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ADAPTIVE NONCONFORMING CROUZEIX-RAVIART FEM FOR EIGENVALUE PROBLEMS

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Abstract. The nonconforming approximation of eigenvalues is of high practical interest because it allows for guaranteed upper and lower eigenvalue bounds and for a convenient computation via a consistent diagonal mass matrix in 2D. The first main result is a comparison which states equivalence of the error of the nonconforming eigenvalue approximation with its best-approximation error and its error in a conforming computation on the same mesh. The second main result is optimality of an adaptive algorithm for the effective eigenvalue computation for the Laplace operator with optimal convergence rates in terms of the number of degrees of freedom relative to the concept of a nonlinear approximation class. The analysis includes an inexact algebraic eigenvalue computation on each level of the adaptive algorithm which requires an iterative algorithm and a controlled termination criterion. The analysis is carried out for the first eigenvalue in a Laplace eigenvalue model problem in 2D.

1. Introduction

Given a bounded simply connected Lipschitz domain Ω with polygonal boundary $\partial\Omega$, the weak form of the eigenvalue problem $-\Delta u = \lambda u$ with homogenous boundary conditions seeks the first eigenpair $(\lambda, u) \in \mathbb{R} \times V$ such that $\|u\|_{L^2(\Omega)} = 1$ and

$$a(u, v) = \lambda b(u, v) \quad \text{for all } v \in V := H^1_0(\Omega). \tag{1.1}$$

Here and throughout this paper, standard notation is employed on Lebesgue and Sobolev spaces, and the scalar products a and b read

$$a(v, w) := \int_\Omega \nabla v \cdot \nabla w \, dx \quad \text{for any } v, w \in V \equiv H^1_0(\Omega),$$
$$b(v, w) := \int_\Omega v \, w \, dx \quad \text{for any } v, w \in L^2(\Omega)$$

with induced norms $|\!|\!|\cdot|\!|\!| := a(\cdot, \cdot)^{1/2}$ and $\|\cdot\|_{L^2(\Omega)} = b(\cdot, \cdot)^{1/2}$. The Crouzeix-Raviart finite element space of piecewise linear polynomials (denoted by $P_1(\mathcal{T})$) with continuity condition at the interior edges' midpoints and corresponding zero boundary conditions along $\partial\Omega$ for some shape-regular triangulation $\mathcal{T}$ of Ω into closed triangles $T \in \mathcal{T}$ with interior edges $\mathcal{E}(\Omega)$, boundary edges $\mathcal{E}(\partial\Omega)$ and midpoints $\mathrm{mid}(E)$

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for $E \in \mathcal{E} := \mathcal{E}(\Omega) \cup \mathcal{E}(\partial\Omega)$ reads

$$\mathrm{CR}^1_0(\mathcal{T}) := \left\{ v \in P_1(\mathcal{T}) \left| \begin{array}{l} v \text{ is continuous in } \operatorname{mid}(E) \text{ for all } E \in \mathcal{E}(\Omega) \\ \& \; v(\operatorname{mid}(E)) = 0 \text{ for all } E \in \mathcal{E}(\partial\Omega) \end{array} \right. \right\}.$$

The piecewise gradient ∇_{NC} (with respect to the triangulation $\mathcal{T}$) defines the discrete scalar product

$$a_{\mathrm{NC}}(v_{\mathrm{CR}}, w_{\mathrm{CR}}) := \int_\Omega \nabla_{\mathrm{NC}} v_{\mathrm{CR}} \cdot \nabla_{\mathrm{NC}} w_{\mathrm{CR}} \, dx \quad \text{for any } v_{\mathrm{CR}}, w_{\mathrm{CR}} \in V + \mathrm{CR}^1_0(\mathcal{T})$$

with induced norm $|||\cdot|||_{\mathrm{NC}} := a_{\mathrm{NC}}(\cdot,\cdot)^{1/2}$. The discrete eigenvalue problem reads: Seek $(\lambda_{\mathrm{CR}}, u_{\mathrm{CR}}) \in \mathbb{R} \times \mathrm{CR}^1_0(\mathcal{T})$ such that $\lambda_{\mathrm{CR}} > 0$ is minimal, $\|u_{\mathrm{CR}}\|_{L^2(\Omega)} = 1$, and

$$a_{\mathrm{NC}}(u_{\mathrm{CR}}, v_{\mathrm{CR}}) = \lambda_{\mathrm{CR}} b(u_{\mathrm{CR}}, v_{\mathrm{CR}}) \quad \text{for all } v_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T}). \tag{1.2}$$

The nonconforming finite element approximation has recently become highly attractive because of the guaranteed lower and upper eigenvalue bounds [19]. The lowest eigenvalue λ of (1.1) and its Crouzeix-Raviart approximation λ_{CR} satisfy

$$\frac{\lambda_{\mathrm{CR}}}{1 + 0.1931\lambda_{\mathrm{CR}}\|h_0\|_\infty^2} \le \lambda \le |||v_{\mathrm{C}}|||^2 \tag{1.3}$$

for any postprocessing $v_{\mathrm{C}} \in V$ of the computed $u_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T})$ with L^2 norm one and the maximal mesh-size $\|h_0\|_\infty$. This is one striking advantage of the Crouzeix-Raviart discretisation, another advantage is the diagonal mass matrix in 2D. The first main result of this paper compares the energy norm errors of the discrete first eigenfunction computed by the nonconforming and the conforming P_1 finite element schemes. For sufficiently small mesh-size $\|h_0\|_\infty$, Theorem 3.1 asserts the equivalence of the errors of the nonconforming Crouzeix-Raviart solution u_{CR} and the conforming P_1 solution u_{C} with the L^2 projection $\Pi_0 \nabla u$ of the gradient onto piecewise constants,

$$|||u - u_{\mathrm{C}}||| \approx |||u - u_{\mathrm{CR}}|||_{\mathrm{NC}} \approx \|\nabla u - \Pi_0 \nabla u\|_{L^2(\Omega)}.$$

In conclusion, the nonconforming approximation is not worse than the conforming one and has the advantage of a consistent diagonal mass matrix in 2D and that of guaranteed error bounds (1.3).

The reliability and efficiency of the error estimator

$$\mu_\ell^2(T) := |T| \, \|\lambda_{\mathrm{CR}} u_{\mathrm{CR}}\|^2_{L^2(T)} + |T|^{1/2} \sum_{E \in \mathcal{E}(T)} \|[\partial u_{\mathrm{CR}}/\partial s]_E\|^2_{L^2(E)}$$

have been established [24] up to higher-order terms (for more details cf. Subsection 4.2). This and the recent work [19] motivate an adaptive algorithm Acrevfem with successive loops on the level $\ell = 0, 1, 2 \ldots$ of the form

(inexact Solve & Estimate) $\to$ Mark $\to$ Refine.

On each of those levels ℓ, the algebraic eigenvalue solver computes an approximation $(\tilde\lambda_\ell, \tilde u_\ell)$ to the discrete eigenpair (λ_ℓ, u_ℓ) up to any tolerance monitored in terms of the error estimator η_ℓ with respect to inexact solve and some parameter $0 < \kappa < 1/2$. A sufficiently fine initial mesh $\mathcal{T}_0$ allows for some quasi-orthogonality, which leads to the contraction property for the inexact eigenpair approximations. The second main result of this paper asserts quasi-optimal convergence towards the eigenpair (λ, u) of the smallest eigenvalue λ with respect to the discrete energy norm $|||\cdot|||_{\mathrm{NC}}$ in the sense that

$$(|\mathcal{T}_\ell| - |\mathcal{T}_0|)^\sigma |||u - \tilde{u}_\ell|||_{\mathrm{NC}} \leq C_{\mathrm{opt}} \, |u|_{\mathcal{A}_\sigma} \quad \text{for all } \ell = 1, 2, 3, \dots .$$

For conforming finite element discretisations optimal convergence rates are proven in [18, 23]. Subsection 4.3 presents the details on the approximation seminorm $|u|_{\mathcal{A}_\sigma}$ and the optimal convergence rate $\sigma > 0$ and states optimality up to the factor $1 \leq C_{\mathrm{opt}} < \infty$ under the condition that $\mathcal{T}_0$ is sufficiently fine and the bulk parameter θ as well as the control parameter κ for the inexact solve are sufficiently small. All constants C_{opt} and upper bounds on κ and θ depend exclusively on the initial triangulation $\mathcal{T}_0$ and on the parameter $\sigma > 0$.

Optimality of adaptive algorithms for the nonconforming finite element discretisation is well studied for the Poisson problem [5, 29, 30] the Stokes equations [4, 21, 27] and the Navier-Lamé equaitons [16].

One technical difficulty behind the treatment of the nonlinearity is the L^2 error control for possibly singular solutions u in $H^1_0(\Omega) \setminus H^{3/2}(\Omega)$. The standard duality technique has to circumvent the fact that the discrete solutions are not allowed as test functions on the continuous level and lead to jump terms times normal derivatives of the dual solution along edges. Their analysis can be found in textbooks [8, 12] for convex domains outside of the main application for adaptive mesh-refinement. Instead, this paper shows an alternative L^2 error control for arbitrarily small regularity $s > 0$ (compare with $s > 1/2$ required for the traces of normal derivatives to exist). A similar approach has independently been developed in [28].

The remaining parts of this paper are organised as follows. Section 2 establishes the L^2 control for the eigenfunctions and convergence rates for the eigenvalues and provides the framework for the balance of higher-order terms that arise from the nonlinearity of the eigenvalue problem. Section 3 compares the error of the conforming first-order method with the errors of the nonconforming approximation and best-approximation. This equivalence enables the subsequent analysis of the optimal convergence of the adaptive algorithm ACREVFEM of Section 4 with respect to some equivalent approximation class. The quasi-orthogonality and convergence in the sense of a contraction property will be proven in Section 5. Section 6 provides the discrete reliability and the quasi-optimal convergence of the algorithm.

Throughout this paper, standard notation on Lebesgue and Sobolev spaces and their norms is employed; $\fint$ denotes the integral mean. The formula $A \lesssim B$ represents an inequality $A \leq CB$ for some mesh-size independent, positive generic constant C; $A \approx B$ abbreviates $A \lesssim B \lesssim A$. By convention, all generic constants $C \approx 1$ do not depend on the mesh-size but may depend on the fixed coarse triangulation $\mathcal{T}_0$ and its interior angles. The measure $|\cdot|$ is context-sensitive and refers to the number of elements of some finite set (e.g. the number $|\mathcal{T}|$ of triangles in a triangulation $\mathcal{T}$) or the length $|E|$ of an edge E or the area $|T|$ of some domain T and not just the modulus of a real number or the Euclidean length of a vector. The piecewise constant function $h_{\mathcal{T}}$ with $h_{\mathcal{T}}|_T := |T|^{1/2}$ on the triangle $T \in \mathcal{T}$ denotes the mesh-size of the triangulation $\mathcal{T}$ with maximum $\|h_{\mathcal{T}}\|_\infty$. The L^2 projection onto piecewise constant functions is denoted by Π_0. The space of piecewise polynomials of degree $\leq k$ is denoted by $P_k(\mathcal{T})$.

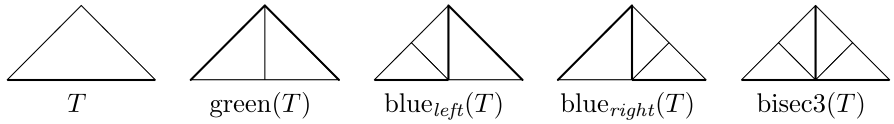

FIGURE 1. Possible refinements of a triangle T in one level within the NVB. The thick lines indicate the refinement edges of the sub-triangles as in [6, 33].

2. L^2 CONTROL

This section is devoted to the L^2 error control of the first nonconforming eigenfunction on a fixed triangulation $\mathcal{T} \in \mathbb{T}$ in the set $\mathbb{T}$ of all regular triangulations that are refinements of the coarse initial triangulation $\mathcal{T}_0$ with maximal mesh-size $\|h_0\|_\infty$ by Newest-Vertex-Bisection (NVB) [6, 33], see Figure 1. The following error estimate is well established for $H^2(\Omega)$ regular domains [7]. That proof might be extendable to $H^{1+s}(\Omega)$ regular domains for $1/2 < s \leq 1$ because of the existence of the normal derivative of the dual solution along interior edges. The proof in this section covers the case of reduced elliptic regularity $0 < s \leq 1$ with some constant $C(s,\Omega) \approx 1$ (which depends on the maximal interior angle ω of the polygon $\partial\Omega$ via $s < \pi/\omega$) for the Laplace equation and pure Dirichlet conditions, such that for all $f \in L^2(\Omega)$ there exists some $z \in H^1_0(\Omega) \cap H^{1+s}(\Omega)$ such that

$$f + \Delta z = 0 \quad \text{in } \mathcal{D}'(\Omega) \quad \text{and} \quad \|z\|_{H^{1+s}(\Omega)} \leq C(s,\Omega)\|f\|_{L^2(\Omega)}. \tag{2.1}$$

Theorem 2.1 (Eigenvalue and L^2 control). *Suppose that the initial mesh-size $\|h_0\|_\infty := \|h_{\mathcal{T}_0}\|_\infty \ll 1$ is sufficiently small, then the first eigenpair (λ, u) and the discrete first eigenpair $(\lambda_{\mathrm{CR}}, u_{\mathrm{CR}}) \in \mathbb{R} \times \mathrm{CR}^1_0(\mathcal{T})$ with $\|u_{\mathrm{CR}}\|_{L^2(\Omega)} = 1$ and $b(u, u_{\mathrm{CR}}) \geq 0$ satisfy*

$$|\lambda - \lambda_{\mathrm{CR}}| + \|u - u_{\mathrm{CR}}\|_{L^2(\Omega)} \lesssim \|h_0\|_\infty^s |||u - u_{\mathrm{CR}}|||_{\mathrm{NC}}.$$

Before the remaining parts of this section are devoted to the proof of this theorem, some conclusion for the discrete eigenpair approximations on two different triangulations is in order.

Corollary 2.2. *Suppose that the initial mesh-size $\|h_0\|_\infty := \|h_{\mathcal{T}_0}\|_\infty \ll 1$ is sufficiently small and that $\mathcal{T}_{\ell+m} \in \mathbb{T}$ is a refinement of $\mathcal{T}_\ell \in \mathbb{T}$. Then some eigenfunction $u_\ell \in \mathrm{CR}^1_0(\mathcal{T}_\ell)$ (resp. $u_{\ell+m} \in \mathrm{CR}^1_0(\mathcal{T}_{\ell+m})$) with $\|u_\ell\| = 1$ (resp. $\|u_{\ell+m}\| = 1$) with respect to the first discrete eigenvalues λ_ℓ (resp. $\lambda_{\ell+m}$) satisfies*

$$\|\lambda_{\ell+m}u_{\ell+m} - \lambda_\ell u_\ell\|_{L^2(\Omega)} \lesssim \|h_0\|_\infty^s (|||u - u_\ell|||_{\mathrm{NC}} + |||u - u_{\ell+m}|||_{\mathrm{NC}}).$$

Proof. Theorem 2.1 proves that there exist eigenfunctions $u_\ell \in \mathrm{CR}^1_0(\mathcal{T}_\ell)$ and $u_{\ell+m} \in \mathrm{CR}^1_0(\mathcal{T}_{\ell+m})$ with $\|u_\ell\|_{L^2(\Omega)} = 1 = \|u_{\ell+m}\|_{L^2(\Omega)}$ and

$$\begin{aligned}
&4\,\|\lambda_{\ell+m}u_{\ell+m} - \lambda_\ell u_\ell\|^2_{L^2(\Omega)} \\
&= (\lambda_{\ell+m} - \lambda_\ell)^2\|u_{\ell+m} + u_\ell\|^2_{L^2(\Omega)} + (\lambda_{\ell+m} + \lambda_\ell)^2\|u_{\ell+m} - u_\ell\|^2_{L^2(\Omega)} \\
&\lesssim \|h_0\|_\infty^{2s}(|||u - u_\ell|||^2_{\mathrm{NC}} + |||u - u_{\ell+m}|||^2_{\mathrm{NC}}).
\end{aligned}$$

The a priori analysis [1, 7, 34] guarantees that λ_ℓ and $\lambda_{\ell+m}$ are bounded for $\|h_0\|_\infty \ll 1$. $\square$

The proofs of the results in this and the following section rely on the design of some novel conforming P_4 companion to the nonconforming discrete solution u_{CR}.

Proposition 2.3. *For any $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ there exists some $J_4 v_{\mathrm{CR}} \in P_4(\mathcal{T}) \cap C_0(\Omega)$ such that (a) $v_{\mathrm{CR}} - J_4 v_{\mathrm{CR}}$ is L^2 orthogonal on the space $P_1(\mathcal{T})$ of piecewise first-order polynomials, (b) it enjoys the integral mean property of the gradient*

$$\Pi_0(\nabla_{\mathrm{NC}}(v_{\mathrm{CR}} - J_4 v_{\mathrm{CR}})) = 0,$$

and (c) it satisfies the approximation and stability property

$$\|h_{\mathcal{T}}^{-1}(v_{\mathrm{CR}} - J_4 v_{\mathrm{CR}})\|_{L^2(\Omega)} + \|\!|v_{\mathrm{CR}} - J_4 v_{\mathrm{CR}}|\!\|_{\mathrm{NC}} \lesssim \min_{v \in V} \|\!|v_{\mathrm{CR}} - v|\!\|_{\mathrm{NC}}. \tag{2.2}$$

Proof. The design follows in three steps.

Step 1. Let $\mathcal{N}$ denote the set of vertices of $\mathcal{T}$ and let $\mathcal{N}(\Omega) := \mathcal{N} \cap \Omega$ be the set of interior vertices. The operator $J_1 : \mathrm{CR}_0^1(\mathcal{T}) \to P_1(\mathcal{T}) \cap C_0(\Omega)$ acts on any function $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ by averaging the function values at each interior node z, i.e.,

$$J_1 v_{\mathrm{CR}}(z) = |\mathcal{T}(z)|^{-1} \sum_{T \in \mathcal{T}(z)} v_{\mathrm{CR}}|_T(z) \quad \text{for all } z \in \mathcal{N}(\Omega) \tag{2.3}$$

for $\mathcal{T}(z) := \{T \in \mathcal{T} \mid z \in T\}$. This operator is also known as enriching operator in the context of fast solvers [11]. The proof of the approximation property

$$\|h_{\mathcal{T}}^{-1}(v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}})\|_{L^2(\Omega)} \lesssim \min_{v \in V} \|\!|v_{\mathrm{CR}} - v|\!\|_{\mathrm{NC}} \tag{2.4}$$

is included in [13, Theorem 5.1]. This and an inverse estimate imply the stability property

$$\|\!|v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}}|\!\|_{\mathrm{NC}} \lesssim \min_{v \in V} \|\!|v_{\mathrm{CR}} - v|\!\|_{\mathrm{NC}}. \tag{2.5}$$

Step 2. Given any edge $E = \mathrm{conv}\{a, b\}$ with nodal P_1 conforming basis functions $\varphi_a, \varphi_b \in P_1(\mathcal{T}) \cap C_0(\Omega)$ (defined by $\varphi_a(a) = 1$ and $\varphi_a(z) = 0$ for $z \in \mathcal{N} \setminus \{a\}$), the quadratic edge-bubble function

$$\flat_E := 6\, \varphi_a \varphi_b$$

has support $\mathrm{supp}(\varphi_a) \cap \mathrm{supp}(\varphi_b)$ and satisfies $⨍_E \flat_E \, ds = 1$. For any function $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ the operator $J_2 : \mathrm{CR}_0^1(\mathcal{T}) \to P_2(\mathcal{T}) \cap C_0(\Omega)$ acts as

$$J_2 v_{\mathrm{CR}} := J_1 v_{\mathrm{CR}} + \sum_{E \in \mathcal{E}(\Omega)} \left(⨍_E (v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}}) \, ds \right) \flat_E.$$

An immediate consequence of this choice reads

$$⨍_E J_2 v_{\mathrm{CR}} \, ds = ⨍_E v_{\mathrm{CR}} \, ds \quad \text{for all } E \in \mathcal{E}.$$

An integration by parts shows the integral mean property of the gradients $\Pi_0 \nabla J_2 = \nabla_{\mathrm{NC}}$, i.e,

$$\int_T \nabla J_2 v_{\mathrm{CR}} \, dx = \int_T \nabla_{\mathrm{NC}} v_{\mathrm{CR}} \, dx \quad \text{for all } T \in \mathcal{T}.$$

An integration by parts shows for the vertex $P_E \in \mathcal{N}(T)\backslash E$ opposite to $E \in \mathcal{E}(T)$ in the triangle T the trace identity

$$\begin{aligned}&\fint_E (v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}})\, ds\\ &\qquad = \fint_T (v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}})\, dx + \frac{1}{2}\fint_T (x - P_E)\cdot \nabla_{\mathrm{NC}}(v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}})\, dx.\end{aligned}$$

The scaling $\|\flat_E\|_{L^2(\Omega)} \lesssim |T|^{1/2}$ shows

$$\begin{aligned}|T|^{-1/2}\Big\| \sum_{E\in\mathcal{E}(T)} \left(\fint_E (v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}})\, ds\right) \flat_E \Big\|_{L^2(T)} &\lesssim \sum_{E\in\mathcal{E}(T)} \Big| \fint_E (v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}})\, ds\Big|\\ &\lesssim |T|^{-1/2}\|v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}}\|_{L^2(T)} + \|\nabla_{\mathrm{NC}}(v_{\mathrm{CR}} - J_1 v_{\mathrm{CR}})\|_{L^2(T)}.\end{aligned}$$

This and the properties (2.4)–(2.5) yield

$$\|h_{\mathcal{T}}^{-1}(v_{\mathrm{CR}} - J_2 v_{\mathrm{CR}})\|_{L^2(\Omega)} \lesssim \min_{v\in V} \|\!| v_{\mathrm{CR}} - v \|\!|_{\mathrm{NC}}.$$

The stability property of J_2 follows with an inverse estimate

$$\|\!| v_{\mathrm{CR}} - J_2 v_{\mathrm{CR}} \|\!|_{\mathrm{NC}} \lesssim \|h_{\mathcal{T}}^{-1}(v_{\mathrm{CR}} - J_2 v_{\mathrm{CR}})\|_{L^2(\Omega)} \lesssim \min_{v\in V} \|\!| v_{\mathrm{CR}} - v \|\!|_{\mathrm{NC}}.$$

Step 3. On any triangle $T = \operatorname{conv}\{a,b,c\}$ with nodal basis functions φ_a, φ_b, φ_c, the cubic volume bubble function reads

$$\flat_T := \varphi_a \varphi_b \varphi_c \in H_0^1(T).$$

The affine functions

$$\phi_{T,z} := \sqrt{40 + 10\sqrt{7}}\, |T|^{-1/2}\big(2 - (7-\sqrt{7})\varphi_z\big) \quad \text{for } z \in \{a,b,c\}$$

are $\flat_T$ orthonormal in the sense that (with the Kronecker δ)

$$\int_T \phi_{T,y}\, \phi_{T,z} \flat_T \, dx = \delta_{yz} \quad \text{for } y,z \in \{a,b,c\}.$$

Define

$$J_4 v_{\mathrm{CR}} := J_2 v_{\mathrm{CR}} + \sum_{T\in\mathcal{T}} \sum_{z\in\mathcal{N}(T)} \left(\int_T (v_{\mathrm{CR}} - J_2 v_{\mathrm{CR}})\phi_{T,z}\, dx\right) \phi_{T,z} \flat_T.$$

The difference $v_{\mathrm{CR}} - J_4 v_{\mathrm{CR}}$ is L^2 orthogonal to all piecewise affine functions. Since $\phi_{T,z}$ vanishes on $E \in \mathcal{E}$, J_4 enjoys the integral mean property of the gradient $\Pi_0 \nabla J_4 = \nabla_{\mathrm{NC}}$. Since

$$\left| \int_T (v_{\mathrm{CR}} - J_2 v_{\mathrm{CR}})\phi_{T,z}\, dx \right| \lesssim \|v_{\mathrm{CR}} - J_2 v_{\mathrm{CR}}\|_{L^2(T)},$$

the scaling $|T|^{1/2}\|\nabla \phi_{T,z}\|_{L^2(T)} \approx \|\nabla \flat_T\|_{L^2(\Omega)} \approx \|\flat_T\|_{L^\infty(\Omega)} \approx |T|^{1/2}\|\phi_{T,z}\|_{L^\infty(T)} \approx 1$ and Step 2 imply the stability property

$$\|\!| v_{\mathrm{CR}} - J_4 v_{\mathrm{CR}} \|\!|_{\mathrm{NC}} \lesssim \min_{v\in V} \|\!| v_{\mathrm{CR}} - v \|\!|_{\mathrm{NC}}.$$

The Poincaré inequality proves the approximation property

$$\|h_{\mathcal{T}}^{-1}(v_{\mathrm{CR}} - J_4 v_{\mathrm{CR}})\|_{L^2(\Omega)} \lesssim \min_{v\in V} \|\!| v_{\mathrm{CR}} - v \|\!|_{\mathrm{NC}}. \qquad \square$$

The proof of Theorem 2.1 starts with arguments from [34] exploited in [17] for conforming FEM.

Let λ_k denote the k-th exact eigenvalue and let $\lambda_{\mathrm{CR}}(\mathcal{K})$ denote the first discrete eigenvalue with respect to $\mathrm{CR}_0^1(\mathcal{K})$. For a sufficiently small mesh-size $\|h_0\|_\infty$ of $\mathcal{T}_0$ the well-established a priori analysis of [1, 7, 34] implies that

$$M := \sup_{\mathcal{K}\in\mathbb{T}} \sup_{k=2,3,4,\dots} \frac{\lambda_{\mathrm{CR}}(\mathcal{K})}{|\lambda_{\mathrm{CR}}(\mathcal{K}) - \lambda_k|} < \infty. \tag{2.6}$$

Lemma 2.4. *Let $G_{\mathrm{CR}}u \in \mathrm{CR}_0^1(\mathcal{T})$ denote the nonconforming finite element solution of the Poisson problem with right-hand side λu, i.e.,*

$$a_{\mathrm{NC}}(G_{\mathrm{CR}}u, v_{\mathrm{CR}}) = b(\lambda u, v_{\mathrm{CR}}) \quad \textit{for all } v_{\mathrm{CR}}\ \mathrm{CR}_0^1(\mathcal{T}). \tag{2.7}$$

Any eigenfunction $u_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ corresponding to λ_{CR} and $u \in V$ corresponding to λ such that $\|u\|_{L^2(\Omega)} = 1 = \|u_{\mathrm{CR}}\|_{L^2(\Omega)}$ and $b(u, u_{\mathrm{CR}}) \geq 0$ satisfy (with the constant M from (2.6)) that

$$\|u - u_{\mathrm{CR}}\|_{L^2(\Omega)} \leq \sqrt{2}(1+M)\|u - G_{\mathrm{CR}}u\|_{L^2(\Omega)}.$$

Proof. Some algebra with $\|u_{\mathrm{CR}}\|_{L^2(\Omega)} = 1 = \|u\|_{L^2(\Omega)}$ and $b(u, u_{\mathrm{CR}}) \geq 0$ proves

$$\frac{\|u - u_{\mathrm{CR}}\|^2_{L^2(\Omega)}}{2} = \frac{\|u - b(u, u_{\mathrm{CR}})u_{\mathrm{CR}}\|^2_{L^2(\Omega)}}{1 + |b(u, u_{\mathrm{CR}})|}. \tag{2.8}$$

Note that $b(u, u_{\mathrm{CR}})u_{\mathrm{CR}}$ is the L^2 projection onto $\mathrm{span}\{u_{\mathrm{CR}}\}$. The combination of (2.8) with the triangle inequality proves

$$\begin{aligned} \frac{\|u - u_{\mathrm{CR}}\|_{L^2(\Omega)}}{\sqrt{2}} &\leq \|u - b(u, u_{\mathrm{CR}})u_{\mathrm{CR}}\|_{L^2(\Omega)} = \min_{t\in\mathbb{R}}\|u - t\,u_{\mathrm{CR}}\|_{L^2(\Omega)} \\ &\leq \|u - b(G_{\mathrm{CR}}u, u_{\mathrm{CR}})u_{\mathrm{CR}}\|_{L^2(\Omega)} \\ &\leq \|u - G_{\mathrm{CR}}u\|_{L^2(\Omega)} + \|G_{\mathrm{CR}}u - b(G_{\mathrm{CR}}u, u_{\mathrm{CR}})u_{\mathrm{CR}}\|_{L^2(\Omega)}. \end{aligned} \tag{2.9}$$

It remains to estimate the second term on the right-hand side of (2.9).

Set $v_{\mathrm{CR}} := G_{\mathrm{CR}}u - b(G_{\mathrm{CR}}u, u_{\mathrm{CR}})u_{\mathrm{CR}}$ and $N := \dim(\mathrm{CR}_0^1(\mathcal{T}))$. Since the discrete eigenfunctions $(u_{\mathrm{CR},j} \mid j = 1, \dots, N)$ form an L^2-orthonormal basis of $\mathrm{CR}_0^1(\mathcal{T})$ and v_{CR} is L^2 orthogonal on $\mathrm{span}\{u_{\mathrm{CR}}\} \equiv \mathrm{span}\{u_{\mathrm{CR},1}\}$, there exist coefficients $(\alpha_j \mid j = 2, \dots N)$ such that

$$v_{\mathrm{CR}} = \sum_{j=2}^N \alpha_j u_{\mathrm{CR},j} \quad \text{and} \quad \sum_{j=2}^N \alpha_j^2 = \|v_{\mathrm{CR}}\|^2_{L^2(\Omega)}.$$

The definition of G_{CR} shows that

$$(\lambda_{\mathrm{CR},j} - \lambda)b(G_{\mathrm{CR}}u, u_{\mathrm{CR},j}) = \lambda b(u - G_{\mathrm{CR}}u, u_{\mathrm{CR},j}).$$

Therefore the orthogonality and the preceding identities lead to

$$\|v_{\mathrm{CR}}\|^2_{L^2(\Omega)} = b(G_{\mathrm{CR}}u, \sum_{j=2}^N \alpha_j u_{\mathrm{CR},j}) = b(u - G_{\mathrm{CR}}u, \sum_{j=2}^N \alpha_j \frac{\lambda}{\lambda_{\mathrm{CR},j} - \lambda} u_{\mathrm{CR},j}).$$

The Cauchy inequality, the estimate (2.6) and the L^2-orthogonality of the discrete eigenfunctions therefore shows

$$\|v_{\mathrm{CR}}\|_{L^2(\Omega)} \leq M\|u - G_{\mathrm{CR}}u\|_{L^2(\Omega)}.$$

The combination with (2.9) concludes the proof. □

One of the difficulties in the proof of Theorem 2.1 is the fact that the right-hand side $u - G_{\mathrm{CR}}u$ in the duality argument does *not* belong to V. This difficulty is circumvented by the use of the companion operator of Proposition 2.3. A similar result has been derived independently in [28].

Lemma 2.5 (L^2 control for $u - G_{\mathrm{CR}}u$)**.** *The first exact and discrete eigenfunctions satisfy*

$$\|u - G_{\mathrm{CR}}u\|_{L^2(\Omega)} \lesssim \|h_0\|_\infty^s \||u - u_{\mathrm{CR}}\||_{\mathrm{NC}}.$$

Proof. Let $e := u - G_{\mathrm{CR}}u$ and let $z \in V$ denote the solution of the following linear Poisson problem

$$a(z, v) = b(e, v) \quad \text{for all } v \in V.$$

Since $\Pi_0(G_{\mathrm{CR}}u - J_4 G_{\mathrm{CR}}u) = 0$, it holds

$$\begin{aligned}\|e\|_{L^2(\Omega)}^2 &= b(J_4 G_{\mathrm{CR}}u - G_{\mathrm{CR}}u, e) + b(e, u - J_4 G_{\mathrm{CR}}u)\\ &= b(J_4 G_{\mathrm{CR}}u - G_{\mathrm{CR}}u, (1 - \Pi_0)e) + a(z, u - J_4 G_{\mathrm{CR}}u).\end{aligned} \tag{2.10}$$

Piecewise Poincaré inequalities and (2.2) lead to

$$b(J_4 G_{\mathrm{CR}}u - G_{\mathrm{CR}}u, (1 - \Pi_0)e) \lesssim \|h_0\|_\infty^2 \||e\||_{\mathrm{NC}}^2.$$

Since e is perpendicular to the conforming finite element functions in $P_1(\mathcal{T}) \cap V$ and since $\Pi_0 \nabla_{\mathrm{NC}}(G_{\mathrm{CR}}u - J_4 G_{\mathrm{CR}}u) = 0$, the Scott-Zhang [31] quasi-interpolation $z_{\mathrm{C}} \in P_1(\mathcal{T}) \cap V$ of z satisfies

$$a(z, u - J_4 G_{\mathrm{CR}}u) = a_{\mathrm{NC}}(e, z - z_{\mathrm{C}}) + a_{\mathrm{NC}}(G_{\mathrm{CR}}u - J_4 G_{\mathrm{CR}}u, z - z_{\mathrm{C}}).$$

Standard a priori estimates [12] and elliptic regularity imply

$$\||z - z_{\mathrm{C}}\|| \lesssim \|h_0\|_\infty^s \|z\|_{H^{1+s}(\Omega)} \lesssim \|h_0\|_\infty^s \|e\|_{L^2(\Omega)}$$

The combination of the above estimates with (2.2) proves

$$\|e\|_{L^2(\Omega)} \lesssim \|h_0\|_\infty^s \||e\||_{\mathrm{NC}}.$$

The following best-approximation for the nonconforming approximation of the Poisson problem (here with right-hand side λu) can be found in [15, 26, 28]. The improved oscillation term on the right-hand side

$$\||e\||_{\mathrm{NC}} \lesssim \min_{v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})} \||u - v_{\mathrm{CR}}\||_{\mathrm{NC}} + \min_{p \in P_1(\mathcal{T})} \|h_{\mathcal{T}}(\lambda u - p)\|_{L^2(\Omega)}$$

can be obtained by a refined efficiency analysis as in [28]. The combination of the foregoing two displayed inequalities leads to

$$\|u - G_{\mathrm{CR}}u\|_{L^2(\Omega)} \lesssim \|h_0\|_\infty^s (\||u - u_{\mathrm{CR}}\||_{\mathrm{NC}} + \|h_0\|_\infty \lambda \|u - u_{\mathrm{CR}}\|_{L^2(\Omega)}).$$

The discrete Friedrichs inequality [12, Theorem 10.6.12] concludes the proof. $\square$

Proof of Theorem 2.1. Lemmas 2.4–2.5 prove

$$\|u - u_{\mathrm{CR}}\|_{L^2(\Omega)} \le \sqrt{2}(1 + M)\|u - G_{\mathrm{CR}}u\|_{L^2(\Omega)} \lesssim \|h_0\|_\infty^s \||u - u_{\mathrm{CR}}\||_{\mathrm{NC}}.$$

For the proof of the eigenvalue error bound, elementary algebra with $\||u\||^2 = \lambda$ and $\||u_{\mathrm{CR}}\||_{\mathrm{NC}}^2 = \lambda_{\mathrm{CR}}$ and $\|u\|_{L^2(\Omega)} = 1 = \|u_{\mathrm{CR}}\|_{L^2(\Omega)}$ proves

$$\lambda - \lambda_{\mathrm{CR}} + \||u - u_{\mathrm{CR}}\||_{\mathrm{NC}}^2 = \lambda \|u - u_{\mathrm{CR}}\|_{L^2(\Omega)}^2 + 2\big(\lambda b(u, u_{\mathrm{CR}}) - a_{\mathrm{NC}}(u, u_{\mathrm{CR}})\big). \tag{2.11}$$

The eigenvalue problem proves for the last contribution that

$$\lambda b(u, u_{\mathrm{CR}}) - a_{\mathrm{NC}}(u, u_{\mathrm{CR}}) = \lambda b(u, u_{\mathrm{CR}} - J_4 u_{\mathrm{CR}}) - a_{\mathrm{NC}}(u, u_{\mathrm{CR}} - J_4 u_{\mathrm{CR}}).$$

Since $u_{\mathrm{CR}} - J_4 u_{\mathrm{CR}}$ is L^2-orthogonal on u_{CR} and since $\nabla_{\mathrm{NC}}(u_{\mathrm{CR}} - J_4 u_{\mathrm{CR}})$ is L^2-orthogonal on $\Pi_0 \nabla u$, this equals

$$\lambda b(u - u_{\mathrm{CR}}, u_{\mathrm{CR}} - J_4 u_{\mathrm{CR}}) - \int_\Omega (1 - \Pi_0)\nabla u \cdot \nabla_{\mathrm{NC}}(u_{\mathrm{CR}} - J_4 u_{\mathrm{CR}})\, dx.$$

The estimates (2.2) and $\|(1-\Pi_0)\nabla u\|_{L^2(\Omega)} \lesssim \|h_0\|_\infty^s$ therefore prove

$$|\lambda b(u, u_{\mathrm{CR}}) - a_{\mathrm{NC}}(u, u_{\mathrm{CR}})| \lesssim \|u - u_{\mathrm{CR}}\|_{L^2(\Omega)} \|h_0\|_\infty |||u - u_{\mathrm{CR}}|||_{\mathrm{NC}} + \|h_0\|_\infty^s |||u - u_{\mathrm{CR}}|||.$$

The combination with (2.11) and the fact [19] that $\lambda_{\mathrm{CR}} \le \lambda$ for $\|h_0\|_\infty \ll 1$ conclude the proof. $\square$

3. Comparison Results

This section states an equivalence result for the errors of the eigenfunction approximations by conforming and nonconforming finite element methods as initiated in [9, 15] for linear problems. This comparison result is utilised in Section 6 to change the approximation seminorm and so enables the optimality proof.

Theorem 3.1. *Let $\Pi_0 \nabla u$ denote the L^2 best-approximation of the gradient of an exact eigenfunction u corresponding to the first exact eigenvalue λ onto piecewise constants. For sufficiently small $\|h_0\|_\infty \ll 1$, the discrete eigenfunctions u_{CR} and u_{C} with $b(u, u_{\mathrm{C}}) > 0$ and $b(u, u_{\mathrm{CR}}) > 0$ and $\|u_{\mathrm{C}}\|_{L^2(\Omega)} = 1 = \|u_{\mathrm{CR}}\|_{L^2(\Omega)}$ satisfy*

$$|||u - u_{\mathrm{C}}||| \approx |||u - u_{\mathrm{CR}}|||_{\mathrm{NC}} \approx \|\nabla u - \Pi_0 \nabla u\|_{L^2(\Omega)}. \tag{3.1}$$

Proof. The nonconforming interpolation operator I_{CR} is defined by

$$(\mathrm{I}_{\mathrm{CR}}\, v)(\mathrm{mid}(E)) = \fint_E v\, ds \quad \text{for all } E \in \mathcal{E} \text{ and all } v \in V. \tag{3.2}$$

An integration by parts proves the integral mean property of the gradients

$$\Pi_0 \nabla = \nabla_{\mathrm{NC}}\, \mathrm{I}_{\mathrm{CR}}\,. \tag{3.3}$$

The proof of comparison departs with the split

$$\begin{aligned} &|||u - u_{\mathrm{CR}}|||_{\mathrm{NC}}^2 \\ &= a_{\mathrm{NC}}(u, J_4 u_{\mathrm{CR}} - u_{\mathrm{CR}}) + a(u, u - J_4 u_{\mathrm{CR}}) - a_{\mathrm{NC}}(u_{\mathrm{CR}}, \mathrm{I}_{\mathrm{CR}}\, u - u_{\mathrm{CR}}). \end{aligned} \tag{3.4}$$

The integral mean property of the gradient $\Pi_0 \nabla J_4 = \nabla_{\mathrm{NC}}$ shows

$$\begin{aligned} a_{\mathrm{NC}}(u, J_4 u_{\mathrm{CR}} - u_{\mathrm{CR}}) &= a_{\mathrm{NC}}(u - \mathrm{I}_{\mathrm{CR}}\, u, J_4 u_{\mathrm{CR}} - u_{\mathrm{CR}}) \\ &\le |||u - \mathrm{I}_{\mathrm{CR}}\, u|||_{\mathrm{NC}} |||u_{\mathrm{CR}} - J_4 u_{\mathrm{CR}}|||_{\mathrm{NC}}. \end{aligned}$$

This, the projection property (3.3) of I_{CR}, and the stability property (2.2) of J_4 imply

$$a_{\mathrm{NC}}(u, J_4 u_{\mathrm{CR}} - u_{\mathrm{CR}}) \lesssim \|\nabla u - \Pi_0 \nabla u\|_{L^2(\Omega)} |||u - u_{\mathrm{CR}}|||_{\mathrm{NC}}. \tag{3.5}$$

The eigenvalue problem on the continuous and discrete level plus some algebra imply for the last and second last term of (3.4) that

$$\begin{aligned} &a_{\mathrm{NC}}(u, u - J_4 u_{\mathrm{CR}}) - a_{\mathrm{NC}}(u_{\mathrm{CR}}, \mathrm{I}_{\mathrm{CR}}\, u - u_{\mathrm{CR}}) \\ &\quad = b(\lambda u, u - J_4 u_{\mathrm{CR}}) - b(\lambda_{\mathrm{CR}} u_{\mathrm{CR}}, \mathrm{I}_{\mathrm{CR}}\, u - u_{\mathrm{CR}}) \\ &\quad = b(\lambda u - \lambda_{\mathrm{CR}} u_{\mathrm{CR}}, u - J_4 u_{\mathrm{CR}}) \\ &\qquad + b(\lambda_{\mathrm{CR}} u_{\mathrm{CR}}, u - \mathrm{I}_{\mathrm{CR}}\, u) + b(\lambda_{\mathrm{CR}} u_{\mathrm{CR}}, u_{\mathrm{CR}} - J_4 u_{\mathrm{CR}}). \end{aligned} \tag{3.6}$$

By the design of J_4, the term $u_{\mathrm{CR}} - J_4 u_{\mathrm{CR}}$ is orthogonal on piecewise affine functions. Thus,

$$b(\lambda_{\mathrm{CR}} u_{\mathrm{CR}}, u_{\mathrm{CR}} - J_4 u_{\mathrm{CR}}) = 0.$$

The arguments from Corollary 2.2 with $(\lambda_{\ell+m}, u_{\ell+m})$ replaced by (λ, u) show

$$b(\lambda u - \lambda_{\mathrm{CR}} u_{\mathrm{CR}}, u - J_4 u_{\mathrm{CR}}) \lesssim \|h_0\|_\infty^s \| u - u_{\mathrm{CR}} \|_{\mathrm{NC}} \, \|u - J_4 u_{\mathrm{CR}}\|_{L^2(\Omega)}.$$

For sufficiently small $\|h_0\|_\infty \ll 1$, u_{C} and u_{CR} satisfy $b(u_{\mathrm{C}}, u_{\mathrm{CR}}) \geq 0$ and therefore Theorem 2.1 plus a triangle inequality followed by (2.2) show

$$\begin{aligned} \|u - J_4 u_{\mathrm{CR}}\|_{L^2(\Omega)} &\lesssim \|u - u_{\mathrm{CR}}\|_{L^2(\Omega)} \|u_{\mathrm{CR}} - J_4 u_{\mathrm{CR}}\|_{L^2(\Omega)} \\ &\lesssim \|h_0\|_\infty^s \| u - u_{\mathrm{CR}} \|_{\mathrm{NC}} + \|h_0\|_\infty \| u - u_{\mathrm{CR}} \|_{\mathrm{NC}}. \end{aligned}$$

Hence,

$$b(\lambda u - \lambda_{\mathrm{CR}} u_{\mathrm{CR}}, u - J_4 u_{\mathrm{CR}}) \lesssim \|h_0\|_\infty^{2s} \| u - u_{\mathrm{CR}} \|_{\mathrm{NC}}^2. \tag{3.7}$$

The L^2 error estimate [12, 14] for the nonconforming interpolation reads

$$\|u - \mathrm{I}_{\mathrm{CR}}\, u\|_{L^2(\Omega)} \lesssim \| h_{\mathcal{T}} (u - \mathrm{I}_{\mathrm{CR}}\, u) \|_{\mathrm{NC}}. \tag{3.8}$$

This and the projection property $\nabla_{\mathrm{NC}} \mathrm{I}_{\mathrm{CR}} = \Pi_0 \nabla$ lead to

$$b(\lambda_{\mathrm{CR}} u_{\mathrm{CR}}, u - \mathrm{I}_{\mathrm{CR}}\, u) \lesssim \|h_{\mathcal{T}} \lambda_{\mathrm{CR}} u_{\mathrm{CR}}\|_{L^2(\Omega)} \|\nabla u - \Pi_0 \nabla u\|_{L^2(\Omega)}. \tag{3.9}$$

The efficiency of the term $\|h_{\mathcal{T}} \lambda_{\mathrm{CR}} u_{\mathrm{CR}}\|_{L^2(\Omega)}$ is discussed in Subsection 4.2 based on [24]. Independently of Section 3, Theorem 4.4 shows

$$\|h_{\mathcal{T}} \lambda_{\mathrm{CR}} u_{\mathrm{CR}}\|_{L^2(\Omega)} \lesssim \| u - u_{\mathrm{CR}} \|_{\mathrm{NC}}. \tag{3.10}$$

The combination of (3.4)–(3.10) leads to

$$\| u - u_{\mathrm{CR}} \|_{\mathrm{NC}} \lesssim \|\nabla u - \Pi_0 \nabla u\|_{L^2(\Omega)} + \|h_0\|_\infty^{2s} \| u - u_{\mathrm{CR}} \|_{\mathrm{NC}}.$$

For $\|h_0\|_\infty \ll 1$, the second term can be absorbed. This proves

$$\| u - u_{\mathrm{CR}} \|_{\mathrm{NC}} \lesssim \|\nabla u - \Pi_0 \nabla u\|_{L^2(\Omega)}.$$

The comparison of $\| u - u_{\mathrm{C}} \|$ with $\| u - u_{\mathrm{CR}} \|_{\mathrm{NC}}$ is inspired by [15] for the Poisson problem. The inclusion $P_1(\mathcal{T}) \cap C_0(\Omega) \subset \mathrm{CR}_0^1(\mathcal{T})$ implies for

$$v_{\mathrm{C}} := \operatorname*{argmin}_{w_{\mathrm{C}} \in P_1(\mathcal{T}) \cap C_0(\Omega)} \| u_{\mathrm{CR}} - w_{\mathrm{C}} \|_{\mathrm{NC}}$$

that

$$\begin{aligned} \| u_{\mathrm{CR}} - u_{\mathrm{C}} \|_{\mathrm{NC}}^2 &= a_{\mathrm{NC}}(u_{\mathrm{CR}} - u_{\mathrm{C}}, u_{\mathrm{CR}} - v_{\mathrm{C}}) \\ &\quad + b(\lambda_{\mathrm{CR}} u_{\mathrm{CR}} - \lambda_{\mathrm{C}} u_{\mathrm{C}}, v_{\mathrm{C}} - u_{\mathrm{C}}) \\ &\leq \| u_{\mathrm{CR}} - u_{\mathrm{C}} \|_{\mathrm{NC}} \, \| u_{\mathrm{CR}} - v_{\mathrm{C}} \|_{\mathrm{NC}} \\ &\quad + \|\lambda_{\mathrm{CR}} u_{\mathrm{CR}} - \lambda_{\mathrm{C}} u_{\mathrm{C}}\|_{L^2(\Omega)} \, \|v_{\mathrm{C}} - u_{\mathrm{C}}\|_{L^2(\Omega)}. \end{aligned} \tag{3.11}$$

The bound for the eigenvalues $\lambda_{\mathrm{CR}} \leq \lambda_{\mathrm{C}} \lesssim 1$ and the normalisation $\|u_{\mathrm{C}}\|_{L^2(\Omega)} = 1$ yield

$$\|\lambda_{\mathrm{CR}} u_{\mathrm{CR}} - \lambda_{\mathrm{C}} u_{\mathrm{C}}\|_{L^2(\Omega)} \lesssim \|u_{\mathrm{CR}} - u_{\mathrm{C}}\|_{L^2(\Omega)} + |\lambda_{\mathrm{CR}} - \lambda_{\mathrm{C}}|.$$

Therefore, the triangle and Young inequalities control the last term in (3.11) as

$$\begin{aligned} &\|\lambda_{\mathrm{CR}} u_{\mathrm{CR}} - \lambda_{\mathrm{C}} u_{\mathrm{C}}\|_{L^2(\Omega)} \, \|v_{\mathrm{C}} - u_{\mathrm{C}}\|_{L^2(\Omega)} \\ &\qquad \lesssim \|u_{\mathrm{CR}} - u_{\mathrm{C}}\|_{L^2(\Omega)}^2 + |\lambda_{\mathrm{CR}} - \lambda_{\mathrm{C}}|^2 + \|v_{\mathrm{C}} - u_{\mathrm{CR}}\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.12}$$

Known a priori results [7, 34] for conforming eigenvalue approximations read

$$|\lambda - \lambda_{\mathrm{C}}| + \|u - u_{\mathrm{C}}\|_{L^2(\Omega)} \lesssim \|h_0\|_\infty^s |\!|\!| u - u_{\mathrm{C}} |\!|\!|. \tag{3.13}$$

This, Theorem 2.1 and the triangle and Young inequalities bound the right-hand side of (3.12) by

$$\|h_0\|_\infty^{2s} |\!|\!| u - u_{\mathrm{C}} |\!|\!|^2 + \|h_0\|_\infty^{2s} |\!|\!| u - u_{\mathrm{CR}} |\!|\!|_{\mathrm{NC}}^2 + \|v_{\mathrm{C}} - u_{\mathrm{CR}}\|_{L^2(\Omega)}^2.$$

The discrete Friedrichs inequality [12, Theorem 10.6.12] reads $\|v_{\mathrm{C}} - u_{\mathrm{CR}}\|_{L^2(\Omega)} \lesssim |\!|\!| v_{\mathrm{C}} - u_{\mathrm{CR}} |\!|\!|_{\mathrm{NC}}$. It is known [13, Theorem 5.1] that

$$|\!|\!| u_{\mathrm{CR}} - v_{\mathrm{C}} |\!|\!|_{\mathrm{NC}} \approx \min_{v \in V} |\!|\!| u_{\mathrm{CR}} - v |\!|\!|_{\mathrm{NC}} \leq |\!|\!| u_{\mathrm{CR}} - u |\!|\!|_{\mathrm{NC}}.$$

The preceding two displayed formulas and (3.11) yield

$$|\!|\!| u_{\mathrm{CR}} - u_{\mathrm{C}} |\!|\!|_{\mathrm{NC}}^2 \lesssim |\!|\!| u_{\mathrm{CR}} - u |\!|\!|_{\mathrm{NC}} |\!|\!| u_{\mathrm{CR}} - u_{\mathrm{C}} |\!|\!|_{\mathrm{NC}} + |\!|\!| u_{\mathrm{CR}} - u |\!|\!|_{\mathrm{NC}}^2 + \|h_0\|_\infty^{2s} |\!|\!| u - u_{\mathrm{C}} |\!|\!|^2.$$

The term $|\!|\!| u_{\mathrm{CR}} - u_{\mathrm{C}} |\!|\!|_{\mathrm{NC}}$ on the right-hand side can be absorbed. This plus the triangle inequality and $\|h_0\|_\infty \ll 1$ prove the assertion

$$|\!|\!| u - u_{\mathrm{C}} |\!|\!| \leq |\!|\!| u - u_{\mathrm{CR}} |\!|\!|_{\mathrm{NC}} + |\!|\!| u_{\mathrm{CR}} - u_{\mathrm{C}} |\!|\!| \lesssim |\!|\!| u - u_{\mathrm{CR}} |\!|\!|_{\mathrm{NC}}.$$

The remaining inequalities are obvious. □

4. Algorithm and Optimality

This section presents the adaptive algorithm ACREVFEM and its optimality in terms of the approximation seminorm. The section adopts the notation of the previous sections for a sequence of regular triangulations $\mathcal{T}_\ell$ with mesh-size $h_\ell := h_{\mathcal{T}_\ell}$ and interior edges $\mathcal{E}_\ell(\Omega)$, boundary edges $\mathcal{E}_\ell(\partial\Omega)$ and $\mathcal{E}_\ell := \mathcal{E}_\ell(\Omega) \cup \mathcal{E}_\ell(\partial\Omega)$. The notation for the piecewise gradient $\nabla_{\mathrm{NC}(\ell)}$ and the discrete scalar product $a_{\mathrm{NC}(\ell)}$ depends on the triangulation $\mathcal{T}_\ell$ and, hence, on the level ℓ. The index ℓ is dropped, whenever there is no risk of confusion. The first discrete eigenpair on the level ℓ is denoted by $(\lambda_\ell, u_\ell) \in \mathbb{R} \times \mathrm{CR}_0^1(\mathcal{T}_\ell)$.

4.1. **Adaptive algorithm ACREVFEM.**

INPUT. Given an initial triangulation $\mathcal{T}_0$ with maximal mesh-size $\|h_0\|_\infty$ (and refinement edges $RE(\mathcal{T}_0)$ as in [6, 33]), the bulk parameter $0 < \theta \leq 1$, and $0 < \kappa < 1/2$, the adaptive algorithm ACREVFEM runs the following loop.

For $\ell = 0, 1, 2, \ldots$ (until termination) **do**

INEXACT SOLVE. Throughout this paper, the algebraic eigenvalue problem (1.2) is solved approximately with some known discrete approximation $(\tilde{\lambda}_\ell, \tilde{u}_\ell) \in \mathbb{R} \times \mathrm{CR}_0^1(\mathcal{T}_\ell)$ such that $\|\tilde{u}_\ell\|_{L^2(\Omega)} = 1$ and

$$|\!|\!| u_\ell - \tilde{u}_\ell |\!|\!|_{\mathrm{NC}}^2 + |\lambda_\ell - \tilde{\lambda}_\ell|^2 \leq \kappa \min\{\eta_\ell^2, \eta_{\ell-1}^2\} \tag{4.1}$$

for $\ell \in \mathbb{N}_0$ (with $\eta_{-1} := +\infty$).

Remark 4.1. The inexact solve is unavoidable in iterative procedures for the algebraic eigenvalue problem. The interaction of ESTIMATE and SOLVE breaks with the traditional AFEM loop in that the tolerance $\kappa\eta_\ell^2$ is not known in (4.1) when the termination is applied. In other words, the assumption (4.1) cannot be implemented straight away but needs to be linked in an internal loop with the computation of η_ℓ in (4.2). We refer to [2, 3] for the analysis of a similar algorithm for linear problems and to [18] for an example of a practical realisation in the context of conforming FEMs for eigenvalue problems.

That paper [18] furthermore illustrates that optimal complexity of an overall strategy can in fact be expected under realistic assumptions on the performance of the algebraic solver in SOLVE. This paper focusses on the convergence analysis of the discretisation and, hence, omits further algorithmic details on the algebraic eigenvalue problem.

ESTIMATE. For any interior edge $E = T_+ \cap T_-$ shared by the two triangles $T_\pm \in \mathcal{T}_\ell$ with edge-patch $\omega_E := \operatorname{int}(T_+ \cup T_-)$, let $[\cdot]_E := \cdot|_{T_+} - \cdot|_{T_-}$ denote the jump across E. For $E \in \mathcal{E}_\ell(\partial\Omega)$, the jump is defined as $[\cdot]_E := \cdot|_{T_+}$ for the one element T_+ with $E \subset T_+$ and $\omega_E := \operatorname{int}(T_+)$ owing to the homogeneous boundary conditions. For any $T \in \mathcal{T}_\ell$ with set $\mathcal{E}_\ell(T)$ of edges and the known approximations $\tilde\lambda_\ell$ and $\tilde u_\ell$ with (4.1), set $\eta_\ell^2 := \eta_\ell^2(\mathcal{T}_\ell) := \sum_{T\in\mathcal{T}_\ell} \eta_\ell^2(T)$, where, for any $T \in \mathcal{T}$,

$$\eta_\ell^2(T) := |T|\, \|\tilde\lambda_\ell \tilde u_\ell\|_{L^2(T)}^2 + |T|^{1/2} \sum_{E\in\mathcal{E}_\ell(T)} \|[\partial \tilde u_\ell/\partial s]_E\|_{L^2(E)}^2. \tag{4.2}$$

MARK. The bulk criterion [25] selects an (almost) minimal subset $\mathcal{M}_\ell \subset \mathcal{T}_\ell$ of triangles with

$$\theta \eta_\ell^2 \le \eta_\ell^2\,(\mathcal{M}_\ell) := \sum_{T\in\mathcal{M}_\ell} \eta_\ell^2(T). \tag{4.3}$$

REFINE. Given the marked edges $\mathcal{M}_\ell$ in $\mathcal{T}_\ell$, refine the triangulation with newest-vertex bisection (NVB) [6, 33] of Figure 1 and generate a minimal regular triangulation $\mathcal{T}_{\ell+1}$ in which at least the marked edges are refined. The refinement edge $RE : \mathcal{T}_0 \to \mathcal{E}_0$, with $RE(T) \in \mathcal{E}_0(T)$ for any $T \in \mathcal{T}_0$, is fixed for the initial triangulation $\mathcal{T}_0$; the configuration of the refinement edges in refined triangles is depicted in Figure 1. The result $\mathcal{T}_{\ell+1}$ of REFINE is the smallest regular refinement of $\mathcal{T}_\ell$ from NVB, where at least the refinement edges of the triangles in $\mathcal{M}_\ell$ are bisected [10].
od

OUTPUT. Sequence of triangulations $(\mathcal{T}_\ell)_\ell$ and discrete approximations $(\tilde\lambda_\ell, \tilde u_\ell)_\ell$ with $b(u, \tilde u_\ell) > 0$.

Remark 4.2. The analysis of the following sections relies on the assumption of a sufficiently fine initial mesh $\mathcal{T}_0$ with mesh-size $\|h_0\|_\infty \ll 1$ such that the results from Sections 2–3 are valid.

Remark 4.3. The discussion of the next subsection (cf. Remark 4.5) shows that a proper choice of κ and a sufficiently fine initial mesh-size guarantee $b(u, \tilde u_\ell) \neq 0$. Hence, the output of the adaptive algorithm is uniquely defined.

4.2. **Efficiency and reliability of the error estimator.** Recall that the parameter $0 < s \le 1$ describes the elliptic regularity of the Poisson problem as in (2.1) and $\|h_0\|_\infty$ denotes the maximal mesh-size of $\mathcal{T}_0$.

Theorem 4.4 (efficiency and reliability [24])**.** *The error estimator* $\mu_\ell := \mu_\ell(\mathcal{T}_\ell) := \left(\sum_{T\in\mathcal{T}_\ell} \mu_\ell^2(T)\right)^{1/2}$ *with respect to the exact discrete eigenpair* (λ_ℓ, u_ℓ) *with* $b(u, u_\ell) > 0$*, namely*

$$\mu_\ell^2(T) := |T|\, \|\lambda_\ell u_\ell\|_{L^2(T)}^2 + |T|^{1/2} \sum_{E\in\mathcal{E}_\ell(T)} \|[\partial u_\ell/\partial s]_E\|_{L^2(E)}^2 \quad \textit{for all } T \in \mathcal{T},$$

is reliable and efficient in the sense that

$$\begin{aligned} |||u - u_\ell|||_{\mathrm{NC}} &\lesssim \mu_\ell + \|h_0\|_\infty^s |||u - u_\ell|||_{\mathrm{NC}}, \\ \mu_\ell &\lesssim (1 + \|h_0\|_\infty^s) |||u - u_\ell|||_{\mathrm{NC}}. \end{aligned} \tag{4.4}$$

Proof. It is proven in [24] that

$$\begin{aligned} \|\!|u-u_\ell|\!\|_{\mathrm{NC}} &\lesssim \mu_\ell + |\lambda-\lambda_\ell| + (\lambda\lambda_\ell)^{1/2}\|u-u_\ell\|_{L^2(\Omega)}, \\ \mu_\ell &\lesssim \|\!|u-u_\ell|\!\|_{\mathrm{NC}} + \|h_\ell(\lambda u - \lambda_\ell u_\ell)\|_{L^2(\Omega)}. \end{aligned} \tag{4.5}$$

It is stated in [24] that, according to known a priori estimates, the additional terms in (4.5) are of higher order. Indeed, the results from Section 2 prove

$$|\lambda-\lambda_\ell| + (\lambda\lambda_\ell)^{1/2}\|u-u_\ell\|_{L^2(\Omega)} + \|h_\ell(\lambda u-\lambda_\ell u_\ell)\|_{L^2(\Omega)} \lesssim \|h_0\|_\infty^s \|\!|u-u_\ell|\!\|_{\mathrm{NC}}. \quad \square$$

The following lemma plus the triangle inequality and (4.1) imply efficiency and reliability of the error estimator $\eta_\ell^2 = \sum_{T\in\mathcal{T}_\ell}\eta_\ell^2(T)$ with an approximate eigenpair $(\tilde\lambda_\ell, \tilde u_\ell)$ and

$$\eta_\ell^2(T) = |T|\ \|\tilde\lambda_\ell \tilde u_\ell\|^2_{L^2(T)} + |T|^{1/2}\sum_{E\in\mathcal{E}_\ell(T)} \|[\partial \tilde u_\ell/\partial s]_E\|^2_{L^2(E)}.$$

The inexact discrete solutions $(\tilde\lambda_\ell, \tilde u_\ell)$ satisfy for, sufficiently small $\kappa \ll 1$, that

$$\|\!|u-\tilde u_\ell|\!\|_{\mathrm{NC}} \le C_{\mathrm{rel}}\left(\eta_\ell + \|h_0\|_\infty^s \|\!|u-\tilde u_\ell|\!\|_{\mathrm{NC}}\right), \tag{4.6}$$

$$\eta_\ell^2 \le C_{\mathrm{eff}}(1+\|h_0\|_\infty^{2s})\|\!|u-u_\ell|\!\|^2_{\mathrm{NC}}. \tag{4.7}$$

Remark 4.5. In particular this plus the L^2 control from Theorem 2.1 and the tolerance (4.1) imply

$$\|u-\tilde u_\ell\| \lesssim \left(\|h_0\|_\infty^s + \kappa(1+\|h_0\|_\infty^{2s})\right)\|\!|u-u_\ell|\!\|_{\mathrm{NC}}$$

and, therefore, $1 \lesssim b(u,\tilde u_\ell)$ for sufficiently small $\|h_0\|_\infty^{2s}$ and κ.

Lemma 4.6 (continuity of the error estimator)**.** *There exists $C_{\mathrm{cont}} \approx 1$ such that any subset $\mathcal{M}\subset\mathcal{T}_\ell$ satisfies*

$$|\eta_\ell(\mathcal{M}) - \mu_\ell(\mathcal{M})| \le C_{\mathrm{cont}}(\|\!|u_\ell - \tilde u_\ell|\!\|_{\mathrm{NC}} + |\lambda_\ell - \tilde\lambda_\ell|).$$

Proof. One triangle inequality in $\mathbb{R}^{4|\mathcal{M}|}$ is followed by another in $L^2(T)$ for any $T\in\mathcal{M}$ to verify

$$\begin{aligned}
&|\eta_\ell(\mathcal{M}) - \mu_\ell(\mathcal{M})| \\
&\quad = \left| \left(\sum_{T\in\mathcal{M}} \Big(|T|\ \|\tilde\lambda_\ell\tilde u_\ell\|^2_{L^2(T)} + |T|^{1/2}\sum_{E\in\mathcal{E}_\ell(T)} \|[\partial\tilde u_\ell/\partial s]_E\|^2_{L^2(E)} \Big)\right)^{1/2} \right. \\
&\qquad\qquad \left. - \left(\sum_{T\in\mathcal{M}} \Big(|T|\ \|\lambda_\ell u_\ell\|^2_{L^2(T)} + |T|^{1/2}\sum_{E\in\mathcal{E}_\ell(T)} \|[\partial u_\ell/\partial s]_E\|^2_{L^2(E)} \Big)\right)^{1/2} \right| \\
&\quad \le \left(\sum_{T\in\mathcal{M}} \Big(|T| \Big(\|\tilde\lambda_\ell\tilde u_\ell\|_{L^2(T)} - \|\lambda_\ell u_\ell\|_{L^2(T)} \Big)^2 \right. \\
&\qquad\qquad \left. + |T|^{1/2}\sum_{E\in\mathcal{E}_\ell(T)} \Big(\|[\partial\tilde u_\ell/\partial s]_E\|_{L^2(E)} - \|[\partial u_\ell/\partial s]_E\|_{L^2(E)} \Big)^2 \Big) \right)^{1/2} \\
&\quad \le \left(\sum_{T\in\mathcal{M}} \Big(|T| \|\tilde\lambda_\ell\tilde u_\ell - \lambda_\ell u_\ell\|^2_{L^2(T)} + |T|^{1/2} \sum_{E\in\mathcal{E}_\ell(T)} \|[\partial(\tilde u_\ell - u_\ell)/\partial s]_E\|^2_{L^2(E)} \Big)\right)^{1/2}.
\end{aligned}$$

The discrete Friedrichs inequality [12, Theorem 10.6.12] controls the first term

$$\sum_{T\in\mathcal{M}} |T| \|\tilde{\lambda}_\ell \tilde{u}_\ell - \lambda_\ell u_\ell\|^2_{L^2(T)} \lesssim |||u_\ell - \tilde{u}_\ell|||^2_{\mathrm{NC}} + |\lambda_\ell - \tilde{\lambda}_\ell|^2.$$

The trace inequality [12, p.282] leads to

$$\sum_{T\in\mathcal{M}} |T|^{1/2} \sum_{E\in\mathcal{E}_\ell(T)} \|[\partial(\tilde{u}_\ell - u_\ell)/\partial s]_E\|^2_{L^2(E)} \lesssim |||u_\ell - \tilde{u}_\ell|||^2_{\mathrm{NC}}. \qquad \square$$

4.3. **Approximation class.** Given an initial triangulation $\mathcal{T}_0$, a triangulation $\mathcal{T}_\ell$ is called an admissible triangulation, written $\mathcal{T}_\ell \in \mathbb{T}$, if there exist regular triangulations $\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_\ell$ such that, for $k = 1, \dots, \ell$, each $\mathcal{T}_k$ is generated from $\mathcal{T}_{k-1}$ with refinements from Figure 1.

The set of all such admissible triangulations is denoted by $\mathbb{T}$, while $\mathbb{T}(N)$ denotes the subset of all admissible triangulations with at most $|\mathcal{T}_0| + N$ triangles. For any $\mathcal{T} \in \mathbb{T}(N)$, let $\Pi_\mathcal{T}$ denote the L^2 best-approximation onto piecewise constants with respect to $\mathcal{T}$. For the first eigenpair (λ, u) and $\sigma > 0$ define

$$|u|_{\mathcal{A}_\sigma} := \sup_{N\in\mathbb{N}} N^\sigma \inf_{\mathcal{T}\in\mathbb{T}(N)} \|\nabla u - \Pi_\mathcal{T} \nabla u\|_{L^2(\Omega)}.$$

It is the comparison of Theorem 3.1 that allows the conclusion that $|u|_{\mathcal{A}_\sigma} < \infty$ for the first eigenpair (λ, u) leads to discrete eigenvalues which converge of the same rate σ (with respect to the optimal admissible meshes) and so enables the optimality analysis of this paper.

Optimal convergence rates means that $|u|_{\mathcal{A}_\sigma} < \infty$ for some $0 < \sigma < \infty$ implies the rate for the output (λ_ℓ, u_ℓ) of the adaptive algorithm (with an appropriate choice of u_ℓ amongst all eigenvectors of the minimal discrete eigenvalue) even on any level ℓ with $N_\ell := |\mathcal{T}_\ell| - |\mathcal{T}_0|$ in the sense that

$$N_\ell^\sigma \sup_{\ell\in\mathbb{N}_0} |||u - u_\ell|||_{\mathrm{NC}} \leq C_{\mathrm{opt}}\, |u|_{\mathcal{A}_\sigma}.$$

The point is that the constant $C_{\mathrm{opt}} \geq 1$ is bounded from above, $C_{\mathrm{opt}} < \infty$.

4.4. **Asymptotic optimality.** The following theorem states the quasi-optimal convergence of the adaptive algorithm; its proof follows at the end of Section 6.

Theorem 4.7 (quasi-optimal convergence). *Let Ω be simply connected. For sufficiently small $0 < \theta \ll 1$, $0 < \kappa \ll 1$, $0 < \|h_0\|_\infty := \|h_{\mathcal{T}_0}\|_{L^\infty(\Omega)} \ll 1$ and any $\sigma > 0$ with $|u|_{\mathcal{A}_\sigma} < \infty$,* AREVFEM *computes sequences of triangulations $(\mathcal{T}_\ell)_\ell$ and discrete solutions $(\tilde{\lambda}_\ell, \tilde{u}_\ell)_\ell$ of optimal rate of convergence in the sense that*

$$(|\mathcal{T}_\ell| - |\mathcal{T}_0|)^\sigma |||u - \tilde{u}_\ell|||_{\mathrm{NC}} \leq C_{\mathrm{opt}}\, |u|_{\mathcal{A}_\sigma} \quad \textit{for all } \ell = 0, 1, 2, \dots$$

5. Contraction Property

This section is devoted to the proof of the contraction property, which implies the convergence of the adaptive algorithm.

Theorem 5.1 (contraction property). *For sufficiently small $\|h_0\|_\infty$ and $0 < \kappa \ll 1$, there exist positive constants $0 < \beta$, $\gamma < \infty$ and $0 < \rho_2 < 1$ (which depend in addition on $\mathcal{T}_0$) such that, for any $\ell \in \mathbb{N}_0$ the following holds. The solution (λ_ℓ, u_ℓ), its approximation $(\tilde{\lambda}_\ell, \tilde{u}_\ell)$, the error estimator η_ℓ from (4.2) with respect to the triangulation $\mathcal{T}_\ell$ generated by* AREVFEM*, and the term*

$$\xi_\ell^2 := \eta_\ell^2 + \beta\, |||u - \tilde{u}_\ell|||^2_{\mathrm{NC}} + \gamma \|h_\ell \lambda_\ell u_\ell\|^2_{L^2(\Omega)}$$

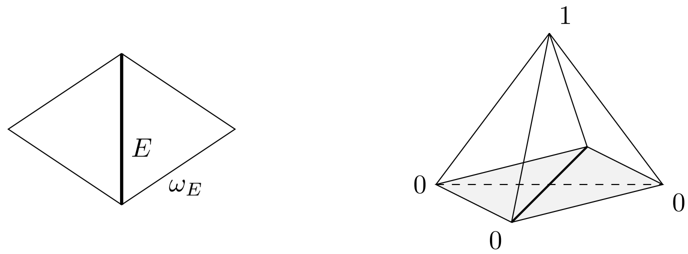

FIGURE 2. Edge patch ω_E and plot of the function ϕ_E.

satisfy

$$\xi_{\ell+1}^2 \leq \rho_2 \xi_\ell^2 \quad \textit{for } \ell = 0, 1, 2, \ldots \tag{5.1}$$

The proof is based on the error estimator reduction property.

Theorem 5.2 (error estimator reduction). *There exist constants $0 < \rho_1 < 1$ and $0 < \Lambda < \infty$ which depend only on $\mathcal{T}_0$, such that for the refinement $\mathcal{T}_{\ell+1}$ of $\mathcal{T}_\ell$ generated by* ACREVFEM *on two consecutive levels ℓ and $\ell+1$, the respective discrete approximations $\tilde{u}_\ell \in \mathrm{CR}_0^1(\mathcal{T}_\ell)$ and $\tilde{u}_{\ell+1} \in \mathrm{CR}_0^1(\mathcal{T}_{\ell+1})$ satisfy*

$$\begin{aligned} \eta_{\ell+1}^2 \leq \rho_1 \eta_\ell^2 + \Lambda\big(&|||\tilde{u}_{\ell+1} - \tilde{u}_\ell|||_{\mathrm{NC}}^2 + |\lambda_{\ell+1} - \tilde{\lambda}_{\ell+1}|^2 + |\lambda_\ell - \tilde{\lambda}_\ell|^2 \\ &+ |||u_{\ell+1} - \tilde{u}_{\ell+1}|||_{\mathrm{NC}}^2 + |||u_\ell - \tilde{u}_\ell|||_{\mathrm{NC}}^2\big). \end{aligned} \tag{5.2}$$

The proof employs the following lemma, which generalises [20, Theorem 4.1].

Lemma 5.3 (local discrete efficiency). *Any $v_\ell \in \mathrm{CR}_0^1(\mathcal{T}_\ell)$ and any edge $E \in \mathcal{E}_\ell \setminus \mathcal{E}_{\ell+m}$ with edge-patch ω_E satisfy*

$$|E|^{1/2} \, \|[\partial v_\ell/\partial s]_E\|_{L^2(E)} \lesssim \min_{v_{\ell+m} \in \mathrm{CR}_0^1(\mathcal{T}_{\ell+m})} \|\nabla_{\mathrm{NC}}(v_\ell - v_{\ell+m})\|_{L^2(\omega_E)}.$$

Proof. Let $\phi_E \in P_1(\mathcal{T}_{\ell+m}) \cap C(\Omega)$ be the piecewise affine continuation of $\phi_E(\mathrm{mid}(E)) = 1$ and $\phi_E = 0$ on $\partial\omega_E$ as in Figure 2. An integration by parts and the L^2 orthogonality of $\mathrm{Curl}\,\phi_E := (-\partial\phi_E/\partial x_2, \partial\phi_E/\partial x_1)$ on $\nabla_{\mathrm{NC}}\,\mathrm{CR}_0^1(\mathcal{T}_{\ell+m})$ prove

$$\begin{aligned} \pm|E|^{1/2}\|[\partial v_\ell/\partial s]_E\|_{L^2(E)} &= |E| \, [\partial v_\ell/\partial s]_E = 2\int_E [\partial v_\ell/\partial s]_E \phi_E \, ds \\ &= 2\int_{\omega_E} \nabla_{\mathrm{NC}} v_\ell \cdot \mathrm{Curl}\,\phi_E \, dx = 2\int_{\omega_E} \nabla_{\mathrm{NC}}(v_\ell - v_{\ell+m}) \cdot \mathrm{Curl}\,\phi_E \, dx. \end{aligned}$$

A Cauchy inequality plus a scaling argument for $\|\mathrm{Curl}\,\phi_E\|_{L^2(\omega_E)} \lesssim 1$ conclude the proof. □

Lemma 5.4 (discrete Friedrichs inequality on two levels). *Let $\mathcal{T}_{\ell+1}$ be some refinement of $\mathcal{T}_\ell$ generated by* ACREVFEM. *Any functions $v_{\ell+1} \in \mathrm{CR}_0^1(\mathcal{T}_{\ell+1})$ and $v_\ell \in \mathrm{CR}_0^1(\mathcal{T}_\ell)$ satisfy*

$$\|v_{\ell+1} - v_\ell\|_{L^2(\Omega)} \lesssim |||v_{\ell+1} - v_\ell|||_{\mathrm{NC}}.$$

Proof. The discrete Friedrichs inequality [12, Theorem 10.6.12] reads

$$\|v_{\ell+1} - v_\ell\|_{L^2(\Omega)}^2 \lesssim \sum_{F \in \mathcal{E}_{\ell+1}} \left| \fint_F [v_{\ell+1} - v_\ell]_F \, ds \right|^2 + |||v_{\ell+1} - v_\ell|||_{\mathrm{NC}}^2.$$

Note that each edge $E \in \mathcal{E}_\ell \setminus \mathcal{E}_{\ell+1}$ is bisected and, hence, $[v_\ell]_F$ vanishes at $\operatorname{mid}(E) \in F$, for $F \subset E$, $F \in \mathcal{E}_{\ell+1} \setminus \mathcal{E}_\ell$, while $\int_F [v_{\ell+1}]_F \, ds = 0$. Hence, the Friedrichs inequality along each edge $F \in \mathcal{E}_{\ell+1}$ yield

$$\begin{aligned}\sum_{F\in\mathcal{E}_{\ell+1}} \left| \fint_F [v_\ell - v_{\ell+1}]_F \, ds \right|^2 &= \sum_{F\in\mathcal{E}_{\ell+1}\setminus\mathcal{E}_\ell} \left| \fint_F [v_\ell]_F \, ds \right|^2 \\ \leq \sum_{F\in\mathcal{E}_{\ell+1}\setminus\mathcal{E}_\ell} |F|^{-1} \|[v_\ell]_F\|^2_{L^2(F)} &\lesssim \sum_{F\in\mathcal{E}_{\ell+1}\setminus\mathcal{E}_\ell} |F| \, \|[\partial v_\ell/\partial s]_F\|^2_{L^2(F)} \\ &\leq \sum_{E\in\mathcal{E}_\ell\setminus\mathcal{E}_{\ell+1}} |E| \, \|[\partial v_\ell/\partial s]_E\|^2_{L^2(E)}.\end{aligned}$$

This and Lemma 5.3 prove

$$\sum_{E\in\mathcal{E}_{\ell+1}} \left| \fint_E [v_{\ell+1} - v_\ell]_E ds \right|^2 \lesssim \|v_{\ell+1} - v_\ell\|^2_{\mathrm{NC}}. \qquad \square$$

Proof of Theorem 5.2. Let $\varrho(K) := 1/2$ if $K \in \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$ and $\varrho(K) := 1$ if $K \in \mathcal{T}_\ell \cap \mathcal{T}_{\ell+1}$. The triangle inequality implies for $K \in \mathcal{T}_\ell$ and all $T \in \mathcal{T}_{\ell+1}(K) := \{T' \in \mathcal{T}_{\ell+1} \,|\, T' \subseteq K\}$ that $|T| \leq \varrho(K)|K|$. Hence, it follows for all $0 < \mu < \infty$ that

$$\begin{aligned}&\sum_{T\in\mathcal{T}_{\ell+1}(K)} |T| \, \|\tilde\lambda_{\ell+1}\tilde u_{\ell+1}\|^2_{L^2(T)} \\ &\qquad \leq (1 + 1/\mu) \sum_{T\in\mathcal{T}_{\ell+1}(K)} |T| \, \|\tilde\lambda_{\ell+1}\tilde u_{\ell+1} - \tilde\lambda_\ell \tilde u_\ell\|^2_{L^2(T)} \\ &\qquad\quad + (1 + \mu) \sum_{T\in\mathcal{T}_{\ell+1}(K)} \varrho(K) \, |K| \, \|\tilde\lambda_\ell \tilde u_\ell\|^2_{L^2(T)}.\end{aligned}$$

Since $b(\tilde u_{\ell+1} + \tilde u_\ell, \tilde u_{\ell+1} - \tilde u_\ell) = 0$,

$$\begin{aligned}&4\|\tilde\lambda_{\ell+1}\tilde u_{\ell+1} - \tilde\lambda_\ell \tilde u_\ell\|^2_{L^2(\Omega)} \\ &\quad = (\tilde\lambda_{\ell+1} + \tilde\lambda_\ell)^2 \, \|\tilde u_{\ell+1} - \tilde u_\ell\|^2_{L^2(\Omega)} + (\tilde\lambda_{\ell+1} - \tilde\lambda_\ell)^2 \, \|\tilde u_{\ell+1} + \tilde u_\ell\|^2_{L^2(\Omega)} \\ &\quad \leq (\tilde\lambda_{\ell+1} + \tilde\lambda_\ell)^2 \, \|\tilde u_{\ell+1} - \tilde u_\ell\|^2_{L^2(\Omega)} + 4(\tilde\lambda_{\ell+1} - \tilde\lambda_\ell)^2.\end{aligned}$$

Since $\|u_\ell\|^2_{\mathrm{NC}} = \lambda_\ell$ and $\|u_{\ell+1}\|^2_{\mathrm{NC}} = \lambda_{\ell+1}$ are bounded, it holds

$$|\lambda_{\ell+1} - \lambda_\ell| = |a_{\mathrm{NC}}(u_{\ell+1} + u_\ell, u_{\ell+1} - u_\ell)| \lesssim \|u_{\ell+1} - u_\ell\|_{\mathrm{NC}}.$$

The triangle inequality therefore proves

$$\begin{aligned}|\tilde\lambda_{\ell+1} - \tilde\lambda_\ell| \lesssim \|\tilde u_{\ell+1} - \tilde u_\ell\|_{\mathrm{NC}} &+ \|u_{\ell+1} - \tilde u_{\ell+1}\|_{\mathrm{NC}} + \|u_\ell - \tilde u_\ell\|_{\mathrm{NC}} \\ &+ \left|\lambda_{\ell+1} - \tilde\lambda_{\ell+1}\right| + \left|\lambda_\ell - \tilde\lambda_\ell\right|.\end{aligned}$$

The combination of the above estimates with Lemma 5.4 for $\tilde u_\ell$ and $\tilde u_{\ell+1}$ plus $(\tilde\lambda_{\ell+1} + \tilde\lambda_\ell)^2 \lesssim 1$ yield

$$\begin{aligned}&\left(\sum_{T\in\mathcal{T}_{\ell+1}(K)} |T| \, \|\tilde\lambda_{\ell+1}\tilde u_{\ell+1}\|^2_{L^2(T)} \right) - (1 + \mu)\varrho(K) \, |K| \, \|\tilde\lambda_\ell \tilde u_\ell\|^2_{L^2(K)} \\ &\qquad \lesssim (1 + 1/\mu) \, |K| \big(\|\tilde u_\ell - \tilde u_{\ell+1}\|^2_{\mathrm{NC}} + |\tilde\lambda_{\ell+1} - \lambda_{\ell+1}|^2 + |\tilde\lambda_\ell - \lambda_\ell|^2 \\ &\qquad\qquad + \|\tilde u_{\ell+1} - u_{\ell+1}\|^2_{\mathrm{NC}} + \|\tilde u_\ell - u_\ell\|^2_{\mathrm{NC}} \big).\end{aligned}$$

The triangle inequality used for the second summand of the estimator and the trace inequality [12] with constant C_{tr} lead, for $K \in \mathcal{T}_\ell$, to

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_{\ell+1}(K)} |T|^{1/2} \sum_{E\in\mathcal{E}_{\ell+1}(T)} \|[\partial\tilde{u}_{\ell+1}/\partial s]_E\|^2_{L^2(E)}\\
&\qquad \le (1+1/\mu) \sum_{T\in\mathcal{T}_{\ell+1}(K)} |T|^{1/2} \sum_{E\in\mathcal{E}_{\ell+1}(T)} \|[\partial\tilde{u}_{\ell+1}/\partial s - \partial\tilde{u}_\ell/\partial s]_E\|^2_{L^2(E)}\\
&\qquad\quad + (1+\mu) \sum_{T\in\mathcal{T}_{\ell+1}(K)} |T|^{1/2} \sum_{E\in\mathcal{E}_{\ell+1}(T)} \|[\partial\tilde{u}_\ell/\partial s]_E\|^2_{L^2(E)}\\
&\qquad \le (1+1/\mu) \sum_{T\in\mathcal{T}_{\ell+1}(K)} C_{\mathrm{tr}}|T|^{1/2} \sum_{E\in\mathcal{E}_{\ell+1}(T)} |E|^{-1}\|\nabla\tilde{u}_{\ell+1} - \nabla\tilde{u}_\ell\|^2_{L^2(T)}\\
&\qquad\quad + (1+\mu)\,\varrho(K)^{1/2}|K|^{1/2} \sum_{E\in\mathcal{E}_\ell(K)} \|[\partial\tilde{u}_\ell/\partial s]_E\|^2_{L^2(E)}.
\end{aligned}$$

Since $|T|^{1/2}|E|^{-1} \approx 1$, the sum over all triangles in $\mathcal{T}_\ell$ yields

$$\begin{aligned}
&\eta^2_{\ell+1} - (1+\mu)\left(\eta^2_\ell(\mathcal{T}_\ell\cap\mathcal{T}_{\ell+1}) + \eta^2_\ell(\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+1})/\sqrt{2}\right)\\
&\qquad \lesssim (1+1/\mu)\big(\|\tilde{u}_\ell - \tilde{u}_{\ell+1}\|^2_{\mathrm{NC}} + |\tilde{\lambda}_{\ell+1} - \lambda_{\ell+1}|^2\\
&\qquad\qquad + |\tilde{\lambda}_\ell - \lambda_\ell|^2 + \|\tilde{u}_{\ell+1} - u_{\ell+1}\|^2_{\mathrm{NC}} + \|\tilde{u}_\ell - u_\ell\|^2_{\mathrm{NC}}\big).
\end{aligned}$$

The bulk criterion assures $\theta\eta^2_\ell \le \eta^2_\ell(\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+1})$ whence

$$\eta^2_\ell(\mathcal{T}_\ell\cap\mathcal{T}_{\ell+1}) + \eta^2_\ell(\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+1})/\sqrt{2} \le (1-\theta(1-1/\sqrt{2}))\eta^2_\ell$$

The combination of the preceding two estimates imply (5.2) with $\rho_1 := (1+\mu)(1-\theta(1-1/\sqrt{2})) < 1$ and some $\Lambda \approx (1+1/\mu)$ for sufficiently small $\mu > 0$. □

Quasi-orthogonality is the second main ingredient for the contraction property.

Theorem 5.5 (quasi-orthogonality)**.** *There exists some positive constant $C_{\mathrm{qo}} \approx 1$ which solely depends on $\mathcal{T}_0$ such that, for any refinement $\mathcal{T}_{\ell+m}$ of $\mathcal{T}_\ell$, the exact solution (λ, u) and the discrete solutions $(\lambda_{\ell+m}, u_{\ell+m})$ and (λ_ℓ, u_ℓ) (with respect to $\mathcal{T}_{\ell+m}$ and $\mathcal{T}_\ell$) with inexact approximations $(\tilde{\lambda}_{\ell+m}, \tilde{u}_{\ell+m})$ and $(\tilde{\lambda}_\ell, \tilde{u}_\ell)$ with $\|\tilde{u}_\ell\|_{L^2(\Omega)} = 1 = \|\tilde{u}_{\ell+m}\|_{L^2(\Omega)}$ satisfy*

$$\begin{aligned}
&|a_{\mathrm{NC}}(u-\tilde{u}_{\ell+m}, \tilde{u}_\ell - \tilde{u}_{\ell+m})|\\
&\quad \le C_{\mathrm{qo}}\big(\|h_\ell\lambda_\ell u_\ell\|_{L^2(\cup(\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m}))}\|u - \tilde{u}_{\ell+m}\|_{\mathrm{NC}}\\
&\qquad + \|u - \tilde{u}_{\ell+m}\|_{\mathrm{NC}}(\|u_\ell - \tilde{u}_\ell\|_{\mathrm{NC}} + \|u_{\ell+m} - \tilde{u}_{\ell+m}\|_{\mathrm{NC}})\\
&\qquad + \|u_{\ell+m} - \tilde{u}_{\ell+m}\|^2_{\mathrm{NC}} + \|h_0\|^{2s}_\infty(\|u - u_\ell\|^2_{\mathrm{NC}} + \|u - u_{\ell+m}\|^2_{\mathrm{NC}})\big).
\end{aligned} \tag{5.3}$$

Proof. Some elementary algebra plus the Cauchy inequality show

$$\begin{aligned}
&a_{\mathrm{NC}}(u - \tilde{u}_{\ell+m}, \tilde{u}_\ell - \tilde{u}_{\ell+m})\\
&= a_{\mathrm{NC}}(u - \tilde{u}_{\ell+m}, \tilde{u}_\ell - u_\ell)\\
&\qquad + a_{\mathrm{NC}}(u - \tilde{u}_{\ell+m}, u_{\ell+m} - \tilde{u}_{\ell+m}) + a_{\mathrm{NC}}(u - \tilde{u}_{\ell+m}, u_\ell - u_{\ell+m})\\
&\le \|u - \tilde{u}_{\ell+m}\|_{\mathrm{NC}}\big(\|u_\ell - \tilde{u}_\ell\|_{\mathrm{NC}} + \|u_{\ell+m} - \tilde{u}_{\ell+m}\|_{\mathrm{NC}}\big) + a_{\mathrm{NC}}(u - \tilde{u}_{\ell+m}, u_\ell - u_{\ell+m}).
\end{aligned}$$

It remains to bound the last term $a_{\mathrm{NC}}(u - \tilde{u}_{\ell+m}, u_\ell - u_{\ell+m})$.

Let I_ℓ (resp. $\mathrm{I}_{\ell+m}$) denote the nonconforming interpolation operator from (3.2) with respect to the triangulation $\mathcal{T}_\ell$ (resp. $\mathcal{T}_{\ell+m}$). Note that the interpolation operator I_ℓ is well-defined also for functions $v_{\ell+m} \in \mathrm{CR}_0^1(\mathcal{T}_{\ell+m})$ by

$$\mathrm{I}_\ell\, v_{\ell+m}(\mathrm{mid}(E)) = \fint_E v_{\ell+m}\, ds \quad \text{for any } E \in \mathcal{E}_\ell.$$

The piecewise integration by parts shows that the analogue of (3.3) holds in the form

$$\int_T \nabla_{\mathrm{NC}} v_{\ell+m}\, dx = \int_T \nabla\, \mathrm{I}_\ell\, v_{\ell+m}\, dx \quad \text{for any } T \in \mathcal{T}_\ell. \tag{5.4}$$

This and the discrete Friedrichs inequality [12, Theorem 10.6.12] eventually lead to the approximation result [30]

$$\|v_{\ell+m} - \mathrm{I}_\ell\, v_{\ell+m}\|_{L^2(T)} \lesssim |T|^{1/2} \|\nabla_{\mathrm{NC}} v_{\ell+m}\|_{L^2(T)} \quad \text{for } T \in \mathcal{T}_\ell. \tag{5.5}$$

The orthogonality (5.4) implies the Pythagoras theorem

$$|||v_{\ell+m} - \mathrm{I}_\ell\, v_{\ell+m}|||_{\mathrm{NC}}^2 + |||\mathrm{I}_\ell\, v_{\ell+m}|||_{\mathrm{NC}}^2 = |||v_{\ell+m}|||_{\mathrm{NC}}^2. \tag{5.6}$$

This shows stability of $\mathrm{I}_\ell : \mathrm{CR}_0^1(\mathcal{T}_{\ell+m}) \to \mathrm{CR}_0^1(\mathcal{T}_\ell)$. The projection properties (3.3) and (5.4) of the nonconforming interpolation operators I_ℓ and $\mathrm{I}_{\ell+m}$ on the levels ℓ and $\ell+m$ and the discrete problem (1.2) prove

$$\begin{aligned}
&a_{\mathrm{NC}}(u_\ell - u_{\ell+m}, u - \tilde{u}_{\ell+m})\\
&\quad = a_{\mathrm{NC}}(u_\ell, \mathrm{I}_\ell(u - \tilde{u}_{\ell+m})) - a_{\mathrm{NC}}(u_{\ell+m}, \mathrm{I}_{\ell+m}\, u - \tilde{u}_{\ell+m})\\
&\quad = \lambda_\ell b(u_\ell, \mathrm{I}_\ell(u - \tilde{u}_{\ell+m})) - \lambda_{\ell+m} b(u_{\ell+m}, \mathrm{I}_{\ell+m}\, u - \tilde{u}_{\ell+m})\\
&\quad = b(\lambda_\ell u_\ell, (\mathrm{I}_{\ell+m} - \mathrm{I}_\ell)(\tilde{u}_{\ell+m} - u)) + b(\lambda_{\ell+m} u_{\ell+m} - \lambda_\ell u_\ell, \mathrm{I}_{\ell+m}(\tilde{u}_{\ell+m} - u)).
\end{aligned}$$

Since the action of the nonconforming interpolation operators I_ℓ and $\mathrm{I}_{\ell+m}$ on the levels ℓ and $\ell+m$ is the same on the triangles $\mathcal{T}_\ell \cap \mathcal{T}_{\ell+m}$, the approximation property (5.5) and the stability property (5.6) of I_ℓ and the projection property (3.3) of $\mathrm{I}_{\ell+m}$ for the gradient prove

$$\begin{aligned}
&b(\lambda_\ell u_\ell, (\mathrm{I}_{\ell+m} - \mathrm{I}_\ell)(\tilde{u}_{\ell+m} - u))\\
&= b(\lambda_\ell u_\ell, \mathrm{I}_{\ell+m}(\tilde{u}_{\ell+m} - u) - \mathrm{I}_\ell(\mathrm{I}_{\ell+m}(\tilde{u}_{\ell+m} - u)))\\
&\lesssim \|h_\ell \lambda_\ell u_\ell\|_{L^2(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+m})} |||u - \tilde{u}_{\ell+m}|||_{\mathrm{NC}}.
\end{aligned}$$

The Cauchy and the Young inequalities prove

$$\begin{aligned}
&2\, b(\lambda_{\ell+m} u_{\ell+m} - \lambda_\ell u_\ell, \mathrm{I}_{\ell+m}(\tilde{u}_{\ell+m} - u))\\
&\qquad \le \|\lambda_{\ell+m} u_{\ell+m} - \lambda_\ell u_\ell\|_{L^2(\Omega)}^2 + \|\mathrm{I}_{\ell+m}\, u - \tilde{u}_{\ell+m}\|_{L^2(\Omega)}^2.
\end{aligned} \tag{5.7}$$

The first term on the right-hand side has been bounded in Corollary 2.2. For the second term on the right-hand side of (5.7), the triangle inequality reveals

$$\begin{aligned}
&\|\mathrm{I}_{\ell+m}\, u - \tilde{u}_{\ell+m}\|_{L^2(\Omega)}\\
&\qquad \le \|u - u_{\ell+m}\|_{L^2(\Omega)} + \|u_{\ell+m} - \tilde{u}_{\ell+m}\|_{L^2(\Omega)} + \|u - \mathrm{I}_{\ell+m}\, u\|_{L^2(\Omega)}.
\end{aligned}$$

Theorem 2.1 proves

$$\|u - u_{\ell+m}\|_{L^2(\Omega)} \lesssim \|h_0\|_\infty^s |||u - u_{\ell+m}|||_{\mathrm{NC}}.$$

The discrete Friedrichs inequality [12, Theorem 10.6.12] shows

$$\|u_{\ell+m} - \tilde{u}_{\ell+m}\|_{L^2(\Omega)} \lesssim |||u_{\ell+m} - \tilde{u}_{\ell+m}|||_{\mathrm{NC}}.$$

The estimate for the nonconforming interpolation (3.8) and the projection property (3.3) prove

$$\|u - \mathrm{I}_{\ell+m}\, u\|_{L^2(\Omega)} \lesssim \|h_0\|_\infty \vert\!\vert\!\vert u - u_{\ell+m}\vert\!\vert\!\vert_{\mathrm{NC}}.$$

The combination of the previous arguments shows

$$\begin{aligned} &b(\lambda_{\ell+m} u_{\ell+m} - \lambda_\ell u_\ell, \mathrm{I}_{\ell+m}(\tilde u_{\ell+m} - u)) \\ &\qquad\lesssim \|h_0\|_\infty^{2s} \vert\!\vert\!\vert u - u_{\ell+m}\vert\!\vert\!\vert_{\mathrm{NC}}^2 + \vert\!\vert\!\vert u_{\ell+m} - \tilde u_{\ell+m}\vert\!\vert\!\vert_{\mathrm{NC}}^2 + \|h_0\|_\infty^{2s} \vert\!\vert\!\vert u - u_\ell\vert\!\vert\!\vert_{\mathrm{NC}}^2. \end{aligned}$$

This concludes the proof. □

Proof of Theorem 5.1. The estimator reduction property (5.2) and the binomial formula for $\tilde u_{\ell+1} - \tilde u_\ell = (\tilde u_{\ell+1} - u) + (u - \tilde u_\ell)$ yield

$$\begin{aligned} \eta_{\ell+1}^2 \le \rho_1 \eta_\ell^2 + \Lambda\Big(&\vert\!\vert\!\vert u - \tilde u_\ell\vert\!\vert\!\vert_{\mathrm{NC}}^2 - \vert\!\vert\!\vert u - \tilde u_{\ell+1}\vert\!\vert\!\vert_{\mathrm{NC}}^2 + |\lambda_{\ell+1} - \tilde\lambda_{\ell+1}|^2 + |\lambda_\ell - \tilde\lambda_\ell|^2 \\ &+ \vert\!\vert\!\vert u_{\ell+1} - \tilde u_{\ell+1}\vert\!\vert\!\vert_{\mathrm{NC}}^2 + \vert\!\vert\!\vert u_\ell - \tilde u_\ell\vert\!\vert\!\vert_{\mathrm{NC}}^2 - 2a_{\mathrm{NC}}(u - \tilde u_{\ell+1}, \tilde u_{\ell+1} - \tilde u_\ell)\Big). \end{aligned}$$

This, the quasi-orthogonality (5.3), and the Young inequality lead to

$$\begin{aligned} \eta_{\ell+1}^2 \le \rho_1 \eta_\ell^2 + \Lambda\Bigg(&(1 + 4C_{\mathrm{qo}}\|h_0\|_\infty^{2s}) \vert\!\vert\!\vert u - \tilde u_\ell\vert\!\vert\!\vert_{\mathrm{NC}}^2 \\ &- (1 - 4C_{\mathrm{qo}}\|h_0\|_\infty^{2s}) \vert\!\vert\!\vert u - \tilde u_{\ell+1}\vert\!\vert\!\vert_{\mathrm{NC}}^2 + |\lambda_{\ell+1} - \tilde\lambda_{\ell+1}|^2 \\ &+ |\lambda_\ell - \tilde\lambda_\ell|^2 + (1 + 2C_{\mathrm{qo}} + 4C_{\mathrm{qo}}\|h_0\|_\infty^{2s}) \vert\!\vert\!\vert u_{\ell+1} - \tilde u_{\ell+1}\vert\!\vert\!\vert_{\mathrm{NC}}^2 \\ &+ (1 + 4C_{\mathrm{qo}}\|h_0\|_\infty^{2s}) \vert\!\vert\!\vert u_\ell - \tilde u_\ell\vert\!\vert\!\vert_{\mathrm{NC}}^2 \\ &+ 2C_{\mathrm{qo}}\big(\|h_\ell \lambda_\ell u_\ell\|_{L^2(\cup(\mathcal T_\ell\setminus\mathcal T_{\ell+1}))} \vert\!\vert\!\vert u - \tilde u_{\ell+1}\vert\!\vert\!\vert_{\mathrm{NC}} \\ &+ \vert\!\vert\!\vert u - \tilde u_{\ell+1}\vert\!\vert\!\vert_{\mathrm{NC}}(\vert\!\vert\!\vert u_\ell - \tilde u_\ell\vert\!\vert\!\vert_{\mathrm{NC}} + \vert\!\vert\!\vert u_{\ell+1} - \tilde u_{\ell+1}\vert\!\vert\!\vert_{\mathrm{NC}})\big)\Bigg). \end{aligned} \tag{5.8}$$

The Young inequality asserts, for any $0 < \mu < 1$, that

$$\begin{aligned} &2C_{\mathrm{qo}}\|h_\ell \lambda_\ell u_\ell\|_{L^2(\cup(\mathcal T_\ell\setminus\mathcal T_{\ell+1}))} \vert\!\vert\!\vert u - \tilde u_{\ell+1}\vert\!\vert\!\vert_{\mathrm{NC}} \\ &\qquad\le \frac{2C_{\mathrm{qo}}^2}{\mu}\|h_\ell \lambda_\ell u_\ell\|_{L^2(\cup(\mathcal T_\ell\setminus\mathcal T_{\ell+1}))}^2 + \frac{\mu}{2}\vert\!\vert\!\vert u - \tilde u_{\ell+1}\vert\!\vert\!\vert_{\mathrm{NC}}^2. \end{aligned}$$

Similarly

$$\begin{aligned} &2C_{\mathrm{qo}}\vert\!\vert\!\vert u - \tilde u_{\ell+1}\vert\!\vert\!\vert_{\mathrm{NC}}\big(\vert\!\vert\!\vert u_\ell - \tilde u_\ell\vert\!\vert\!\vert_{\mathrm{NC}} + \vert\!\vert\!\vert u_{\ell+1} - \tilde u_{\ell+1}\vert\!\vert\!\vert_{\mathrm{NC}}\big) \\ &\qquad\le \frac{\mu}{2}\vert\!\vert\!\vert u - \tilde u_{\ell+1}\vert\!\vert\!\vert_{\mathrm{NC}}^2 + \frac{4C_{\mathrm{qo}}^2}{\mu}\vert\!\vert\!\vert u_\ell - \tilde u_\ell\vert\!\vert\!\vert_{\mathrm{NC}}^2 + \frac{4C_{\mathrm{qo}}^2}{\mu}\vert\!\vert\!\vert u_{\ell+1} - \tilde u_{\ell+1}\vert\!\vert\!\vert_{\mathrm{NC}}^2. \end{aligned}$$

The combination with (5.8) plus some rearrangements lead to

$$\begin{aligned}\eta_{\ell+1}^2 \leq \rho_1\eta_\ell^2 + \Lambda\Bigg(&(1+4C_{\mathrm{qo}}\|h_0\|_\infty^{2s})\|\!|u-\tilde u_\ell|\!\|_{\mathrm{NC}}^2\\ &-(1-4C_{\mathrm{qo}}\|h_0\|_\infty^{2s}-\mu)\|\!|u-\tilde u_{\ell+1}|\!\|_{\mathrm{NC}}^2\\ &+\frac{2C_{\mathrm{qo}}^2}{\mu}\|h_\ell\lambda_\ell u_\ell\|_{L^2(\cup(\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+1}))}^2+|\lambda_{\ell+1}-\tilde\lambda_{\ell+1}|^2\\ &+|\lambda_\ell-\tilde\lambda_\ell|^2+\Big(1+\frac{4C_{\mathrm{qo}}^2}{\mu}+4C_{\mathrm{qo}}\|h_0\|_\infty^{2s}\Big)\|\!|u_\ell-\tilde u_\ell|\!\|_{\mathrm{NC}}^2\\ &+\Big(1+\frac{4C_{\mathrm{qo}}^2}{\mu}+4C_{\mathrm{qo}}\|h_0\|_\infty^{2s}+2C_{\mathrm{qo}}\Big)\|\!|u_{\ell+1}-\tilde u_{\ell+1}|\!\|_{\mathrm{NC}}^2)\Bigg).\end{aligned}\tag{5.9}$$

For the third contribution to $\xi_{\ell+1}$, the triangle inequality followed by the Young inequality implies, for any $0<\delta<\infty$, that

$$\begin{aligned}&\|h_{\ell+1}\lambda_{\ell+1}u_{\ell+1}\|_{L^2(\Omega)}^2\\ \leq&(1+\delta)\|h_{\ell+1}(\lambda_{\ell+1}u_{\ell+1}-\lambda_\ell u_\ell)\|_{L^2(\Omega)}^2+(1+1/\delta)\|h_{\ell+1}\lambda_\ell u_\ell\|_{L^2(\Omega)}^2.\end{aligned}$$

A moment's reflection shows that

$$1/2\|h_\ell\lambda_\ell u_\ell\|_{L^2(\cup(\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+1}))}^2+\|h_{\ell+1}\lambda_\ell u_\ell\|_{L^2(\Omega)}^2\leq\|h_\ell\lambda_\ell u_\ell\|_{L^2(\Omega)}^2.$$

The combination of the previous two estimates with Corollary 2.2 yields

$$\begin{aligned}\|h_{\ell+1}\lambda_{\ell+1}u_{\ell+1}\|_{L^2(\Omega)}^2&+(1+1/\delta)/2\,\|h_\ell\lambda_\ell u_\ell\|_{L^2(\cup(\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+1}))}^2\\ &\leq(1+\delta)(C/2)\|h_0\|_\infty^{2+2s}(\|\!|u-u_\ell|\!\|_{\mathrm{NC}}^2+\|\!|u-u_{\ell+1}|\!\|_{\mathrm{NC}}^2)\\ &\qquad+(1+1/\delta)\|h_\ell\lambda_\ell u_\ell\|_{L^2(\Omega)}^2\\ &\leq(1+\delta)C\|h_0\|_\infty^{2+2s}(\|\!|u-\tilde u_\ell|\!\|_{\mathrm{NC}}^2+\|\!|u-\tilde u_{\ell+1}|\!\|_{\mathrm{NC}}^2)\\ &\qquad+(1+1/\delta)\|h_\ell\lambda_\ell u_\ell\|_{L^2(\Omega)}^2\\ &\qquad+(1+\delta)C\|h_0\|_\infty^{2+2s}(\|\!|u_\ell-\tilde u_\ell|\!\|_{\mathrm{NC}}^2+\|\!|u_{\ell+1}-\tilde u_{\ell+1}|\!\|_{\mathrm{NC}}^2).\end{aligned}\tag{5.10}$$

with some $C\approx 1$. For $\beta:=\Lambda(1-4C_{\mathrm{qo}}\|h_0\|_\infty^{2s}-\mu-4\delta C_{\mathrm{qo}}^2C\|h_0\|_\infty^{2+2s}/\mu)$ and $\gamma:=4\Lambda\delta C_{\mathrm{qo}}^2/(\mu(\delta+1))$, the estimates (5.9) and (5.10) eventually imply

$$\begin{aligned}\eta_{\ell+1}^2+&\beta\|\!|u-\tilde u_{\ell+1}|\!\|_{\mathrm{NC}}^2+\gamma\|h_{\ell+1}\lambda_{\ell+1}u_{\ell+1}\|_{L^2(\Omega)}^2\\ \leq&\rho_1\eta_\ell^2+\Lambda\left(1+4C_{\mathrm{qo}}\|h_0\|_\infty^{2s}+4\delta C_{\mathrm{qo}}^2C\|h_0\|_\infty^{2+2s}/\mu\right)\|\!|u-\tilde u_\ell|\!\|_{\mathrm{NC}}^2\\ &+\Lambda\left(4C_{\mathrm{qo}}^2/\mu\right)\|h_\ell\lambda_\ell u_\ell\|_{L^2(\Omega)}^2+\Lambda|\tilde\lambda_{\ell+1}-\lambda_{\ell+1}|^2+\Lambda|\lambda_\ell-\tilde\lambda_\ell|^2\\ &+\Lambda\left(1+4\delta CC_{\mathrm{qo}}^2\|h_0\|_\infty^{2+2s}/\mu+4C_{\mathrm{qo}}^2/\mu+4C_{\mathrm{qo}}\|h_0\|_\infty^{2s}\right)\\ &\qquad\times(\|\!|u_\ell-\tilde u_\ell|\!\|_{\mathrm{NC}}^2+\|\!|u_{\ell+1}-\tilde u_{\ell+1}|\!\|_{\mathrm{NC}}^2)\\ &+2\Lambda C_{\mathrm{qo}}\|\!|u_{\ell+1}-\tilde u_{\ell+1}|\!\|_{\mathrm{NC}}^2.\end{aligned}$$

Lemma 5.4 (for $v_{\ell+1}\equiv 0$) leads to

$$\|h_\ell(\lambda_\ell u_\ell-\tilde\lambda_\ell\tilde u_\ell)\|_{L^2(\Omega)}^2\lesssim\|h_0\|_\infty^2\|\!|\lambda_\ell u_\ell-\tilde\lambda_\ell\tilde u_\ell|\!\|_{\mathrm{NC}}^2.$$

Hence, a triangle inequality and the tolerance (4.1) guarantee the existence of some $\widetilde C$ such that

$$\|h_\ell\lambda_\ell u_\ell\|_{L^2(\Omega)}^2\leq 2\kappa\|h_0\|_\infty^2\widetilde C\eta_\ell^2+2\|h_\ell\tilde\lambda_\ell\tilde u_\ell\|_{L^2(\Omega)}^2.$$

Since $\|h_\ell \tilde\lambda_\ell \tilde u_\ell\|^2_{L^2(\Omega)} \le \eta_\ell^2$, this proves

$$\|h_\ell \lambda_\ell u_\ell\|^2_{L^2(\Omega)} \le (2 + 2\kappa \|h_0\|^2_\infty \widetilde{C})\eta_\ell^2.$$

The reliability (4.6) of η_ℓ and the choice of the tolerance (4.1) lead to

$$\begin{aligned}
&\eta^2_{\ell+1} + \beta |\!|\!| u - \tilde u_{\ell+1} |\!|\!|^2_{\mathrm{NC}} + \gamma \|h_{\ell+1}\lambda_{\ell+1}u_{\ell+1}\|^2_{L^2(\Omega)} \\
&\le \Big(\rho_1 + (2 + 4C_{\mathrm{rel}}^2)\mu\Lambda + 2\kappa\Lambda\Big(2 + 4\delta C C^2_{\mathrm{qo}}\|h_0\|^{2+2s}_\infty/\mu + 4C^2_{\mathrm{qo}}/\mu \\
&\qquad + 4C_{\mathrm{qo}}\|h_0\|^{2s}_\infty + C_{\mathrm{qo}} + \|h_0\|^2_\infty \widetilde{C}\mu\Big)\Big)\eta_\ell^2 \\
&\quad + \Lambda\left(1 + 4C_{\mathrm{qo}}\|h_0\|^{2s}_\infty + 4\delta C^2_{\mathrm{qo}} C \|h_0\|^{2+2s}_\infty/\mu - 2\mu + 4C^2_{\mathrm{rel}}\|h_0\|^{2s}_\infty \mu\right) |\!|\!| u - \tilde u_\ell |\!|\!|^2_{\mathrm{NC}} \\
&\quad + \Lambda\left(4C^2_{\mathrm{qo}}/\mu - \mu\right)\|h_\ell\lambda_\ell u_\ell\|^2_{L^2(\Omega)}.
\end{aligned}$$

This is smaller than or equal to $\rho_2\big(\eta_\ell^2 + \beta |\!|\!| u - \tilde u_\ell |\!|\!|^2_{\mathrm{NC}} + \gamma\|h_\ell\lambda_\ell u_\ell\|^2_{L^2(\Omega)}\big)$ for

$$\begin{aligned}
\rho_2 := \max\Big\{&\rho_1 + (2 + 4C_{\mathrm{rel}}^2)\mu\Lambda + 2\kappa\Lambda\Big(2 + 4\delta C C^2_{\mathrm{qo}}\|h_0\|^{2+2s}_\infty/\mu + 4C^2_{\mathrm{qo}}/\mu \\
&\quad + 4C_{\mathrm{qo}}\|h_0\|^{2s}_\infty + C_{\mathrm{qo}} + \|h_0\|^2_\infty \widetilde{C}\mu\Big), \\
&\Lambda\left(1 + 4C_{\mathrm{qo}}\|h_0\|^{2s}_\infty + 4\delta C^2_{\mathrm{qo}} C\|h_0\|^{2+2s}_\infty/\mu - 2\mu + 4C^2_{\mathrm{rel}}\|h_0\|^{2s}_\infty\mu\right)/\beta, \\
&\Lambda\left(4C^2_{\mathrm{qo}}/\mu - \mu\right)/\gamma\Big\}.
\end{aligned}$$

For sufficiently small μ, κ, and $\|h_0\|_\infty$ with $\delta := 4C^2_{\mathrm{qo}}/\mu^2$ it follows $\rho_2 < 1$. □

6. Optimality Analysis

This section is devoted to the proof of Theorem 4.7 with the discrete reliability.

Theorem 6.1 (discrete reliability)**.** *Let Ω be simply connected. For sufficiently small mesh-size $\|h_0\|_\infty \ll 1$, there exists a constant $C_{\mathrm{drel}} \lesssim 1$ such that any refinement $\mathcal{T}_{\ell+m}$ of $\mathcal{T}_\ell$ in $\mathbb{T}$ and their respective discrete solutions $(\lambda_{\ell+m}, u_{\ell+m})$ and (λ_ℓ, u_ℓ) from the adaptive algorithm satisfy*

$$|\!|\!| u_{\ell+m} - u_\ell |\!|\!|^2_{\mathrm{NC}} \le C_{\mathrm{drel}}\Big(\mu^2_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+m}) + \|h_0\|^{2s}_\infty\left(|\!|\!| u - u_{\ell+m}|\!|\!|^2_{\mathrm{NC}} + |\!|\!| u - u_\ell |\!|\!|^2_{\mathrm{NC}}\right)\Big).$$

The proof of Theorem 6.1 splits the left-hand side in two orthogonal terms. One of these terms, the nonconformity residual, is bounded by the tangential jumps in the following consequence of [30].

Theorem 6.2 (discrete reliability of nonconformity residual)**.** *If Ω is simply connected, any refinement $\mathcal{T}_{\ell+m}$ of $\mathcal{T}_\ell$ and any function $v_\ell \in \mathrm{CR}^1_0(\mathcal{T}_\ell)$ satisfy*

$$\min_{v_{\ell+m}\in \mathrm{CR}^1_0(\mathcal{T}_{\ell+m})} |\!|\!| v_\ell - v_{\ell+m} |\!|\!|^2_{\mathrm{NC}} \lesssim \sum_{T\in\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m}} \sum_{E\in\mathcal{E}_\ell(T)} |E|\,\|[\partial v_\ell/\partial s]_E\|^2_{L^2(E)}.$$

Proof. The use of the discrete Helmholtz decomposition as in [5] yields the existence of $\alpha_{\ell+m} \in \mathrm{CR}^1_0(\mathcal{T}_{\ell+m})$ and $\beta_{\ell+m} \in P_1(\mathcal{T}_{\ell+m}) \cap C(\Omega)$ such that

$$v_\ell = \nabla_{\mathrm{NC}}\alpha_{\ell+m} + \operatorname{Curl}\beta_{\ell+m}.$$

The orthogonality of the decomposition implies

$$\min_{v_{\ell+m}\in \mathrm{CR}^1_0(\mathcal{T}_{\ell+m})} |\!|\!| v_\ell - v_{\ell+m} |\!|\!|^2_{\mathrm{NC}} = \|\operatorname{Curl}\beta_{\ell+m}\|^2_{L^2(\Omega)}.$$

The proof of [30, Theorem 2.1] shows

$$\|\operatorname{Curl}\beta_{\ell+m}\|_{L^2(\Omega)}^2 \lesssim \sum_{T\in\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+m}} \sum_{E\in\mathcal{E}_\ell(T)} |E|\, \|[\partial v_\ell/\partial s]_E\|_{L^2(E)}^2. \qquad \square$$

Proof of Theorem 6.1. Let $P_{\ell+m}u_\ell \in \mathrm{CR}_0^1(\mathcal{T}_{\ell+m})$ denote the best approximation of u_ℓ in $\mathrm{CR}_0^1(\mathcal{T}_{\ell+m})$ with respect to $|\!|\!|\cdot|\!|\!|_{\mathrm{NC}}$ and $\mathcal{T}_{\ell+m}$ and set $v_{\ell+m} := u_{\ell+m} - P_{\ell+m}u_\ell \in \mathrm{CR}_0^1(\mathcal{T}_{\ell+m})$. The Pythagoras theorem reads

$$|\!|\!|u_{\ell+m} - u_\ell|\!|\!|_{\mathrm{NC}}^2 = |\!|\!|v_{\ell+m}|\!|\!|_{\mathrm{NC}}^2 + |\!|\!|u_\ell - P_{\ell+m}u_\ell|\!|\!|_{\mathrm{NC}}^2. \tag{6.1}$$

Since $a_{\mathrm{NC}}(P_{\ell+m}u_\ell, v_{\ell+m}) = a_{\mathrm{NC}}(u_\ell, v_{\ell+m}) = a_{\mathrm{NC}}(u_\ell, \mathrm{I}_\ell\, v_{\ell+m})$, the discrete problem (1.2) (on the levels $\ell+m$ and ℓ) implies

$$\begin{aligned} &|\!|\!|v_{\ell+m}|\!|\!|_{\mathrm{NC}}^2 = |\!|\!|u_{\ell+m} - P_{\ell+m}u_\ell|\!|\!|_{\mathrm{NC}}^2 = a_{\mathrm{NC}}(u_{\ell+m}, v_{\ell+m}) - a_{\mathrm{NC}}(u_\ell, \mathrm{I}_\ell\, v_{\ell+m}) \\ &\quad = \lambda_{\ell+m} b(u_{\ell+m} - u_\ell, v_{\ell+m}) + (\lambda_{\ell+m} - \lambda_\ell) b(u_\ell, v_{\ell+m}) + \lambda_\ell b(u_\ell, v_{\ell+m} - \mathrm{I}_\ell\, v_{\ell+m}). \end{aligned}$$

The Cauchy and discrete Friedrichs [12, Theorem 10.6.12] inequalities prove

$$\begin{aligned} &\lambda_{\ell+m} b(u_{\ell+m} - u_\ell, v_{\ell+m}) + (\lambda_{\ell+m} - \lambda_\ell) b(u_\ell, v_{\ell+m}) \\ &\lesssim \big(\lambda_{\ell+m}\, \|u_{\ell+m} - u_\ell\|_{L^2(\Omega)} + |\lambda_{\ell+m} - \lambda_\ell|\, \|u_\ell\|_{L^2(\Omega)}\big) |\!|\!|v_{\ell+m}|\!|\!|_{\mathrm{NC}}. \end{aligned}$$

The fact $v_{\ell+m} - \mathrm{I}_\ell\, v_{\ell+m} = 0$ on all $T \in \mathcal{T}_\ell \cap \mathcal{T}_{\ell+m}$ and the approximation property (5.5) of I_ℓ lead to

$$\lambda_\ell b(u_\ell, v_{\ell+m} - \mathrm{I}_\ell\, v_{\ell+m}) \lesssim \|h_\ell \lambda_\ell u_\ell\|_{L^2(\cup(\mathcal{T}_{\ell+m}\setminus\mathcal{T}_\ell))} |\!|\!|v_{\ell+m}|\!|\!|_{\mathrm{NC}}.$$

The combination of the preceding estimates results in

$$|\!|\!|v_{\ell+m}|\!|\!|_{\mathrm{NC}} \lesssim \lambda_{\ell+m}\, \|u_{\ell+m} - u_\ell\|_{L^2(\Omega)} + |\lambda_{\ell+m} - \lambda_\ell| + \|h_\ell \lambda_\ell u_\ell\|_{L^2(\cup(\mathcal{T}_{\ell+m}\setminus\mathcal{T}_\ell))}.$$

This, the triangle inequality, and Theorem 2.1 lead to

$$|\!|\!|v_{\ell+m}|\!|\!|_{\mathrm{NC}} \lesssim \mu_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+m}) + \|h_0\|_\infty^s (|\!|\!|u - u_\ell|\!|\!|_{\mathrm{NC}} + |\!|\!|u - u_{\ell+m}|\!|\!|_{\mathrm{NC}}). \tag{6.2}$$

This and (6.1) plus Theorem 6.2 conclude the proof. $\square$

Proof of Theorem 4.7. Theorem 3.1 implies that the approximation seminorm $|u|_{\mathcal{A}_\sigma}$ is equivalent to the following modified version

$$|u|_{\mathcal{A}'_\sigma} := \sup_{N\in\mathbb{N}} N^\sigma \inf_{\mathcal{T}\in\mathbb{T}(N)} |\!|\!|u - u_{\mathcal{T}}|\!|\!|_{\mathrm{NC}},$$

where $u_{\mathcal{T}} \in \mathrm{CR}_0^1(\mathcal{T})$ denotes the nonconforming discrete normalised eigenfunction with $\|u_{\mathcal{T}}\|_{L^2(\Omega)} = 1$ with minimal distance $|\!|\!|u - u_{\mathcal{T}}|\!|\!|_{\mathrm{NC}}$. The proof of quasi-optimality will rely on this characterisation. The proof is structured into Claim A–D and excludes the pathological case $\xi_0 = 0$ for

$$\xi_\ell^2 := \eta_\ell^2 + \beta\, |\!|\!|u - \tilde{u}_\ell|\!|\!|_{\mathrm{NC}}^2 + \gamma\, \|h_\ell \lambda_\ell u_\ell\|_{L^2(\Omega)}^2 \quad \text{for all } \ell = 0, 1, 2, \dots$$

from Theorem 5.1. Choose $0 < \tau \le |u|_{\mathcal{A}'_\sigma}^2 / \xi_0^2$, and set $\varepsilon^2(\ell) := \tau \xi_\ell^2$. Let $N(\ell) \in \mathbb{N}$ be minimal with the property

$$|u|_{\mathcal{A}'_\sigma} \le \varepsilon(\ell)\, N(\ell)^\sigma. \tag{6.3}$$

Claim A. *It holds*

$$N(\ell) \le 2\, |u|_{\mathcal{A}'_\sigma}^{1/\sigma}\, \varepsilon(\ell)^{-1/\sigma} \quad \textit{for all } \ell \in \mathbb{N}_0. \tag{6.4}$$

Proof of Claim A. For $N(\ell) = 1$, (6.3) and the contraction property (5.1) imply

$$|u|^2_{\mathcal{A}'_\sigma} \le \varepsilon(\ell)^2 = \tau\xi_\ell^2 \le \tau\xi_0^2 \le |u|^2_{\mathcal{A}'_\sigma},$$

whence $|u|^2_{\mathcal{A}'_\sigma} = \varepsilon(\ell)^2$. For $N(\ell) \ge 2$, the minimality of $N(\ell)$ in (6.3) yields

$$\varepsilon(\ell)\big(N(\ell)-1\big)^\sigma < |u|_{\mathcal{A}'_\sigma}.$$

Therefore,

$$N(\ell) \le 2\big(N(\ell)-1\big) \le 2\,|u|^{1/\sigma}_{\mathcal{A}'_\sigma}\,\varepsilon(\ell)^{-1/\sigma}. \qquad \square$$

The definition of $|u|_{\mathcal{A}'_\sigma}$ as a supremum over N shows for $N = N(\ell)$ that there exists some optimal triangulation $\overline{\mathcal{T}}_\ell$ (which is possibly not related to $\mathcal{T}_\ell$) of cardinality $|\overline{\mathcal{T}}_\ell| \le |\mathcal{T}_0| + N(\ell)$ with discrete solution $(\overline{\lambda}_\ell, \overline{u}_\ell) \in \mathbb{R} \times \mathrm{CR}^1_0(\overline{\mathcal{T}}_\ell)$ and

$$\|\!|u - \overline{u}_\ell|\!\|^2_{\mathrm{NC}} \le N(\ell)^{-2\sigma}\,|u|^2_{\mathcal{A}'_\sigma} \le \varepsilon(\ell)^2. \tag{6.5}$$

The overlay $\widehat{\mathcal{T}}_\ell := \mathcal{T}_\ell \otimes \overline{\mathcal{T}}_\ell$ is defined as the smallest common refinement of $\mathcal{T}_\ell$ and $\overline{\mathcal{T}}_\ell$. It is known [22, 33] that

$$|\widehat{\mathcal{T}}_\ell| - |\mathcal{T}_\ell| \le |\overline{\mathcal{T}}_\ell| - |\mathcal{T}_0| \le N(\ell).$$

The number of triangles in $\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell$ can be estimated as

$$|\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell| \le \sum_{K \in \mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell} \big(|\widehat{\mathcal{T}}_\ell(K)| - 1\big) = |\widehat{\mathcal{T}}_\ell \setminus \mathcal{T}_\ell| - |\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell| = |\widehat{\mathcal{T}}_\ell| - |\mathcal{T}_\ell|.$$

Thus

$$|\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell| \le N(\ell) \le 2\,|u|^{1/\sigma}_{\mathcal{A}'_\sigma}\,\varepsilon(\ell)^{-1/\sigma}. \tag{6.6}$$

Claim B. *For sufficiently small $\|h_0\|_\infty \ll 1$ there exists $C_1 \approx 1$ such that the discrete solution $\widehat{u}_\ell \in \mathrm{CR}^1_0(\widehat{\mathcal{T}}_\ell)$ with respect to $\widehat{\mathcal{T}}_\ell$ satisfies*

$$\|\!|u - \widehat{u}_\ell|\!\|^2_{\mathrm{NC}} \le C_1 \varepsilon^2(\ell). \tag{6.7}$$

Proof of Claim B. The quasi-orthogonality of Theorem 5.5 shows

$$\begin{aligned}
\|\!|u - \widehat{u}_\ell|\!\|^2_{\mathrm{NC}} &= \|\!|u - \overline{u}_\ell|\!\|^2_{\mathrm{NC}} - \|\!|\overline{u}_\ell - \widehat{u}_\ell|\!\|^2_{\mathrm{NC}} + 2a_{\mathrm{NC}}(u - \widehat{u}_\ell, \overline{u}_\ell - \widehat{u}_\ell)\\
&\le (1 + 2C_{\mathrm{qo}}\|h_0\|^{2s}_\infty)\,\|\!|u - \overline{u}_\ell|\!\|^2_{\mathrm{NC}} - \|\!|\overline{u}_\ell - \widehat{u}_\ell|\!\|^2_{\mathrm{NC}}\\
&\qquad + 2C^2_{\mathrm{qo}}\|h_\ell\overline{\lambda}_\ell\overline{u}_\ell\|^2_{L^2(\cup(\overline{\mathcal{T}}_\ell\setminus\widehat{\mathcal{T}}_\ell))} + (1/2 + 2C_{\mathrm{qo}}\|h_0\|^{2s}_\infty)\|\!|u - \widehat{u}_\ell|\!\|^2_{\mathrm{NC}}.
\end{aligned}$$

The efficiency $\|h_\ell\overline{\lambda}_\ell\overline{u}_\ell\|^2_{L^2(\Omega)} \le C_{\mathrm{eff}}(1 + \|h_0\|^{2s}_\infty)\|\!|u - \overline{u}_\ell|\!\|^2_{\mathrm{NC}}$ from (4.4) implies

$$\begin{aligned}
&(1/2 - 2C_{\mathrm{qo}}\|h_0\|^{2s}_\infty)\|\!|u - \widehat{u}_\ell|\!\|^2_{\mathrm{NC}} + \|\!|\overline{u}_\ell - \widehat{u}_\ell|\!\|^2_{\mathrm{NC}}\\
&\quad\le (1 + 2C_{\mathrm{qo}}\|h_0\|^{2s}_\infty)\|\!|u - \overline{u}_\ell|\!\|^2_{\mathrm{NC}} + 2C^2_{\mathrm{qo}}\|h_\ell\overline{\lambda}_\ell\overline{u}_\ell\|^2_{L^2(\cup(\overline{\mathcal{T}}_\ell\setminus\widehat{\mathcal{T}}_\ell))}\\
&\quad\le (1 + 2C_{\mathrm{qo}}\|h_0\|^{2s}_\infty + 2C^2_{\mathrm{qo}}C_{\mathrm{eff}}(1 + \|h_0\|^{2s}_\infty))\|\!|u - \overline{u}_\ell|\!\|^2_{\mathrm{NC}}.
\end{aligned}$$

This and (6.5) conclude the proof for

$$C_1 := (1 + 2C_{\mathrm{qo}}\|h_0\|^{2s}_\infty + 2C^2_{\mathrm{qo}}C_{\mathrm{eff}}(1 + \|h_0\|^{2s}_\infty))/(1/2 - 2C_{\mathrm{qo}}\|h_0\|^{2s}_\infty). \qquad \square$$

Claim C. *There exists $C_2 \approx 1$ with*

$$\eta^2_\ell \le C_2\,\eta^2_\ell(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell). \tag{6.8}$$

Proof of Claim C. Since $\kappa \ll 1$, the efficiency from (4.7) shows

$$\eta_\ell^2 / C_{\mathrm{eff}} \leq (1 + \|h_0\|_\infty^{2s}) \|u - u_\ell\|_{\mathrm{NC}}^2. \tag{6.9}$$

The quasi-orthogonality leads to

$$\begin{aligned}\|u - u_\ell\|_{\mathrm{NC}}^2 &= \|u - \widehat{u}_\ell\|_{\mathrm{NC}}^2 + \|\widehat{u}_\ell - u_\ell\|_{\mathrm{NC}}^2 + 2a_{\mathrm{NC}}(u - \widehat{u}_\ell, \widehat{u}_\ell - u_\ell)\\ &\leq (2 + 2C_{\mathrm{qo}}\|h_0\|_\infty^{2s}) \|u - \widehat{u}_\ell\|_{\mathrm{NC}}^2 + 2C_{\mathrm{qo}}\|h_0\|_\infty^{2s} \|u - u_\ell\|_{\mathrm{NC}}^2\\ &\quad + \|\widehat{u}_\ell - u_\ell\|_{\mathrm{NC}}^2 + C_{\mathrm{qo}}^2 \|h_\ell \lambda_\ell u_\ell\|_{L^2(\cup(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell))}^2.\end{aligned}$$

This and the discrete reliability from Theorem 6.1 with constant C_{drel} lead to

$$\begin{aligned}&(1 - 2C_{\mathrm{qo}}\|h_0\|_\infty^{2s} - C_{\mathrm{drel}}\|h_0\|_\infty^{2s}) \|u - u_\ell\|_{\mathrm{NC}}^2\\ &\quad \leq (2 + 2C_{\mathrm{qo}}\|h_0\|_\infty^{2s} + C_{\mathrm{drel}}\|h_0\|_\infty^{2s}) \|u - \widehat{u}_\ell\|_{\mathrm{NC}}^2 + (C_{\mathrm{drel}} + C_{\mathrm{qo}}^2)\mu_\ell^2(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell).\end{aligned} \tag{6.10}$$

Lemma 4.6 and the choice of the tolerance (4.1) yields

$$\mu_\ell^2(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell) \leq 2\eta_\ell^2(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell) + 4C_{\mathrm{cont}}^2 \kappa \eta_\ell^2. \tag{6.11}$$

The combination of (6.7) and (6.9)–(6.11) leads to

$$\frac{\eta_\ell^2}{C_{\mathrm{eff}}} \leq \frac{C_2}{2\,C_{\mathrm{eff}}} \eta_\ell^2(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell) + \frac{\varrho}{2\,C_{\mathrm{eff}}}\, \eta_\ell^2$$

with

$$C_2 := 4\,C_{\mathrm{eff}}(1 + \|h_0\|_\infty^{2s})(1 - 2C_{\mathrm{qo}}\|h_0\|_\infty^{2s} - C_{\mathrm{drel}}\|h_0\|_\infty^{2s})^{-1}(C_{\mathrm{drel}} + C_{\mathrm{qo}}^2)$$

and

$$\begin{aligned}\varrho &:= 2C_{\mathrm{eff}}(1 + \|h_0\|_\infty^{2s})(1 - C_{\mathrm{qo}}\|h_0\|_\infty^{2s} - C_{\mathrm{drel}}\|h_0\|_\infty^{2s})^{-1}\\ &\quad \times \big((2 + 2C_{\mathrm{qo}}\|h_0\|_\infty^{2s} + C_{\mathrm{drel}}\|h_0\|_\infty^{2s})C_{\mathrm{eq}}C_1\tau + 4C_{\mathrm{cont}}^2\kappa(C_{\mathrm{drel}} + C_{\mathrm{qo}}^2)\big)\end{aligned}$$

with equivalence constant C_{eq} from $\eta_\ell^2 \leq \xi_\ell^2 \leq C_{\mathrm{eq}}\eta_\ell^2$ (for $\|h_0\|_\infty \ll 1$). The choice of

$$0 < \tau < \frac{1 - C_{\mathrm{qo}}\|h_0\|_\infty^{2s} - C_{\mathrm{drel}}\|h_0\|_\infty^{2s}}{4\,C_{\mathrm{eff}}C_{\mathrm{eq}}C_1(2 + 2C_{\mathrm{qo}}\|h_0\|_\infty^{2s} + C_{\mathrm{drel}}\|h_0\|_\infty^{2s})(1 + \|h_0\|_\infty^{2s})}$$

and of sufficiently small κ leads to $\varrho < 1$ and, hence, to (6.8). □

Claim D. *The choice* $0 < \theta \leq 1/C_2$ *implies*

$$\big(|\mathcal{T}_\ell| - |\mathcal{T}_0|\big)^\sigma \|u - \tilde{u}_\ell\|_{\mathrm{NC}} \leq C(\sigma)\, |u|_{\mathcal{A}'_\sigma}\,.$$

Proof of Claim D. Mark selects $\mathcal{M}_\ell \subset \mathcal{T}_\ell$ with minimal cardinality $|\mathcal{M}_\ell|$ such that $\theta\eta_\ell^2 \leq \eta_\ell^2(\mathcal{M}_\ell)$. Since

$$\theta\eta_\ell^2 \leq \eta_\ell^2/C_2 \leq \eta_\ell^2(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell),$$

$\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell$ also satisfies the bulk criterion and the minimality of $\mathcal{M}_\ell$ proves, with (6.6) and the definition of $\varepsilon(\ell)$ from the very beginning of the proof, that

$$|\mathcal{M}_\ell| \leq |\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell| \leq 2\,|u|_{\mathcal{A}'_\sigma}^{1/\sigma}\, \varepsilon(\ell)^{-1/\sigma} = 2\,|u|_{\mathcal{A}'_\sigma}^{1/\sigma}\, \tau^{-1/(2\sigma)} \xi_\ell^{-1/\sigma}$$

with $\tau \approx 1$ and for all $\ell \in \mathbb{N}_0$. It is known [6, Theorem 2.4] (see also [33, Theorem 6.1]) that newest-vertex bisection and proper initialisation of refinement edges leads to a constant $C_{\mathrm{BDV}} \approx 1$ with

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \leq C_{\mathrm{BDV}} \sum_{k=0}^{\ell-1} |\mathcal{M}_k| \leq 2C_{\mathrm{BDV}}\, |u|_{\mathcal{A}'_\sigma}^{1/\sigma}\, \tau^{-1/(2\sigma)} \sum_{k=0}^{\ell-1} \xi_k^{-1/\sigma}.$$

The contraction property of Theorem 5.1 $\xi_{k+1}^2 \le \rho_2 \xi_k^2$ for all $k \in \mathbb{N}_0$ and mathematical induction prove

$$\xi_\ell^2 \le \rho_2^{\ell-k} \xi_k^2 \quad \text{for } 0 \le k \le \ell.$$

Since $0 < \rho_2 < 1$,

$$\sum_{k=0}^{\ell-1} \xi_k^{-1/\sigma} \le \xi_\ell^{-1/\sigma} \sum_{k=0}^{\ell-1} \rho_2^{(\ell-k)/(2\sigma)} \le \xi_\ell^{-1/\sigma} \rho_2^{1/(2\sigma)} / (1 - \rho_2^{1/(2\sigma)}).$$

Altogether,

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \le 2C_{\mathrm{BDV}}\, |u|_{\mathcal{A}'_\sigma}^{1/\sigma}\, \tau^{-1/(2\sigma)} \xi_\ell^{-1/\sigma} \rho_2^{1/(2\sigma)} / \big(1 - \rho_2^{1/(2\sigma)}\big).$$

This and $|\!|\!| u - u_\ell |\!|\!|_{\mathrm{NC}} \le \xi_\ell$ conclude the proof of Claim D and of Theorem 4.7. $\square$

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