

COMPUTATIONAL SURVEY ON A POSTERIORI ERROR ESTIMATORS FOR THE CROUZEIX-RAVIART NONCONFORMING FINITE ELEMENT METHOD FOR THE STOKES PROBLEM*

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ABSTRACT. This survey compares different strategies for guaranteed error control for the lowest-order nonconforming Crouzeix-Raviart finite element method for the Stokes equations. The upper error bound involves the minimal distance of the computed piecewise gradient $D_{\text{NC}} u_{\text{CR}}$ to the gradients of Sobolev functions with exact boundary conditions. Several improved suggestions for the cheap computation of such test functions compete in five benchmark examples. This paper provides numerical evidence that guaranteed error control of the nonconforming FEM is indeed possible for the Stokes equations with overall effectivity indices between 1 to 4.

1. INTRODUCTION

The a posteriori error analysis of conforming FEM is well established and contained even in textbooks [BS08, Bra07, AO00, Ver96]. Although a unified framework is established [Car05, CH07], much less is known about a posteriori error analysis for nonconforming lowest-order Crouzeix-Raviart finite element methods [DDP95, AD05, Ain04, AR08, Joh98, BCJ02]. This paper concerns the 2D Stokes equations: Given a right-hand side $f \in L^2(\Omega; \mathbb{R}^2)$ and Dirichlet data $u_D \in H^1(\Omega; \mathbb{R}^2)$ with $\int_{\partial\Omega} u_D \cdot \nu \, ds = 0$, seek a pressure $p \in L_0^2(\Omega)$ with $\int_{\Omega} p \, dx = 0$ and a velocity field $u \in H^1(\Omega; \mathbb{R}^2)$ with

$$(1.1) \quad -\Delta u + \nabla p = f \quad \text{and} \quad \operatorname{div} u = 0 \quad \text{in } \Omega \quad \text{while} \quad u = u_D \quad \text{on } \partial\Omega.$$

The primal variable u will be discretised with the nonconforming Crouzeix-Raviart FEM on some shape-regular triangulation \mathcal{T} with discrete solution u_{CR} . This paper discusses and compares different a posteriori error estimators for the error $e = u - u_{\text{CR}}$ in the (nonconforming) energy norm

$$\|e\|_{\text{NC}}^2 := \|D_{\text{NC}} e\|_{L^2(\Omega)}^2 := \sum_{T \in \mathcal{T}} \|D e\|_{L^2(T)}^2.$$

The decomposition from [AD05] allows for a split of this error into

$$\|e\|_{\text{NC}}^2 \leq \eta^2 + \left(\|D_{\text{NC}}(u_{\text{CR}} - v)\|_{L^2(\Omega)} + 1/c_0 \|\operatorname{div} v\|_{L^2(\Omega)} \right)^2.$$

The first term on the right-hand side (with $j_{1,1} \geq 3.8317$)

$$\eta := \|f_{\mathcal{T}}/2 \otimes (\bullet - \operatorname{mid}(\mathcal{T}))\|_{L^2(\Omega)} + 1/j_{1,1} \operatorname{osc}(f, \mathcal{T})$$

Key words and phrases. nonconforming finite element method, Crouzeix-Raviart finite element method, adaptive finite element method, a posteriori error estimation.

*This work was supported by DFG Research Center MATHEON and by the World Class University (WCU) program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology R31-2008-000-10049-0. The second author was also supported by the German Academic Exchange Service (DAAD, D/10/44641) during his stay at the Yonsei University in 2010.

involves contributions of the data f and is directly computable (up to quadrature errors). The second term employs *any* $v \in H^1(\Omega; \mathbb{R}^2)$ with $v = u_D$ along $\partial\Omega$, e.g., componentwise interpolation by [Ain04] or novel interpolations on the red-refined triangulation $\text{red}(\mathcal{T})$ from [CM13]. The constant c_0 depends only on the domain Ω and equals the smallest eigenvalue of some general eigenvalue problem [Sto99], cf. Section 3 below. This paper compares several possible designs of v and applies them in the five benchmark examples of Table 1.1.

| Subsect. | Short name | Problem data & features | c_0 |
|----------|----------------------|---|--------|
| 5.1 | Smooth Solution | $f \neq 0, u_D \neq 0$ | 0.3826 |
| 5.2 | 2nd Smooth Solution | $f \neq 0, u_D \neq 0$ | 0.3826 |
| 5.3 | Colliding Flow | $f \equiv 0, u_D \neq 0$ | 0.3826 |
| 5.4 | L-shaped Domain | $f \equiv 0, u_D \neq 0$, corner singularity | 0.3 |
| 5.5 | Backward facing step | $f \equiv 0, u_D \neq 0$, corner singularity | 0.3 |

TABLE 1.1. Benchmark examples and subsection references.

The remaining parts of this paper are outlined as follows. Section 2 introduces the necessary notation, preliminaries and our adaptive mesh refinement algorithm. Section 3 presents the a posteriori error analysis and derives a sharp upper bound for the energy error with explicit constants. Section 4 explains the realisations of the guaranteed upper bounds. Section 5 gives details on the error estimator competition for the five benchmark problems from Table 1.1. Section 6 draws some conclusions from the theoretical and numerical results of this paper.

Throughout this paper we use standard notation for Lebesgue and Sobolev spaces and their norms: $V := H_0^1(\Omega; \mathbb{R}^2)$ is endowed with the energy norm $\|\cdot\| := \|\nabla \cdot\|_{L^2(\Omega)} = |\cdot|_{H^1(\Omega)}$. Finally $a \lesssim b$ abbreviates $a \leq Cb$ for some generic constant C that depends only on the shape regularity of the triangulation while $a \approx b$ stands for $a \lesssim b \lesssim a$.

2. NOTATION AND PRELIMINARIES

2.1. Nonconforming finite element spaces. Given a regular triangulation \mathcal{T} of the bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^2$ into closed triangles in the sense of Ciarlet with edges \mathcal{E} , nodes \mathcal{N} and free nodes \mathcal{M} , the midpoints of all edges are $\text{mid}(\mathcal{E}) := \{\text{mid}(E) \mid E \in \mathcal{E}\}$ and $\mathcal{E}(\partial\Omega)$ denotes the edges along the boundary $\partial\Omega$. The set $\mathcal{E}(T)$ contains the three edges of a triangle $T \in \mathcal{T}$. With the elementwise first-order polynomials $P_1(\mathcal{T}; \mathbb{R}^2)$, the nonconforming Crouzeix-Raviart finite element spaces are defined by

$$\begin{aligned} \text{CR}^1(\mathcal{T}; \mathbb{R}^2) &:= \{v \in P_1(\mathcal{T}; \mathbb{R}^2) \mid v \text{ is continuous at } \text{mid}(\mathcal{E})\}, \\ \text{CR}_0^1(\mathcal{T}; \mathbb{R}^2) &:= \{v \in \text{CR}^1(\mathcal{T}; \mathbb{R}^2) \mid \forall E \in \mathcal{E}(\partial\Omega), v(\text{mid}(E)) = 0\}. \end{aligned}$$

The Crouzeix-Raviart finite elements form a subspace of the broken Sobolev functions

$$H^1(\mathcal{T}) := \{v \in L^2(\Omega) \mid \forall T \in \mathcal{T}, v|_T \in H^1(T)\}.$$

The diameter $\text{diam}(T)$ of $T \in \mathcal{T}$ is denoted by h_T and $h_{\mathcal{T}}$ denotes their piecewise constant values with $h_{\mathcal{T}}|_T := h_T := \text{diam}(T)$ for all $T \in \mathcal{T}$. The integral mean of a function $f \in L^2(\omega)$ (or any vector $f \in L^2(\omega; \mathbb{R}^2)$) over some open set ω is denoted by

$$f_{\omega} := \fint_{\omega} f \, dx := \int_{\omega} f \, dx / |\omega|.$$

The oscillations of $f \in L^2(\Omega)$ (as well as of vectors $f \in L^2(\Omega; \mathbb{R}^2)$) read

$$\text{osc}(f, \mathcal{T}) := \left(\sum_{T \in \mathcal{T}} \|h_T(f - f_T)\|_{L^2(T)}^2 \right)^{1/2} = \|h_{\mathcal{T}}(f - f_{\mathcal{T}})\|_{L^2(\Omega)}.$$

2.2. Crouzeix-Raviart FEM for the Stokes equations. The discrete bilinear form reads

$$a_{\text{NC}}(u_{\text{CR}}, v_{\text{CR}}) := \sum_{T \in \mathcal{T}} \int_T \mathbf{D} u_{\text{CR}} : \mathbf{D} v_{\text{CR}} dx$$

for all $u_{\text{CR}}, v_{\text{CR}} \in X_h := \text{CR}^1(\mathcal{T}; \mathbb{R}^2) \subseteq H^1(\mathcal{T}; \mathbb{R}^2)$ with $A : B := \sum_{j,k=1,2} A_{jk} B_{jk}$ for all 2×2 matrices $A, B \in \mathbb{R}^{2 \times 2}$. The particular choice of

$$Y_h := P_0(\mathcal{T}) \cap L_0^2(\Omega) \quad \text{and} \quad b_{\text{NC}}(v, q) := \int_{\Omega} q \operatorname{div}_{\text{NC}} v dx,$$

with the piecewise divergence operator $\operatorname{div}_{\text{NC}}$, leads to the discrete counterpart

$$Z_{\text{NC}} := \{v_{\text{CR}} \in \text{CR}_0^1(\mathcal{T}; \mathbb{R}^2) \mid \operatorname{div}_{\text{NC}} v_{\text{CR}} = 0\}$$

of the space of divergence-free functions

$$Z := \{v \in H_0^1(\Omega; \mathbb{R}^2) \mid \operatorname{div} v = 0\}.$$

The nonconforming representation of the Stokes problem reads: Given $f \in L^2(\Omega; \mathbb{R}^2)$, seek $u_{\text{CR}} \in \text{CR}^1(\mathcal{T}; \mathbb{R}^2)$ with $\operatorname{div}_{\text{NC}} u_{\text{CR}} = 0$,

$$u_{\text{CR}}(\operatorname{mid}(E)) = \int_E u_D ds \quad \text{for all } E \in \mathcal{E}(\partial\Omega),$$

and

$$a_{\text{NC}}(u_{\text{CR}}, v_{\text{CR}}) = F(v_{\text{CR}}) := \int_{\Omega} f \cdot v_{\text{CR}} dx \quad \text{for all } v_{\text{CR}} \in Z_{\text{NC}}.$$

In other words, up to boundary conditions, u_{CR} is computed from the Riesz representation of a linear functional (given as right-hand side plus boundary modifications) in the Hilbert space $(Z_{\text{NC}}, a_{\text{NC}})$. The actual implementation uses unconstrained Crouzeix-Raviart elements $v_{\text{CR}} \in \text{CR}^1(\mathcal{T}; \mathbb{R}^2)$ and another Lagrange multiplier to enforce the global constraint

$$\operatorname{div}_{\text{NC}} u_{\text{CR}} = 0 \quad \text{a.e. in } \Omega.$$

2.3. Adaptive mesh refinement algorithm.

Algorithm 2.1 (ACRFEM). INPUT coarse regular triangulation \mathcal{T}_0 and $0 < \Theta \leq 1$.

For level $\ell = 0, 1, 2, \dots$ until termination do

COMPUTE discrete solution u_{CR} on \mathcal{T}_{ℓ} with N_{ℓ} degrees of freedom. For any v_{xyz} for $\text{xyz} \in \{\text{A, PMA, MAred, PMred, MP1, MP2, MP2CG5, MP1red, MP1redCG3}\}$, compute the refinement indicators

$$\begin{aligned} \eta(T, v_{\text{xyz}})^2 &:= \|f_{\mathcal{T}}/2 \otimes (\bullet - \operatorname{mid}(T))\|_{L^2(T)}^2 + 1/j_{1,1}^2 \|h_T(f - f_T)\|_{L^2(T)}^2 \\ &+ \|\mathbf{D}_{\text{NC}}(u_{\text{CR}} - v_{\text{xyz}})\|_{L^2(T)}^2 + 1/c_0^2 \|\operatorname{div} v_{\text{xyz}}\|_{L^2(T)}^2 + \|h_{\mathcal{E}}^{3/2} \partial_{\mathcal{E}}^2 u_D / \partial s^2\|_{L^2(\partial\Omega \cap \partial T)}^2. \end{aligned}$$

With the tangential jump $[\mathbf{D}_{\text{NC}} u_{\text{CR}} \cdot \tau_E]_E$ for interior edges and $[\mathbf{D}_{\text{NC}} u_{\text{CR}} \cdot \tau_E]_E := \|\partial u_D / \partial s - \mathbf{D}_{\text{NC}} u_{\text{CR}} \cdot \tau_E\|_{L^2(E)}$ for boundary edges $E \in \mathcal{E}(\partial\Omega)$, the residual-based refinement indicators read

$$\eta_{\text{R}}(T)^2 := |T| \|f\|_{L^2(T)}^2 + |T|^{1/2} \sum_{E \in \mathcal{E}(T)} \|[\mathbf{D}_{\text{NC}} u_{\text{CR}} \cdot \tau_E]_E\|_{L^2(E)}^2.$$

Estimate

$$\begin{aligned} \eta(v_{xyz})^2 := & \left(\|f_{\mathcal{T}}/2 \otimes (\bullet - \text{mid}(\mathcal{T}))\|_{L^2(\Omega)} + 1/j_{1,1} \text{osc}(f, \mathcal{T}) \right)^2 \\ & + \left(\|D_{\text{NC}}(u_{\text{CR}} - v_{xyz})\|_{L^2(\Omega)} + \|h_{\mathcal{E}}^{3/2} \partial_{\mathcal{E}}^2 u_D / \partial s^2\|_{L^2(\partial\Omega)} + 1/c_0 \|\text{div } v_{xyz}\|_{L^2(\Omega)} \right)^2. \end{aligned}$$

MARK a minimal subset \mathcal{M}_ℓ of \mathcal{T}_ℓ such that

$$\Theta \sum_{T \in \mathcal{T}_\ell} \eta_{xyz}(T)^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta_{xyz}(T)^2.$$

REFINE \mathcal{T}_ℓ by *red*-refinement of triangles in \mathcal{M}_ℓ and *red-green-blue*-refinement of further triangles to avoid hanging nodes and compute $\mathcal{T}_{\ell+1}$. od

OUTPUT Sequence of meshes $\mathcal{T}_0, \mathcal{T}_1, \dots$ with respective discrete solution u_{CR} and residual-based error estimator

$$\eta_{\text{R}}(\mathcal{T}_\ell) := \|h_{\mathcal{T}_\ell} f\|_{L^2(\Omega)} + \left(\sum_{E \in \mathcal{E}_\ell} |E| \| [D_{\text{NC}} u_{\text{CR}} \cdot \tau_E]_E \|_{L^2(E)}^2 \right)^{1/2}.$$

3. A POSTERIORI ERROR ESTIMATION FOR STOKES EQUATIONS

This section is devoted to guaranteed upper bounds for the error in the nonconforming energy norm for the Stokes problem based on a decomposition from [AD05] with a slightly sharper upper bound. The general reliability result involves the computable term

$$\eta := \|f_{\mathcal{T}}/2 \otimes (\bullet - \text{mid}(\mathcal{T}))\|_{L^2(\Omega)} + 1/j_{1,1} \text{osc}(f, \mathcal{T}).$$

Here, $j_{1,1}$ is the first positive root of the first Bessel function J_1 . For every triangle $T \in \mathcal{T}$ with the set of its edges $\mathcal{E}(T)$ and $s(T)^2 := \sum_{E \in \mathcal{E}(T)} |E|^2$, an elementary calculation shows

$$\|f_{\mathcal{T}}/2 \otimes (\bullet - \text{mid}(\mathcal{T}))\|_{L^2(T)}^2 := |f_{\mathcal{T}}/2|^2 \| \bullet - \text{mid}(\mathcal{T}) \|_{L^2(T)}^2 = |f_{\mathcal{T}}|^2 |T| s(T)^2 / 144.$$

The constant c_0 in the inf-sup condition

$$0 < c_0 := \inf_{q \in L_0^2(\Omega)} \sup_{v \in H^1(\Omega; \mathbb{R}^2) / \mathbb{R}^2} \frac{\int_{\Omega} q \text{div}(v) \, dx}{\|D v\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}}$$

depends only on the domain Ω and equals the smallest eigenvalue of some general eigenvalue problem [Sto99]. Their values are given in Table 1.1.

The following a posteriori error estimate resembles the upper bound from [AD05, Theorem 1]

$$\begin{aligned} \|e\|_{\text{NC}} \leq & \|f_{\mathcal{T}}/2 \otimes (\bullet - \text{mid}(\mathcal{T}))\|_{L^2(\Omega)} + C \text{osc}(f, \mathcal{T}) \\ & + \|D_{\text{NC}}(u_{\text{CR}} - v)\|_{L^2(\Omega)} + 1/c_0 \|\text{div } v\|_{L^2(\Omega)} \end{aligned}$$

and gives a refined version with an explicit value for $C \lesssim 1$.

Theorem 3.1. *Any $v \in H^1(\Omega; \mathbb{R}^2)$ with $v = u_D$ on $\partial\Omega$ satisfies*

$$\|e\|_{\text{NC}}^2 \leq \eta^2 + \left(\|D_{\text{NC}}(u_{\text{CR}} - v)\|_{L^2(\Omega)} + 1/c_0 \|\text{div } v\|_{L^2(\Omega)} \right)^2.$$

Proof. The analysis follows [AD05] and is repeated here for convenient reading to stress the little differences to [AD05, Theorem 1]. The point of departure is the orthogonal split

$$D_{\text{NC}} e = D z + y$$

into $z \in Z$ with

$$\int_{\Omega} D z : D v \, dx = \int_{\Omega} D_{\text{NC}} e : D v \, dx \quad \text{for all } v \in Z,$$

and the remainder

$$y \in Y := \left\{ y \in L^2(\Omega; \mathbb{R}^{2 \times 2}) \mid \int_{\Omega} y : D v \, dx = 0 \quad \text{for all } v \in Z \right\}.$$

Orthogonality holds in the sense of

$$\|e\|_{\text{NC}}^2 = \|z\|_{\text{NC}}^2 + \|y\|_{L^2(\Omega)}^2 = \int_{\Omega} D_{\text{NC}} e : D z \, dx + \int_{\Omega} D_{\text{NC}} e : y \, dx.$$

To estimate $\int_{\Omega} D_{\text{NC}} e : D z \, dx$, employ the nonconforming interpolation $z_{\text{NC}} \in \text{CR}^1(\mathcal{T}; \mathbb{R}^2)$ of z defined by

$$z_{\text{NC}}(\text{mid}(E)) := \fint_E z \, ds \quad \text{for all } E \in \mathcal{E}.$$

An integration by parts yields $z_{\text{NC}} \in Z_{\text{NC}}$ and

$$(3.1) \quad \int_T D(z - z_{\text{NC}}) \, dx = 0 \quad \text{for all } T \in \mathcal{T}.$$

This allows for the following computation

$$\begin{aligned} \int_{\Omega} D_{\text{NC}} e : D z \, dx &= F(z) - a_{\text{NC}}(u_{\text{CR}}, z) \\ &= F(z - z_{\text{NC}}) - a_{\text{NC}}(u_{\text{CR}}, z) + F(z_{\text{NC}}) \\ &= \int_{\Omega} f \cdot (z - z_{\text{NC}}) \, dx - a_{\text{NC}}(u_{\text{CR}}, z - z_{\text{NC}}). \end{aligned}$$

Equation (3.1) and $\nabla u_{\text{CR}} \in P_0(\mathcal{T})$ yield $a_{\text{NC}}(u_{\text{CR}}, z - z_{\text{NC}}) = 0$. Hence,

$$(3.2) \quad \begin{aligned} \int_{\Omega} D_{\text{NC}} e : D z \, dx &= \int_{\Omega} f \cdot (z - z_{\text{NC}}) \, dx \\ &= \int_{\Omega} f_{\mathcal{T}} \cdot (z - z_{\text{NC}}) \, dx + \int_{\Omega} (f - f_{\mathcal{T}}) \cdot (z - z_{\text{NC}}) \, dx. \end{aligned}$$

For every $T \in \mathcal{T}$, consider the function $q_{\mathcal{T}}|_T \in H(\text{div}, T; \mathbb{R}^{2 \times 2})$ defined by

$$q_{\mathcal{T}}(x)|_T := -f_{\mathcal{T}}/2 \otimes (x - \text{mid}(T)) \quad \text{for } x \in T.$$

An integration by parts and some basic calculations show, for any $v \in H_0^1(\Omega; \mathbb{R}^2)$, that

$$\begin{aligned} \int_T q_{\mathcal{T}} : \nabla v \, dx &= \int_{\partial T} q_{\mathcal{T}} \nu \cdot v \, ds - \int_T \text{div } q_{\mathcal{T}} \cdot v \, dx \\ &= - \sum_{E \in \mathcal{E}(\partial T)} f_{\mathcal{T}} |T| / (3|E|) \int_E v \, ds + \int_T f_{\mathcal{T}} \cdot v \, dx \\ &= \int_T f_{\mathcal{T}} \cdot (v - v_{\text{NC}}) \, dx. \end{aligned}$$

Moreover, the $P_0(\mathcal{T})$ orthogonality of $f - f_{\mathcal{T}}$ and a Poincaré inequality with constant $\text{diam}(T)/j_{1,1}$ from [LS10] yield

$$\begin{aligned} \int_{\Omega} (f - f_{\mathcal{T}}) \cdot (z - z_{\text{NC}}) dx &= \int_{\Omega} (f - f_{\mathcal{T}}) \cdot (z - z_{\text{NC}} - (z - z_{\text{NC}})_{\mathcal{T}}) dx \\ &\leq \sum_{T \in \mathcal{T}} \|f - f_{\mathcal{T}}\|_{L^2(T)} \|z - z_{\text{NC}} - (z - z_{\text{NC}})_{\mathcal{T}}\|_{L^2(T)} \\ &\leq 1/j_{1,1} \text{osc}(f, \mathcal{T}) \|D(z - z_{\text{NC}})\|_{L^2(\Omega)}. \end{aligned}$$

Hence, (3.2) reads

$$\begin{aligned} \int_{\Omega} D_{\text{NC}} e : D z dx &= \int_{\Omega} q_{\mathcal{T}} : \nabla z dx + \int_{\Omega} (f - f_{\mathcal{T}}) \cdot (z - z_{\text{NC}}) dx \\ &\leq \|q_{\mathcal{T}}\|_{L^2(\Omega)} \|z\|_{\text{NC}} + 1/j_{1,1} \text{osc}(f, \mathcal{T}) \|z - z_{\text{NC}}\|_{\text{NC}}. \end{aligned}$$

Notice that (3.1) yields

$$\begin{aligned} \|D(z - z_{\text{NC}})\|_{L^2(T)}^2 &= \int_T |D z|^2 dx - 2 \int_T D z : D z_{\text{NC}} dx + \int_T |D z_{\text{NC}}|^2 dx \\ &= \int_T |D z|^2 dx - \int_{\Omega} |D z_{\text{NC}}|^2 dx \leq \|D z\|_{L^2(T)}^2. \end{aligned}$$

It remains to estimate $\int_{\Omega} D_{\text{NC}} e : y dx$. Recall from [AD05] that, for each $y \in Y$, there exists some $w \in L_0^2(\Omega) := \{q \in L^2(\Omega) \mid \int_{\Omega} q dx = 0\}$ with

$$(3.3) \quad \int_{\Omega} y : D v dx = \int_{\Omega} w \text{div} v dx \quad \text{for all } v \in H^1(\Omega; \mathbb{R}^2)$$

and

$$\|w\|_{L^2(\Omega)} \leq 1/c_0 \|y\|_{L^2(\Omega)}.$$

Hence, given any $v \in H^1(\Omega; \mathbb{R}^2)$ with $u - v = 0$ on $\partial\Omega$, it holds

$$\begin{aligned} \int_{\Omega} D_{\text{NC}} e : y dx &= \int_{\Omega} D_{\text{NC}}(u_{\text{CR}} - v) : y dx + \int_{\Omega} D(v - u) : y dx \\ &\leq \|D_{\text{NC}}(u_{\text{CR}} - v)\|_{L^2(\Omega)} \|y\|_{L^2(\Omega)} + \int_{\Omega} \text{div}(v - u) w dx \\ &\leq \left(\|D_{\text{NC}}(u_{\text{CR}} - v)\|_{L^2(\Omega)} + 1/c_0 \|\text{div} v\|_{L^2(\Omega)} \right) \|y\|_{L^2(\Omega)}. \end{aligned}$$

The combination of all mentioned results concludes the proof. \square

4. REALISATIONS OF GUARANTEED UPPER BOUNDS

The subsequent Subsections 4.1-4.3 discuss nine designs for v and the estimation of $\|e\|$ via Theorem 3.1 with

$$(4.1) \quad \mu(v) := \|D_{\text{NC}}(u_{\text{CR}} - v)\|_{L^2(\Omega)} + 1/c_0 \|\text{div} v\|_{L^2(\Omega)}.$$

The significant difference to [CM13] on the Poisson problem lies in the additional divergence term which leads to a sum of L^2 norms and Algorithms 4.1 and 4.2.

4.1. Interpolation after Ainsworth. This subsection introduces the interpolation after Ainsworth [Ain04] that designs some piecewise linear $v_A \in H^1(\Omega; \mathbb{R}^2)$ with respect to the original triangulation \mathcal{T} ,

$$v_A(z) := \begin{cases} u_D(z) & \text{if } z \in \mathcal{N} \setminus \mathcal{M}, \\ \left(\sum_{T \in \mathcal{T}(z)} u_{\text{CR}|T}(z) \right) / |\mathcal{T}(z)| & \text{if } z \in \mathcal{M}. \end{cases}$$

The related error estimator reads

$$\eta_A^2 := \eta^2 + \mu(v_A)^2.$$

4.2. Modified interpolation operator. This subsection introduces an improved interpolation that designs some piecewise linear $v_{\text{red}} \in H_0^1(\Omega; \mathbb{R}^2)$ with respect to the red refined triangulation $\text{red}(\mathcal{T})$. The nodes of $\text{red}(\mathcal{T})$ consists of the nodes \mathcal{N} and the edge midpoints $\text{mid}(\mathcal{E})$ of \mathcal{T} . At the boundary the interpolation equals the nodal interpolation of u_D and on all interior edge midpoints it equals u_{CR} ;

$$(4.2) \quad v_{\text{red}}(z) := \begin{cases} u_{\text{CR}}(z) & \text{for } z \in \text{mid}(\mathcal{E}) \setminus \text{mid}(\mathcal{E}(\partial\Omega)), \\ u_D(z) & \text{for } z \in (\mathcal{N} \cup \text{mid}(\mathcal{E})) \cap \partial\Omega, \\ v_z & \text{for } z \in \mathcal{M}. \end{cases}$$

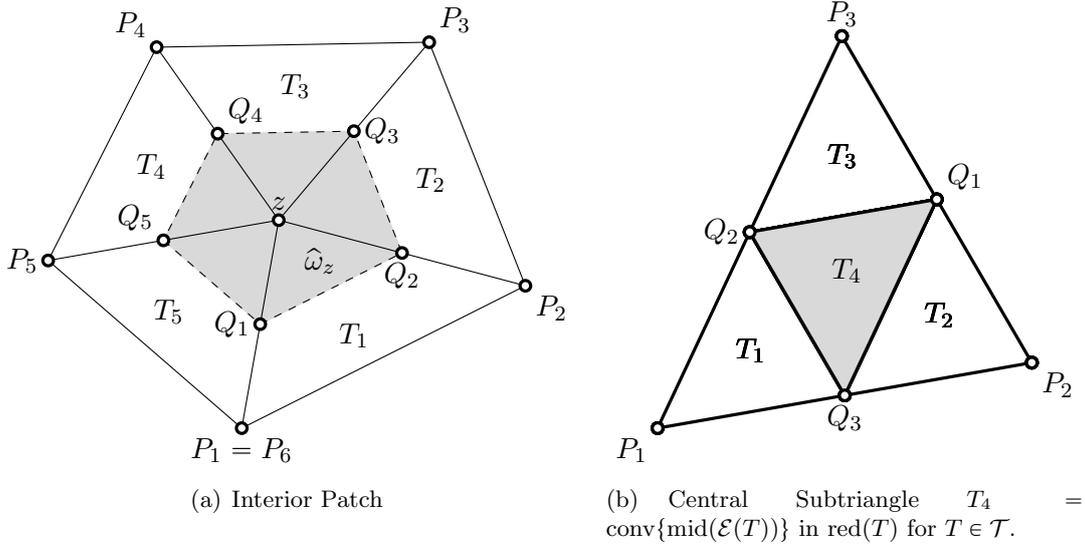


FIGURE 4.1. Notation for red-refinements.

In this way, the interpolation v_{red} equals u_{CR} on all central subtriangles like T_4 in Figure 4.1(b) and it remains to determine the values v_z at the free nodes $z \in \mathcal{M}$. They may be chosen as in the design of v_A , but we suggest to choose them locally optimal with Algorithm 4.1 by solving local problems around each node patch \hat{w}_z with respect to the red-refined triangulation as in Figure 4.1(a) under the side condition of the fixed values at the edge midpoints Q_j of the adjacent edges. The value v_z at z remains the only degree of freedom in this local problem.

Algorithm 4.1 (Patchwise minimisation). INPUT $u_{\text{CR}} \in \text{CR}^1(\mathcal{T}; \mathbb{R}^2)$ and $c_0 > 0$. Set $\lambda := 1$. For $j = 1, 2, \dots$ until termination compute (a) & (b).

$$(a) \text{ For all } z \in \mathcal{N}(\Omega), \quad v_0 := \sum_{E \in \mathcal{E}} v_{\text{red}}(\text{mid}(E)) \varphi_{\text{mid}(E)}^{\text{red}} \quad \text{and}$$

$$v_z := \underset{w \in \mathbb{R}^2}{\text{argmin}} \left((1 + \lambda) \left\| \mathbf{D}_{\text{NC}}(u_{\text{CR}} - v_0 - w \varphi_z^{\text{red}}) \right\|_{L^2(\hat{\omega}_z)}^2 \right. \\ \left. + (1 + 1/\lambda)/c_0^2 \left\| \text{div}(v_0 - w \varphi_z^{\text{red}}) \right\|_{L^2(\hat{\omega}_z)}^2 \right).$$

$$(b) \text{ Update} \quad v_{\text{red}} := v_0 + \sum_{z \in \mathcal{N}(\Omega)} v_z \varphi_z^{\text{red}},$$

$$\lambda := \left\| \text{div } v_{\text{red}} \right\|_{L^2(\Omega)} / \left(c_0 \left\| \mathbf{D}_{\text{NC}}(u_{\text{CR}} - v_{\text{red}}) \right\|_{L^2(\Omega)} \right),$$

$$\eta_{\text{PMred}(j)}^2 := \eta^2 + \mu(v_{\text{red}})^2.$$

OUTPUT $\eta_{\text{PMred}(j)}$ for $j = 1, 2, \dots$

We distinguish between the optimal version $\eta_{\text{PMred}(j)}$ from Algorithm 4.1, and η_{MAred} with the suboptimal choice v_z as in Subsection 4.1. This can be seen as a modification v_{MAred} of v_{A} at the edge midpoints.

Some numerical examples below suggest that the fixed values at the edge midpoints from v_{red} lead to unexpectedly large divergence terms. Indeed, even the interpolation v_{A} may lead to better results than v_{MAred} on fine meshes as displayed in Section 5.4. Of course, it is possible to describe other values at the edge midpoints and substitute (4.2) by, e.g.,

$$(4.3) \quad v_{\text{red}}(z) := \begin{cases} v_{\text{A}}(z) & \text{for } z \in \text{mid}(\mathcal{E}) \setminus \text{mid}(\mathcal{E}(\partial\Omega)), \\ u_{\text{D}}(z) & \text{for } z \in (\mathcal{N} \cup \text{mid}(\mathcal{E})) \cap \partial\Omega, \\ v_z & \text{for } z \in \mathcal{M}. \end{cases}$$

| ndof | 13 | 57 | 241 | 993 | 4033 | 16257 | 65281 | 261633 |
|--------------------------|---------|---------|---------|---------|---------|---------|---------|---------|
| η_{A} | 1817.92 | 699.646 | 276.868 | 112.429 | 46.5926 | 19.7549 | 8.59524 | 3.83932 |
| $\eta_{\text{PMA}(1)}$ | 719.926 | 290.836 | 131.632 | 61.9372 | 29.1130 | 13.6895 | 6.48276 | 3.10096 |
| $\eta_{\text{PMA}(2)}$ | 719.926 | 290.773 | 131.611 | 61.9337 | 29.1126 | 13.6894 | 6.48274 | 3.10096 |
| $\eta_{\text{PMA}(3)}$ | 719.926 | 290.773 | 131.611 | 61.9337 | 29.1126 | 13.6894 | 6.48274 | 3.10096 |
| $\eta_{\text{PMA}(4)}$ | 719.926 | 290.773 | 131.611 | 61.9337 | 29.1126 | 13.6894 | 6.48274 | 3.10096 |
| $\eta_{\text{PMA}(5)}$ | 719.926 | 290.773 | 131.611 | 61.9337 | 29.1126 | 13.6894 | 6.48274 | 3.10096 |
| η_{MAred} | 719.926 | 286.729 | 122.127 | 55.5387 | 25.5059 | 11.7695 | 5.48898 | 2.59459 |
| $\eta_{\text{PMred}(1)}$ | 719.926 | 286.684 | 121.653 | 54.4621 | 24.4546 | 11.0798 | 5.10119 | 2.39005 |
| $\eta_{\text{PMred}(2)}$ | 719.926 | 286.677 | 121.632 | 54.4186 | 24.4286 | 11.0678 | 5.09563 | 2.38739 |
| $\eta_{\text{PMred}(3)}$ | 719.926 | 286.677 | 121.632 | 54.4183 | 24.4281 | 11.0674 | 5.09542 | 2.38728 |
| $\eta_{\text{PMred}(4)}$ | 719.926 | 286.677 | 121.632 | 54.4183 | 24.4281 | 11.0674 | 5.09542 | 2.38728 |
| $\eta_{\text{PMred}(5)}$ | 719.926 | 286.677 | 121.632 | 54.4183 | 24.4281 | 11.0674 | 5.09542 | 2.38728 |

TABLE 4.1. Error estimators η_{A} , η_{MAred} , $\eta_{\text{PMA}(j)}$ and $\eta_{\text{PMred}(j)}$ for $j = 1, 2, 3, 5, 10$ in Algorithm 4.1 and uniform mesh refinement in the example of Section 5.3 (with Dirichlet data error contribution from Section 4.4).

The optimal value v_z can again be computed similar to Algorithm 4.1 with v_{red} replaced by v_A and output $\eta_{\text{PMA}(j)}$. Since $v_z = v_A(z)$ is admissible in this optimisation, $\eta_{\text{PMA}(j)}$ can only lead to better results than the interpolation η_A .

Table 4.1 compares the outcome $\eta_{\text{PMred}(j)}$ (resp. $\eta_{\text{PMA}(j)}$) of Algorithm 4.1 for $j = 1, 2, 3, 5, 10$ with the edge values (4.2) (resp. (4.3)). There is no significant improvement for coarse meshes and only little improvement on fine meshes. Surprisingly, the design of η_{PMA} is somehow insensitive for $j \geq 2$ in Algorithm 4.1.

4.3. Optimal choices. The global minimisers v_{MP1} in $P_1(\mathcal{T}) \cap C(\Omega)$, v_{MP2} in $P_2(\mathcal{T}) \cap C(\Omega)$ and v_{MP1red} in $P_1(\text{red}(\mathcal{T})) \cap C(\Omega)$ on the red-refined triangulation $\text{red}(\mathcal{T})$ of the functional μ from (4.1) are computed by Algorithm 4.2.

Algorithm 4.2 (Global minimisation). INPUT $u_{\text{CR}} \in \text{CR}^1(\mathcal{T}; \mathbb{R}^2)$, $W(\mathcal{T}) \in \{P_1(\mathcal{T}) \cap C(\Omega), P_1(\text{red}(\mathcal{T})) \cap C(\Omega), P_2(\mathcal{T}) \cap C(\Omega)\}$ and $c_0 > 0$. Set $\lambda := 1$.

For $j = 1, 2, \dots$ until termination do

$$(4.4) \quad v_{W(\mathcal{T})} := \underset{v \in W(\mathcal{T})}{\operatorname{argmin}} \left((1 + \lambda) \|D_{\text{NC}}(u_{\text{CR}} - v)\|_{L^2(\Omega)}^2 + (1 + 1/\lambda) \|\operatorname{div} v\|_{L^2(\Omega)/c_0}^2 \right),$$

$$\eta_{j,W(\mathcal{T})}^2 := \eta^2 + \mu(v_{W(\mathcal{T})})^2,$$

$$\lambda := \left\| \operatorname{div} v_{W(\mathcal{T})} \right\|_{L^2(\Omega)} / \left(c_0 \|D_{\text{NC}}(u_{\text{CR}} - v_{W(\mathcal{T})})\|_{L^2(\Omega)} \right). \quad \text{od}$$

OUTPUT $\eta_{\text{MP1}(j)} := \eta_{j,P_1(\mathcal{T}) \cap C(\Omega)}$, $\eta_{\text{MP1red}(j)} := \eta_{j,P_1(\text{red}(\mathcal{T})) \cap C(\Omega)}$, $\eta_{\text{MP2}(j)} := \eta_{j,P_2(\mathcal{T}) \cap C(\Omega)}$ for $j = 1, 2, \dots$

Table 4.2 displays values of $\eta_{\text{MP2}(j)}$ for $j = 1, 2, \dots, 5$ and suggests that there is a more significant improvement by Algorithm 4.2 compared to Algorithm 4.1 for the local designs. In the computational examples of Section 5, the termination of Algorithms 4.1 and 4.2 is with $j = 3$.

To reduce the computational costs of (4.4) one might use v_{MAred} as an initial guess for some iterative solver to draw near the minimiser of (4.4) for $W(\mathcal{T}) = P_1(\text{red}(\mathcal{T})) \cap C(\Omega)$. We use a preconditioned conjugate gradients scheme and stop at the third iterate. The preconditioner is the diagonal of the system matrix named after Jacobi. This approximation of (4.4) in Algorithm (4.4) with $j = 3$ results in the estimator $\eta_{\text{MP1redCG3}}$. Similarly, the nodal values of v_{MAred} define some piecewise quadratic function and hence an initial value for some PCG algorithm for the approximation of the minimiser of (4.4) for $W(\mathcal{T}) = P_2(\mathcal{T}) \cap C(\Omega)$. The truncation of the minimisation in (4.4) after five PCG iterations and $j = 3$ in Algorithm 4.2 defines the error estimator η_{MP2CG5} .

| ndof | 13 | 57 | 241 | 993 | 4033 | 16257 | 65281 | 261633 |
|------------------------|---------|---------|---------|---------|---------|---------|---------|----------|
| $\eta_{\text{MP2}(1)}$ | 516.780 | 171.508 | 50.5265 | 18.2578 | 7.88212 | 3.72095 | 1.82154 | 0.903860 |
| $\eta_{\text{MP2}(2)}$ | 516.747 | 169.398 | 49.1931 | 17.2531 | 7.24667 | 3.36511 | 1.63548 | 0.809433 |
| $\eta_{\text{MP2}(3)}$ | 516.747 | 169.159 | 48.9606 | 17.1022 | 7.14785 | 3.30029 | 1.59558 | 0.787235 |
| $\eta_{\text{MP2}(4)}$ | 516.747 | 169.144 | 48.9070 | 17.0820 | 7.13818 | 3.29419 | 1.59092 | 0.783648 |
| $\eta_{\text{MP2}(5)}$ | 516.747 | 169.143 | 48.8919 | 17.0791 | 7.13742 | 3.29387 | 1.59069 | 0.783424 |

TABLE 4.2. Error estimator $\eta_{\text{MP2}(j)}$ for $j = 1, \dots, 5$ in Algorithm 4.2 and uniform mesh refinement in the example of Section 5.3 (with Dirichlet data error contribution from Section 4.4).

4.4. Inhomogeneous Dirichlet boundary conditions. In case of inhomogeneous boundary conditions, the designs of v_{xyz} from the previous subsections do not satisfy $u - v_{\text{xyz}} = 0$ on $\partial\Omega$ in general. To overcome this difficulty consider $w_D \in H^1(\Omega; \mathbb{R}^2)$ from [CM13, BCD04] with

$$w_D|_{\partial\Omega} = u_D|_{\partial\Omega} - v_{\text{xyz}}|_{\partial\Omega}.$$

Then, Theorem 3.1 for $v = v_{\text{xyz}} - w_D$ yields $u - v \in H_0^1(\Omega)$ and

$$\begin{aligned} \|e\|_{\text{NC}}^2 &\leq \eta^2 + \left(\|D_{\text{NC}}(u_{\text{CR}} - v - w_D)\|_{L^2(\Omega)} + 1/c_0 \|\operatorname{div}(v + w_D)\|_{L^2(\Omega)} \right)^2 \\ &\leq \eta^2 + \left(\|D_{\text{NC}}(u_{\text{CR}} - v)\|_{L^2(\Omega)} + 1/c_0 \|\operatorname{div} v\|_{L^2(\Omega)} + (1 + 1/c_0) \|w_D\| \right)^2. \end{aligned}$$

In case of $v_{\text{xyz}}|_{\partial\Omega} = \mathcal{I}u_D|_{\partial\Omega} := \sum_{z \in \mathcal{N} \setminus \mathcal{M}} u_D(z) \varphi_z|_{\partial\Omega}$, it holds [CM13, BCD04]

$$\|w_D\| \leq C_\gamma \|h_{\mathcal{E}}^{3/2} \delta_{\mathcal{E}}^2 (u_D - v_{\text{xyz}}) / \partial s^2\|_{L^2(\partial\Omega)}.$$

Remark 4.1 ([CM13]). For right isosceles triangles numerical calculations suggest the constant $C_\gamma = 0.4980$. If $v_{\text{xyz}}|_{\partial\Omega}$ is the nodal interpolation of $u_D|_{\partial\Omega}$ on the red-refined triangulation, w_D can be designed on the red-refined triangulation with halved edge lengths and the constant reduces to $C_\gamma = 0.4980/2^{3/2} = .1761$. The same holds if v_{xyz} equals the P_2 interpolation of u_D on \mathcal{T} along $\partial\Omega$.

5. NUMERICAL EXPERIMENTS

This section discusses the five benchmark examples from Table 1.1.

5.1. Smooth example. The first benchmark problem employs the right-hand side $f(x, y) = (4\pi^2 \sin(\pi(x - y)), 0)$ and inhomogeneous Dirichlet boundary data u_D with exact solution

$$u(x, y)_j = \sin(\pi x) \cos(\pi y) - \cos(\pi x) \sin(\pi y) \quad \text{for } j = 1, 2$$

on the square domain $\Omega = (-1, 1)^2$ with $c_0 = 0.3826$ from [Sto99].

Figure 5.1 shows the efficiency indices in case of uniform and adaptive mesh refinement in the range of 1 and 3. While η_{MAred} is superior to η_{A} in case of Poisson problems from [CM13], this example shows that this must not be the case for Stokes problems. The piecewise minimal improvement η_{PMred} closes the gap between η_{A} and η_{MAred} , but barely leads to more efficient upper bounds than η_{A} at least for uniform mesh refinement. Here, η_{PMA} performs better and converges to efficiency indices close to 1.5 for uniform mesh refinement and efficiency indices below 2.5 for adaptive mesh refinement. The error estimator $\eta_{\text{MP1redCG3}}$ and its optimal limit η_{MP1red} with efficiency indices between 1.25 and 1.75 only lose to η_{MP2} which allows for efficiency indices close to 1. The residual error estimator η_{R} with efficiency indices above 8 is not displayed. Figure 5.2 shows that none of the error estimators leads to significantly better refined meshes.

5.2. Second smooth example. The second benchmark problems from [AD05] employs the right-hand side $f(x, y) = (-4y, 4x)$ and inhomogeneous Dirichlet boundary data u_D that matches the exact solution

$$u(x, y) = [x(1 - x)(1 - 2y), -y(1 - y)(1 - 2x)]$$

on the square domain $\Omega = (0, 1)^2$ with $c_0 = 0.3826$ from [Sto99].

The efficiency indices displayed in Figure 5.3 scatter more than in the first example. The error estimator η_{MP1} and η_{A} yield almost identical efficiency indices of larger than 5 for uniform mesh refinement. The red(\mathcal{T})-based interpolation error estimators η_{MAred} and

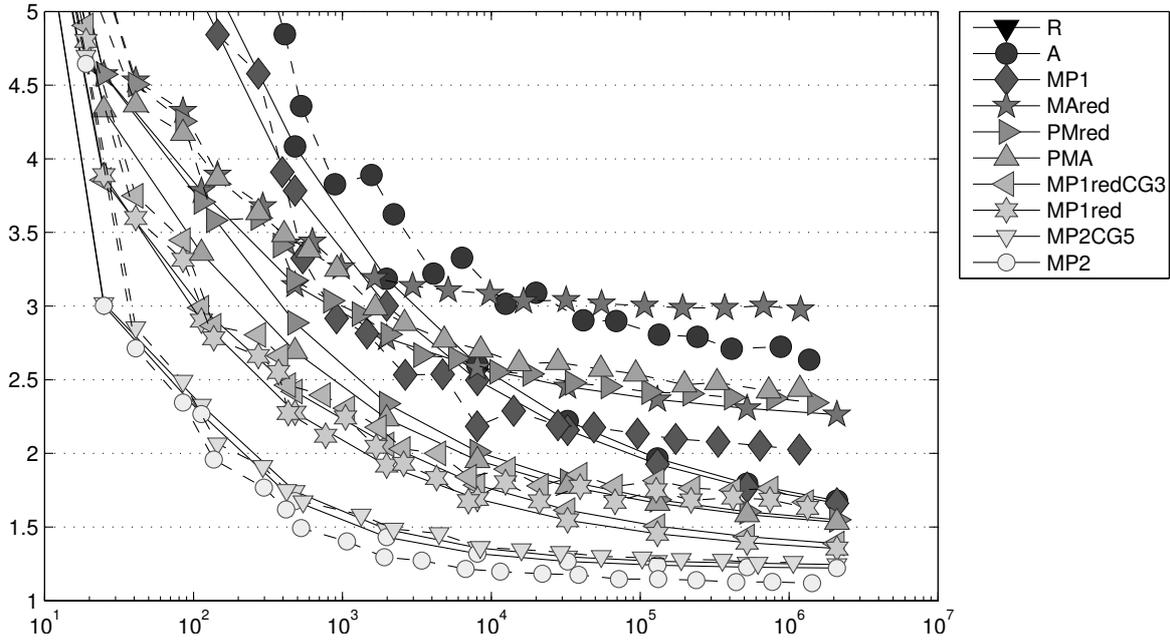


FIGURE 5.1. History of efficiency indices $\eta_{xyz}/\|e\|_{NC}$ of various a posteriori error estimators η_{xyz} labelled xyz in the figure as functions of the number of unknowns on adaptive (dashed lines) and uniform meshes (solid lines) in Subsection 5.1.

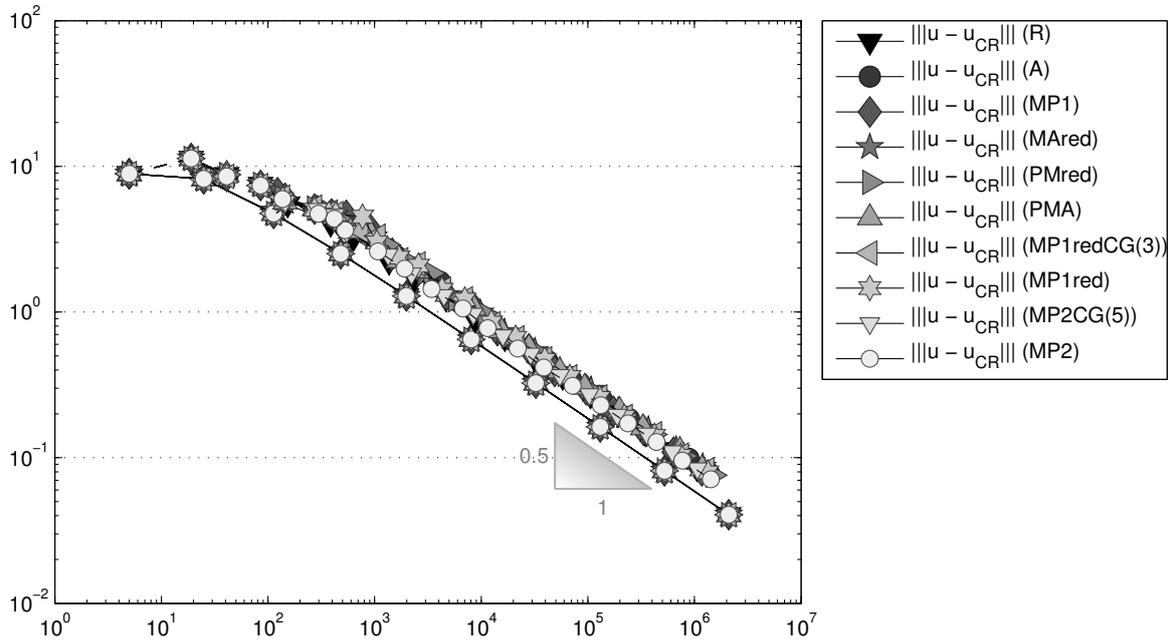


FIGURE 5.2. Convergence history of the energy error for uniform (solid lines) and adaptive (dashed lines) mesh refinements in Subsection 5.1.

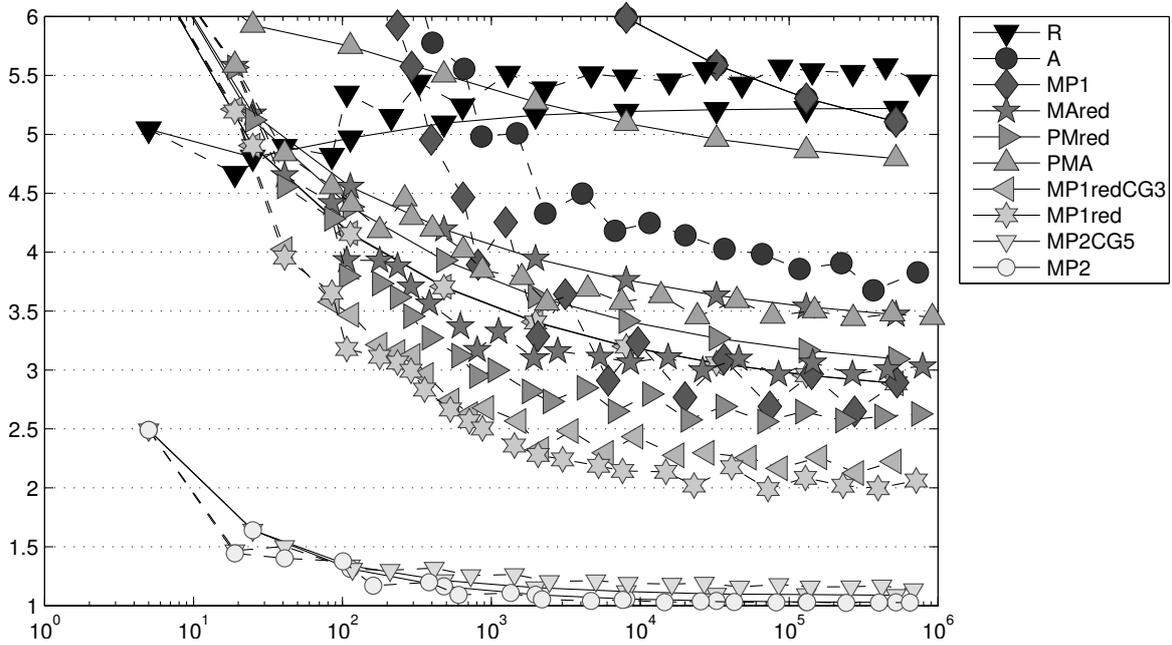


FIGURE 5.3. History of efficiency indices $\eta_{xyz}/\|e\|_{NC}$ of various a posteriori error estimators η_{xyz} labelled xyz in the figure as functions of the number of unknowns on adaptive (dashed lines) and uniform meshes (solid lines) in Subsection 5.2.

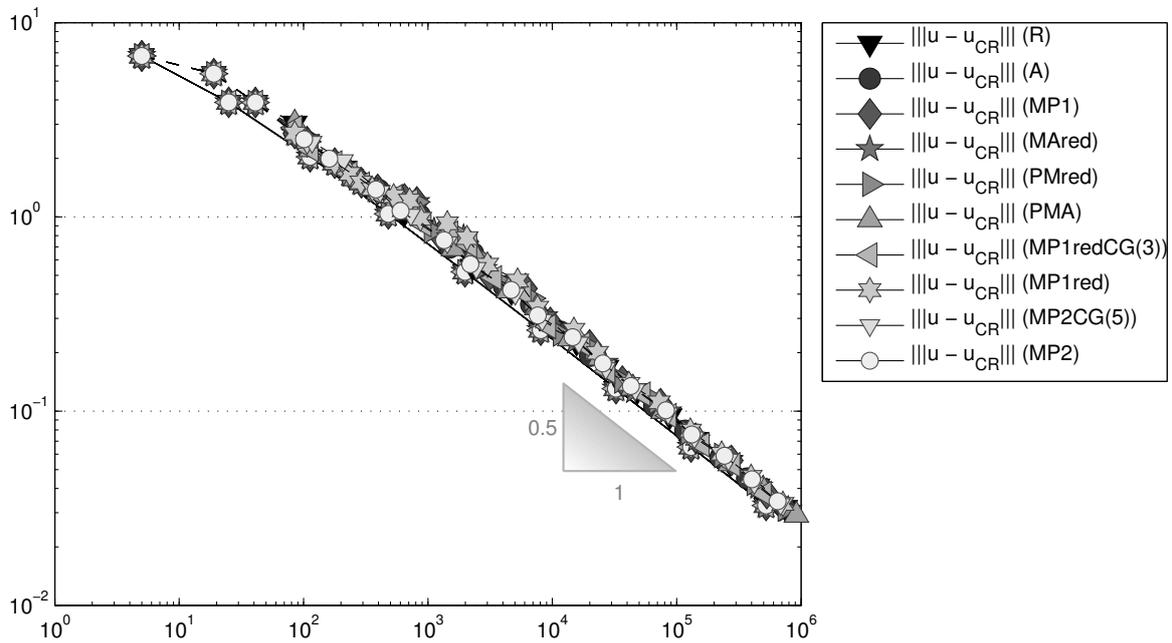


FIGURE 5.4. Convergence history of the energy error for uniform (solid lines) and adaptive (dashed lines) mesh refinements in Subsection 5.2.

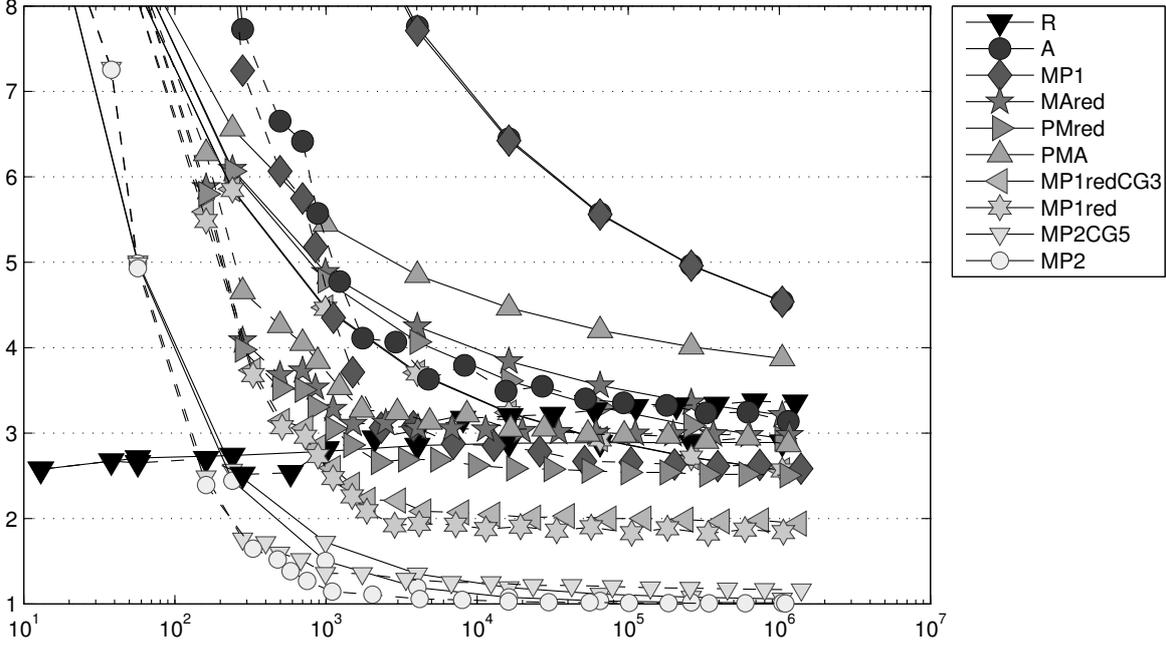


FIGURE 5.5. History of efficiency indices $\eta_{xyz}/\|e\|_{NC}$ of various a posteriori error estimators η_{xyz} labelled xyz in the figure as functions of the number of unknowns on adaptive (dashed lines) and uniform meshes (solid lines) in Subsection 5.3.

η_{PMred} yield efficiency indices of about 3, while η_{MP1red} and $\eta_{MP1redCG3}$ allow efficiency indices around 2. Again, the most accurate error estimator are η_{MP2} and η_{MP2CG5} .

5.3. Colliding flow example. The third benchmark problem employs $f(x, y) \equiv 0$ and the exact solution $u(x, y) = (20xy^4 - 4x^5, 20x^4y - 4y^5)$ on the square domain $\Omega = (-1, 1)^2$ with $c_0 = 0.3826$ from [Sto99].

The efficiency indices for adaptive mesh refinement displayed in Figure 5.5 are in the range between almost 1 in case of η_{MP2} to 3.5 in case of η_A . The piecewise minimal interpolation η_{PMred} yields efficiency indices around 2.5 which is significantly better than η_{PMA} and also better than η_{MP1} . The error estimator $\eta_{MP1redCG3}$ is almost as efficient as the optimal η_{MP1red} with around 2.

5.4. Example on L-shaped domain. The fourth benchmark problem employs $f(x, y) \equiv 0$ and u_D matching the exact solution

$$u(r, \varphi) = r^\alpha \begin{pmatrix} (\alpha + 1) \sin(\varphi) \psi(\varphi) + \cos(\varphi) \psi'(\varphi) \\ -(\alpha + 1) \cos(\varphi) \psi(\varphi) + \sin(\varphi) \psi'(\varphi) \end{pmatrix}^T$$

on the L-shaped domain $\Omega = (-1, 1)^2 \setminus ((0, 1) \times (-1, 0))$ with

$$\begin{aligned} \psi(\varphi) = & 1/(\alpha + 1) \sin((\alpha + 1)\varphi) \cos(\alpha\omega) - \cos((\alpha + 1)\varphi) \\ & + 1/(\alpha - 1) \sin((\alpha - 1)\varphi) \cos(\alpha\omega) + \cos((\alpha - 1)\varphi) \end{aligned}$$

for $\alpha = 856399/1572864 \approx 0.54$, $\omega = 3\pi/2$ from [Ver89]. For the estimator we set $c_0 = 0.3$ from [Sto99].

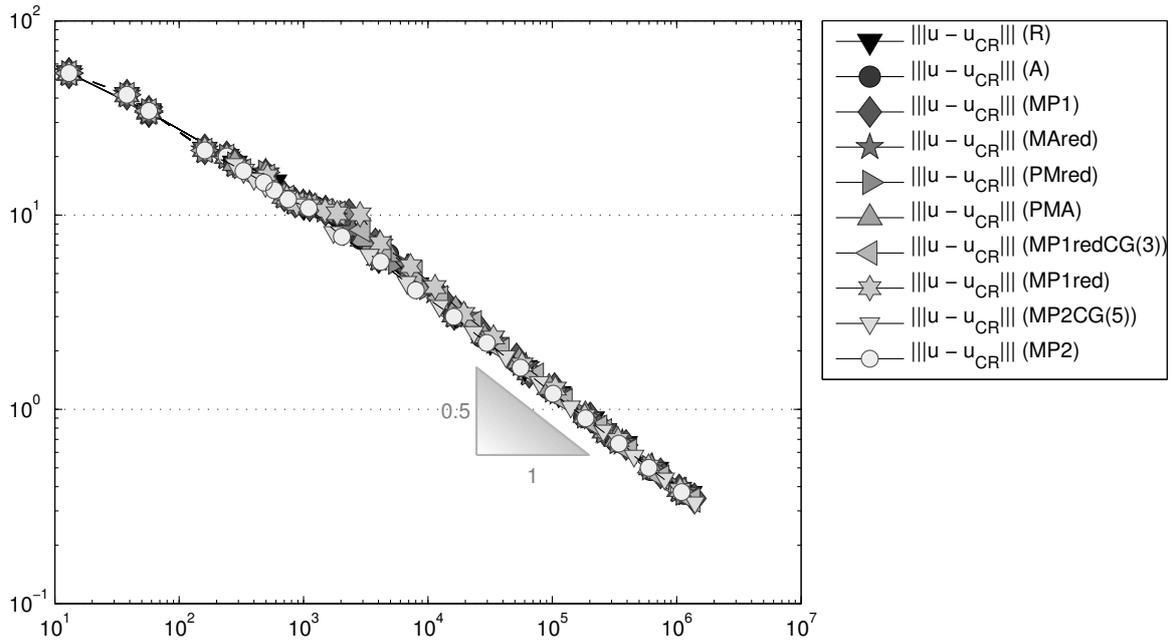


FIGURE 5.6. Convergence history of the energy error for uniform (solid lines) and adaptive (dashed lines) mesh refinements in Subsection 5.3.

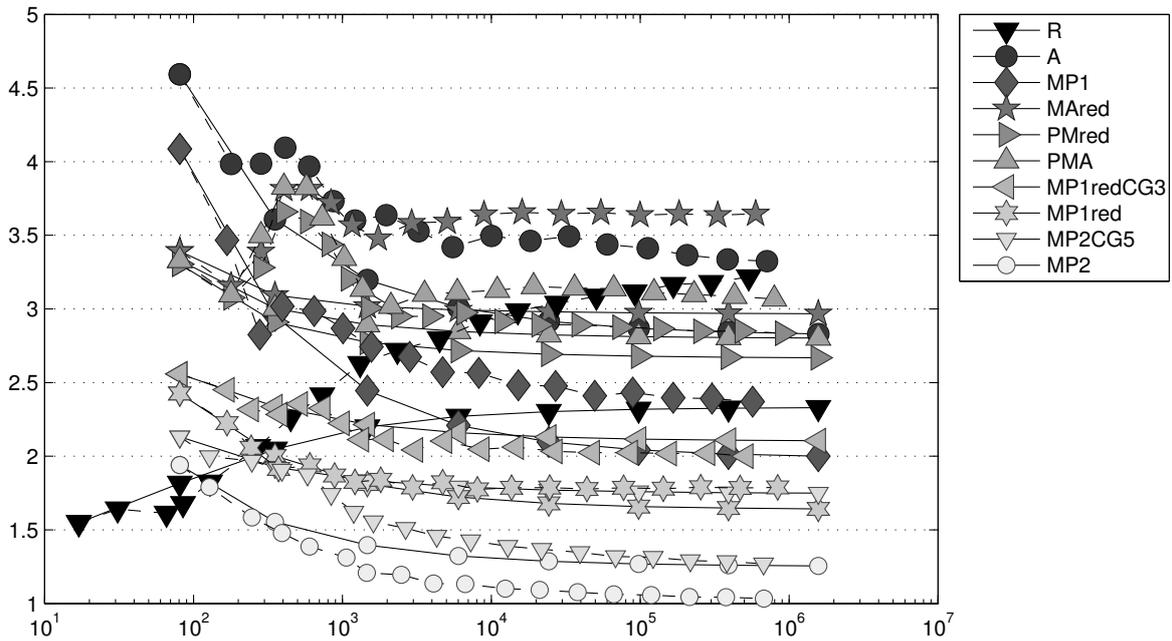


FIGURE 5.7. History of efficiency indices $\eta_{xyz}/\|e\|_{NC}$ of various a posteriori error estimators η_{xyz} labelled xyz in the figure as functions of the number of unknowns on adaptive (dashed lines) and uniform meshes (solid lines) in Subsection 5.4.

Figure 5.8 shows the convergence history of the energy error for uniform and adaptive mesh refinement. The singularity reduces the convergence speed for uniform mesh refinement

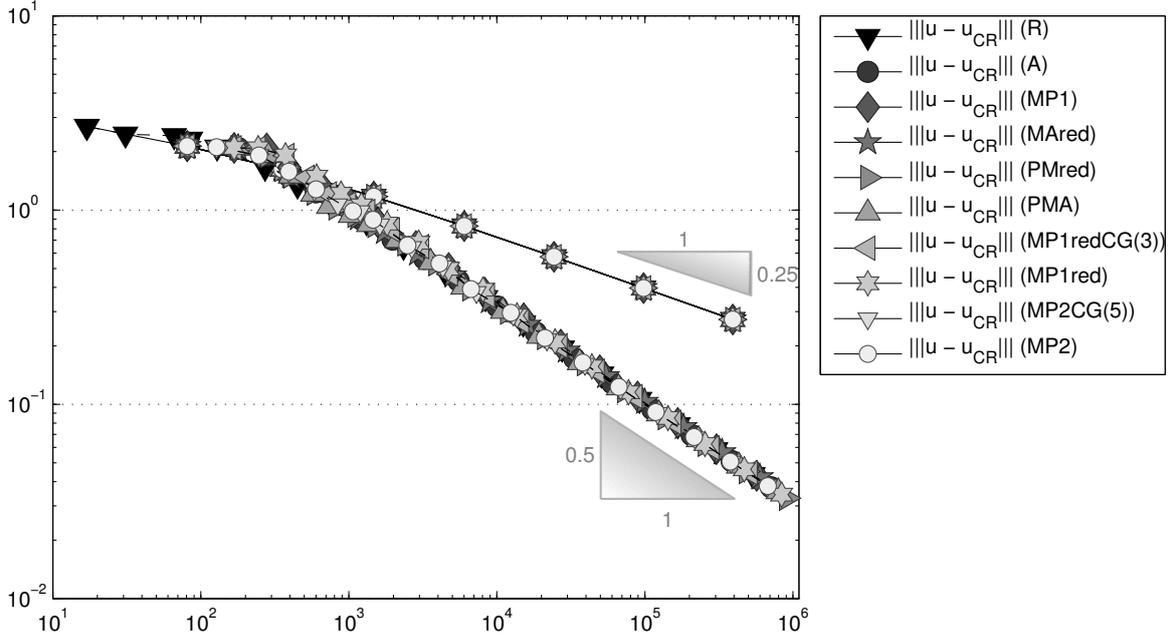


FIGURE 5.8. Convergence history of the energy error for uniform (solid lines) and adaptive (dashed lines) mesh refinements in Subsection 5.4.

significantly. The adaptive mesh refinement algorithm from Section 2.3 leads to the optimal convergence speed, independently of the chosen refinement indicators. This is also true for all other examples so far.

The efficiency indices are displayed in Figure 5.7 and appear similar to the examples before in the range of 1 to 4.

5.5. Backward facing step example. The last example employs the backstep domain $\Omega = ((-2, 8) \times (-1, 1)) \setminus ((-2, 0) \times (-1, 0))$, the right-hand side $f \equiv 0$ and the inhomogeneous boundary data

$$u_D(x, y) = \begin{cases} (-y(y - 1)/10, 0) & \text{at } x = -2, \\ (-y^2 - 1)/80, 0) & \text{at } x = 8. \end{cases}$$

There is no known reference solution, but the example is well-understood [BW91, CF01]. The error estimators yield guaranteed upper bounds that are displayed in Figure 5.9 for the energy error with $c_0 = 0.3$ (not justified). Again, optimal minimisation leads to significantly smaller bounds than the local interpolation designs.

6. CONCLUSIONS

The theoretical and practical results of this paper support the following observations.

6.1. Explicit error estimator sufficient for effective mesh design. The adaptive mesh refinement may be steered by simple η_R -based marking. It does not appear to be favourable to spend more computational time for more laborious refinement rules.

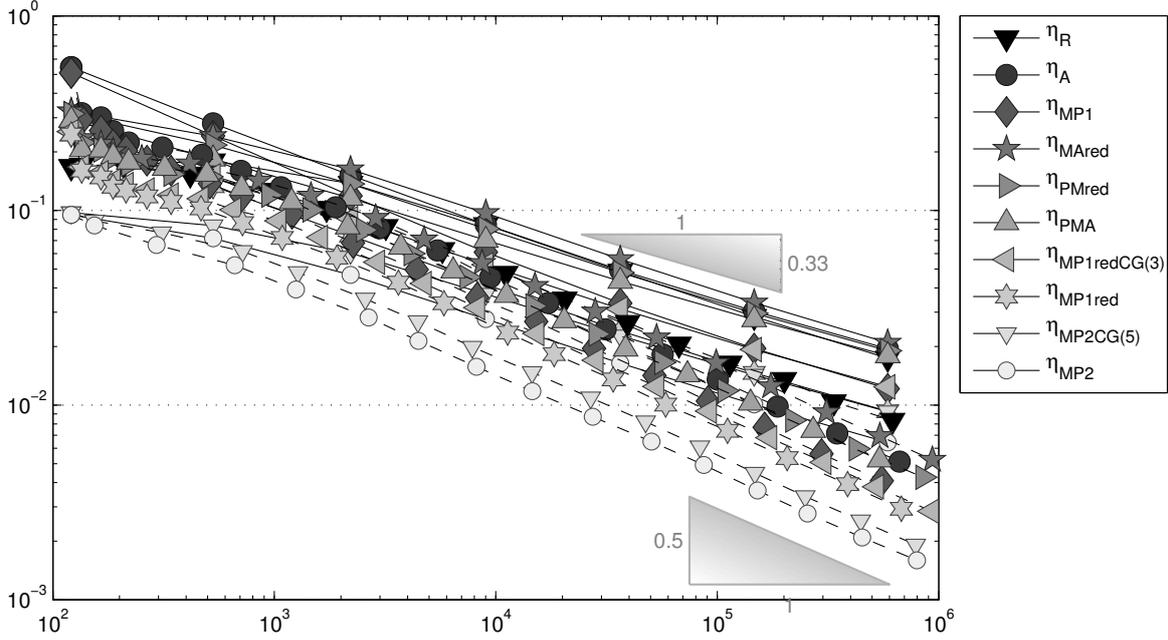


FIGURE 5.9. Convergence history of the error estimators for uniform (solid lines) and adaptive (dashed lines) mesh refinements in Subsection 5.5.

6.2. No explicit error estimation for reliable error control. The explicit residual-based error estimator η_R is no guaranteed upper bound for the exact error, since sharp reliability constants are unknown or hard to calculate. The shown efficiency indices for η_R are therefore not comparable with those of the other guaranteed error bounds. However, from related situations on the Poisson problem, we expect that the displayed η_R from Subsection 2.3 with $C_{\text{rel}} = 1$ is an optimistic approximation [CF99, VV09]. Therefore, the values of η_R are plotted in all the figures and display that those explicit residual-based error estimates are less competitive.

6.3. Accurate and cheap error control via η_{PMred} . The experience for Poisson problems in [CM13] is that the modification v_{red} of v_A is superior in all benchmark examples. This is not true for Stokes problems, since η_A sometimes is more accurate than η_{MAred} . However, the associated piecewise minimal error estimator η_{PMred} is better than η_{PMA} in all benchmark examples with efficiency indices between 2 and 3.

6.4. Performance of Algorithm 4.1 and 4.2. Since the error estimators of Theorem 3.1, as well as that of [AD05], involves the sum of norms and not the sum of their squares, some alternating direction minimisation in the variable v and λ is suggested in Algorithm 4.1 and 4.2. The numerical experiments reported in Table 4.1 and ?? suggest that the value $\lambda = 1$ is already a good approximation. Hence, an expensive outer loop over various j does not appear to be necessary.

6.5. More accurate error control via η_{MP2} or η_{MP1red} . Global Minimisation on the red-refined triangulation $\text{red}(\mathcal{T})$ leads to the error estimator η_{MP1red} with efficiency indices between 1.5 to 2. In most benchmark examples, its approximation $\eta_{\text{MP1redCG3}}$ leads to only slightly less accurate results. However, the optimisation with piecewise quadratic polynomials η_{MP2} allows the best error control with efficiency indices below 1.5, often close to 1. The

error estimator η_{MP2CG5} is a very good approximation towards η_{MP2} and even yields better efficiency indices than η_{MP1red} .

6.6. Suggested approximation of η_{MP1red} with $\eta_{\text{MP1redCG3}}$ or η_{MP2} with η_{MP2CG5} .

The PCG approximation $\eta_{\text{MP1redCG3}}$ of η_{MP1red} is computed by three iterations of some conjugate gradient scheme with initial value v_{MAred} in direction of the minimiser of the sum of squares. So λ is set to 1 and there is no outer loop of the minimisation as discussed in Subsection 6.4. The error estimator η_{MP2CG5} also uses the nodal values of v_{MAred} as coefficients for the P_2 ansatz functions on \mathcal{T} and performs five PCG iterations to draw near η_{MP2} with $\lambda = 1$.

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