transmission problem

$\theta = 0.2$		
N	$\frac{\Vert u_1 - u_N \Vert_{H^1(\Omega)}}{\Vert u_1 \Vert_{H^1(\Omega)}}$	$\frac{\log\left(\frac{\Vert u_1 - u_{N_1}\Vert}{\Vert u_1 - u_{N_2}\Vert}\right)}{\log \frac{N_2}{N_1}}$
16	0.25281	
		0.00881
22	0.25352	
		0.55156
38	0.18754	
		0.26539
56	0.16920	
		1.18407
74	0.12164	
		0.34385
110	0.10614	
		0.60987
152	0.08714	
		0.84846
170	0.07925	
		0.52938
236	0.06661	
		0.45343
278	0.06185	
		0.82056
334	0.05320	
		0.47952
392	0.04951	

Table 3. Numerical results for the nonlinear transmission problem

$\theta = 0.4$		
N	$\frac{\Vert u_1 - u_N \Vert_{H^1(\Omega)}}{\Vert u_1 \Vert_{H^1(\Omega)}}$	$\frac{\log\left(\frac{\Vert u_1 - u_{N_1} \Vert}{\Vert u_1 - u_{N_2} \Vert}\right)}{\log \frac{N_2}{N_1}}$
16	0.25281	
		0.07237
20	0.24876	
		0.54832
28	0.20685	
		0.39568
46	0.16996	
		0.67566
60	0.14203	
		1.38666
78	0.11940	
		0.37952
88	0.10827	
		0.60628
110	0.09457	
		0.49228
130	0.08710	
		0.65764
152	0.07859	
		0.80326
168	0.07252	
		0.76238
194	0.06499	
		0.35093
254	0.05912	
		0.46000
302	0.05460	
		0.67717
348	0.04960	
		0.48282
410	0.04582	

Table 4. Numerical results for the nonlinear transmission problem

$\theta = 0.6$		
N	$\frac{\Vert u_1 - u_N \Vert_{H^1(\Omega)}}{\Vert u_1 \Vert_{H^1(\Omega)}}$	$\frac{\log\left(\frac{\Vert u_1 - u_{N_1} \Vert}{\Vert u_1 - u_{N_2} \Vert}\right)}{\log \frac{N_2}{N_1}}$
16	0.25281	
		0.07237
20	0.24876	
		0.92354
24	0.21021	
		0.10453
28	0.20685	
		0.25132
42	0.18681	
		0.84729
56	0.14640	
		1.24741
66	0.11927	
		0.90066
72	0.11028	
		1.06361
80	0.09859	
		0.32314
104	0.09058	
		0.39653
124	0.08447	
		0.42860
144	0.07923	
		0.49460
166	0.07385	
		0.67173
186	0.06842	
		0.47885
224	0.06259	
		0.39081
280	0.05736	
		0.55940
336	0.05268	
		0.70794
372	0.04798	

Table 5. Numerical results for the nonlinear transmission problem

$\theta = 0.8$		
N	$\frac{\Vert u_1 - u_N \Vert_{H^1(\Omega)}}{\Vert u_1 \Vert_{H^1(\Omega)}}$	$\frac{\log\left(\frac{\Vert u_1 - u_{N_1}\Vert}{\Vert u_1 - u_{N_2}\Vert}\right)}{\log \frac{N_2}{N_1}}$
16	0.25281	
		0.63723
18	0.23453	
		0.63019
22	0.20667	
		0.19519
24	0.21021	
		0.18019
26	0.20720	
		0.18842
32	0.19925	
		0.26904
44	0.18289	
		0.09819
46	0.18369	
		1.18070
62	0.12913	
		1.05527
72	0.11028	
		1.06361
80	0.09859	
		0.70532
84	0.09525	
		0.22608
104	0.09076	
		0.43513
128	0.08292	
		0.40064
146	0.07866	
		0.53578
166	0.07344	
		0.58319
182	0.06960	
		0.71381
212	0.06242	
		0.35890
270	0.05723	
		0.38711
312	0.05411	
		0.52817
346	0.05124	

Table 6. Numerical results for the nonlinear transmission problem

$\theta = 1.0$		
N	$\frac{\Vert u_1 - u_N \Vert_{H^1(\Omega)}}{\Vert u_1 \Vert_{H^1(\Omega)}}$	$\frac{\log\left(\frac{\Vert u_1 - u_{N_1}\Vert}{\Vert u_1 - u_{N_2}\Vert}\right)}{\log \frac{N_2}{N_1}}$
16	0.25281	
		0.63723
18	0.23453	
		1.17641
20	0.20719	
		0.02637
22	0.20667	
		0.19519
24	0.21021	
		0.18019
26	0.20720	
		0.02281
28	0.20685	
		0.28034
32	0.19925	
		0.36072
34	0.19494	
		0.23764
36	0.19231	
		0.52977
40	0.18187	
		0.06189
42	0.18242	
		0.16561
44	0.18102	
		0.32939
46	0.18369	
		1.66691
48	0.17111	
		0.64698
50	0.16665	
		1.28970
52	0.15843	
		1.42936
54	0.15011	
		0.82558
56	0.14567	
		1.18413
60	0.12913	
		0.83262
62	0.12913	

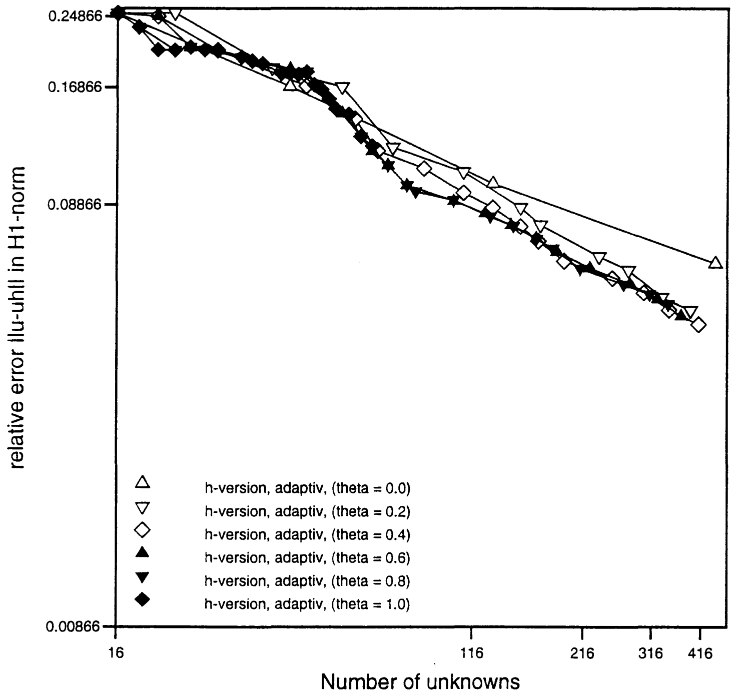

Fig. 3. Numerical results for the nonlinear transmission problem

In order to compare the adaptive algorithms for various parameters we compress the data in the sequel using Fig. 3. In Fig. 3 an entry corresponds to a symbol depending on the parameter θ. The entries belonging to the same parameter are connected by a straight line. The x-coordinate of a symbol is $\log(N)$ where N is the number of degrees of freedom corresponding to a mesh. The y-coordinate of the symbol is $\log(e_N)$. A straight line with the slope $-\alpha$ corresponds to an algebraic convergence of order α.

6.4. Practical example

In this section we describe the following tunnel problem where the exact solution is unknown. An infinite elastic plane of steel is considered with a rectangular hole and a socket of rubber in it. The hole in the socket is also rectangular and one side is loaded by a constant force, the other sides are fixed (see Fig. 4). The task is to compute the displacements. In a more mathematical formulation it is the following transmission problem. Two different materials are connected. In the interior domain we have rubber ($E = 2\,\mathrm{Ncm}^{-1}, \nu = 0.45$) which has nonlinear character and in the exterior domain we have steel ($E = 2 \cdot 10^7\,\mathrm{Ncm}^{-1}, \nu = 0.3$) which behaves almost linear. This classical situation leads to the coupling of finite elements and boundary elements for the transmission problem described as

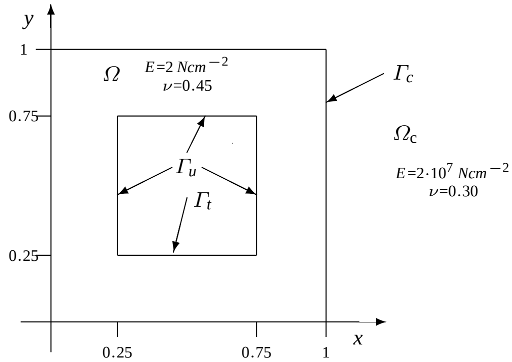

Fig. 4.

$$\begin{array}{ll} -\operatorname{div} A(\epsilon u) = f & \text{in } \Omega := (0,1)^2 \setminus \left(\frac{1}{4}, \frac{3}{4}\right)^2 \\ -\Delta^* v = 0 & \text{in } \Omega_c := I\!R^2 \setminus (0,1)^2 \\ u = v \text{ and } A(\epsilon u_1)n = T_2(u_2) & \text{on } \Gamma_c := \partial\Omega_c \\ u = 0 & \text{on } \Gamma_u := \partial\Omega \setminus (\Gamma_c \cup \Gamma_t) \\ A(\epsilon u)\cdot n = g & \text{on } \Gamma_t := \left(\frac{1}{4}, \frac{3}{4}\right) \times \left\{\frac{1}{4}\right\} \end{array}$$

in the strong formulation. Following the description above we can reformulate this transmission problem with mixed boundary conditions on the noncoupling boundary as the following variational problem:
Given $(f, g) \in L^2(\Omega; \mathbb{R}^2) \times L^2(\Gamma_t; \mathbb{R}^2)$ *find* $(u, \phi) \in H^1_u(\Omega) \times H_0^{-1/2}(\Gamma_c)$ *with*

$$\begin{aligned} &\int_\Omega A(\epsilon u)\cdot \epsilon\eta \, d\Omega \\ &\quad + < W\gamma u + (K' - 1/2)\phi, \gamma\eta > + < \psi, V\phi + (1/2 - K)\gamma u > \\ &= \int_\Omega f \cdot \eta \, d\Omega + \int_{\Gamma_t} g \cdot \gamma\eta \, ds \end{aligned}$$

for all $(\eta, \psi) \in H^1_u(\Omega) \times H_0^{-1/2}(\Gamma_c)$ *where* $H^1_u(\Omega) := \{u \in H^1(\Omega; \mathbb{R}^2) |\, u = 0$ *on* $\Gamma_u\}$.

Remark 13. With the Dirichlet boundary $\Gamma_u \neq \emptyset$ we need not look for solutions u in $H^1_0(\Omega) \cap H^1_u(\Omega)$; instead the variational problem above, is even coercive on $H^1_u(\Omega) \times H_0^{1/2}(\Gamma_c)$.

Corresponding to this variational formulation we get the contributions a_j, b_k to the right hand side of the a posteriori error estimate (30) in Theorem 4 as follows:

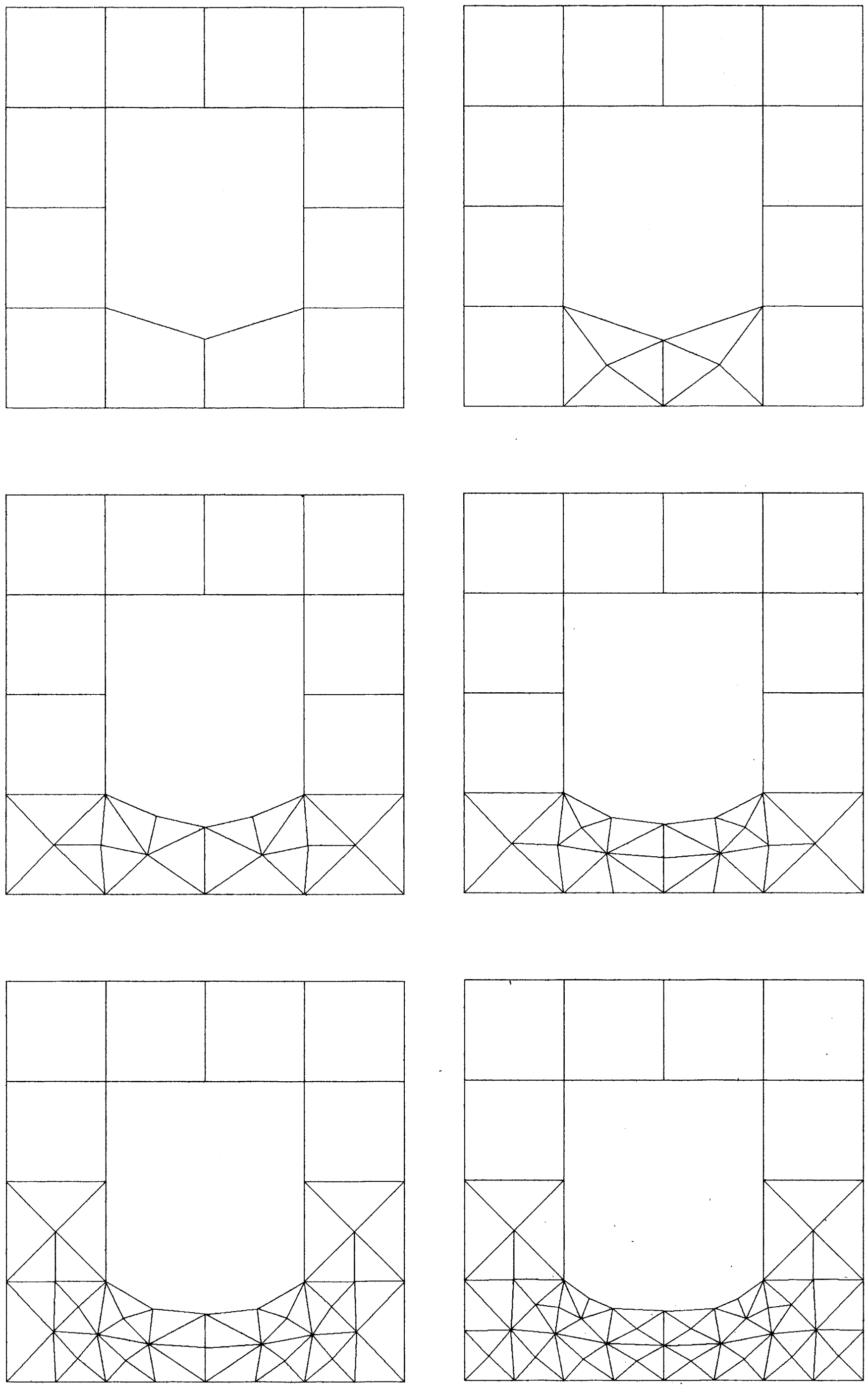

Fig. 5. Adapted meshes for the nonlinear transmission problem

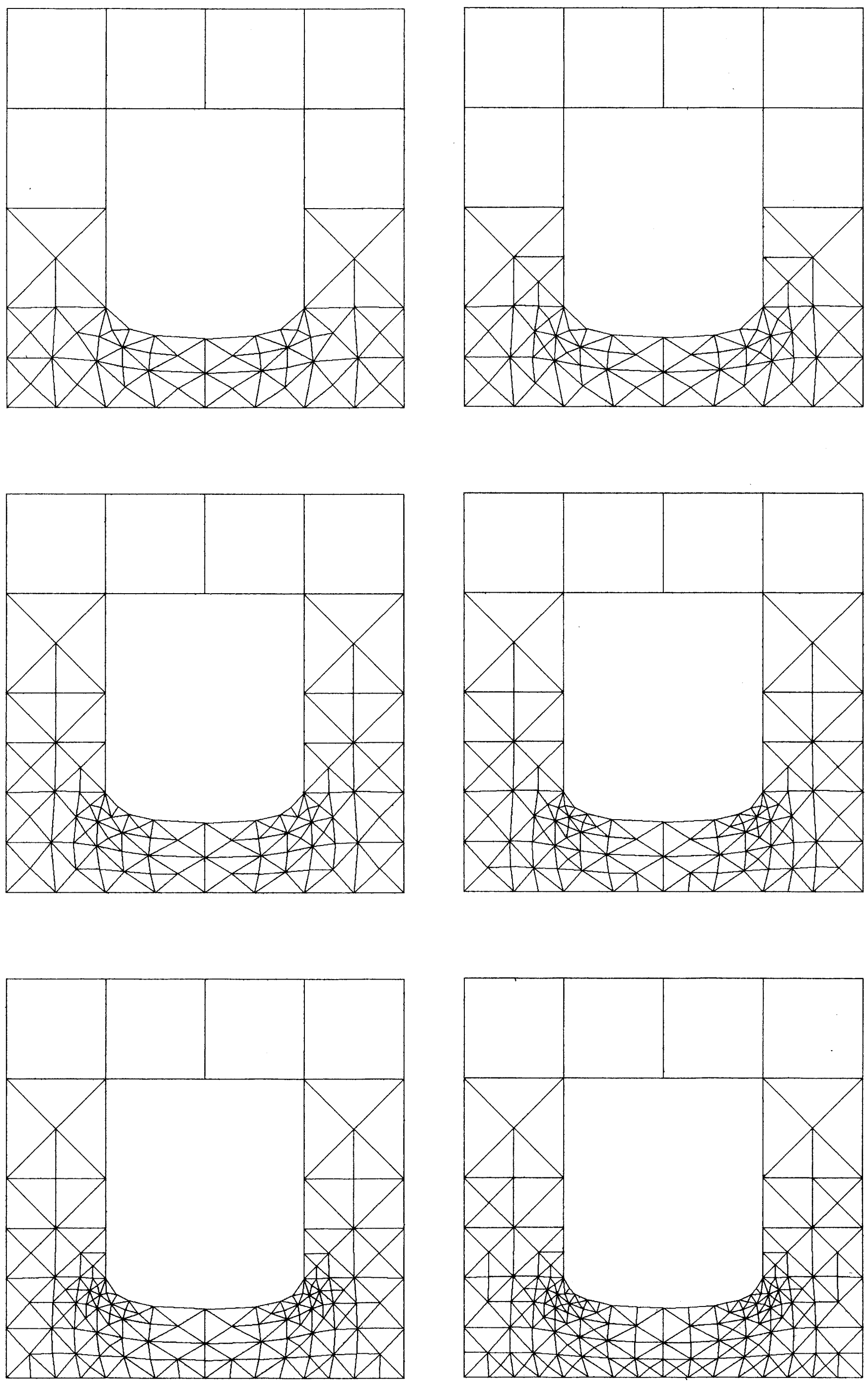

Fig. 5. (continued)

$$
\begin{aligned}
a_j^2 \quad := \quad & \operatorname{diam}(\Delta_j)^2 \cdot \int_{\Delta_j} |f + \operatorname{div} A(\epsilon u_h)|^2 \, d\Omega \\
+ \quad & \sum_{E \in \mathcal{S}_h^0, E \subseteq \partial\Delta_j} \operatorname{diam}(E) \cdot \int_E |[A(\epsilon u_h) \cdot n]|^2 \, ds \\
+ \quad & \operatorname{diam}(\Gamma_c \cap \partial\Delta_j) \cdot \| t_0 - A(\epsilon u_h) \cdot n + W(u_0 - \gamma u_h) \\
& \qquad -(K' - 1/2)\phi_h \|^2_{L^2(\Gamma_c \cap \partial\Delta_j; \mathbb{R}^2)} \\
+ \quad & \operatorname{diam}(\Gamma_t \cap \partial\Delta_j) \cdot \| g - A(\epsilon u) \cdot n \|^2_{L^2(\Gamma_t \cap \partial\Delta_j; \mathbb{R}^2)} \\
b_k \quad := \quad & \operatorname{diam}(\Gamma_k \cap \Gamma_c)^{1/2} \cdot \| \frac{\partial}{\partial s} \{(1/2 - K)(u_0 - \gamma u_h) - V\phi_h\} \|_{L^2(\Gamma_k; \mathbb{R}^2)}.
\end{aligned}
$$

Note the error indicator in our adaptive algorithm is then given as in (36). In our numerical example we neglect the body forces, i.e. $f = 0$ in Ω and the load density $g = 2Ncm^{-1}$ on Γ_t. Our coupling procedure was performed with piecewise linear finite elements in Ω and piecewise constant boundary elements on Γ_c. We start the adaptive algorithm with 10 subsquares of equal size. The resulting displacements and a sequence of meshes are shown in Fig. 5.

In Fig. 5 the sum of node-coordinates and node-displacements is plotted. The meshes are refined at corners with mixed boundary conditions and on the loaded side. The resulting displacements at the coupling boundary Γ_c are almost zero. As Fig. 5 shows both the refinement ot the mesh and the resulting displacements are symmetric to a parallel of the y-axis. Hence, the adaptive algorithm is a robust procedure which produces well refined meshes.

6.5. Conclusion

From the numerical experiments reported in the previous subsections, we claim that adaptive methods are important tools for an efficient numerical solution of transmission or interface problems via a coupling of finite elements and boundary elements. The asymptotic convergence rates are improved as well as the quality of the Galerkin solutions corresponding to only a few degrees of freedom.

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