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Fast parallel solvers for symmetric boundary element domain decomposition equations*

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Dedicated to Prof. Dr.-Ing. Erwin Stein on the occasion of his 65th birthday

Summary. The boundary element method (BEM) is of advantage in many applications including far-field computations in magnetostatics and solid mechanics as well as accurate computations of singularities. Since the numerical approximation is essentially reduced to the boundary of the domain under consideration, the mesh generation and handling is simpler than, for example, in a finite element discretization of the domain. In this paper, we discuss fast solution techniques for the linear systems of equations obtained by the BEM (BE-equations) utilizing the non-overlapping domain decomposition (DD). We study parallel algorithms for solving large scale Galerkin BE–equations approximating linear potential problems in plane, bounded domains with piecewise homogeneous material properties. We give an elementary spectral equivalence analysis of the BEM Schur complement that provides the tool for constructing and analysing appropriate preconditioners. Finally, we present numerical results obtained on a massively parallel machine using up to 128 processors, and we sketch further applications to elasticity problems and to the coupling of the finite element method (FEM) with the boundary element method. As shown theoretically and confirmed by the numerical experiments, the methods are of $O(h^{-2})$ algebraic complex-

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ity and of high parallel efficiency, where h denotes the usual discretization parameter.

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1 Introduction

Besides the finite element method (FEM), the boundary element method (BEM) is one of the main tools in the numerical approximation to solutions of partial differential equations. The FEM and the BEM have certain complementary properties. The FEM is quite flexible and applies to non-linear problems with varying coefficients but (in its standard form) is restricted to a bounded domain Ω which is partitioned during the performance of the FEM. In far field computations, which are of special interest in magnetostatics, electrostatics, thermodynamics, solid mechanics, fluid mechanics, etc., the domain Ω is unbounded but its boundary Γ may be bounded. The BEM applies in those situations where Γ is bounded and the equation in Ω is simple enough such that we know a fundamental solution. Another advantage of the BEM over the FEM consists in the adaptation of the discretization to singularities since only the skeleton has to be discretized. The same is true if one needs the solution and the derivatives on some skeleton only.

In this paper we consider model potential problems with piecewise constant, positive weight functions $a \in L^\infty(\Omega)$ in bounded domains Ω. The generalization of the approach to unbounded domains is straightforward [5, 23].

Definition 1 (Strong formulation). Find $u \in H^1(\Omega)$ such that

$$-\operatorname{div}(a\nabla u) = 0 \quad \text{in } \Omega \quad \text{and} \quad u = g \quad \text{on } \Gamma := \partial\Omega, \tag{1}$$

with given $g \in H^{1/2}(\Gamma)$.

Let us further assume that $a(x) = a_i > 0$ for almost all $x \in \Omega_i$ and $\Omega_1, \ldots, \Omega_p$ is a partition of Ω into Lipschitz domains. Since the equation in (1) is piecewise a (weighted) Laplace equation, a fundamental solution is known and both, the FEM and the BEM can be applied to compute a numerical approximation to the solution u.

In either the FEM or the BEM, the use of parallel computers leads to enormous time-reductions of the calculation. The main tools for parallelizing the generation and the solution of finite element schemes are the domain decomposition (DD) techniques. There are different variants of overlapping and non-overlapping domain decomposition methods (see, e.g., the proceedings of the annual DD–conferences since 1986). G.C. Hsiao and

W.L. Wendland utilized the non-overlapping domain decomposition for the derivation of symmetric boundary element equations living on the skeleton of the decomposition [25]. This formulation has been used as starting point for the construction of various parallel solvers [25, 28–30, 42, 44]. In this context, we consider the following interface formulation of problem (1).

Definition 2 (Interface problem). Find $(u_1, ..., u_p) \in H^1(\Omega_1) \times \cdots \times H^1(\Omega_p)$ such that

$$-\operatorname{div}(a_i \nabla u_i) = 0 \quad \text{in } \Omega_i, \tag{2}$$

$$u_i = g \quad \text{on } \Gamma \cap \partial\Omega_i, \tag{3}$$

$$u_i = u_j \quad \text{on } \partial\Omega_i \cap \partial\Omega_j, \tag{4}$$

$$a_i \cdot \partial u_i/\partial n = a_j \cdot \partial u_j/\partial n \quad \text{on } \partial\Omega_i \cap \partial\Omega_j \tag{5}$$

for all $i, j \in \{1, \ldots, p\}$, where $g \in H^{1/2}(\Gamma)$ is given, and n denotes the unit outer normal on $\partial\Omega_i$.

The aim of this paper is to provide, to analyse and to test numerically fast parallel solvers for the discrete equations related to the boundary integral formulation of (2)–(5).

The outline of this paper is as follows. Equivalent weak forms and the boundary integral equation related to (2)–(5) are recalled in Sect. 2. The Galerkin discretization of the resulting boundary integral formulation is introduced in Sect. 3 in notations which stress the parallel algorithms below. The preconditioned conjugate gradient algorithm is described in Sect. 4, where we obtain a convergence rate which is mesh-independent provided that we can construct some spectral equivalent preconditioner. This preconditioner is basically defined by two block operators preconditioning the discrete single layer potential operator and the discrete (BE-) Schur-complement. Preconditioners for the discrete single layer potential operator are well understood now [36, 42]. In Subsect. 5.6, we briefly discuss those preconditioners that have been used in our numerical experiments. The construction of preconditioners that are spectrally equivalent to the BE-Schur-complement with spectral equivalence constants which are independent of the discretization parameter h and, possibly, other "bad" parameters, such as the number p of subdomains seems to be more difficult. However, a careful analysis of the discrete Schur-complement and its relation to the discretized (integral operator) Schur-complement provides us with necessary tools for designing and analysing appropriate BE-Schur-complement preconditioners. In Sect. 5, we prove that the discrete Schur-complement is spectrally equivalent to the discretization of the Schur-complement on both an abstract (Subsects. 5.1 and 5.2) and a specialized level (Subsects. 5.3 and 5.4). From these results, we derive two special BE-Schur-complement preconditioners (Subsect. 5.5) that have been also used in our numerical experiments.

Roughly speaking, any preconditioner which is equivalent to some weighted $H^{1/2}$-norm, or to the discrete hypersingular integral operator on the skeleton may be applied to precondition the BE-Schur-complement. In Sect. 6, we present the results of our numerical experiments on distributed memory computers with up to 128 processors. These results confirm that the number of iterations is independent of the discretization parameter h, the number p of subdomains (see first example) and the mesh grading (see second example). A further example shows that this kind of preconditioning works for pure boundary element and coupled boundary–finite element discretizations of real-life linear elasticity problems as well.

2 The symmetric Galerkin DD-formulation

The potential problem from Sect. 1 can be rewritten in a weak form and in an equivalent integral form. Finally, the integral formulation yields a Problem (P), which is rewritten below, of seeking $(\lambda, u) \in \Lambda \times \mathbf{U}_g$ that satisfies

$$a(\lambda, u; \eta, v) = 0 \qquad \forall\, (\eta, v) \in \Lambda \times \mathbf{U}_0. \tag{6}$$

The straightforward numerical treatment of this Problem (P) described in Sect. 3 leads to a discrete Problem (P_h). The fast solution of Problem (P_h) is the main topic of this paper.

We recall that $\Omega_1, \ldots, \Omega_p$ are p pairwise disjoint bounded Lipschitz domains in the plane which separate Ω, $\overline{\Omega} = \cup_{j=1}^{p} \overline{\Omega}_j$ and $a_1, \ldots, a_p$ are positive numbers. Then, the weak form is straightforward.

Definition 3 (Variational problem). Find $u \in \mathbf{V}_g := \{v \in H^1(\Omega) : v = g \text{ on } \Gamma\}$ such that

$$\sum_{i=1}^{p} a_i \int_{\Omega_i} \nabla^{\mathrm{T}} u \nabla v \, dx = 0 \quad \forall v \in \mathbf{V}_0 := \overset{o}{H}{}^1(\Omega), \tag{7}$$

with given $g \in H^{1/2}(\Gamma)$.

The variational problem can be rewritten as a system of boundary integral equations as in [25] where we define

$$\Lambda := H^{-1/2}(\Gamma_1) \times \cdots \times H^{-1/2}(\Gamma_p), \tag{8}$$

$$\mathbf{U}_g := \{u \in L^2(\Gamma_{\mathrm{BE}}) : u|_{\Gamma} = g \text{ and, } u|_{\Gamma_i} \in H^{1/2}(\Gamma_i) \quad \forall i = 1, \ldots, p\}, \tag{9}$$

and let $\langle ., . \rangle_{\Gamma_i}$ be the duality product between $H^{-1/2}(\Gamma_i)$ and $H^{1/2}(\Gamma_i)$ extended from the scalar product in $L^2(\Gamma_i)$. Here, $\Gamma_i := \partial\Omega_i$ and $u_i = u|_{\Gamma_i}$, $\lambda_i = \partial u/\partial n|_{\Gamma_i}$ for $i \in \{1, \ldots, p\}$ and $\Gamma_{\mathrm{BE}} := \cup_{i=1}^{p} \Gamma_i$.

Definition 4 (Integral equation). Find $\lambda = (\lambda_1, \ldots, \lambda_p) \in \Lambda$ and $u \in \mathbf{U}_g$ such that, for all $v \in \mathbf{U}_0$, $\eta = (\eta_1, \ldots, \eta_p) \in \Lambda$, the equations

$$\sum_{i=1}^{p} a_i \{ \langle \mathcal{D}_i u_i, v_i \rangle_{\Gamma_i} + \frac{1}{2} \langle \lambda_i, v_i \rangle_{\Gamma_i} + \langle \lambda_i, \mathcal{K}_i v_i \rangle_{\Gamma_i} \} = 0, \tag{10}$$

$$-\frac{a_i}{2} \langle \eta_i, u_i \rangle_{\Gamma_i} - a_i \langle \eta_i, \mathcal{K}_i u_i \rangle_{\Gamma_i} + a_i \langle \eta_i, \mathcal{V}_i \lambda_i \rangle_{\Gamma_i} = 0, \quad i = 1, \ldots, p, \tag{11}$$

are satisfied.

The integral operators $\mathcal{V}_i$, $\mathcal{K}_i$, $\mathcal{D}_i$ denote the single layer potential, the double layer potential and the hypersingular operators, respectively, and are defined by

$$\mathcal{V}_i \lambda_i(x) := \int_{\Gamma_i} \mathcal{E}(x,y) \lambda_i(y)\, ds_y, \quad \mathcal{V}_i \in \mathcal{L}(H^{-1/2}(\Gamma_i); H^{1/2}(\Gamma_i)),$$

$$\mathcal{K}_i v_i(x) := \int_{\Gamma_i} \partial_{n_y} \mathcal{E}(x,y) v_i(y)\, ds_y, \quad \mathcal{K}_i \in \mathcal{L}(H^{1/2}(\Gamma_i); H^{1/2}(\Gamma_i)),$$

$$\mathcal{D}_i u_i(x) := -\partial_{n_x} \int_{\Gamma_i} \partial_{n_y} \mathcal{E}(x,y) u_i(y)\, ds_y,$$

$$\mathcal{D}_i \in \mathcal{L}(H^{1/2}(\Gamma_i); H^{-1/2}(\Gamma_i)).$$

Here, $\mathcal{E}(x,y)$ is the fundamental solution of the differential operator (e.g., $-\frac{1}{2\pi} \log|x-y|$ for the 2d-Laplacian) and ∂_{n_x} denotes the (weak) normal derivative with respect to the variable x. For Banach spaces X and Y we write $\mathcal{L}(X;Y)$ for the Banach space of linear and bounded operators from X to Y endowed with the operator norm. The mapping properties of the boundary integral operators on Sobolev spaces are known from [9].

The equations (10),(11) are equivalently recast to (6) provided the bilinear form a is defined by

$$a(\lambda, u; \eta, v) := \sum_{i=1}^{p} a_i \Big\{ \langle \mathcal{D}_i u_i, v_i \rangle_{\Gamma_i} + \frac{1}{2} \langle \lambda_i, v_i \rangle_{\Gamma_i} + \langle \lambda_i, \mathcal{K}_i v_i \rangle_{\Gamma_i} + \langle \eta_i, \mathcal{V}_i \lambda_i \rangle_{\Gamma_i} - \langle \eta_i, \mathcal{K}_i u_i \rangle_{\Gamma_i} - \frac{1}{2} \langle \eta_i, u_i \rangle_{\Gamma_i} \Big\}, \tag{12}$$

where the space $\mathbf{V} := \Lambda \times \mathbf{U}_0$ is endowed with the norm

$$\|(\lambda, u)\|_{\mathbf{V}} := \left(\|\lambda\|_{\Lambda}^2 + \|u\|_{\mathbf{U}_0}^2 \right)^{1/2}, \quad \text{with} \tag{13}$$

$$\|\lambda\|_{\Lambda}^2 := \sum_{i=1}^{p} \|\lambda_i\|_{H^{-1/2}(\Gamma_i)}^2 \quad \text{and} \quad \|u\|_{\mathbf{U}_0}^2 := \sum_{i=1}^{p} \|u_i\|_{H^{1/2}(\Gamma_i)}^2.$$

Definition 5 (Problem (P)). Find $(\lambda, u) \in \Lambda \times \mathbf{U}_g$ that satisfies (6) for given $g \in H^{1/2}(\Gamma)$.

Theorem 1 (Hsiao, Wendland [25]). *The problems given by Definitions 3–5 are pairwise equivalent. The bilinear form a is $\mathbf{V}$-elliptic and the problems given by Definitions 3–5 have unique solutions.*

3 Galerkin discretization

To describe a Galerkin discretization of Problem (P), we choose finite dimensional subspaces

$$\begin{aligned} \Lambda_h &:= \operatorname{span}\{\psi_1, \psi_2, \ldots, \psi_{N_\Lambda}\}, \\ \mathbf{U}_h &:= \operatorname{span}\{\phi_1, \ldots, \phi_{N_{\mathrm{C}}}\} \end{aligned}$$

of Λ and $\mathbf{U}_0$, respectively, and set

$$\mathbf{V}_h := \Lambda_h \times \mathbf{U}_h.$$

Definition 6 (Problem (P_h)). Find $(\lambda_h, u_h) \in \mathbf{V}_h$ that satisfies

$$a(\lambda_h, u_h; \eta_h, v_h) = f(\eta_h, v_h) \quad \forall (\eta_h, v_h) \in \mathbf{V}_h, \tag{14}$$

for given $f \in \mathbf{V}_h^*$.

Remark 1. In Equation (14), $f(\eta_h, v_h) := -a(0, g; \eta_h, v_h)$, and $g \in \mathbf{U}_g$ is arbitrarily extended from the given data $g|_\Gamma$. Thus, the final approximation is $(\lambda_h, g + u_h) \in \mathbf{\Lambda} \times \mathbf{U}_g$.

To reflect the domain decomposition, we assume a certain order in the basis function. The boundaries $\Gamma_1, \ldots, \Gamma_p$ are split into boundary pieces $\Gamma_{ij} := \bar{\Omega}_i \cap \bar{\Omega}_j$, $1 \le i, j \le p$, which are further partitioned into boundary elements resulting in a full discretization of the skeleton Γ_{BE}. Based upon this discretization we can define the usual nodal BE basis consisting of piecewise polynomial trial functions

$$\Phi := \Psi_\Lambda \cup \Phi_{\mathrm{C}} = [\psi_1, \psi_2, \ldots, \psi_{N_\Lambda}; \phi_1, \ldots, \phi_{N_{\mathrm{C}}}]$$

with $N_\Lambda = \sum_{i=1}^{p} N_{\Lambda,i}$ and

$$\begin{aligned} \Psi_\Lambda &:= \Psi_{\Lambda,1} \cup \cdots \cup \Psi_{\Lambda,p} \\ &= [\psi_1, \ldots, \psi_{N_{\Lambda,1}}] \cup \cdots \cup [\psi_{N_\Lambda - N_{\Lambda,p}+1}, \ldots, \psi_{N_\Lambda}], \\ \Phi_{\mathrm{C}} &:= [\phi_1, \ldots, \phi_{N_{\mathrm{C}}}]. \end{aligned}$$

Utilizing the isomorphism $\Phi : \mathbb{R}^{N_\Lambda+N_\mathrm{C}} \to \mathbf{V}_h$ we are lead to the linear system of equations (which is equivalent to Problem (P_h))

$$\begin{pmatrix} \mathbf{K}_\Lambda & -\mathbf{K}_{\Lambda\mathrm{C}} \\ \mathbf{K}_{\mathrm{C}\Lambda} & \mathbf{K}_\mathrm{C} \end{pmatrix} \begin{pmatrix} \underline{u}_\Lambda \\ \underline{u}_\mathrm{C} \end{pmatrix} = \begin{pmatrix} \underline{f}_\Lambda \\ \underline{f}_\mathrm{C} \end{pmatrix}. \tag{15}$$

The block entries are given by

$$(\mathbf{K}_\Lambda \underline{u}_\Lambda, \underline{v}_\Lambda) := \sum_{i=1}^{p} a_i \langle \eta_i, \mathcal{V}_i \lambda_i \rangle_{\Gamma_i} \quad \text{with } \lambda_i = \Psi_{\Lambda,i}\underline{u}_{\Lambda,i}, \eta_i = \Psi_{\Lambda,i}\underline{v}_{\Lambda,i},$$

$$(\mathbf{K}_{\mathrm{C}\Lambda} \underline{u}_\Lambda, \underline{v}_\mathrm{C}) := \sum_{i=1}^{p} a_i \{ \langle \lambda_i, \mathcal{K}_i v_i \rangle_{\Gamma_i} + \frac{1}{2} \langle \lambda_i, v_i \rangle_{\Gamma_i} \}, \quad \mathbf{K}_{\Lambda\mathrm{C}} := \mathbf{K}_{\mathrm{C}\Lambda}^\mathrm{T},$$

$$(\mathbf{K}_\mathrm{C} \underline{u}_\mathrm{C}, \underline{v}_\mathrm{C}) := \sum_{i=1}^{p} a_i \langle \mathcal{D}_i u_i, v_i \rangle_{\Gamma_i}, \quad \text{with } u_i = \Phi_\mathrm{C}\underline{u}_{\mathrm{C},i},\ v_i = \Phi_\mathrm{C}\underline{v}_{\mathrm{C},i},$$

where the entries of $\underline{u}_{\mathrm{C},i} \in \mathbb{R}^{N_\mathrm{C}}$ (analogously for $\underline{v}_{\mathrm{C},i}$) which act as coefficients for non-zero trial functions on Γ_i are equal to the entries of $\underline{u}_\mathrm{C}$, all other entries of $\underline{u}_{\mathrm{C},i}$ are zero.

Remark 2. Problem (P_h) is equivalent to (15).

Remark 3. In contrast to finite element discretizations, the BE-coefficient matrices are, in general, *not* sparse.

4 Iterative solver and its parallelization

The system matrix of the Galerkin equations (15) is positive definite, but non-symmetric. To solve such systems, one can apply preconditioned Krylov subspace methods (see [37] for a detailed state-of-the-art description). However, either no convergence rate estimates are available or the parallelization of these methods is far from being as efficient as that of the method used in the present paper. Alternatively, multiplying the last block row of (15) by -1 yields an equivalent system with a symmetric, but now indefinite system matrix. Again, there are several iterative methods for solving such systems (see [12, 26, 32, 35]). A comparison of some of the above methods applied to boundary element equations can be found in [42].

Since we are interested in a parallel implementation and in rate estimates, we prefer to use the so-called Bramble-Pasciak transformation [2] in order to transform (15) into the equivalent system

$$\mathbf{M}\underline{u} = \underline{q}, \tag{16}$$

with the symmetric and positive definite (spd) coefficient matrix

$$\mathbf{M} := \begin{pmatrix} \mathbf{K}_\Lambda \mathbf{C}_\Lambda^{-1} \mathbf{K}_\Lambda - \mathbf{K}_\Lambda & -(\mathbf{K}_\Lambda - \mathbf{C}_\Lambda)\, \mathbf{C}_\Lambda^{-1} \mathbf{K}_{\Lambda \mathrm{C}} \\ -\mathbf{K}_{\mathrm{C}\Lambda} \mathbf{C}_\Lambda^{-1} (\mathbf{K}_\Lambda - \mathbf{C}_\Lambda) & \mathbf{K}_{\mathrm{C}\Lambda} \mathbf{C}_\Lambda^{-1} \mathbf{K}_{\Lambda \mathrm{C}} + \mathbf{K}_{\mathrm{C}} \end{pmatrix}.$$

The vectors $\underline{q}$ and $\underline{u}$ are defined as follows

$$\underline{q} := \begin{pmatrix} (\mathbf{K}_\Lambda - \mathbf{C}_\Lambda) \cdot \mathbf{C}_\Lambda^{-1} \underline{f}_\Lambda \\ -\mathbf{K}_{\mathrm{C}\Lambda} \mathbf{C}_\Lambda^{-1} \underline{f}_\Lambda + \underline{f}_{\mathrm{C}} \end{pmatrix} \quad \text{and} \quad \underline{u} := \begin{pmatrix} \underline{u}_\Lambda \\ \underline{u}_{\mathrm{C}} \end{pmatrix},$$

where the spd $(N_\Lambda \times N_\Lambda)$–matrix $\mathbf{C}_\Lambda$ denotes some suitably scaled preconditioner for $\mathbf{K}_\Lambda$, i.e., we assume

$$\underline{\gamma}_\Lambda \cdot \mathbf{C}_\Lambda \le \mathbf{K}_\Lambda \le \overline{\gamma}_\Lambda \cdot \mathbf{C}_\Lambda \quad \text{with } \underline{\gamma}_\Lambda, \overline{\gamma}_\Lambda > 1. \tag{17}$$

Throughout this paper, the relation $\mathbf{A} \le \mathbf{B}$ for spd matrices $\mathbf{A}$ and $\mathbf{B}$ means that $\mathbf{B} - \mathbf{A}$ is positive semi-definite.

The following theorem is equivalent to the results in [2] and gives sufficient conditions for the spectral equivalence of $\mathbf{M}$ with the preconditioner

$$\mathbf{C} := \begin{pmatrix} \mathbf{K}_\Lambda - \mathbf{C}_\Lambda & 0 \\ 0 & \mathbf{C}_{\mathrm{C}} \end{pmatrix}. \tag{18}$$

Theorem 2. *Let $\mathbf{C}_{\mathrm{C}}$ be a spd preconditioner for the BE-Schur-complement $\mathbf{K}_{\mathrm{C}} + \mathbf{K}_{\mathrm{C}\Lambda} \mathbf{K}_\Lambda^{-1} \mathbf{K}_{\Lambda \mathrm{C}}$, i.e., there exist positive constants $\underline{\gamma}_{\mathrm{C}}$ and $\overline{\gamma}_{\mathrm{C}}$ with*

$$\underline{\gamma}_{\mathrm{C}} \cdot \mathbf{C}_{\mathrm{C}} \le \mathbf{K}_{\mathrm{C}} + \mathbf{K}_{\mathrm{C}\Lambda} \mathbf{K}_\Lambda^{-1} \mathbf{K}_{\Lambda \mathrm{C}} \le \overline{\gamma}_{\mathrm{C}} \cdot \mathbf{C}_{\mathrm{C}}. \tag{19}$$

Then the spd preconditioner $\mathbf{C}$ defined in (18) is spectrally equivalent to the matrix $\mathbf{M}$ of the spd system (16), i.e.,

$$\underline{\gamma} \cdot \mathbf{C} \le \mathbf{M} \le \overline{\gamma} \cdot \mathbf{C} \tag{20}$$

with the spectral equivalence constants

$$\begin{aligned} \underline{\gamma} &:= \left(1 + \frac{\alpha}{2} + \sqrt{\alpha + \frac{\alpha^2}{4}}\right)^{-1} \min\{1, \underline{\gamma}_{\mathrm{C}}\}, \\ \overline{\gamma} &:= \frac{1 + \sqrt{\alpha}}{1 - \alpha} \max\{1, \overline{\gamma}_{\mathrm{C}}\}, \end{aligned} \tag{21}$$

where $\alpha := 1 - (1/\overline{\gamma}_\Lambda)$.

We utilize a conjugate gradient (cg) method preconditioned by $\mathbf{C}$, i.e., we apply the cg-iteration to the linear system of equations

$$\mathbf{C}^{-1}\mathbf{M}\underline{u} = \mathbf{C}^{-1}\underline{q}. \tag{22}$$

In the sequel, $\underline{u}$ is the exact solution of (22) and (15) while $\underline{u}^k := (\underline{u}_\Lambda^k, \underline{u}_\mathrm{C}^k)^\mathrm{T}$ denotes the k-th cg-iterate, $\underline{u}^0$ is a given initial vector. The details of the numerical computation of $\underline{u}^{(k)}$ are discussed below. We consider the convergence properties with respect to the energy norm $\|.\|_M^2 := (\mathbf{M}.,.)$.

Theorem 3. *Under the assumptions and notations of Theorem 2 let* $\rho := (1-\sqrt{\xi})/(1+\sqrt{\xi})$ *and* $\xi := \underline{\gamma}/\overline{\gamma}$. *Then,*

$$\|\underline{u} - \underline{u}^k\|_M \leq \frac{2\rho^k}{1+\rho^{2k}} \cdot \|\underline{u} - \underline{u}^0\|_M \,. \tag{23}$$

Proof. The theorem follows from Theorem 2 and standard results on preconditioned conjugate gradient algorithms, see [13, 37, 38] for more details. □

We conclude this section with several remarks on details within a numerical performance of the conjugate gradient method applied to (22) as shown in Table 1.

We define two types of distribution, called overlapping (type 1) and adding (type 2), for the vectors belonging to the inner coupling boundary $\Gamma_\mathrm{C} \subset \Gamma_\mathrm{BE}$. Namely,

Type 1: $\underline{u}_\mathrm{C}, \underline{w}_\mathrm{C}, \underline{s}_\mathrm{C}$ are stored in $P_i : \underline{u}_{\mathrm{C},i} = A_{\mathrm{C},i}\underline{u}_\mathrm{C}$ (analogous $\underline{w}_{\mathrm{C},i}, \underline{s}_{\mathrm{C},i}$),
Type 2: $\underline{r}_\mathrm{C}, \underline{v}_\mathrm{C}, \underline{f}_\mathrm{C}$ are stored in $P_i : \underline{\mathbf{r}}_{\mathrm{C},i}$ such that $\underline{r}_\mathrm{C} = \sum_{i=1}^p A_{\mathrm{C},i}^\mathrm{T}\underline{\mathbf{r}}_{\mathrm{C},i}$ (analogous $\underline{\mathbf{v}}_{\mathrm{C},i}, \underline{\mathbf{f}}_{\mathrm{C},i}$).

Here, P_i stands for the i^th processor and the matrices $A_{\mathrm{C},i}$ are the "C-block" of the Boolean subdomain connectivity matrix A_i which maps some overall vector of nodal parameters into the superelement vector of parameters associated with the subdomain $\bar{\Omega}_i$ only. Then, we introduce the PCG-algorithm with a given accuracy ε as stopping criterion in Table 1, the hat symbol '^' marks the new iterates.

Remark 4. Only those parts of the algorithm which take the full width of the table require communication between the processors; all other parts are performed completely in parallel.

Remark 5. The vectors $\underline{z}_{\Lambda,i}$ and $\underline{h}_{\Lambda,i}$ have been inserted to reduce the number of matrix by vector operations. The vector $\underline{p}_i$ avoids the computation of $\mathbf{C}_{\Lambda,i}\underline{r}_{\Lambda,i}$ which is not necessarily available ($\widetilde{\mathbf{C}}_{\Lambda,i}$ is defined such that the inverse operation $\mathbf{C}_{\Lambda,i}^{-1} \times \underline{w}_{\Lambda,i}$ can be performed easily as in the case of multigrid preconditioners).

Table 1. The pcg-algorithm

For all Ω_i, $i = 1, \ldots, p$
Starting Step
Choose an initial guess $\underline{u} = \underline{u}_0$ $\underline{u}_i = \left[\underline{u}_{\Lambda,i}\ \underline{u}_{\mathrm{C},i}\right]^{\mathrm{T}}$
$\underline{v}_{\Lambda,i} = \underline{f}_{\Lambda,i} - K_{\Lambda,i}\underline{u}_{\Lambda,i} + K_{\Lambda C,i}\underline{u}_{\mathrm{C},i}$ $\underline{r}_{\Lambda,i} = C_{\Lambda,i}^{-1}\underline{v}_{\Lambda,i}$ $\underline{\mathbf{r}}_{\mathrm{C},i} = \underline{\mathbf{f}}_{\mathrm{C},i} - K_{\mathrm{C},i}\underline{u}_{\mathrm{C},i} - K_{\mathrm{C}\Lambda,i}\underline{u}_{\Lambda,i}$ $\underline{\mathbf{r}}_{\mathrm{C},i} = \underline{\mathbf{r}}_{\mathrm{C},i} - K_{\mathrm{C}\Lambda,i}\underline{r}_{\Lambda,i}$
$\underline{w}_{\Lambda,i} = \underline{r}_{\Lambda,i};\ \underline{p}_{\Lambda,i} = \underline{v}_{\Lambda,i}$ $\underline{\mathbf{v}}_{\mathrm{C},i} = \underline{\mathbf{r}}_{\mathrm{C},i};\ \underline{z}_{\Lambda,i} = K_{\Lambda,i}\underline{r}_{\Lambda,i}$
$\underline{w}_{\mathrm{C}} = C_{\mathrm{C}}^{-1}\sum_{i=1}^{p} A_{\mathrm{C},i}^{\mathrm{T}}\underline{\mathbf{v}}_{\mathrm{C},i}$
$\underline{w}_{\mathrm{C},i} = A_{\mathrm{C},i}\underline{w}_{\mathrm{C}};\ \underline{s}_i = \underline{w}_i$
$\sigma_i = \underline{\mathbf{r}}_{\mathrm{C},i}^{\mathrm{T}}\underline{w}_{\mathrm{C},i} + \underline{w}_{\Lambda,i}^{\mathrm{T}}(\underline{z}_{\Lambda,i} - \underline{p}_i)$
$\sigma = \sigma^0 = \sum_{i=1}^{p}\sigma_i$

Iteration
$\underline{w}_{\Lambda,i} = \underline{z}_{\Lambda,i} - K_{\Lambda C,i}\underline{s}_{\mathrm{C},i}$ $\underline{v}_{\Lambda,i} = C_{\Lambda,i}^{-1}\underline{w}_{\Lambda,i}$ $\underline{\mathbf{v}}_{\mathrm{C},i} = K_{\mathrm{C},i}\underline{s}_{\mathrm{C},i} + K_{\mathrm{C}\Lambda,i}(\underline{s}_{\Lambda,i} - \underline{v}_{\Lambda,i})$
$\delta_i = \underline{\mathbf{v}}_{\mathrm{C},i}^{\mathrm{T}}\underline{s}_{\mathrm{C},i} + \underline{v}_{\Lambda,i}^{\mathrm{T}}\underline{z}_{\Lambda,i} - \underline{w}_{\Lambda,i}^{\mathrm{T}}\underline{s}_{\Lambda,i}$
$\delta = \sum_{i=1}^{p}\delta_i;\ \alpha = \sigma/\delta$
$\hat{\underline{u}}_i = \underline{u}_i + \alpha\underline{s}_i$ $\hat{\underline{\mathbf{r}}}_{\mathrm{C},i} = \underline{\mathbf{r}}_{\mathrm{C},i} - \alpha\underline{\mathbf{v}}_{\mathrm{C},i}$ $\hat{\underline{r}}_{\Lambda,i} = \underline{r}_{\Lambda,i} - \alpha\underline{v}_{\Lambda,i}$ $\hat{\underline{\mathbf{v}}}_{\mathrm{C},i} = \hat{\underline{\mathbf{r}}}_{\mathrm{C},i};\ \hat{\underline{h}}_{\Lambda,i} = K_{\Lambda,i}\hat{\underline{r}}_{\Lambda,i}$
$\underline{w}_{\mathrm{C}} = C_{\mathrm{C}}^{-1}\sum_{i=1}^{p} A_{\mathrm{C},i}^{\mathrm{T}}\underline{\mathbf{v}}_{\mathrm{C},i}$
$\underline{w}_{\mathrm{C},i} = A_{\mathrm{C},i}\underline{w}_{\mathrm{C}}$ $\hat{\underline{w}}_{\Lambda,i} = \hat{\underline{r}}_{\Lambda,i};\ \hat{\underline{p}}_{\Lambda,i} = \underline{p}_{\Lambda,i} - \alpha\hat{\underline{w}}_{\Lambda,i}$
$\sigma_i = \hat{\underline{\mathbf{r}}}_{\mathrm{C},i}^{\mathrm{T}}\hat{\underline{w}}_{\mathrm{C},i} + \underline{h}_{\Lambda,i}^{\mathrm{T}}\hat{\underline{r}}_{\Lambda,i} - \hat{\underline{p}}_{\Lambda,i}^{\mathrm{T}}\hat{\underline{r}}_{\Lambda,i}$
$\hat{\sigma} = \sum_{i=1}^{p}\sigma_i;\ \beta = \hat{\sigma}/\sigma$
$\hat{\underline{s}}_i = \hat{\underline{w}}_i + \beta\underline{s}_i;\ \hat{\underline{z}}_{\Lambda,i} = \underline{h}_{\Lambda,i} + \beta\underline{z}_{\Lambda,i}$
If $\hat{\sigma} \le \epsilon^2 * \sigma^0$, then STOP else start next iteration.

5 Preconditioning

In order to construct BE-Schur-complement preconditioners $\mathbf{C}_{\mathrm{C}}$ with the property (19) we prove in this section that the BE-Schur-complement norm is equivalent to some weighted $H^{1/2}$-norm with h-independent equivalence constants. The first two subsections treat this question on a quite general level and follow partly arguments from [6,8]. The third subsection collects some properties of the Poincaré-Steklov operator. Then, we consider the particular case at hand in the fourth subsection. Subsect. 5.5 is devoted to two quite efficient realizations of $\mathbf{C}_{\mathrm{C}}$ on the basis of multilevel and multigrid ideas. In Subsect. 5.6, we briefly discuss two preconditioners for the discrete single layer potential operator and their appropriate scaling. The final subsection summarizes the results.

5.1 Preliminaries I: Spectral equivalence of Schur complements

This section presents some auxiliary results for the spectral equivalence utilized in the subsequent subsections to construct optimal preconditioners for the BE-Schur-complement.

We use the following general notation. Let X and Y be reflexive Banach spaces and let $U \subset X$ be a closed subspace with the canonical embedding $i_U : U \hookrightarrow X$ and its dual $i_U^* : X^* \hookrightarrow U^*$.

We consider $\mathcal{A} \in \mathcal{L}(X; X^*)$ and $\mathcal{B} \in \mathcal{L}(Y; X^*)$ and define

$$\mathcal{A}_U := i_U^* \mathcal{A} i_U \in \mathcal{L}(U; U^*). \tag{24}$$

We assume that $\mathcal{A}$ is symmetric and X-elliptic, i.e., there exists $\alpha > 0$ such that for all $x, y \in X$

$$(\mathcal{A}x)(y) = (\mathcal{A}y)(x) \quad \text{and} \quad (\mathcal{A}x)(x) \geq \alpha \cdot \|x\|_X^2. \tag{25}$$

The Lax-Milgram lemma applies to $\mathcal{A}$ and $\mathcal{A}_U$ and shows that $\mathcal{A}^{-1}$ and $\mathcal{A}_U^{-1}$ exist and are bounded. Moreover, we have the following estimate.

Lemma 1. *There holds*

$$(\mathcal{B}^* i_U \mathcal{A}_U^{-1} i_U^* \mathcal{B} y)(y) \leq (\mathcal{B}^* \mathcal{A}^{-1} \mathcal{B} y)(y)\,. \tag{26}$$

Proof. Given $y \in Y$ define $f := \mathcal{B}y$, $f_U := i_U^* f$, $x := \mathcal{A}^{-1} f$, and $x_U := \mathcal{A}_U^{-1} f_U$. The so-called Galerkin-orthogonality states, for all $u \in U \subset X$, that $(\mathcal{A}(x - x_U))(u) = (f - f_U)(u) = 0$. According to this we compute

$$\begin{aligned}(\mathcal{B}^* \mathcal{A}^{-1} \mathcal{B} y)(y) - (\mathcal{B}^* i_U \mathcal{A}_U^{-1} i_U^* \mathcal{B} y)(y) &= f(x - x_U) = (\mathcal{A}x)(x - x_U) \\ &= (\mathcal{A}(x - x_U))(x) = (\mathcal{A}(x - x_U))(x - x_U) \geq 0\end{aligned}$$

because of $x_U \in U$ and (25). □

The next result is closely related to the construction of norms well-known in the context of the Bramble-Hilbert lemma.

Let $\mathcal{C}, \mathcal{D} \in \mathcal{L}(Y; Y^*)$ and $\gamma, \delta > 0$ satisfy for all $y \in Y$ and $w \in \ker \mathcal{C}$

$$\begin{aligned}&(\mathcal{C}y)(y) \geq \gamma \cdot \inf_{c \in \ker \mathcal{C}} \|y - c\|_Y^2\,,\ (\mathcal{D}y)(y) \geq 0 \\ &\text{and, } (\mathcal{D}w)(w) \geq \delta \cdot \|w\|_Y^2\,.\end{aligned} \tag{27}$$

Furthermore, assume that $\mathcal{D}$ is symmetric, i.e., $(\mathcal{D}u)(v) = (\mathcal{D}v)(u), \forall u, v \in Y$.

Lemma 2. *Let $\mathcal{C}, \mathcal{D}$ satisfy (27). Then, for $\beta := \frac{1}{4}\Big\{\delta + \gamma + \|\mathcal{D}\|_{\mathcal{L}(Y;Y^*)} - \Big((\delta + \gamma + \|\mathcal{D}\|_{\mathcal{L}(Y;Y^*)})^2 - 4\delta\gamma\Big)^{1/2}\Big\}$ and all $y \in Y$ we have $\beta > 0$ and*

$$((\mathcal{C} + \mathcal{D})(y))(y) \geq \beta \cdot \|y\|_Y^2. \tag{28}$$

Proof. Given $y \in Y$ and $\epsilon > 0$ we find $w \in \ker \mathcal{C}$ with $\inf_{c\in\ker\mathcal{C}} \|y-c\|_Y^2 \geq \|y-w\|_Y^2 - \epsilon/\gamma$ and define $z := y - w$. According to (27) and Cauchy's inequality, we find, for all $\lambda \in (0,1)$,

$$\begin{aligned}
&((\mathcal{C}+\mathcal{D})y)(y) + \epsilon \geq \gamma\cdot\|z\|_Y^2 + (\mathcal{D}(w+z))(w+z)\\
&= \gamma\cdot\|z\|_Y^2 + (\mathcal{D}w)(w) + 2(\mathcal{D}z)(w) + (\mathcal{D}z)(z)\\
&\geq \gamma\cdot\|z\|_Y^2 + (\mathcal{D}w)(w) - \lambda\cdot(\mathcal{D}w)(w) - \lambda^{-1}\cdot(\mathcal{D}z)(z) + (\mathcal{D}z)(z)\\
&\geq \gamma\cdot\|z\|_Y^2 + (1-\lambda)\cdot\delta\cdot\|w\|_Y^2 - (\lambda^{-1}-1)\cdot(\mathcal{D}z)(z)\\
&\geq \Big(\gamma - (\lambda^{-1}-1)\cdot\|\mathcal{D}\|_{\mathcal{L}(Y;Y^*)}\Big)\cdot\|z\|_Y^2 + (1-\lambda)\cdot\delta\cdot\|w\|_Y^2\\
&\geq 2\beta\cdot(\|z\|_Y^2 + \|w\|_Y^2)\\
&\geq \beta\cdot\|z+w\|_Y^2 = \beta\cdot\|y\|_Y^2,
\end{aligned}$$

where we set $\beta := \frac{1}{2}\min\{\Big(\gamma - (\lambda^{-1}-1)\cdot\|\mathcal{D}\|_{\mathcal{L}(Y;Y^*)}\Big), (1-\lambda)\cdot\delta\}$. Finally, the lemma follows by choosing λ as the positive root of $\lambda^2 + \frac{1}{\delta}(\gamma + \|\mathcal{D}\|_{\mathcal{L}(Y;Y^*)} - \delta)\cdot\lambda - \|\mathcal{D}\|_{\mathcal{L}(Y;Y^*)}/\delta$ which belongs to $(0,1)$ and letting $\epsilon \to 0$. □

The preceding two lemmas imply the following estimate where we assume that

$$\inf_{c\in\ker\mathcal{C}\setminus\{0\}} \frac{(\mathcal{B}^* i_U \mathcal{A}_U^{-1} i_U^* \mathcal{B}c)(c)}{\|c\|_Y^2} =: \delta_U > 0 \tag{29}$$

and define

$$\begin{aligned}
\beta := \frac{1}{4}\Big\{&\delta_U + \gamma + \|\mathcal{B}^* i_U \mathcal{A}_U^{-1} i_U^* \mathcal{B}\|_{\mathcal{L}(Y;Y^*)}\\
&- \Big((\delta_U + \gamma + \|B^* i_U A_U^{-1} i_U^* \mathcal{B}\|_{\mathcal{L}(Y;Y^*)})^2 - 4\delta\gamma\Big)^{1/2}\Big\} > 0.
\end{aligned} \tag{30}$$

Theorem 4. *For all $y \in Y$, we have*

$$\begin{aligned}
&\beta\cdot\|y\|_Y^2 \leq ((\mathcal{C} + \mathcal{B}^* i_U \mathcal{A}_U^{-1} i_U^* \mathcal{B})y)(y)\\
&\quad\leq ((\mathcal{C} + \mathcal{B}^* \mathcal{A}^{-1} \mathcal{B})y)(y) \leq \|\mathcal{C} + \mathcal{B}^* \mathcal{A}^{-1} \mathcal{B}\|_{\mathcal{L}(Y;Y^*)}\cdot\|y\|_Y^2.
\end{aligned} \tag{31}$$

Proof. The first inequality follows from Lemma 2 with $\mathcal{D} := \mathcal{B}^* i_U \mathcal{A}_U^{-1} i_U^* \mathcal{B}$, the second from Lemma 1 and the last inequality in (31) is the continuity of $\mathcal{C} + \mathcal{B}^* \mathcal{A}^{-1} \mathcal{B}$. □

Remark 6. The condition (29) follows from $\inf_{c\in\ker\mathcal{C}\setminus\{0\}} \frac{\|i_U^*\mathcal{B}c\|_{U^*}}{\|c\|_Y} =: \delta_U' > 0$ and (25) with $\delta_U \geq (\delta_U')^2/\alpha > 0$.

Remark 7. The operator norm $\mathcal{B}^* i_U \mathcal{A}_U^{-1} i_U^* \mathcal{B}$ in condition (30) is bounded by $\|\mathcal{B}\|_{\mathcal{L}(X;Y^*)}^2/\alpha$. For example,

$$\|\mathcal{C} + \mathcal{B}^* \mathcal{A}^{-1} \mathcal{B}\|_{\mathcal{L}(Y;Y^*)} \leq \|\mathcal{C}\|_{\mathcal{L}(Y;Y^*)} + \|\mathcal{B}\|_{\mathcal{L}(X;Y^*)}^2/\alpha. \tag{32}$$

5.2 Preliminaries II: Matrix equivalence of Schur complements

A computer realization of the Schur complement $\mathcal{C} + \mathcal{B}^*\mathcal{A}^{-1}\mathcal{B}$ requires a basis of X and Y, which are assumed to be finite dimensional in this subsection. We prove that the stiffness matrix of the Schur complement $\mathcal{C} + \mathcal{B}^*\mathcal{A}^{-1}\mathcal{B}$ is the Schur complement of the corresponding stiffness matrices.

Throughout this section we consider

$$X = \operatorname{span}\{\chi_1, \dots, \chi_M\} \quad \text{and} \quad Y = \operatorname{span}\{\eta_1, \dots, \eta_N\} \tag{33}$$

and operators $\mathcal{A} \in \mathcal{L}(X; X^*)$ with bounded inverse $\mathcal{A}^{-1} \in \mathcal{L}(X^*; X)$, $\mathcal{B} \in \mathcal{L}(Y; X^*)$, $\mathcal{C} \in \mathcal{L}(Y; Y^*)$, and define $\mathcal{S} := \mathcal{C} + \mathcal{B}^*\mathcal{A}^{-1}\mathcal{B}$. We assume that $(\chi_1, \dots, \chi_M)$ is a basis of X and $(\eta_1, \dots, \eta_N)$ is a basis of Y and define the corresponding stiffness matrices by

$$\mathbf{A}_{i,j} := (\mathcal{A}\chi_j)(\chi_i) \qquad (i, j = 1, \dots, M), \tag{34}$$

$$\mathbf{B}_{i,j} := (\mathcal{B}\eta_j)(\chi_i) \qquad (i = 1, \dots, M; j = 1, \dots, N), \tag{35}$$

$$\mathbf{C}_{i,j} := (\mathcal{C}\eta_j)(\eta_i) \qquad (i, j = 1, \dots, N), \tag{36}$$

$$\mathbf{S}_{i,j} := (\mathcal{S}\eta_j)(\eta_i) \qquad (i, j = 1, \dots, N). \tag{37}$$

Theorem 5. *The stiffness matrix* $\mathbf{S}$ *of the Schur complement* $\mathcal{S}$ *is the Schur complement of the stiffness matrices, i.e.,*

$$\mathbf{S} = \mathbf{C} + \mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{B}, \tag{38}$$

under the assumptions made above.

Proof. Given $i, j \in \{1, \dots, N\}$, it is sufficient to prove that $((\mathcal{B}^*\mathcal{A}^{-1}\mathcal{B})\eta_j) \cdot(\eta_i) = (\mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{B})_{i,j}$. Let $b_j := \mathcal{B}\eta_j \in X^*$ and $x_j := \mathcal{A}^{-1}b_j \in X$. If $\chi_1^*, \dots, \chi_M^*$ denotes a basis of X^* which is dual to $\chi_1, \dots, \chi_M$, i.e., $\chi_i^*(\chi_j) = \delta_{ij}$ (Kronecker's δ), then we find coefficients $\xi = (\xi_1, \dots, \xi_M)^{\mathrm{T}}$ and $\zeta = (\zeta_1, \dots, \zeta_M)^{\mathrm{T}}$ with

$$x_j = \xi_1\chi_1 + \cdots + \xi_M\chi_M \quad \text{and} \quad b_j = \zeta_1\chi_1^* + \cdots + \zeta_M\chi_M^*.$$

Note that $\zeta_\ell := b_j(\chi_\ell) = \mathbf{B}_{\ell,j}$ $(j = 1, \dots, M)$. From $\mathcal{A}^{-1}b_j = \sum_{k=1}^M \xi_k\chi_k$ we conclude $b_j = \sum_{k=1}^M \xi_k\mathcal{A}\chi_k$ and so

$$\zeta_\ell = \sum_{k=1}^{M} \mathbf{A}_{\ell,k}\xi_k \qquad (\ell = 1, \dots, M).$$

Since $\mathcal{A}^{-1}$ exists, $\mathbf{A}$ is regular and we infer

$$\xi_k = (\mathbf{A}^{-1}\zeta)_k = (\mathbf{A}^{-1}\mathbf{B})_{k,j} \qquad (k = 1, \dots, M).$$

Using this we compute

$$\begin{aligned}(\mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{B})_{i,j} = (\mathbf{B}^{\mathrm{T}}\xi)_i &= \sum_{k=1}^{M} \mathbf{B}_{k,i}\xi_k = \sum_{k=1}^{M} (\mathcal{B}\eta_i)(\chi_k)\xi_k \\ &= (\mathcal{B}\eta_i)(x_j) = (\mathcal{B}^* x_j)(\eta_i) \\ &= (\mathcal{B}^*\mathcal{A}^{-1}\mathcal{B}\eta_j)(\eta_i)\end{aligned}$$

which concludes the proof. □

5.3 Preliminaries III: The Poincaré-Steklov operator

In this subsection we summarize some known properties of the Poincaré-Steklov operator only for convenient reading.

Definition 7. Assuming that the conformal radius of Γ_i is smaller than 1, we define the Poincaré-Steklov operator

$$\mathcal{S}_i := \mathcal{D}_i + (\mathcal{K}_i^* + \tfrac{1}{2})\mathcal{V}_i^{-1}(\mathcal{K}_i + \tfrac{1}{2}). \tag{39}$$

Lemma 3. *The Poincaré-Steklov operator $\mathcal{S}_i \in \mathcal{L}(H^{1/2}(\Gamma_i); H^{-1/2}(\Gamma_i))$ is well-defined, symmetric, positive semi-definite and $H^{1/2}(\Gamma_i)/\mathbb{R}$-elliptic.*

Proof. As it is well-known, the single-layer potential $\mathcal{V}_i$ is positive definite under the restriction on Γ_i. Hence the Poincaré-Steklov operator $\mathcal{S}_i$ is well-defined. Then, the assertions follow from the mapping properties of the boundary integral operators involved [9]. Finally, the definiteness is a consequence of Lemma 4. □

Lemma 4. *The Poincaré-Steklov operator $\mathcal{S}_i$ acts as a Dirichlet-Neumann map, i.e., given data $v \in H^{1/2}(\Gamma_i)$ there exists exactly one $u \in H^1(\Omega_i)$ with $\Delta u = 0$ and $u|_{\Gamma_i} = v$ and its Neumann data are $Sv = \partial_n u|_{\Gamma_i}$.*

Proof. The assertion is well-known to the experts, so we only sketch the proof. The representation formula

$$u(x) = -\int_{\Gamma_i} \partial_{n_y}\mathcal{E}(x,y)v(y)\,ds_y + \int_{\Gamma_i} \mathcal{E}(x,y)\partial_n u(y)\,ds_y \qquad (x \in \Omega_i)$$

describes the solution $u \in H^1(\Omega_i)$ to $\Delta u = 0$ and $u|_{\Gamma_i} = v$ in Ω_i. Letting $x \to \Gamma_i$ in the representation formula and taking the normal derivative we compute the Cauchy data of u, i.e.,

$$v = u|_{\Gamma_i} = (\tfrac{1}{2} - \mathcal{K}_i)v + \mathcal{V}_i\partial_n u, \tag{40}$$

$$\partial_n u = \mathcal{D}_i v + (\tfrac{1}{2} + \mathcal{K}_i^*)\partial_n u, \tag{41}$$

almost everywhere on Γ_i. We emphasize that the classical jump relations are used here and we assume that Γ_i is piecewise smooth [10]. From the first equation we obtain $\partial_n u = \mathcal{V}_i^{-1}(\mathcal{K}_i + \frac{1}{2})v$ which we insert in the right-hand side of the second equation. This results in $\partial_n u = \mathcal{S}_i v$ as claimed. □

Lemma 5. *For all $u_i \in H^1(\Omega_i)$ with $\Delta u_i = 0$ there holds*

$$\langle \mathcal{S}_i u_i|_{\Gamma_i}, u_i|_{\Gamma_i}\rangle_{\Gamma_i} = \int_{\Omega_i} |\nabla u_i|^2 \, dx. \tag{42}$$

Proof. Since $\Delta u_i = 0$, integration by parts shows that $\int_{\Gamma_i} u_i|_{\Gamma_i} \partial_n u|_{\Gamma_i} \, ds = \|\nabla u_i\|_{L^2(\Omega_i)}$ so that Lemma 4 proves (42). □

The representation (42) gives rise to consider the energy norm on $\mathbf{U}_0$.

Definition 8. For $u \in \mathbf{U}_0$ let

$$\|u\|_a := \Big(\sum_{i=1}^{p} a_i \cdot \langle \mathcal{S}_i u_i|_{\Gamma_i}, u_i|_{\Gamma_i}\rangle_{\Gamma_i}\Big)^{1/2}. \tag{43}$$

Remark 8. Each $u \in \mathbf{U}_0$ can be extended to $E_1 u, E_2 u$ in $H_0^1(\Omega)$ by defining $\tilde{u}_i := E_1 u|_{\Omega_i}, \hat{u}_i := E_2 u|_{\Omega_i} \in H^1(\Omega_i)$ as the unique (weak) solution of the Dirichlet problems

$$-a_i \Delta \tilde{u}_i = 0 \quad \text{respectively} \quad -\Delta \hat{u}_i + \hat{u}_i = 0 \tag{44}$$

with boundary data $\hat{u}_i|_{\Gamma_i} = u|_{\Gamma_i} = \tilde{u}_i|_{\Gamma_i}$ Then, the piecewise defined functions fit together and we have indeed $E_1 u$, $E_2 u \in H_0^1(\Omega)$. Moreover, the equations in (44) are strong forms of Euler-Lagrange equations which correspond to minimization of norms. Indeed,

$$\|a^{1/2}\nabla(E_1 u)\|_{L^2(\Omega)} = \|u\|_a \quad \text{and} \quad \|E_2 u\|_{H_0^1(\Omega)} = \|u\|_{\mathbf{U}_0}. \tag{45}$$

This explains the name energy-norm for $\|\cdot\|_a$ and links to the equivalence of the variational problem (Definition 3) to the integral equation (Definition 4).

Remark 9. The norms $\|\cdot\|_{\mathbf{U}_0}$ and $\|\cdot\|_a$ are equivalent, but the equivalence constants depend on a_i and Ω_i as well as on p.

5.4 Spectral equivalences of the BE-Schur-complement

Let us adopt the notations from the previous sections.

Theorem 6. *If all p subdomains Ω_i can be mapped to a fixed (p-independent) number of normalized master domains by scaling, then there exist a positive, h- and p-independent constant c_1 such that the equivalence inequalities*

$$c_1 \cdot \|u_h\|_a^2 \le \underline{u}_{\mathrm{C}}^{\mathrm{T}} (\mathbf{K}_{\mathrm{C}} + \mathbf{K}_{\mathrm{C}\Lambda}\mathbf{K}_{\Lambda}^{-1}\mathbf{K}_{\Lambda\mathrm{C}})\underline{u}_{\mathrm{C}} \le \|u_h\|_a^2 \quad \textit{and} \tag{46}$$

$$\underline{u}_{\mathrm{C}}^{\mathrm{T}}\mathbf{K}_{\mathrm{C}}\underline{u}_{\mathrm{C}} \le \underline{u}_{\mathrm{C}}^{\mathrm{T}} (\mathbf{K}_{\mathrm{C}} + \mathbf{K}_{\mathrm{C}\Lambda}\mathbf{K}_{\Lambda}^{-1}\mathbf{K}_{\Lambda\mathrm{C}})\underline{u}_{\mathrm{C}} \le c_1^{-1} \cdot \underline{u}_{\mathrm{C}}^{\mathrm{T}}\mathbf{K}_{\mathrm{C}}\underline{u}_{\mathrm{C}} \tag{47}$$

hold for all $\underline{u}_{\mathrm{C}} \in \mathbb{R}^{N_{\mathrm{C}}}$ with $u_h = \Phi\underline{u}_{\mathrm{C}}$.

Proof. By considering Subsect. 5.2 with $X = X_i = \mathrm{span}\{\lambda|_{\Gamma_i} : \lambda \in \mathbf{\Lambda}_h\}$, $Y = Y_i = \mathbf{U}_h$, and $\mathcal{A} := j_{X_i}^* \mathcal{V}_i j_{X_i}$ with the canonical embedding $j_{X_i} : X_i \hookrightarrow H^{-1/2}(\Gamma_i)$ etc., we finally conclude that

$$\begin{aligned} &\underline{u}_{\mathrm{C}}^{\mathrm{T}}(\mathbf{K}_{\mathrm{C}} + \mathbf{K}_{\mathrm{C}\Lambda}\mathbf{K}_{\Lambda}^{-1}\mathbf{K}_{\Lambda\mathrm{C}})\underline{u}_{\mathrm{C}} \\ &= \sum_{i=1}^{p} a_i \left\langle (\mathcal{D}_i + (\mathcal{K}_i^* + \tfrac{1}{2})j_{X_i}(j_{X_i}^*\mathcal{V}_i j_{X_i})^{-1} \right. \\ &\qquad \left. \times j_{X_i}^*(\mathcal{K}_i + \tfrac{1}{2}))u_h|_{\Gamma_i}, u_h|_{\Gamma_i} \right\rangle_{\Gamma_i} . \end{aligned} \tag{48}$$

Utilizing Lemma 1, we infer

$$\begin{aligned} 0 &\le \left\langle (\mathcal{K}_i^* + \tfrac{1}{2})j_{X_i}(j_{X_i}^*\mathcal{V}_i j_{X_i})^{-1} j_{X_i}^*(\mathcal{K}_i + \tfrac{1}{2})u_h|_{\Gamma_i}, u_h|_{\Gamma_i} \right\rangle_{\Gamma_i} \\ &\le \left\langle (\mathcal{K}_i^* + \tfrac{1}{2})\mathcal{V}_i^{-1}(\mathcal{K}_i + \tfrac{1}{2}))u_h|_{\Gamma_i}, u_h|_{\Gamma_i} \right\rangle_{\Gamma_i} \end{aligned} \tag{49}$$

and obtain

$$\begin{aligned} &\left\langle (\mathcal{D}_i + (\mathcal{K}_i^* + \tfrac{1}{2})j_{X_i}(j_{X_i}^*\mathcal{V}_i j_{X_i})^{-1} j_{X_i}^*(\mathcal{K}_i + \tfrac{1}{2}))u_h|_{\Gamma_i}, u_h|_{\Gamma_i} \right\rangle_{\Gamma_i} \\ &\qquad \le \langle \mathcal{S}_i u_h|_{\Gamma_i}, u_h|_{\Gamma_i} \rangle_{\Gamma_i} . \end{aligned} \tag{50}$$

This proves the second inequality in (46).

To prove the remaining first inequality, we consider (48) and, by (49), observe that it suffices to prove

$$\langle \mathcal{S}_i u_i|_{\Gamma_i}, u_i|_{\Gamma_i} \rangle_{\Gamma_i} \le c_1^{-1} \cdot \langle \mathcal{D}_i u_i|_{\Gamma_i}, u_i|_{\Gamma_i} \rangle_{\Gamma_i}. \tag{51}$$

It is known that the hypersingular operator $\mathcal{D}_i$ as the Poincaré-Steklov operator $\mathcal{S}_i$ lead to scalar products on $H^{1/2}(\Gamma_i)/\mathbb{R}$ which are equivalent to the norm in $H^{1/2}(\Gamma_i)/\mathbb{R}$. However, the equivalence constants depend on the boundary Γ_i. But a scaling argument, especially the invariance of the $H^{1/2}(\Gamma_i)/\mathbb{R}$-norms induced by $\mathcal{D}_i$ and $\mathcal{S}_i$ with respect to the scaling of the Ω_i's, shows that the equivalence constant c_1 depends only on the shape of the Ω_i's and is independent upon p or h. This verifies (46).

The first inequality in (47) is obvious. To prove the second inequality in (47) we conclude from (48), (49) and (51) that

$$\begin{aligned}
&\underline{u}_{\mathrm{C}}^{\mathrm{T}}(\mathbf{K}_{\mathrm{C}} + \mathbf{K}_{\mathrm{C}\Lambda}\mathbf{K}_{\Lambda}^{-1}\mathbf{K}_{\Lambda\mathrm{C}})\,\underline{u}_{\mathrm{C}} \\
&= \sum_{i=1}^{p} \underline{u}_{\mathrm{C},i}^{\mathrm{T}}(\mathbf{K}_{\mathrm{C},i} + \mathbf{K}_{\mathrm{C}\Lambda,i}\mathbf{K}_{\Lambda,i}^{-1}\mathbf{K}_{\Lambda C,i})\,\underline{u}_{\mathrm{C},i} \\
&\le \sum_{i=1}^{p} a_i \,\langle \mathcal{S}_i u_h|_{\Gamma_i}, u_h|_{\Gamma_i}\rangle_{\Gamma_i} \le c_1^{-1} \sum_{i=1}^{p} \langle \mathcal{D}_i u_h|_{\Gamma_i}, u_h|_{\Gamma_i}\rangle_{\Gamma_i} \\
&= c_1^{-1} \cdot \underline{u}_{\mathrm{C}}^{\mathrm{T}}\mathbf{K}_{\mathrm{C}}\,\underline{u}_{\mathrm{C}}, \qquad (52)
\end{aligned}$$

which gives (47). □

Remark 10. The constant c_1 in (39) can be sharpened. If the underlying grid in X is the coarsest grid under consideration then only

$$\begin{aligned}
&c_1 \cdot \langle \mathcal{S}_i u_i|_{\Gamma_i}, u_i|_{\Gamma_i}\rangle_{\Gamma_i} \\
&\le \langle \mathcal{D}_i + (\mathcal{K}_i^* + \tfrac{1}{2}) j_{X_i}(j_{X_i}^* \mathcal{V}_i j_{X_i})^{-1} j_{X_i}^* (\mathcal{K}_i + \tfrac{1}{2}))u_h|_{\Gamma_i}, u_h|_{\Gamma_i}\rangle_{\Gamma_i}
\end{aligned}$$

is required.

Remark 11. For $p = 1$ we recover a result in [39]. But, here we extended it for *any* discretization (not only for piecewise linears as in [39]).

We extend the results to the case that one domain is infinite as, for example, in far field computations. Assume the following model situation where Ω_1 is a fixed unbounded domain, say the complement of a compact region as is Ω and we adopt the remaining notations from Sect. 1 (and add a radiation condition to the problem in Ω_1). We assume that boundaries $\Gamma_1, \ldots, \Gamma_p$, have diameters smaller than one. In this situation the analog of Theorem 6 holds.

Theorem 7. *There exist a constant $c_1 > 0$ such that for all $u_h \in \mathbf{U}_h$ with coefficient vector $\underline{u}_{\mathrm{C}} = \Phi^{-1} u_h$ there holds (46). Moreover, the Poincaré-Steklov with respect to the unbounded domain Ω_1 is positive definite on $H^{1/2}(\Gamma_1)$; its approximations are uniformly positive definite.*

Proof. The proof is analogous to the proof of Theorem 6 so we focus on the positive definiteness of $\mathcal{S}_1$ and only emphasize the difference to the above arguments.

From (40) and (41) we conclude $\mathcal{K}_i 1 = -1/2$ for $i \ge 2$ (take the Cauchy data of the constant function). Here, Ω_i is the bounded component, the normal points into the unbounded component of $\mathbb{R}^2 \setminus \Gamma_i$. The situation is different with $\mathcal{K}_1$ because the normal on Γ_1 that is used in the definition of $\mathcal{K}_1$ points into the bounded component of $\mathbb{R}^2 \setminus \Gamma_1$ and so has a different

sign. Therefore, we conclude $\mathcal{K}_1 1 = 1/2$. Note that the change of the normal affects only the double layer potential operator and its dual but not the single layer potential or hypersingular operator.

Now, the kernel of $\mathrm{C} = \mathcal{D}_1$ is $\mathbb{R}$ and Theorem 4 applies to $\mathcal{B} := \mathcal{K}_1 + \frac{1}{2}$ and $\mathcal{A} := \mathcal{V}_1$ because of $\mathcal{B}1 = 1 \in X_h$. This concludes the proof. □

5.5 BE-Schur-complement preconditioners $\mathbf{C}_\mathrm{C}$

We propose to use

$$\begin{aligned} \mathbf{C}_\mathrm{C}^{(1)} &:= \text{S–BPX} \quad \text{(Schur-complement BPX)} \quad \text{and} \\ \mathbf{C}_\mathrm{C}^{(2)} &:= \mathbf{K}_\mathrm{C}\,(\mathbf{I}_\mathrm{C} - \mathbf{M}_\mathrm{C})^{-1} \quad \text{(multigrid preconditioner)} \end{aligned} \tag{53}$$

as two quite efficient realizations for $\mathbf{C}_\mathrm{C}$. The Schur-complement BPX preconditioner (S-BPX) and its implementation is well known [43]. The S–BPX generalizes the hierarchical basis Schur-complement preconditioner studied earlier in [18, 40] to the multilevel "basis" case. The multigrid (mg) preconditioner $\mathbf{C}_\mathrm{C}^{(2)}$ is defined by the iteration operator $\mathbf{M}_\mathrm{C}$ of the symmetric multigrid method used. The action $(\mathbf{C}_\mathrm{C}^{(2)})^{-1} \times \underline{d}_\mathrm{C}$ means the application of one symmetric multigrid cycle (e.g., one symmetric W–cycle) to the defect system $\mathbf{K}_\mathrm{C}\underline{w}_\mathrm{C} = \underline{d}_\mathrm{C}$ with the initial guess $\underline{w}_\mathrm{C}^0 = 0$ (see [27] for details), where $\mathbf{K}_\mathrm{C}$ results from the globally assembled boundary element discretization of the hypersingular integral operator which is symmetric and positive definite after implementing the Dirichlet boundary conditions. The mg preconditioner is spd provided that the presmoothing error operators are adjoint to the postsmoothing error operators in the $\mathbf{K}_\mathrm{C}$–energy inner product and the restriction matrices are transposed to the interpolation matrices. Further we assume that the coarse grid matrices are obtained by Galerkin projection and systems arising on the coarsest grid are solved by some direct method. Under these assumptions, the multigrid iteration operator $\mathbf{M}_\mathrm{C}$ is selfadjoint and non-negative with respect to the $\mathbf{K}_\mathrm{C}$–energy inner product [27]. In the numerical experiments, we use a symmetric V–cycle with 1 Jacobi presmoothing step and 1 Jacobi postsmoothing step (see Sect. 6).

Lemma 6. *Under the assumptions made above, there exist h-independent constants* $\underline{\gamma}_\mathrm{C}$ *and* $\overline{\gamma}_\mathrm{C}$ *such that (19) is valid for either choice of* $\mathbf{C}_\mathrm{C}$.

Proof. The equivalence of $\underline{u}_\mathrm{C}^\mathrm{T}\mathbf{C}_\mathrm{C}^{(1)}\underline{u}_\mathrm{C}$ to $\|u_h\|_a^2$ follows from [31]. This equivalence and inequalities (46) prove the spectral equivalence between the S–BPX preconditioner $\mathbf{C}_\mathrm{C}^{(1)}$ and the BE–Schur-Complement $\mathbf{K}_\mathrm{C} + \mathbf{K}_{\mathrm{C}\Lambda}\mathbf{K}_\Lambda^{-1}\mathbf{K}_{\Lambda\mathrm{C}}$. From [27] we conclude the spectral equivalence inequalities

$$(1-\eta)\cdot\mathbf{K}_\mathrm{C}\,(\mathbf{I}_\mathrm{C} - \mathbf{M}_\mathrm{C})^{-1} \;\le\; \mathbf{K}_\mathrm{C} \;\le\; \mathbf{K}_\mathrm{C}\,(\mathbf{I}_\mathrm{C} - \mathbf{M}_\mathrm{C})^{-1}, \tag{54}$$

where η is the spectralradius of the multigrid iteration operator $\mathbf{M}_{\mathrm{C}}$ that can be supposed to be independent of h [22,33]. Combining inequalities (47) and (54), we arrive at the spectral equivalence inequalities (19) with $\underline{\gamma}_{\mathrm{C}} = 1 - \eta$ and $\overline{\gamma}_{\mathrm{C}} = c_2$. $\square$

Remark 12. Since matrix-by-vector operations of the form $\mathbf{K}_{\mathrm{C}} \times \underline{u}_{\mathrm{C}}$ are involved in the case of $\mathbf{C}_{\mathrm{C}}^{(2)}$, the arithmetic work is of one order higher ($\mathcal{O}(h^{-2})$) than for $\mathbf{C}_{\mathrm{C}}^{(1)}$ ($\mathcal{O}(h^{-1})$).

Finally, let us mention that on the basis of Theorem 6 we can utilize every efficient FE-Schur-complement preconditioner and every good preconditioner for the discrete hypersingular integral operator $\mathbf{K}_{\mathrm{C}}$ as BE-Schur-complement preconditioner as well. FE-Schur-complement preconditioners have been studied by the finite element domain decomposition community very extensively. Beside the S-BPX and the multigrid preconditioners, we have also used the BPS preconditioner proposed in [3] and the hierarchical Schur-complement preconditioner in our numerical experiments. However, the S–BPX and the multigrid preconditioners are superior to these other preconditioners theoretically and practically as well. Preconditioners for the discrete hypersingular integral operator $\mathbf{K}_{\mathrm{C}}$ were also proposed and analyzed in [28].

5.6 Preconditoners $\mathbf{C}_\Lambda$ for the discrete single layer potential operator

The matrix $\mathbf{K}_\Lambda$ is block-diagonal with blocks $\mathbf{K}_{\Lambda,i}$ $(i = 1, \ldots, p)$. Thus, in order to construct a preconditioner for $\mathbf{K}_\Lambda$, it is sufficient to find preconditioners for each $\mathbf{K}_{\Lambda,i}$. Following [1,27,34,42] we propose to use

$$\mathbf{C}_\Lambda^{(1)} = \mathrm{diag}[(\mathbf{C}_{\Lambda,i}^{(1)})]_{i=1,\ldots,p}, \text{ with } (\mathbf{C}_{\Lambda,i}^{(1)})^{-1} := \alpha_i \mathbf{T}_i^{\mathrm{T}} \tilde{\mathbf{M}}_{h,i}^{-1} \mathbf{K}_{\mathrm{C},i} \tilde{\mathbf{M}}_{h,i}^{-1} \mathbf{T}_i,$$

$$\mathbf{C}_\Lambda^{(2)} = \mathrm{diag}[(\mathbf{C}_{\Lambda,i}^{(2)})]_{i=1,\ldots,p}, \text{ with } \mathbf{C}_{\Lambda,i}^{(2)} := \alpha_i \mathbf{K}_{\Lambda,i} (\mathbf{I}_{\Lambda,i} - \mathbf{M}_{\Lambda,i})^{-1}, \tag{55}$$

as scaled preconditioner $\mathbf{C}_\Lambda$ for the discrete single layer potential operator $\mathbf{K}_\Lambda$. In (55), $\mathbf{T}_i$ is some basis transformation, $\tilde{\mathbf{M}}_{h,i}$ stands for a modified mass matrix (see [42] for details), and $\mathbf{M}_{\Lambda,i}$ is the multigrid iteration operator satisfying the same conditions as formulated above for the multigrid preconditioner $\mathbf{C}_{\mathrm{C}}^{(2)}$ (cf. also [27]). In the numerical experiments (see Sect. 6), we use a symmetric V–cycle with 2 pre- and 2 postsmoothing steps which is defined as proposed in [1,34]. The preconditioner $\mathbf{C}_\Lambda^{(1)}$ can only be applied if piecewise linear trial functions have been used for both displacements and tractions. Under this condition, we conclude the following lemma.

Lemma 7. *There exist h-independent constants $\underline{\gamma}_\Lambda$ and $\overline{\gamma}_\Lambda$ such that (17) is valid for either choice of C_Λ .*

Proof. The result follows from [42] and [1,27,34] for $\mathbf{C}_\Lambda^{(1)}$ and $\mathbf{C}_\Lambda^{(2)}$, respectively. □

Remark 13. The local scaling factors α_i in (55) have to be chosen such that $\underline{\gamma}_\Lambda > 1$ in (17). In the case of $C_\Lambda^{(2)}$ this is an easy task if the local multigrid convergence rates are available. The use of $C_\Lambda^{(1)}$ requires to estimate the lowest eigenvalue of $(C_{\Lambda,i}^{(1)})^{-1}K_{\Lambda,i}$ for $i = 1,\ldots,p$. This can be done by applying the algorithm given in [38] to the local problems.

5.7 Final results

Now let us summarize the results of this section in the spectral equivalence theorem.

Theorem 8. *For our choice of the block-preconditioners C_Λ and C_{C} the spectral equivalence constants $\overline{\gamma}_\Lambda, \underline{\gamma}_{\mathrm{C}}$ and $\overline{\gamma}_{\mathrm{C}}$ determining $\underline{\gamma}$ and $\overline{\gamma}$ in (20) are independent of the discretization parameter h.*

Proof. These results follow directly from Lemma 6, Lemma 7 together with Theorem 6 (resp. Theorem 7) . □

As a consequence of Theorem 8, we can expect constant iteration numbers for all h resp. l (cf. Sect. 6).

6 Numerical experiments on parallel computers

6.1 An academic test problem

First, we consider the Laplace equation in a rectangle $\Omega = (0,2) \times (0,1)$ under inhomogeneous Dirichlet boundary conditions, with $g(x) := 4\,(x_1 - x_2),\ (x = (x_1, x_2))$, as a rather academical test problem for our algorithm. We decompose Ω into $p = 4 \times 2$, 8×4 and 16×8 squares, the data of which are distributed to 8, 32 and 128 processors, respectively. Here, and in Example 2 (see Sect. 6.2), both, the potential u on $\bigcup_{i=1}^p \Gamma_i$ and the normal derivatives $\lambda_i = \partial u/\partial n|_{\Gamma_i}$, $i = 1,\ldots,p$, are being approximated by piecewise linear functions, and the corresponding discrete BE-operators were computed semi-analytically [41].

The discretization of the first level $(l = 1)$ is illustrated for $p = 8$ in Fig. 1. There, nodes with unknown values of $\underline{u}_{\mathrm{C}}$ approximating $u|_{\Gamma_{\mathrm{C}}}$ are marked by a solid point, and some part of a circle in any Ω_i stands for an unknown component of $\underline{u}_\Lambda$ approximating λ_i.

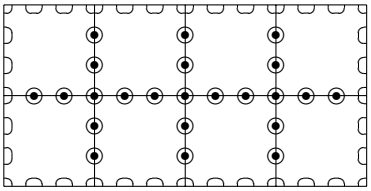

Fig. 1. The discretization of the 1^{st} grid for $p = 8$ in Example 1

Table 2. Number of unknowns (N), iteration count ($I(\epsilon)$, $\epsilon = 10^{-6}$), CPU time in seconds and scaled efficiency according to the CPU time for the 6th level. The experiments were carried out on a GC-PowerPlus using one processor per subdomain

p	4×2			8×4			16×8		
l	N	I(ϵ)	CPU	N	I(ϵ)	CPU	N	I(ϵ)	CPU
1	119	14	0.34	509	15	0.80	2105	16	2.44
2	245	16	0.41	1049	17	0.91	4337	16	2.78
3	497	17	0.55	2129	17	1.05	8801	17	3.27
4	1001	18	1.05	4289	18	1.63	17729	18	4.21
5	2009	18	2.87	8609	18	3.49	35585	18	6.35
6	4025	19	10.14	17249	18	11.00	71297	18	14.09
4×2		1.0			0.99			0.80	
8×4					1.0			0.81	

The preconditioners C_Λ and C_{C} have been chosen as follows:

$$\text{(56)} \qquad \mathbf{C}_\Lambda := \mathbf{C}_\Lambda^{(1)} \ (\text{"}\mathbf{K}_{\mathrm{C}}^{-1}\text{"}) \quad \text{and} \quad \mathbf{C}_{\mathrm{C}} := \mathbf{C}_{\mathrm{C}}^{(1)} \ \text{(S-BPX) [43].}$$

In Table 2 and below, $I(\epsilon)$ denotes the number of iterations needed in order to reduce the initial error by the factor $\epsilon = 10^{-s} \in (0, 1)$. We observe vertically (for all l) and horizontally (for all p) constant iteration numbers. For a fixed l, the local problem size is constant for all p.

The scaled efficiencies have been computed by

$$\text{(57)} \qquad S_{\text{eff}}(p_1, p_2) := \frac{p_1 \cdot t_1}{N_1} \,/\, \frac{p_2 \cdot t_2}{N_2},$$

where, for $i = 1, 2$, t_i is the time required for solving the problem with N_i unknowns on p_i processors with $p_2 \geq p_1$. For the 6th grid, the value S_{eff} has been computed for $(p_1, p_2) = (8, 32), (8, 128), (32, 128)$, obviously we have $S_{\text{eff}} = 1$ for $p_1 = p_2$. The scaled efficiencies $S_{\text{eff}}(p_1, p_2)$ are greater than 0.8 in all cases. Since the ratio between the local problem size and the effort (run–time) for realizing one global communication decreases with increasing p we observe a decreasing efficiency. The lower efficiencies obtained for $p_2 = 128$ are also due to the special architecture of the

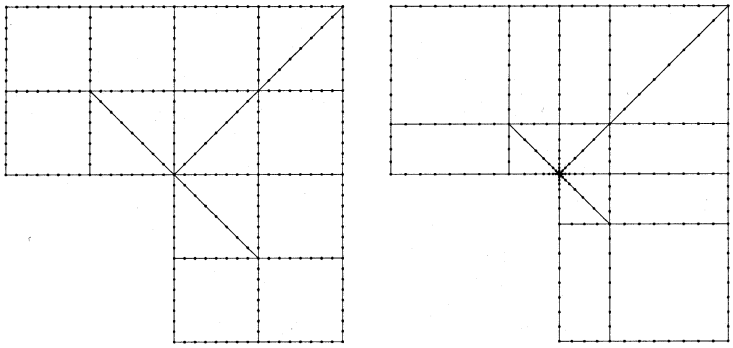

Fig. 2. The subdomains and the BE discretization of the $1^{\rm st}$ level using a uniform grid (left) and a graded grid (right)

PowerGC, which has a link of lower capacity between two 64-processor clusters compared to the links within the clusters. This is confirmed by the numerical results presented in [29], where the unit square was divided into $p = 2 \times 2$, 4×4 and 8×8 squares. For larger local problems, i.e., for greater l ($l = 7, 8, \ldots$) we could expect even higher efficiencies than stated in the table.

Remark 14. The 71297 BE-unknowns on the $6^{\rm th}$ grid for $p = 128$ correspond to about 1.180.000 unknowns of a standard FE-formulation with the same discretization parameter h.

6.2 A L-shape problem using graded grids

Now let us consider a more complicated problem, the potential equation in a L-shaped domain (re-entrant corner at $(0, 0)$) with the inhomogeneous Dirichlet boundary conditions $g(x) := \ln|x - y_0|$, $y_0 := (-0.01, -0.01)$ using uniform and graded grids as shown in Fig. 2. The graded $1^{\rm st}$-level grid has been defined such that $h_{\max}/h_{\min} > 10$. This example was proposed by the Priority Research Programme "Boundary Element Methods" of the German Research Foundation as benchmark for testing boundary element solvers [44]. We will use 16 processors for the corresponding 16 subdomains.

Now, we choose the block preconditioners C_Λ and $C_{\rm C}$ as follows:

$$\text{(58)}\quad \mathbf{C}_\Lambda := \mathbf{C}_\Lambda^{(1)} \ \ (\text{"}\mathbf{K}_{\rm C}^{-1}\text{"}) \quad \text{and} \quad \mathbf{C}_{\rm C} := \mathbf{C}_{\rm C}^{(2)} \ \ \text{(mg preconditioner)}.$$

Looking at Table 3 we conclude that the algorithm is robust with respect to both, the number of levels l and the grading of the grid.

Remark 15. The use of $C_{\rm C}^{(1)}$ instead of $C_{\rm C}^{(2)}$ in Example 2 leads to higher, but still constant (with respect to l) iteration numbers.

Table 3. Iteration count ($I(\epsilon)$, $\epsilon = 10^{-6}$) and CPU time in seconds on a POWER-Xplorer using 16 processors

	uniform grid			graded grid		
l	N	I(ϵ)	CPU	N	I(ϵ)	CPU
1	593	16	1.04	497	18	1.09
2	1181	17	1.56	989	18	1.57
3	2357	17	2.81	1973	18	2.81
4	4709	17	6.86	3941	18	6.86
5	9413	18	22.64	7877	18	22.31
6	18821	18	82.71	15749	18	82.03

6.3 The dam problem

We are now going to apply the ideas discussed above to a plane strain linear elasticity problem in which the displacement vector $u(x) \in \mathbf{V}_0 := \{v = (v_1, v_2)^{\mathrm{T}} \in H^1(\Omega) \times H^1(\Omega) : v = 0 \text{ on } \Gamma_{\mathrm{D}}\}$ satisfies the weak formulation

$$\int_{\Omega} \sum_{i,j,k,l=1}^{2} \varepsilon_{ij}(u(x)) D_{ijkl}(x) \varepsilon_{kl}(v(x))\, dx = \int_{\Gamma_{\mathrm{N}}} \sum_{k=1}^{2} g_k(x) v_k(x)\, ds \tag{59}$$

of Lamé's equations for all test functions $v \in \mathbf{V}_0$, under mixed boundary conditions on the boundary $\Gamma := \partial\Omega = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}$ ($\Gamma_{\mathrm{D}} \neq \emptyset$, $\Gamma_{\mathrm{N}} \neq \emptyset$) of the bounded Lipschitz domain Ω, where $\varepsilon_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, $D_{ijkl} = \lambda(x)\delta_{ij}\delta_{kl} + \mu(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, δ_{ij} denotes Kronecker's delta, λ and μ are the Lamé coefficients of the elastic materials involved, $g = (g_1, g_2)^{\mathrm{T}}$ is the vector of boundary tractions prescribed on Γ_{N}.

From (59), one can now easily derive DD boundary integral formulations (for pure BE discretizations) as above (Sect. 2), DD domain integral variational formulations (for pure FE discretizations, see [16, 23]), or coupled DD boundary and domain integral variational formulations (for BE-FE coupling see [5, 16, 30] for details).

As a test problem we consider a dam filled with water as sketched in Fig. 3. As indicated there, Dirichlet boundary conditions (b.c.) are given on Γ_{D} (zero displacement) and Neumann b.c. on Γ_{N} (the tractions are equal to zero, or they are chosen according to the water pressure). The Lamé constants are given for rock (I-II) by $\mu_{\mathrm{r}} = 7.265\mathrm{e}\,4\mathrm{MPa}$, $\lambda_{\mathrm{r}} = 3.743\mathrm{e}4\,\mathrm{MPa}$ and for concrete (III-VIII) by $\mu_{\mathrm{c}} = 9.2\mathrm{e}6\,\mathrm{MPa}$, $\lambda_{\mathrm{c}} = 9.2\mathrm{e}6\,\mathrm{MPa}$. For the results presented in Table 4, the block preconditioners $\mathbf{C}_{\Lambda}$ and $\mathbf{C}_{\mathrm{C}}$ were chosen as follows:

$$\begin{aligned} \mathbf{C}_{\Lambda} &:= \mathbf{C}_{\Lambda}^{(2)} \quad \text{(mg preconditioner)}, \\ \mathbf{C}_{\mathrm{C}} &:= \mathbf{C}_{\mathrm{C}}^{(2)} \quad \text{(mg preconditioner)}. \end{aligned} \tag{60}$$

Table 4. Levels (l), pure solution time (s), number of unknowns (N), iteration count ($I(\epsilon)$, $\epsilon = 10^{-6}$), CPU time in seconds for the dam-problem. The experiments were carried out on a Power-XPlorer using 8 processors

			FEM:	III-VIII		
	BEM:	I-VIII	BEM:	I-II	FEM:	I-VIII
l	I(ϵ)	CPU	I(ϵ)	CPU	I(ϵ)	CPU
1	26	3.7	22	2.9	20	2.5
2	27	4.9	25	4.3	25	3.9
3	27	8.8	28	8.3	27	7.2
4	28	24.1	30	26.4	31	20.8
5	29	85.9	33	101.4	33	71.5
s(5)		49.5		65.4		54.2
N(5)		6470		78130		119318

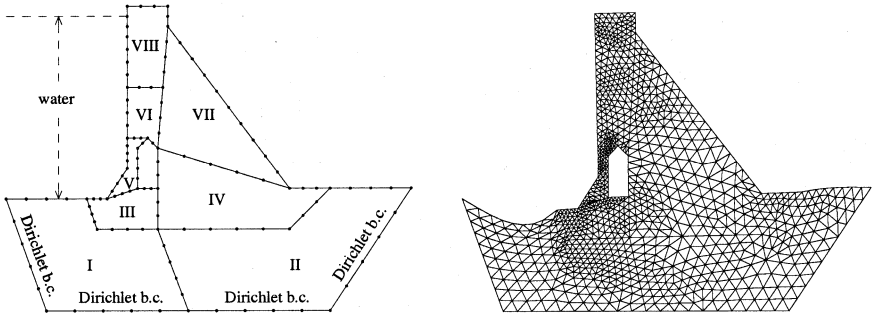

Fig. 3. The subdomains and the BE discretization of the $1^{\rm st}$ level (left) and the deformed (magnification factor 100) FE grid of the $2^{\rm nd}$ level (right)

The BE-matrices were computed fully analytically using piecewise linear functions for the displacements and piecewise constant functions for the tractions [41]. As an extension to the theory presented in this paper, we consider, besides a pure BE discretization, a coupled BE–FE and a pure FE discretization. The BE discretization of the $1^{\rm st}$ level and the FE discretization of the 2^{nd} level (deformed mesh) are shown in Fig. 3. In the coupled model, we defined the domains I and II as BE domains and the remaining as FE domains.

In Table 4, we discuss several combinations of FE/BE discretizations. Looking at the CPU-time we observe that the FE discretization (column 3) leads to the best results. However, if we are interested in the pure solving time $s(.)$ ($s(5)$ for the $5^{\rm th}$ level is given in Table 4) the BE discretization (first column) is of advantage.

Remark 16. Models involving FE subdomains require (besides $\mathbf{C}_A$ and $\mathbf{C}_{\rm C}$) to define additional block operators $\mathbf{C}_{\rm I}$ and $\mathbf{B}_{\rm I}$ which are parts of the Dirichlet DD preconditioner [14, 18, 19]. We have used (V02) (multigrid V-cycle

with 2 postsmoothing steps in the symmetric multiplicative Schwarz method [17]) for $\mathbf{C}_{\mathrm{I}}$ and (HExt) (implicitly defined by the hierarchical extension, formally $\mathbf{E}_{\mathrm{IC},i} = -\mathbf{B}_{\mathrm{I},i}^{-1}\mathbf{K}_{\mathrm{IC},i}$ [14, 19]) for $\mathbf{B}_{\mathrm{I}}$ in the computations documented in Table 4.

7 Concluding remarks

The generalization of the approach to unbounded domains sketched in Subsect. 5.4 and to the coupling of the BEM with the finite element method is more or less straightforward [9, 15, 23, 30]. Moreover, the potential problem can be non-linear in the finite element subdomains [23]. Linear elasticity problems and local plasticity problems can be treated in the same fashion [4–7, 24]. With the exception of plasticity problems, the code FEM⊗BEM [15] can solve these problems on massively parallel computers and workstation clusters (PVM, MPI).

Three dimensional problems can be treated in a very similar way, in particular the preconditioners $\mathbf{C}_{\mathrm{C}}^{(1)}$ (S-BPX) [43] and $\mathbf{C}_{\Lambda}^{(1)}$ ("$\mathbf{K}_{\mathrm{C}}^{-1}$") [42] satisfy the required estimates independently of the dimension of the problem. However, the standard application of the BEM leads to $O(h^{-4})$-complexity, which is not optimal in the 3d-case. For this reason, methods as wavelets (see [11] and the references given there) or the panel clustering method [20, 21] which reduces the complexity for matrix-by-vector operations to $O(h^{-2}\log^7 h^{-1})$ as well as the required memory to $O(h^{-2}\log^3 h^{-1})$ for 3d-Galerkin BE-equations have to be applied.

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