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Weakly over-penalized discontinuous Galerkin schemes for Reissner–Mindlin plates without the shear variable

Paulo Rafael Bösing · Carsten Carstensen

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Abstract This paper introduces a new locking–free formulation that combines the discontinuous Galerkin methods with weakly over-penalized techniques for Reissner–Mindlin plates. We derive optimal *a priori* error estimates in both the energy norm and L^2 norm for polynomials of degree $k = 2$, and we extend the results concerning the energy norm to higher-order polynomial degrees. Numerical tests confirm our theoretical predictions.

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1 Introduction

Given $g \in L^2(\Omega)$ and $\boldsymbol{f} \in L^2(\Omega; \mathbb{R}^2)$, the weak formulation for the Reissner–Mindlin plate model (without the shear variable) reads: seek $(\boldsymbol{\theta}, w) \in H_0^1(\Omega; \mathbb{R}^2) \times H_0^1(\Omega)$ such that

$$
\begin{aligned}
a(\boldsymbol{\theta}, \boldsymbol{\eta}) + t^{-2}\mu(\boldsymbol{\theta} - \nabla w, \boldsymbol{\eta})_\Omega &= (\boldsymbol{f}, \boldsymbol{\eta})_\Omega \quad \text{for all} \quad \boldsymbol{\eta} \in H_0^1(\Omega; \mathbb{R}^2) \\
-t^{-2}\mu(\boldsymbol{\theta} - \nabla w, \nabla v)_\Omega &= (g, v)_\Omega \quad \text{for all} \quad v \in H_0^1(\Omega).
\end{aligned} \tag{1}
$$

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P. R. Bösing (✉)
Department of Mathematics, Federal University of Santa Catarina,
Trindade, Florianópolis, SC 88040-900, Brazil
e-mail: paulo.bosing@ufsc.br

C. Carstensen
Institut für Mathematik, Humboldt–Universität zu Berlin,
Unter den Linden 6, 10099 Berlin, Germany

Here, and throughout this paper, t is the plate thickness, Ω is a convex polygonal domain, and $e(\boldsymbol{\xi})$ is the symmetric part of the gradient of $\boldsymbol{\xi}$,

$$\mathcal{C}e(\boldsymbol{\xi}) = \frac{1}{3}\left[2\mu e(\boldsymbol{\xi}) + \frac{2\mu\lambda}{(2\mu+\lambda)}\operatorname{div}\boldsymbol{\xi}\, I\right]$$

where μ and λ are the Lamé coefficients and I is the identity 2×2 matrix, and

$$a(\boldsymbol{\theta},\boldsymbol{\eta}) = (e(\boldsymbol{\theta}),\mathcal{C}e(\boldsymbol{\eta}))_{\Omega}.$$

Motivated by the simplicity of the weakly over–penalized symmetric interior penalty (WOPSIP) formulation [12], and by the *medius error analysis* of [6,26] applied to study problems under minimal regularity, we introduce in this paper a new locking–free completely discontinuous formulation for (1) that combines the traditional discontinuous Galerkin methods with WOPSIP methods techniques.

This new formulation has the following desirable characteristics: (i) it does not involve the shear stress variable; (ii) it does not need any reduced integration techniques; (iii) it is simpler, in the sense that have less terms; (iv) its number of degrees of freedom is small compared to other discontinuous formulations with the shear stress variable; (v) it allows more freedom in the choosing of the penalty parameters; (vi) it requires only reasonable and standard hypotheses on the domain; (vii) it works for minimal regularity assumptions. Furthermore, we prove optimal *a priori* error estimates in the energy norm and L^2 norm for the symmetric version and low–order approximation.

The first equation of the Reissner–Mindlin model adopts the formulation in [1,2,8,15], while the second equation adopts that in [15], as WOPSIP methods [12]. We highlight that the present method is completely different from that introduced in [15], since here we do not introduce shear as an unknown. The formulation of this paper does not have the interface term that arises from the integration by parts of the second equation. This interface term was treated in [8] using the first equation of the Reissner–Mindlin model to proceed with one substitution, but it is more commonly handled by introducing shear as an extra unknown. With this approach the interior penalty term for the displacement will be over-penalized but, on the other hand, the penalty parameter can be any positive constant. However, for polynomials of degree $k = 2$, for which we have the required theoretical regularity available for the convex domain (see Theorems 6 and 7, and [3,4]), the over-penalization (the power of h) will be as in [8] and [15] and similar to [31–33] for the biharmonic equation.

Other locking–free formulations where completely discontinuous spaces were used for all unknowns can be found in [1,2,8] (see [30] for an overview of the first two articles) and [15]. Many other formulations for the Reissner–Mindlin model combine (with or without bubble function) nonconforming, conforming and fully discontinuous elements [1–3,5,7,13,14,19,20,23,25,27,28,34]. See [24] for a general review of finite element methods for the Reissner–Mindlin and related problems.

In comparison with the formulation in [8], that also does not have the shear stress as a variable, the formulation of this paper is: (i) simpler for high order (less terms); (ii) has more freedom in the choosing of the penalty parameter for the displacement;

(iii) has optimal error for the rotation in L^2 norm for low-order approximation; (iv) for low–order approximation, require only the regularity provided for the solution in the case of a convex polygon domain; (v) the norms of solution and the shear stress present on the right–hand side of the error estimates are uniformly bounded in the case of a convex polygon domain and low–order approximation; (vi) establishes some error estimates for the displacement in L^2 norm and discrete H^1 norms.

As commented in [15], by combining the discontinuous Galerkin methods (dG) with WOPSIP techniques, the resulting discrete formulation is not consistent. This prevents us from obtaining the Galerkin orthogonality and the traditional error analysis of discontinuous Galerkin methods is not applicable. Furthermore, since the consistency term depends on t, we can obtain only suboptimal error estimates (in relation to t) if we apply the WOPSIP analysis techniques.

For the $k = 2$ case we will extend the results of [23] in order to prove the optimal *a priori* error estimates in the energy norm. Applying duality arguments we derive optimal *a priori* error estimates in the L^2 norm for the case of the symmetric version. For $k > 2$, we will establish the convergence of the method in two different ways. Firstly applying WOPSIP error analysis techniques we will extend the results obtained in relation to the energy norm to the case of $k = 2$. This allows us to prove some additional convergence results related to the displacement variable. The other second approach proceeds with the analysis through the residual estimates which are typical of *a posteriori* error analysis [16–18,29] and enriching operators [9,10]. This technique called *medius error analysis* for [6,26] confirms the convergence in a slightly different energy norm.

For this strategy to be successful we need to assum that the Helmholtz decomposition is valid when dealing with the case of $k = 2$. Fortunately, this is the case if Ω is a convex polygon domain. Under this hypothesis our *a priori* error estimates will require only the regularity provided theoretically for the solution in the case of a convex polygon domain (or smooth domain), and the norms of the solution and the shear stress present on the right-hand side are uniformly bounded with respect to t.

The remaining parts of this paper are organized as follows: In the next section we introduce the necessary notation and recall some definitions to deal with discontinuous Galerkin methods. In Sect. 3 we introduce the discrete formulation which combines discontinuous Galerkin methods with WOPSIP techniques. Section 4 is dedicated to the analysis of the case of $k = 2$ while in the Section 5 we treat the case of $k > 2$. In the final section we present some numerical tests consistent with the theoretical findings.

2 Notation and preliminaries

Let $\mathbb{T}$ be a shape-regular family of regular triangulation of $\Omega \subset \mathbb{R}^2$ into closed triangles T, convex, with pairwise disjoint interiors, and such that

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}} T.$$

On the regular triangulation $\mathcal{T} \in \mathbb{T}$, the piecewise constant function $h_{\mathcal{T}}$ is defined by

$$h_{\mathcal{T}|T} = h_T := \mathrm{diam}(T) \quad \text{on} \quad T \in \mathcal{T}$$

and we denote by h the maximum of h_T for $T \in \mathcal{T}$. Let $\mathcal{E}$ be the set of all edges E of all the triangles in $\mathcal{T}$ and let us define the piecewise constant function $h_{\mathcal{E}}$ as

$$h_{\mathcal{E}|E} = h_E := \mathrm{diam}(E) \quad \text{on} \quad E \in \mathcal{E}.$$

$\mathcal{E}(T)$ denotes the set of the three edges of T. The set $\mathcal{E}$ will be divided into two subsets, $\mathcal{E}(\Omega)$ and $\mathcal{E}(\partial\Omega)$, defined by

$$\mathcal{E}(\Omega) = \{E \in \mathcal{E} : E \subset \Omega\} \quad \text{and} \quad \mathcal{E}(\partial\Omega) = \{E \in \mathcal{E} : E \subset \partial\Omega\}.$$

The shape-regularity of $\mathbb{T}$, provides some constant $0 < \gamma(\mathbb{T}) \leq 1$ such that $\forall \mathcal{T} \in \mathbb{T}, \quad \forall T \in \mathcal{T}, \quad \forall E \in \mathcal{E}(T)$

$$\gamma\, h_T \leq h_E \leq h_T.$$

The Sobolev space of real order s of real-valued functions defined on $\omega \subset \Omega$, will be labeled by $H^s(\omega)$. Its inner product, norm and semi-norm will be denoted by $(\cdot,\cdot)_{s,\omega}$, $\|\cdot\|_{s,\omega}$, and $|\cdot|_{s,\omega}$, respectively. In particular, we will write $\|\cdot\|_{\omega}$ and $(\cdot,\cdot)_{\omega}$ instead of $\|\cdot\|_{0,\omega}$ and $(\cdot,\cdot)_{0,\omega}$, respectively. Similarly, for any $E \in \mathcal{E}$ let's denote by $\langle\cdot,\cdot\rangle_E$ and $\|\cdot\|_E$ the inner product and the induced norm in the space $L^2(E)$, respectively. Also, we will denote by $H^s(\omega;\mathbb{R}^2) = H^s(\omega) \times H^s(\omega)$ the Sobolev space of vector functions for which, as in the case of the scalar function, $(\cdot,\cdot)_{s,\omega}$ will denote the inner product. We note that the same notation for the inner product also will be used occasionally for symmetric tensors.

Let

$$H^s(\mathcal{T}) = \{v \in L^2(\Omega) : v_{|T} \in H^s(\mathring{T}) \text{ for all } T \in \mathcal{T}\}$$

be the space of piecewise Sobolev H^s-functions. We denote its inner product, norm and semi–norm by $(\cdot,\cdot)_{s,h}$, $\|\cdot\|_{s,h}$ and $|\cdot|_{s,h}$ respectively. $H^s(\mathcal{T};\mathbb{R}^2) = H^s(\mathcal{T}) \times H^s(\mathcal{T})$ denotes the space of piecewise Sobolev H^s-vector functions.

We use the differential operators $\mathrm{Curl}(v) = (\partial v/\partial y, -\partial v/\partial x)$ for a scalar function v, and $rot(\boldsymbol{\eta}) = \partial\eta_2/\partial x - \partial\eta_1/\partial y$ for a vector function $\boldsymbol{\eta} = (\eta_1, \eta_2)$. We observe that any differential operator defined over a piecewise Sobolev space will be indicated by a subscript h.

For any $T \in \mathcal{T}$ let $\boldsymbol{\nu}_T = (\nu_1, \nu_2)$ be the outer unit normal to the boundary ∂T and let $\boldsymbol{\tau}_T = (-\nu_2, \nu_1)$ be the tangential vector. Let T^- and T^+ be two distinct elements of $\mathcal{T}$ sharing the comon edge $E = T^- \bigcap T^+ \in \mathcal{E}(\Omega)$. We define the jump of $v \in H^1(\mathcal{T})$ on E by

$$[v] = v^-\boldsymbol{\nu}^- + v^+\boldsymbol{\nu}^+,$$

where $v^{\pm} := v|_{T^{\pm}}$ and $\boldsymbol{\nu}^{\pm}$ denotes the outer unit normal $\boldsymbol{\nu}_{T^{\pm}}$ on $T^{\pm}$. For a vector function $\boldsymbol{\eta} \in H^1(\mathcal{T};\mathbb{R}^2)$, define

$$[\boldsymbol{\eta}] = \boldsymbol{\eta}^- \cdot \boldsymbol{\nu}^- + \boldsymbol{\eta}^+ \cdot \boldsymbol{\nu}^+ \quad \text{and} \quad [\![\boldsymbol{\eta}]\!] = \boldsymbol{\eta}^- \odot \boldsymbol{\nu}^- + \boldsymbol{\eta}^+ \odot \boldsymbol{\nu}^+,$$

where $\boldsymbol{\eta} \odot \boldsymbol{\nu} = (\boldsymbol{\eta}\boldsymbol{\nu}^T + \boldsymbol{\nu}\boldsymbol{\eta}^T)/2$. Similarly, for a tensor $\epsilon \in H^1(\Omega; \mathbb{R}^{2\times 2})$ the jump on E is defined by

$$[\![\epsilon]\!] = \epsilon^- \boldsymbol{\nu}^- + \epsilon^+ \boldsymbol{\nu}^+.$$

Note that the jump of a scalar function is a vector. For a vector function $\boldsymbol{\eta}$ the jump $[\boldsymbol{\eta}]$ is a scalar, while the jump $[\![\boldsymbol{\eta}]\!]$ is a symmetric matrix, and for a tensor the jump is a vector. The average of a tensor, scalar function or vector function χ is defined by $\{\chi\} = \frac{1}{2}(\chi^- + \chi^+)$.

On a boundary edge, we define the average $\{\chi\}$ as the trace of χ, while we consider $[v]$ to be $v\boldsymbol{\nu}$, $[\boldsymbol{\eta}]$ to be $\boldsymbol{\eta} \cdot \boldsymbol{\nu}$, $[\![\boldsymbol{\eta}]\!]$ to be $\boldsymbol{\eta} \odot \boldsymbol{\nu}$ and $[\![\epsilon]\!]$ to be $\epsilon\boldsymbol{\nu}$.

Occasionally, we shall use the jump on E in relation to the tangent vector $\boldsymbol{\tau}$, in this case denoted by $[v]_{\boldsymbol{\tau}}$, that is, $[v]_{\boldsymbol{\tau}} = v^-\boldsymbol{\tau}^- + v^+\boldsymbol{\tau}^+$ (idem for a vector function).

For a positive integer k, $\mathcal{P}_k(T)$ will denote the linear space of polynomials on T with a total degree of less than or equal to k, and $\mathcal{P}_k(T; \mathbb{R}^2) := \mathcal{P}_k(T) \times \mathcal{P}_k(T)$. The discrete space for the displacement will be

$$\mathcal{P}_k(\mathcal{T}) = \left\{ v \in L^2(\Omega) : \quad \forall T \in \mathcal{T},\ v_{|T} \in \mathcal{P}_k(T) \right\},$$

and for rotation will be

$$\mathcal{P}_{k-1}(\mathcal{T}; \mathbb{R}^2) = \left\{ \boldsymbol{\eta} \in L^2(\Omega; \mathbb{R}^2) : \forall T \in \mathcal{T}, \quad \boldsymbol{\eta}_{|T} \in \mathcal{P}_{k-1}(T; \mathbb{R}^2) \right\}.$$

Let π_W denote the natural projection onto $\mathcal{P}_k(\mathcal{T})$ (see [1] for definition of π_W). For $w \in H^{k+1}(\Omega)$ let $w^I = \pi_W w$ be the interpolant of w. It then follows that $w^I \in \mathcal{P}_k(\mathcal{T}) \cap H^1(\Omega)$ and that for $0 \le q \le k+1$, there exists a constant c such that

$$\|w - w^I\|_{q,h} \le c h^{k+1-q} \|w\|_{k+1,\Omega} \quad \text{for all } w \in H^{k+1}(\Omega). \tag{2}$$

The rotated Brezzi–Douglas–Marini space of degree $k-1$, i.e. the space of all piecewise polynomial vector fields of degree $k-1$ subject to interelement continuity of the tangential components, will be denoted by $\mathbf{BDM}^R_{k-1}$. Let $\boldsymbol{\pi}_\Theta$ be the natural projection operator of $H^1(\Omega; \mathbb{R}^2)$ into $\mathbf{BDM}^R_{k-1} \subset \mathcal{P}_{k-1}(\mathcal{T}; \mathbb{R}^2)$. For $\boldsymbol{\theta} \in H^k(\Omega; \mathbb{R}^2)$ we define the interpolant $\boldsymbol{\theta}^I$ of $\boldsymbol{\theta}$ by $\boldsymbol{\theta}^I := \boldsymbol{\pi}_\Theta \boldsymbol{\theta}$. With this choice, for $0 \le s \le \ell$ and $1 \le \ell \le k$, we have

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_{s,h} \le c h^{\ell - s} \|\boldsymbol{\theta}\|_{\ell,\Omega} \quad \text{for all } \boldsymbol{\theta} \in H^\ell(\Omega; \mathbb{R}^2). \tag{3}$$

Set $\boldsymbol{\gamma}^I = t^{-2}(\boldsymbol{\theta}^I - \nabla w^I)$ and $\boldsymbol{\gamma} = t^{-2}(\boldsymbol{\theta} - \nabla w)$. The commutative property $\boldsymbol{\pi}_\Theta \nabla w = \nabla \pi_W w$ states

$$\boldsymbol{\pi}_\Theta \boldsymbol{\gamma} = t^{-2} \boldsymbol{\pi}_\Theta (\boldsymbol{\theta} - \nabla w) = t^{-2} (\boldsymbol{\pi}_\Theta \boldsymbol{\theta} - \nabla \pi_W w) = t^{-2} (\boldsymbol{\theta}^I - \nabla w^I) = \boldsymbol{\gamma}^I. \tag{4}$$

Thus $\boldsymbol{\gamma}^I$ interpolates $\boldsymbol{\gamma}$ and for $0 \le s \le \ell$ and $1 \le \ell \le k$ we have

$$\|\boldsymbol{\gamma} - \boldsymbol{\gamma}^I\|_{s,h} \le ch^{\ell-s}\|\boldsymbol{\gamma}\|_{\ell,\Omega} \quad \text{for all } \boldsymbol{\gamma} \in H^{\ell}(\Omega;\mathbb{R}^2). \tag{5}$$

To develop our dG with WOPSIP for the Reissner–Mindlin plate model, we need to define the following auxiliary norms

$$\begin{aligned}
\|v\|_h^2 &= \sum_{T\in\mathcal{T}} \|\nabla_h v\|_T^2 + \sum_{E\in\mathcal{E}} \frac{\sigma_2}{h_E^{\rho}}\|\boldsymbol{\Pi}^{k-1}[v]\|_E^2 \quad \text{for all} \;\; v \in H^1(\mathcal{T});\\
\|\boldsymbol{\eta}\|_h^2 &= \sum_{T\in\mathcal{T}} \left(\|e_h(\boldsymbol{\eta})\|_T^2 + \|\boldsymbol{\eta}\|_T^2\right) + \sum_{E\in\mathcal{E}} \frac{\sigma_1}{h_E}\|[\![\boldsymbol{\eta}]\!]\|_E^2 \quad \text{for all} \quad \boldsymbol{\eta} \in H^1(\mathcal{T};\mathbb{R}^2);\\
\|\boldsymbol{\eta}, v\|_h^2 &= \|\boldsymbol{\eta}\|_h^2 + \|v\|_h^2 \quad \text{for all} \;\; (\boldsymbol{\eta}, v) \in H^1(\mathcal{T};\mathbb{R}^2) \times H^1(\mathcal{T}).
\end{aligned}$$

Here and throughout this paper, ρ, σ_1 and σ_2 are positive constants that will be defined below. The operator $\boldsymbol{\Pi}^{k-1}$ is the orthogonal projections from $L^2(E;\mathbb{R}^2)$ onto $\mathcal{P}_{k-1}(E;\mathbb{R}^2)$ where $\mathcal{P}_{k-1}(E)$ is the space of polynomials of degree less than or equal to $k-1$ on E.

3 Combined formulation of dG and WOPSIP

The new formulation for the Reissner–Mindlin model that combines WOPSIP and dG uses the following bilinear form on $\left(H^{1+\kappa}(\mathcal{T};\mathbb{R}^2) \times H^1(\mathcal{T})\right)^2$ with $\kappa > 1/2$,

$$\mathcal{A}_h(\boldsymbol{\xi}, u; \boldsymbol{\eta}, v) = \mathcal{B}_h(\boldsymbol{\xi}, \boldsymbol{\eta}) + t^{-2}\mu \sum_{T\in\mathcal{T}} (\boldsymbol{\xi} - \nabla_h u, \boldsymbol{\eta} - \nabla_h v)_T + \mathcal{J}(u, v). \tag{6}$$

Here and throughout this paper, for any $\boldsymbol{\xi},\ \boldsymbol{\eta} \in H^{1+\kappa}(\mathcal{T};\mathbb{R}^2)$ and $u, v \in H^1(\mathcal{T})$, set

$$\begin{aligned}
\mathcal{B}_h(\boldsymbol{\xi}, \boldsymbol{\eta}) &:= a_h(\boldsymbol{\xi}, \boldsymbol{\eta}) - \sum_{E\in\mathcal{E}} \langle\{\mathcal{C}e_h(\boldsymbol{\xi})\}, [\![\boldsymbol{\eta}]\!]\rangle_E - \delta \sum_{E\in\mathcal{E}} \langle\{\mathcal{C}e_h(\boldsymbol{\eta})\}, [\![\boldsymbol{\xi}]\!]\rangle_E + \boldsymbol{J}(\boldsymbol{\xi}, \boldsymbol{\eta}),\\
a_h(\boldsymbol{\xi}, \boldsymbol{\eta}) &:= \sum_{T\in\mathcal{T}} (\mathcal{C}e_h(\boldsymbol{\xi}), e_h(\boldsymbol{\eta}))_T,\\
\boldsymbol{J}(\boldsymbol{\xi}, \boldsymbol{\eta}) &:= \sum_{E\in\mathcal{E}} \frac{\sigma_1}{h_E}\langle[\![\boldsymbol{\xi}]\!], [\![\boldsymbol{\eta}]\!]\rangle_E,\\
\mathcal{J}(u, v) &:= \sum_{E\in\mathcal{E}} \frac{\sigma_2}{h_E^{\rho}}\langle\boldsymbol{\Pi}^{k-1}[u], \boldsymbol{\Pi}^{k-1}[v]\rangle_E.
\end{aligned}$$

Moreover, σ_1 and σ_2 are the penalty parameters, and $\rho > 1$ (which depends on k) will be specified below. The parameter $-1 \le \delta \le 1$ is the symmetric/nonsymmetric bilinear form parameter.

This gives the following energy norm associated with the bilinear form $\mathcal{A}_h(\cdot,\cdot;\cdot,\cdot)$

$$\|\!|\boldsymbol{\eta}, v|\!\|^2 = \|e_h(\boldsymbol{\eta})\|_{0,h}^2 + t^{-2}\|\boldsymbol{\eta} - \nabla_h v\|_{0,h}^2 + \boldsymbol{J}(\boldsymbol{\eta},\boldsymbol{\eta}) + \mathcal{J}(v,v) + \sum_{E\in\mathcal{E}} \frac{h_E}{\sigma_1} \|\{\mathcal{C}e_h(\boldsymbol{\eta})\}\|_E^2$$

for all $(\boldsymbol{\eta}, v) \in H^{1+\kappa}(\mathcal{T};\mathbb{R}^2) \times H^1(\mathcal{T})$, and the energy norm

$$\|\!|\boldsymbol{\eta}|\!\|^2 = \|\boldsymbol{\eta}\|_{1,h}^2 + \sum_{E\in\mathcal{E}} h_E \|\{\mathcal{C}e_h(\boldsymbol{\eta})\}\|_E^2 + \boldsymbol{J}(\boldsymbol{\eta},\boldsymbol{\eta}), \tag{7}$$

associated with the bilinear form $\mathcal{B}_h(\cdot,\cdot)$.

The weakly over–penalized interior penalty combined with the discontinuous Galerkin method (dGWOPIP) for the Reissner–Mindlin model reads: Seek $(\boldsymbol{\theta}_h,\ w_h) \in \mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2) \times \mathcal{P}_k(\mathcal{T})$ such that

$$\mathcal{A}_h(\boldsymbol{\theta}_h, w_h; \boldsymbol{\eta}, v) = (g, v)_\Omega + (\boldsymbol{f}, \boldsymbol{\eta})_\Omega \quad \text{for all} \quad (\boldsymbol{\eta}, v) \in \mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)\times\mathcal{P}_k(\mathcal{T}). \tag{8}$$

We note that the dGWOPIP formulation differs from the discontinuous Galerkin formulation introduced in [8] (without the shear stress variable): (a) the penalization term of displacement here involves the projection of the jump while in [8] it involves simply the jump; (b) if $k > 2$, the dGWOPIP formulation does not include the interface terms $\langle [w], \{\mathbf{div}\,\mathcal{C}e_h(\boldsymbol{\eta})\}\rangle_\mathcal{E}$ and $\langle [v], \{\mathbf{div}\,\mathcal{C}e_h(\boldsymbol{\theta})\}\rangle_\mathcal{E}$ present in [8]; (c) if $k > 2$ the over–penalization (the power of h) for the displacement of the dGWOPIP formulation will be greater than that of [8].

Clearly, we have the continuity of $\mathcal{B}_h(\cdot,\cdot)$ over $H^{1+\kappa}(\mathcal{T},\mathbb{R}^2)\times H^{1+\kappa}(\mathcal{T},\mathbb{R}^2)$ with respect the norm (7). The coercivity of $\mathcal{B}_h(\cdot,\cdot)$ over $\mathcal{P}_{k-1}(\mathcal{T},\mathbb{R}^2)$ is established in [2, Proposition 4.7].

Lemma 1 [2, Proposition 4.7] *There exist positive constants $\tilde{\sigma}_a$ and ς independent of h and t such that: if $\sigma_1 > \tilde{\sigma}_a$, then*

$$\varsigma\|\!|\boldsymbol{\eta}|\!\|^2 \le \mathcal{B}_h(\boldsymbol{\eta},\boldsymbol{\eta}) \quad \textit{for all} \quad \boldsymbol{\eta} \in \mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2).$$

In the following we will establish the continuity and coercivity of the bilinear form $\mathcal{A}_h(\cdot,\cdot;\cdot,\cdot)$.

Lemma 2 *Let $\mathcal{T} \in \mathbb{T}$, then there exists a positive constant c independent of h and t, such that for all $((\boldsymbol{\xi}, u), (\boldsymbol{\eta}, v)) \in (H^{1+\kappa}(\mathcal{T};\mathbb{R}^2) \times H^1(\mathcal{T}))^2$ satisfies*

$$|\mathcal{A}_h(\boldsymbol{\xi}, u; \boldsymbol{\eta}, v)| \le c\|\!|\boldsymbol{\xi}, u|\!\|\,\|\!|\boldsymbol{\eta}, v|\!\|.$$

Proof This follows from Cauchy–Schwarz inequality. □

Lemma 3 *Let $\mathcal{T} \in \mathbb{T}$ and assume that the Lamé coefficients are uniformly bounded. Then there exists positive constants $\tilde{\sigma}_b$ and $\varsigma > 0$, such that, $\sigma_1 > \tilde{\sigma}_b$, imply*

$$\varsigma \,|\!|\!| \boldsymbol{\eta}, v |\!|\!|^2 \leq \mathcal{A}_h(\boldsymbol{\eta}, v; \boldsymbol{\eta}, v)$$

for all $(\boldsymbol{\eta}, v) \in \mathcal{P}_{k-1}(\mathcal{T}; \mathbb{R}^2) \times \mathcal{P}_k(\mathcal{T})$ and for any choice of $\sigma_2 > \tilde{\sigma}_2 > 0$ where $\tilde{\sigma}_2$ is arbitrary but fixed.

Proof Let Λ_0, Λ_1 be positive constants such that

$$\Lambda_0 \| e_h(\boldsymbol{\eta}) \|_{0,h}^2 \leq | a_h(\boldsymbol{\eta}, \boldsymbol{\eta}) | \leq \Lambda_1 \| e_h(\boldsymbol{\eta}) \|_{0,h}^2.$$

Then we have

$$\begin{aligned}
&\mathcal{A}_h(\boldsymbol{\eta}, v; \boldsymbol{\eta}, v) - \varsigma \,|\!|\!| \boldsymbol{\eta}, v |\!|\!|^2 \geq (\Lambda_0 - \varsigma) \| e_h(\boldsymbol{\eta}) \|_{0,h}^2 \\
&\quad + (\mu - \varsigma) t^{-2} \| \boldsymbol{\eta} - \nabla_h v \|_{0,h}^2 + (1 - \varsigma) \left(\boldsymbol{J}(\boldsymbol{\eta}, \boldsymbol{\eta}) + \mathcal{J}(v, v) \right) \\
&\quad - (1 + \delta) \sum_{E \in \mathcal{E}} \langle \{ \mathcal{C} e_h(\boldsymbol{\eta}) \}, [\![\boldsymbol{\eta}]\!] \rangle_E - \varsigma \sum_{E \in \mathcal{E}} \frac{h_E}{\sigma_1} \| \{ \mathcal{C} e_h(\boldsymbol{\eta}) \} \|_E^2.
\end{aligned}$$

For any positive constant ϱ the Cauchy–Schwarz inequality and arithmetic-geometric inequality show that

$$- \langle \{ \mathcal{C} e_h(\boldsymbol{\eta}) \}, [\![\boldsymbol{\eta}]\!] \rangle_E \geq - \frac{\varrho}{2} \frac{h_E}{\sigma_1} \| \{ \mathcal{C} e_h(\boldsymbol{\eta}) \} \|_E^2 - \frac{1}{2\varrho} \frac{\sigma_1}{h_E} \| [\![\boldsymbol{\eta}]\!] \|_E^2.$$

With this, and an inverse inequality, we obtain

$$\begin{aligned}
&\mathcal{A}_h(\boldsymbol{\eta}, v; \boldsymbol{\eta}, v) - \varsigma \,|\!|\!| \boldsymbol{\eta}, v |\!|\!|^2 \geq (\mu - \varsigma)\, t^{-2} \| \boldsymbol{\eta} - \nabla_h v \|_{0,h}^2 + (1 - \varsigma) \mathcal{J}(v, v) \\
&\quad + \left(\Lambda_0 - \varsigma \left(1 + \frac{c}{\sigma_1} \right) - (1 + \delta) \frac{\varrho c}{2\sigma_1} \right) \| e_h(\boldsymbol{\eta}) \|_{0,h}^2 \\
&\quad + \left(1 - \varsigma - \frac{(1 + \delta)}{2\varrho} \right) \boldsymbol{J}(\boldsymbol{\eta}, \boldsymbol{\eta}).
\end{aligned}$$

If $\delta \neq -1$, we first choose ϱ such that $1 - \frac{(1+\delta)}{2\varrho} > 0$. In the following we choose $\tilde{\sigma}_b$ such that $\Lambda_0 - (1 + \delta) \frac{\varrho c}{2\tilde{\sigma}_b} > 0$. The assumption follows with $\varsigma > 0$ such that

$$\varsigma < \min \left\{ 1,\ \mu,\ 1 - \frac{(1 + \delta)}{2\varrho},\ \frac{\Lambda_0 - (1 + \delta) \frac{\varrho c}{2\tilde{\sigma}_b}}{1 + \frac{c}{\tilde{\sigma}_b}} \right\}.$$

On the other hand, if $\delta = -1$, the assumption follows for any choice of $\sigma_1 > \tilde{\sigma}_b > 0$, with $\tilde{\sigma}_b$ arbitrary but fixed, if $\varsigma > 0$ be such that

$$\varsigma < \min \left\{ 1,\ \mu,\ \frac{\Lambda_0}{1 + \frac{c}{\tilde{\sigma}_b}} \right\}.$$

□

To ensure the coercivity given by Lemmas 1 and 3 simultaneously, we will consider throughout this paper that $\sigma_1 > \tilde{\sigma}_1 := \max\{\tilde{\sigma}_a,\ \tilde{\sigma}_b\}$.

4 *A priori* error analysis for low order

In this section, we will carry out the *a priori* error analysis for the dGWOPIP formulation when $k = 2$. In order to achieve optimal error estimates we will assume that $\boldsymbol{\gamma}$ has a Helmholtz decomposition in the form

$$\boldsymbol{\gamma} = \nabla\alpha + \mathrm{Curl}(\beta) \quad \text{with} \quad \alpha \in H^2(\Omega) \cap H_0^1(\Omega) \quad \text{and} \quad \beta \in H^2(\Omega)/\mathbb{R}. \tag{9}$$

In addition, we will assume that

$$\|\alpha\|_{2,\Omega} + \|\beta\|_{2,\Omega} \lesssim \|\boldsymbol{\gamma}\|_{1,\Omega}, \quad \text{and} \quad \|\alpha\|_{2,\Omega} + \|\beta\|_{1,\Omega} \lesssim \|\boldsymbol{\gamma}\|_{H(\mathrm{div})}, \tag{10}$$

where $H(\mathrm{div})$ is the space of vectors in $L^2(\Omega; \mathbb{R}^2)$ that have the divergence in $L^2(\Omega)$ and here, and throughout this paper, an inequality $a \lesssim b$ replaces $a \le Cb$ with a multiplicative (t, h_T, h_E)-independent constant C. We note that this result holds if Ω is a convex polygon and if we have H^2 regularity for the Poisson problem $\Delta\alpha = \mathrm{div}(\boldsymbol{\gamma})$.

Recall that the operator $\boldsymbol{\Pi}^{k-1}$ is the orthogonal projection from $L^2(E; \mathbb{R}^2)$ onto $\mathcal{P}_{k-1}(E; \mathbb{R}^2)$, that is, for any $\boldsymbol{\xi} \in L^2(E; \mathbb{R}^2)$

$$\int_E (\boldsymbol{\Pi}^{k-1}\boldsymbol{\xi} - \boldsymbol{\xi}) \cdot \boldsymbol{\eta}\, ds = 0 \quad \forall \quad \boldsymbol{\eta} \in \mathcal{P}_{k-1}(E; \mathbb{R}^2).$$

Let $\pi^{k-1} : L^2(E) \to \mathcal{P}_{k-1}(E)$ be the L^2 orthogonal projection onto $\mathcal{P}_{k-1}(E)$, that is, for any $u \in L^2(E)$

$$\int_E (\pi^{k-1}u - u)v\, ds = 0 \quad \forall \quad v \in \mathcal{P}_{k-1}(E).$$

For simplicity, if $k = 2$ we will write $\boldsymbol{\Pi}$ and π instead of $\boldsymbol{\Pi}^1$ and π^1, respectively.

We will now recall the following lemma proved in [21], which will play an important role in the error analysis below.

Lemma 4 *For any integer m with $0 \le m \le k-1$ and for any $E \in \mathcal{E}(T)$, there exists a constant $c > 0$ such that*

$$\left| \int_E \phi(u - \pi^{k-1}u)\, ds \right| \le c h_T^{m+1} |\phi|_{1,T} |u|_{m+1,T}$$

for all $\phi \in H^1(T)$ and all $u \in H^{m+1}(T)$.

For convenience we rewrite the dGWOPIP formulation (8) as: Seek $(\boldsymbol{\theta}_h,\ w_h) \in \mathcal{P}_1(\mathcal{T}; \mathbb{R}^2) \times \mathcal{P}_2(\mathcal{T})$ such that

$$\mathcal{B}_h(\boldsymbol{\theta}_h, \boldsymbol{\eta}) + \mu \sum_{T \in \mathcal{T}} (\boldsymbol{\gamma}_h, \boldsymbol{\eta})_T = (\boldsymbol{f}, \boldsymbol{\eta})_\Omega \quad \forall \quad \boldsymbol{\eta} \in \mathcal{P}_1(\mathcal{T}; \mathbb{R}^2) \tag{11a}$$

$$-\mu \sum_{T \in \mathcal{T}} (\boldsymbol{\gamma}_h, \nabla_h v)_T + \mathcal{J}(w_h, v) = (g, v)_\Omega \quad \forall \quad v \in \mathcal{P}_2(\mathcal{T}). \tag{11b}$$

Here, and throughout this paper, $\boldsymbol{\gamma}_h = t^{-2}(\boldsymbol{\theta}_h - \nabla_h w_h)$, that is, $\boldsymbol{\gamma}_h$ is the discrete shear stress.

It can easily be observed that the solution $(\boldsymbol{\theta}, w)$ of the Reissner–Mindlin equation satisfies

$$\mathcal{B}_h(\boldsymbol{\theta}, \boldsymbol{\eta}) + \mu \sum_{T \in \mathcal{T}} (\boldsymbol{\gamma}, \boldsymbol{\eta})_T = (\boldsymbol{f}, \boldsymbol{\eta})_\Omega \, \forall \boldsymbol{\eta} \in H^2(\mathcal{T}; \mathbb{R}^2) \tag{12a}$$

$$-\mu \sum_{T \in \mathcal{T}} (\boldsymbol{\gamma}, \nabla_h v)_T + \mathcal{J}(w, v) + \mu \sum_{T \in \mathcal{T}} \int_{\partial T} \boldsymbol{\gamma} \cdot \boldsymbol{\nu} \, v \, ds = (g, v)_\Omega \, \forall v \in H^1(\mathcal{T}). \tag{12b}$$

Subtracting (11a) from (12a) and (11b) from (12b) we have

$$\mathcal{B}_h(\boldsymbol{\theta} - \boldsymbol{\theta}_h, \boldsymbol{\eta}) + \mu \sum_{T \in \mathcal{T}} (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{\eta})_T = 0 \quad \forall \quad \boldsymbol{\eta} \in \mathcal{P}_1(\mathcal{T}; \mathbb{R}^2) \tag{13}$$

$$-\mu \sum_{T \in \mathcal{T}} (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \nabla_h v)_T + \mathcal{J}(w - w_h, v) + \mu \sum_{T \in \mathcal{T}} \int_{\partial T} \boldsymbol{\gamma} \cdot \boldsymbol{\nu} \, v \, ds = 0 \quad \forall \quad v \in \mathcal{P}_2(\mathcal{T}). \tag{14}$$

Denoting

$$\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}_h - \boldsymbol{\theta}^I, \qquad \tilde{w} = w_h - w^I \quad \text{and} \quad \tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma}_h - \boldsymbol{\gamma}^I \tag{15}$$

we have $t^2 \tilde{\boldsymbol{\gamma}} = \tilde{\boldsymbol{\theta}} - \nabla_h \tilde{w}$ and for $\boldsymbol{\eta} = \tilde{\boldsymbol{\theta}}$ and $v = \tilde{w}$ adding (13) and (14) we find that

$$\mathcal{B}_h(\boldsymbol{\theta} - \boldsymbol{\theta}_h, \tilde{\boldsymbol{\theta}}) + t^2 \mu \sum_{T \in \mathcal{T}} (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \tilde{\boldsymbol{\gamma}})_T + \mu \sum_{T \in \mathcal{T}} \int_{\partial T} \boldsymbol{\gamma} \cdot \boldsymbol{\nu} \, \tilde{w} \, ds + \mathcal{J}(w - w_h, \tilde{w}) = 0. \tag{16}$$

From Lemma 1 we have $|||\tilde{\boldsymbol{\theta}}|||^2 \lesssim \mathcal{B}_h(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\theta}}) = \mathcal{B}_h(\boldsymbol{\theta} - \boldsymbol{\theta}^I, \tilde{\boldsymbol{\theta}}) - \mathcal{B}_h(\boldsymbol{\theta} - \boldsymbol{\theta}_h, \tilde{\boldsymbol{\theta}})$. With this and (16) we obtain

$$\begin{aligned} |||\tilde{\boldsymbol{\theta}}|||^2 + t^2 \|\tilde{\boldsymbol{\gamma}}\|_{0,h}^2 + \mathcal{J}(\tilde{w}, \tilde{w}) &\lesssim \mathcal{B}_h(\boldsymbol{\theta} - \boldsymbol{\theta}^I, \tilde{\boldsymbol{\theta}}) + t^2 \sum_{T \in \mathcal{T}} (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \tilde{\boldsymbol{\gamma}})_T \\ &+ \sum_{T \in \mathcal{T}} \int_{\partial T} \boldsymbol{\gamma} \cdot \boldsymbol{\nu} \, \tilde{w} \, ds + \mathcal{J}(w - w_h, \tilde{w}) + t^2 \|\tilde{\boldsymbol{\gamma}}\|_{0,h}^2 + \mathcal{J}(\tilde{w}, \tilde{w}) \\ &= \mathcal{B}_h(\boldsymbol{\theta} - \boldsymbol{\theta}^I, \tilde{\boldsymbol{\theta}}) + t^2 \sum_{T \in \mathcal{T}} (\boldsymbol{\gamma} - \boldsymbol{\gamma}^I, \tilde{\boldsymbol{\gamma}})_T + \sum_{T \in \mathcal{T}} \int_{\partial T} \boldsymbol{\gamma} \cdot \boldsymbol{\nu} \, \tilde{w} \, ds + \mathcal{J}(w - w^I, \tilde{w}). \end{aligned}$$

Applying the Cauchy–Schwarz inequality and continuity of the bilinear form $\mathcal{B}_h(\cdot,\cdot)$, we find that

$$\begin{aligned}|\!|\!|\tilde{\boldsymbol{\theta}}|\!|\!|^2 + t^2\|\tilde{\boldsymbol{\gamma}}\|_{0,h}^2 + \mathcal{J}(\tilde{w},\tilde{w}) &\lesssim |\!|\!|\boldsymbol{\theta}-\boldsymbol{\theta}^I|\!|\!|\,|\!|\!|\tilde{\boldsymbol{\theta}}|\!|\!| + t^2\|\boldsymbol{\gamma}-\boldsymbol{\gamma}^I\|_{0,h}\|\tilde{\boldsymbol{\gamma}}\|_{0,h}\\ &+\left(\mathcal{J}(w-w^I,w-w^I)\right)^{1/2}(\mathcal{J}(\tilde{w},\tilde{w}))^{1/2} + \left|\sum_{T\in\mathcal{T}}\int_{\partial T}\boldsymbol{\gamma}\cdot\boldsymbol{\nu}\,\tilde{w}\,ds\right|. \end{aligned}\tag{17}$$

The inequality (17) is fundamental to prove an *a priori* error estimate. We will first proceed with the analysis of the last term in the next lemma. We observe that the combination of the next lemma with the inequality (17) can be seen as an extension of [23, Lemma 5.2] to the case of discontinuous Galerkin with weak over-penalization.

Lemma 5 *Assuming that the Helmholtz decomposition* (9) *is valid we have that*

$$\begin{aligned}\left|\sum_{T\in\mathcal{T}}\int_{\partial T}\boldsymbol{\gamma}\cdot\boldsymbol{\nu}\,\tilde{w}\,ds\right| &\lesssim ht^2\|\boldsymbol{\gamma}\|_{1,\Omega}\|\tilde{\boldsymbol{\gamma}}\|_{0,h} + h\|\boldsymbol{\gamma}\|_{H(div)}|\!|\!|\tilde{\boldsymbol{\theta}}|\!|\!|\\ &+h^{\frac{\rho-1}{2}}\|\boldsymbol{\gamma}\|_{H(div)}\left(\mathcal{J}(\tilde{w},\tilde{w})\right)^{1/2},\end{aligned}$$

where $\tilde{w}$, $\tilde{\boldsymbol{\theta}}$ and $\tilde{\boldsymbol{\gamma}}$ are defined by (15).

Proof Using the Helmholtz decomposition (9) we have

$$\begin{aligned}\Upsilon &:= \sum_{T\in\mathcal{T}}\int_{\partial T}\boldsymbol{\gamma}\cdot\boldsymbol{\nu}\,\tilde{w}\,ds\\ &= \sum_{T\in\mathcal{T}}\int_{\partial T}\nabla\alpha\cdot\boldsymbol{\nu}\,\tilde{w}\,ds + \sum_{T\in\mathcal{T}}\int_{\partial T}\mathrm{Curl}(\beta)\cdot\boldsymbol{\nu}\,\tilde{w}\,ds =: \Upsilon_1+\Upsilon_2.\end{aligned}$$

We developed the analysis of each part independently. Using the orthogonal projection we decompose the first term in the following way

$$\Upsilon_1 = \sum_{T\in\mathcal{T}}\int_{\partial T}(\nabla\alpha\cdot\boldsymbol{\nu}-\pi(\nabla\alpha\cdot\boldsymbol{\nu}))\,\tilde{w}\,ds + \sum_{T\in\mathcal{T}}\int_{\partial T}\pi(\nabla\alpha\cdot\boldsymbol{\nu})\tilde{w}\,ds =: \Upsilon_{1a}+\Upsilon_{1b}.$$

Applying Lemma 4 with $m = 0$ and noting that $\nabla_h\tilde{w} = \tilde{\boldsymbol{\theta}} - t^2\tilde{\boldsymbol{\gamma}}$ we obtain the following inequality

$$\Upsilon_{1a} \lesssim h\|\alpha\|_{2,\Omega}\|\nabla_h\tilde{w}\|_{0,h} \lesssim h\|\alpha\|_{2,\Omega}\left(\|\tilde{\boldsymbol{\theta}}\|_{0,h} + t^2\|\tilde{\boldsymbol{\gamma}}\|_{0,h}\right).\tag{18}$$

As π is an orthogonal projection, we obtain from the self-adjoint property that

$$\Upsilon_{1b} = \sum_{T\in\mathcal{T}}\int_{\partial T}\nabla\alpha\cdot\boldsymbol{\nu}\,\pi(\tilde{w})\,ds.$$

Since $\nabla\alpha \in H^1(\Omega;\mathbb{R}^2)$, we obtain from the definitions of jumps and orthogonal projections that,

$$\Upsilon_{1b} = \sum_{E\in\mathcal{E}}\int_E \{\nabla\alpha\}\cdot[\pi(\tilde{w})]\,ds = \sum_{E\in\mathcal{E}}\int_E \{\nabla\alpha\}\cdot\boldsymbol{\Pi}[\tilde{w}]\,ds.$$

Applying Cauchy–Schwarz inequality and trace inequality

$$\begin{aligned}\Upsilon_{1b} &= \sum_{E\in\mathcal{E}} h_E^{\rho/2}\|\nabla\alpha\|_E \frac{1}{h_E^{\rho/2}}\|\boldsymbol{\Pi}[\tilde{w}]\|_E\\ &\lesssim \left(\sum_{T\in\mathcal{T}} h_E^{\rho}\left(h_E^{-1}\|\nabla\alpha\|_T^2 + h_E|\nabla\alpha|_{1,T}^2\right)\right)^{1/2}\left(\sum_{E\in\mathcal{E}}\frac{1}{h_E^{\rho}}\|\boldsymbol{\Pi}[\tilde{w}]\|_E^2\right)^{1/2}\end{aligned}$$

from which it directly follows that

$$\Upsilon_{1b} \lesssim h^{\frac{\rho-1}{2}}\left(\|\alpha\|_{1,\Omega} + \|\alpha\|_{2,\Omega}\right)(\mathcal{J}(\tilde{w},\tilde{w}))^{1/2}. \tag{19}$$

Combining (18) with (19) we find that

$$\Upsilon_1 \lesssim h\|\alpha\|_{2,\Omega}\left(\|\tilde{\boldsymbol{\theta}}\|_{0,h} + \|t^2\tilde{\boldsymbol{\gamma}}\|_{0,h}\right) + h^{\frac{\rho-1}{2}}\|\alpha\|_{2,\Omega}(\mathcal{J}(\tilde{w},\tilde{w}))^{1/2}. \tag{20}$$

Let $\beta^I \in \mathcal{P}_1(\mathcal{T})$ be the regularized Clement-type interpolation of β, that is, $\beta^I \in H^1(\Omega)$, $\|\beta-\beta^I\|_{0,h} \lesssim h\|\beta\|_{1,\Omega}$, $\|\beta-\beta^I\|_{1,h} \lesssim h\|\beta\|_{2,\Omega}$ and $\|\beta^I\|_{1,h} \lesssim \|\beta\|_{1,\Omega}$. After integration by parts we obtain

$$\begin{aligned}\Upsilon_2 &= \sum_{T\in\mathcal{T}}\int_T \mathrm{Curl}(\beta)\cdot\nabla_h\tilde{w}\,dx\\ &= \sum_{T\in\mathcal{T}}\int_T \mathrm{Curl}(\beta-\beta^I)\cdot\nabla_h\tilde{w}\,dx + \sum_{T\in\mathcal{T}}\int_T \mathrm{Curl}(\beta^I)\cdot\nabla_h\tilde{w}\,dx =: \Upsilon_{2a} + \Upsilon_{2b}.\end{aligned}$$

Using integration by parts, Cauchy–Schwarz inequality and the fact that $\mathrm{Curl}(\beta^I) \in H(div)$ we get

$$\begin{aligned}\Upsilon_{2b} &= \sum_{T\in\mathcal{T}}\int_{\partial T}\mathrm{Curl}(\beta^I)\cdot\boldsymbol{\nu}\,\tilde{w}\,ds = \sum_{E\in\mathcal{E}}\int_E\left\{\mathrm{Curl}(\beta^I)\right\}\cdot[\tilde{w}]\,ds\\ &= \sum_{E\in\mathcal{E}}\int_E\left\{\mathrm{Curl}(\beta^I)\right\}\cdot\boldsymbol{\Pi}[\tilde{w}]\,ds \lesssim \sum_{E\in\mathcal{E}} h_E^{\rho/2}\|\mathrm{Curl}(\beta^I)\|_E\frac{1}{h_E^{\rho/2}}\|\boldsymbol{\Pi}[\tilde{w}]\|_E.\end{aligned}$$

Finally, from inverse inequality, Cauchy–Schwarz inequality and the properties of β^I

$$\Upsilon_{2b} \lesssim h^{\frac{\rho-1}{2}}\|\beta^I\|_{1,h}(\mathcal{J}(\tilde{w},\tilde{w}))^{1/2} \lesssim h^{\frac{\rho-1}{2}}\|\beta\|_{1,\Omega}(\mathcal{J}(\tilde{w},\tilde{w}))^{1/2}. \tag{21}$$

Considering that $\nabla_h \tilde{w} = \tilde{\boldsymbol{\theta}} - t^2 \tilde{\boldsymbol{\gamma}}$ and using integration by parts

$$\begin{aligned}\Upsilon_{2a} &= \sum_{T\in\mathcal{T}} \int_T \operatorname{Curl}(\beta - \beta^I) \cdot \tilde{\boldsymbol{\theta}}\, dx - t^2 \sum_{T\in\mathcal{T}} \int_T \operatorname{Curl}(\beta - \beta^I) \cdot \tilde{\boldsymbol{\gamma}}\, dx \\ &= \sum_{T\in\mathcal{T}} \int_T (\beta - \beta^I) rot(\tilde{\boldsymbol{\theta}})\, dx - t^2 \sum_{T\in\mathcal{T}} \int_T \operatorname{Curl}(\beta - \beta^I) \cdot \tilde{\boldsymbol{\gamma}}\, dx \\ &\quad - \sum_{T\in\mathcal{T}} \int_{\partial T} (\beta - \beta^I) \tilde{\boldsymbol{\theta}} \cdot \boldsymbol{\tau}\, ds.\end{aligned}$$

Applying Cauchy–Schwarz inequality and observing that $(\beta - \beta^I) \in H^1(\Omega)$ we obtain

$$\begin{aligned}|\Upsilon_{2a}| &\lesssim \sum_{T\in\mathcal{T}} \|\beta - \beta^I\|_T \|rot(\tilde{\boldsymbol{\theta}})\|_T + t^2 \sum_{T\in\mathcal{T}} \|\operatorname{Curl}(\beta - \beta^I)\|_T \|\tilde{\boldsymbol{\gamma}}\|_T \\ &\quad + \left| \sum_{E\in\mathcal{E}} \int_E \{\beta - \beta^I\} [\tilde{\boldsymbol{\theta}}]_{\boldsymbol{\tau}}\, ds \right| \lesssim h \|\beta\|_{1,\Omega} \|rot(\tilde{\boldsymbol{\theta}})\|_{0,h} \\ &\quad + h t^2 \|\beta\|_{2,\Omega} \|\tilde{\boldsymbol{\gamma}}\|_{0,h} + \sum_{E\in\mathcal{E}} h_E^{1/2} \|\{\beta - \beta^I\}\|_E \frac{\sigma_1}{h_E^{1/2}} \|[\tilde{\theta}]_\tau\|_E.\end{aligned}$$

For the last term applying Cauchy–Schwarz inequality, trace inequality and noting that $\|[\tilde{\boldsymbol{\theta}}]_{\boldsymbol{\tau}}\|_E \lesssim \|[\![\tilde{\boldsymbol{\theta}}]\!]\|_E$ we have that

$$\begin{aligned}&\sum_{E\in\mathcal{E}} h_E^{1/2} \|\{\beta - \beta^I\}\|_E \frac{1}{h_E^{1/2}} \|[\tilde{\boldsymbol{\theta}}]_{\boldsymbol{\tau}}\|_E \\ &\lesssim \left(\sum_{E\in\mathcal{E}} h_E \|\{\beta - \beta^I\}\|_E^2 \right)^{1/2} \left(\sum_{E\in\mathcal{E}} \frac{1}{h_E} \|[\![\tilde{\boldsymbol{\theta}}]\!]\|_E^2 \right)^{1/2} \\ &\lesssim \left(\sum_{T\in\mathcal{T}} h_E \left(h_E^{-1} \|\beta - \beta^I\|_T^2 + h_E |\beta - \beta^I|_T^2 \right) \right)^{1/2} \interleave \tilde{\boldsymbol{\theta}} \interleave \lesssim h \|\beta\|_{1,\Omega} \interleave \tilde{\boldsymbol{\theta}} \interleave.\end{aligned}$$

Where we consider that $|\beta - \beta^I|_{1,h}^2 \lesssim \|\beta\|_{1,h}^2 + \|\beta^I\|_{1,h}^2 \lesssim \|\beta\|_{1,\Omega}^2$. With this,

$$|\Upsilon_{2a}| \lesssim h \|\beta\|_{1,\Omega} \interleave \tilde{\boldsymbol{\theta}} \interleave + h t^2 \|\beta\|_{2,\Omega} \|\tilde{\boldsymbol{\gamma}}\|_{0,h} + h \|\beta\|_{1,\Omega} \interleave \tilde{\boldsymbol{\theta}} \interleave. \tag{22}$$

Combining (21) and (22) we obtain

$$\Upsilon_2 \lesssim h^{\frac{\rho-1}{2}} \|\beta\|_{1,\Omega} (\mathcal{J}(\tilde{w}, \tilde{w}))^{1/2} + h \|\beta\|_{1,\Omega} \interleave \tilde{\boldsymbol{\theta}} \interleave + h t^2 \|\beta\|_{2,\Omega} \|\tilde{\boldsymbol{\gamma}}\|_{0,h}. \tag{23}$$

Finally, from (20) and (23) and the definition of $\interleave \cdot \interleave$ we have that

$$\Upsilon \lesssim ht^2 \left(\|\alpha\|_{2,\Omega} + \|\beta\|_{2,\Omega}\right) \|\tilde{\boldsymbol{\gamma}}\|_{0,h} + h \left(\|\alpha\|_{2,\Omega} + \|\beta\|_{1,\Omega}\right) \|\!|\tilde{\boldsymbol{\theta}}|\!\| \\ +h^{\frac{\rho-1}{2}} \left(\|\alpha\|_{2,\Omega} + \|\beta\|_{1,\Omega}\right) (\mathcal{J}(\tilde{w}, \tilde{w}))^{1/2}. \tag{24}$$

The result follows from the estimates (10). □

In the next theorem we combine the result of Lemma 5, inequality (17) and the definitions of $\tilde{\boldsymbol{\theta}}$, $\tilde{w}$ and $\tilde{\boldsymbol{\gamma}}$ given by (15), to establish the energy norm error estimate. We observe that a similar proof can be found in [1, Theorem 6].

Theorem 6 *Let* $(\boldsymbol{\theta}, w)$ *be the solution of* (1)*, and let* $(\boldsymbol{\theta}_h, w_h)$ *be the solution of the dGWOPIP formulation* (11a)–(11b) (*or* (8) *with* $k = 2$). *Assume that* $\boldsymbol{f} \in L^2(\Omega; \mathbb{R}^2)$ *and* $g \in L^2(\Omega)$. *Moreover, assume that the Helmholtz decomposition* (9) *is valid, then if* $\rho = 3$ *we have the following error estimate*

$$\|\!|\boldsymbol{\theta} - \boldsymbol{\theta}_h|\!\| + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,h} + (\mathcal{J}(w - w_h, w - w_h))^{1/2} \\ \lesssim h \left(\|\boldsymbol{\theta}\|_{2,\Omega} + t\|\boldsymbol{\gamma}\|_{1,\Omega} + \|\boldsymbol{\gamma}\|_{H(div)}\right).$$

Proof Applying Lemma 5 in (17) we obtain from Cauchy–Schwarz inequality that

$$\|\!|\tilde{\boldsymbol{\theta}}|\!\|^2 + t^2\|\tilde{\boldsymbol{\gamma}}\|_{0,h}^2 + \mathcal{J}(\tilde{w}, \tilde{w}) \lesssim \left(\|\!|\tilde{\boldsymbol{\theta}}|\!\|^2 + t^2\|\tilde{\boldsymbol{\gamma}}\|_{0,h}^2 + \mathcal{J}(\tilde{w}, \tilde{w})\right)^{1/2} \\ \times \Big(\|\!|\boldsymbol{\theta} - \boldsymbol{\theta}^I|\!\|^2 + t^2\|\boldsymbol{\gamma} - \boldsymbol{\gamma}^I\|_{0,h}^2 + \mathcal{J}(w - w^I, w - w^I) + h^2t^2\|\boldsymbol{\gamma}\|_{1,\Omega}^2 \\ + h^2\|\boldsymbol{\gamma}\|_{H(div)}^2 + h^{\rho-1}\|\boldsymbol{\gamma}\|_{H(div)}^2\Big)^{1/2}.$$

It follows directly from this inequality that

$$\|\!|\tilde{\boldsymbol{\theta}}|\!\|^2 + t^2\|\tilde{\boldsymbol{\gamma}}\|_{0,h}^2 + \mathcal{J}(\tilde{w}, \tilde{w}) \lesssim \|\!|\boldsymbol{\theta} - \boldsymbol{\theta}^I|\!\|^2 + t^2\|\boldsymbol{\gamma} - \boldsymbol{\gamma}^I\|_{0,h}^2 \\ +\mathcal{J}(w - w^I, w - w^I) + h^2t^2\|\boldsymbol{\gamma}\|_{1,\Omega}^2 + h^2\|\boldsymbol{\gamma}\|_{H(div)}^2 + h^{\rho-1}\|\boldsymbol{\gamma}\|_{H(div)}^2. \tag{25}$$

Using the definitions of $\tilde{\boldsymbol{\theta}}$, $\tilde{w}$ and $\tilde{\boldsymbol{\gamma}}$ given by (15), triangle inequality and (25) we obtain

$$\|\!|\boldsymbol{\theta} - \boldsymbol{\theta}_h|\!\|^2 + t^2\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,h}^2 + \mathcal{J}(w - w_h, w - w_h) \lesssim h^2t^2\|\boldsymbol{\gamma}\|_{1,\Omega}^2 + h^2\|\boldsymbol{\gamma}\|_{H(div)}^2 \\ +h^{\rho-1}\|\boldsymbol{\gamma}\|_{H(div)}^2 + \|\!|\boldsymbol{\theta} - \boldsymbol{\theta}^I|\!\|^2 + t^2\|\boldsymbol{\gamma} - \boldsymbol{\gamma}^I\|_{0,h}^2 + \mathcal{J}(w - w^I, w - w^I).$$

Finally, choosing $\rho = 3$ and noting that $w^I \in H^1(\Omega) \cap \mathcal{P}_2(\mathcal{T})$, we get from (3) and (5) that

$$\|\!|\boldsymbol{\theta} - \boldsymbol{\theta}_h|\!\|^2 + t^2\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,h}^2 + \mathcal{J}(w - w_h, w - w_h) \\ \lesssim h^2 \left(\|\boldsymbol{\theta}\|_{2,\Omega}^2 + t^2\|\boldsymbol{\gamma}\|_{1,\Omega}^2 + \|\boldsymbol{\gamma}\|_{H(div)}^2\right).$$

□

For the symmetric version of (8) with $k = 2$ using the traditional duality argument we derive in the next theorem an optimal error estimate in the L^2 norm. For this purpose we need to assume that the dual shear stress also admits a Helmholtz decomposition in the form of (9). Once again, Lemma 5 and inequality (17) will be fundamental in this proof.

Theorem 7 *Let* $(\boldsymbol{\theta}, w)$ *be the solution of* (1)*, and let* $(\boldsymbol{\theta}_h, w_h)$ *be the solution of the dGWOPIP formulation* (11a)–(11b) (*or* (8) *with* $k = 2$) *with* $\delta = 1$ *(symmetric version). Assume that* $\boldsymbol{f} \in L^2(\Omega; \mathbb{R}^2)$ *and* $g \in L^2(\Omega)$*. Moreover, assume that the Helmholtz decomposition* (9) *is valid for primal and dual shear stress. Then, if* $\rho = 3$*, we have the following error estimate*

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_\Omega + \|w - w_h\|_\Omega \lesssim h^2 \left(\|\boldsymbol{\theta}\|_{2,\Omega} + t\|\boldsymbol{\gamma}\|_{1,\Omega} + \|\boldsymbol{\gamma}\|_{H(div)}\right).$$

Proof Let $(\boldsymbol{\varphi}, z) \in H_0^1(\Omega; \mathbb{R}^2) \times H_0^1(\Omega)$ be the solution of

$$\begin{aligned} a(\boldsymbol{\varphi}, \boldsymbol{\eta}) + \mu(\boldsymbol{\zeta}, \boldsymbol{\eta})_\Omega &= (e_{\boldsymbol{\theta}}, \boldsymbol{\eta})_\Omega \quad \forall\, \boldsymbol{\eta} \in H_0^1(\Omega; \mathbb{R}^2) \\ -\mu(\boldsymbol{\zeta}, \nabla v)_\Omega &= (e_w, v)_\Omega \quad \forall\, v \in H_0^1(\Omega), \end{aligned} \tag{26}$$

where here and throughout this paper, $e_{\boldsymbol{\theta}} = \boldsymbol{\theta} - \boldsymbol{\theta}_h$, $e_w = w - w_h$ and $\boldsymbol{\zeta} = t^{-2}(\boldsymbol{\varphi} - \nabla z)$ (the dual shear stress). The regularity result for this dual problem shows that

$$t\|\boldsymbol{\zeta}\|_{1,\Omega} + \|\boldsymbol{\zeta}\|_{H(div)} + \|\boldsymbol{\varphi}\|_{2,\Omega} \lesssim \|e_{\boldsymbol{\theta}}\|_\Omega + \|e_w\|_\Omega. \tag{27}$$

Adding (12a) and (12b) and then applying the result to the dual problem with $\boldsymbol{f} = e_{\boldsymbol{\theta}}$ and $g = e_w$ we obtain

$$\begin{aligned} &\mathcal{B}_h(\boldsymbol{\varphi}, \boldsymbol{\eta}) + \mu \sum_{T\in\mathcal{T}} (\boldsymbol{\zeta}, \boldsymbol{\eta} - \nabla_h v)_T + \mathcal{J}(z, v) + \mu \sum_{T\in\mathcal{T}} \int_{\partial T} \boldsymbol{\zeta} \cdot \boldsymbol{\nu}\, v\, ds \\ &\quad = (e_{\boldsymbol{\theta}}, \boldsymbol{\eta})_\Omega + (e_w, v)_\Omega \quad \forall\, (\boldsymbol{\eta}, v) \in H^2(\mathcal{T}; \mathbb{R}^2) \times H^1(\mathcal{T}). \end{aligned}$$

Setting $\boldsymbol{\eta} = e_{\boldsymbol{\theta}}$, $v = e_w$ and observing that with this choice $\boldsymbol{\eta} - \nabla_h v = t^2(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)$ we obtain

$$\begin{aligned} \|e_{\boldsymbol{\theta}}\|_\Omega^2 + \|e_w\|_\Omega^2 = \mathcal{B}_h(\boldsymbol{\varphi}, e_{\boldsymbol{\theta}}) + t^2\mu \sum_{T\in\mathcal{T}} (\boldsymbol{\zeta}, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h)_T + \mathcal{J}(z, e_w) \\ + \mu \sum_{T\in\mathcal{T}} \int_{\partial T} \boldsymbol{\zeta} \cdot \boldsymbol{\nu}\, e_w\, ds. \end{aligned} \tag{28}$$

Defining $\boldsymbol{\zeta}^I = t^{-2}(\boldsymbol{\varphi}^I - \nabla z^I)$, adding (13) and (14) and setting $\boldsymbol{\eta} = \boldsymbol{\varphi}^I$ and $v = z^I$ we have

$$\mathcal{B}_h(e_{\boldsymbol{\theta}}, \boldsymbol{\varphi}^I) + t^2\mu \sum_{T\in\mathcal{T}} (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{\zeta}^I)_T + \mathcal{J}(e_w, z^I) + \mu \sum_{T\in\mathcal{T}} \int_{\partial T} \boldsymbol{\gamma} \cdot \boldsymbol{\nu}\, z^I\, ds = 0. \tag{29}$$

Using the symmetry of $\mathcal{B}_h(\cdot,\cdot)$ and subtracting (29) from (28) we obtain

$$\|e_{\boldsymbol{\theta}}\|_{\Omega}^2 + \|e_w\|_{\Omega}^2 = \mathcal{B}_h(\boldsymbol{\varphi} - \boldsymbol{\varphi}^I, e_{\boldsymbol{\theta}}) + t^2\mu \sum_{T\in\mathcal{T}} (\boldsymbol{\zeta} - \boldsymbol{\zeta}^I, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h)_T$$
$$+\mathcal{J}(z - z^I, e_w) + \mu \sum_{T\in\mathcal{T}} \int_{\partial T} \boldsymbol{\zeta}\cdot\boldsymbol{\nu}\, e_w\, ds - \mu \sum_{T\in\mathcal{T}} \int_{\partial T} \boldsymbol{\gamma}\cdot\boldsymbol{\nu}\, z^I\, ds. \quad (30)$$

Due to the fact that $z^I, w^I \in \mathcal{P}_2(\mathcal{T}) \cap H^1(\Omega)$ we get that

$$\sum_{T\in\mathcal{T}} \int_{\partial T} \boldsymbol{\gamma}\cdot\boldsymbol{\nu}\, z^I\, ds = \sum_{E\in\mathcal{E}} \int_E \{\boldsymbol{\gamma}\}\cdot[z^I]\, ds = 0,$$

and

$$\sum_{T\in\mathcal{T}} \int_{\partial T} \boldsymbol{\zeta}\cdot\boldsymbol{\nu}\, e_w\, ds = \sum_{T\in\mathcal{T}} \int_{\partial T} \boldsymbol{\zeta}\cdot\boldsymbol{\nu}\,(w - w^I)\, ds - \sum_{T\in\mathcal{T}} \int_{\partial T} \boldsymbol{\zeta}\cdot\boldsymbol{\nu}\,(w_h - w^I)\, ds$$
$$= \sum_{E\in\mathcal{E}} \int_E \{\boldsymbol{\zeta}\}\cdot[w - w^I]\, ds - \sum_{T\in\mathcal{T}} \int_{\partial T} \boldsymbol{\zeta}\cdot\boldsymbol{\nu}\,(w_h - w^I)\, ds$$
$$= -\sum_{T\in\mathcal{T}} \int_{\partial T} \boldsymbol{\zeta}\cdot\boldsymbol{\nu}\, \tilde{w}\, ds.$$

Based on the hypothesis that the Helmholtz decomposition (9) is also valid for $\boldsymbol{\zeta}$, then we have from Lemma 5 that

$$\left|\sum_{T\in\mathcal{T}} \int_{\partial T} \boldsymbol{\zeta}\cdot\boldsymbol{\nu}\, e_w\, ds\right| \lesssim ht^2\|\boldsymbol{\zeta}\|_{1,\Omega}\|\tilde{\boldsymbol{\gamma}}\|_{0,h} + h\|\boldsymbol{\zeta}\|_{H(div)}|\!|\!|\tilde{\boldsymbol{\theta}}|\!|\!|$$
$$+h\|\boldsymbol{\zeta}\|_{H(div)}\left(\mathcal{J}(\tilde{w},\tilde{w})\right)^{1/2},$$

where $\tilde{w}$, $\tilde{\boldsymbol{\theta}}$ and $\tilde{\boldsymbol{\gamma}}$ are defined by (15). Substituting this in (30), applying Cauchy–Schwarz inequality and using the continuity of $\mathcal{B}_h(\cdot,\cdot)$ we obtain

$$\|e_{\boldsymbol{\theta}}\|_{\Omega}^2 + \|e_w\|_{\Omega}^2 \lesssim |\!|\!|\boldsymbol{\varphi} - \boldsymbol{\varphi}^I|\!|\!|\, |\!|\!|e_{\boldsymbol{\theta}}|\!|\!| + t^2\|\boldsymbol{\zeta} - \boldsymbol{\zeta}^I\|_{0,h}\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,h}$$
$$+\left(\mathcal{J}(z - z^I, z - z^I)\right)^{1/2}\left(\mathcal{J}(e_w, e_w)\right)^{1/2} + ht^2\|\boldsymbol{\zeta}\|_{1,\Omega}\|\tilde{\boldsymbol{\gamma}}\|_{0,h} + h\|\boldsymbol{\zeta}\|_{H(div)}|\!|\!|\tilde{\boldsymbol{\theta}}|\!|\!|$$
$$+h\|\boldsymbol{\zeta}\|_{H(div)}\left(\mathcal{J}(\tilde{w},\tilde{w})\right)^{1/2}.$$

Applying Cauchy–Schwarz inequality again gives

$$\|e_{\boldsymbol{\theta}}\|_{\Omega}^2 + \|e_w\|_{\Omega}^2 \lesssim \left(|\!|\!|\boldsymbol{\varphi} - \boldsymbol{\varphi}^I|\!|\!|^2 + t^2\|\boldsymbol{\zeta} - \boldsymbol{\zeta}^I\|_{0,h}^2 + \mathcal{J}(z - z^I, z - z^I)\right)^{1/2}$$
$$\times\left(|\!|\!|e_{\boldsymbol{\theta}}|\!|\!|^2 + t^2\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,h}^2 + \mathcal{J}(e_w, e_w)\right)^{1/2}$$
$$+h\left(t^2\|\boldsymbol{\zeta}\|_{1,\Omega}^2 + \|\boldsymbol{\zeta}\|_{H(div)}^2\right)^{1/2} \times \left(t^2\|\tilde{\boldsymbol{\gamma}}\|_{0,h}^2 + |\!|\!|\tilde{\boldsymbol{\theta}}|\!|\!|^2 + \mathcal{J}(\tilde{w},\tilde{w})\right)^{1/2}. \quad (31)$$

Using the regularity result (27) we have

$$h\left(t^2\|\boldsymbol{\zeta}\|_{1,\Omega}^2+\|\boldsymbol{\zeta}\|_{H(div)}^2\right)^{1/2}\lesssim h\left(\|e_{\boldsymbol{\theta}}\|_{\Omega}+\|e_w\|_{\Omega}\right).$$

Owing to the definition of $\boldsymbol{\zeta}^I$ we have $\boldsymbol{\zeta}^I=\pi_\Theta\boldsymbol{\zeta}$. Then, applying (3) and (5) to $\boldsymbol{\varphi}$ and $\boldsymbol{\zeta}$, respectively, observing that $z^I\in\mathcal{P}_2(\mathcal{T})\cap H^1(\Omega)$ and applying the regularity result (27) we obtain

$$\begin{aligned}&\left(\|\!|\boldsymbol{\varphi}-\boldsymbol{\varphi}^I\|\!|^2+t^2\|\boldsymbol{\zeta}-\boldsymbol{\zeta}^I\|_{0,h}^2+\mathcal{J}(z-z^I,z-z^I)\right)^{1/2}\lesssim h\left(\|\boldsymbol{\varphi}\|_{2,\Omega}+t\|\boldsymbol{\zeta}\|_1\right)\\&\lesssim h\left(\|e_{\boldsymbol{\theta}}\|_{\Omega}+\|e_w\|_{\Omega}\right).\end{aligned}$$

Applying (3) and (5) to the right-hand side of (25) and due to the fact that $w^I\in\mathcal{P}_2(\mathcal{T})\cap H^1(\Omega)$ and $\rho=3$ we have

$$\left(t^2\|\tilde{\boldsymbol{\gamma}}\|_{0,h}^2+\|\!|\tilde{\boldsymbol{\theta}}\|\!|^2+\mathcal{J}(\tilde{w},\tilde{w})\right)^{1/2}\lesssim h\left(\|\boldsymbol{\theta}\|_{2,\Omega}+t\|\boldsymbol{\gamma}\|_{1,\Omega}+\|\boldsymbol{\gamma}\|_{H(div)}\right).$$

Substituting the last three inequalities on the right-hand side of (31) and using Theorem 6 we find that

$$\|e_{\boldsymbol{\theta}}\|_{\Omega}^2+\|e_w\|_{\Omega}^2\lesssim h^2\left(\|\boldsymbol{\theta}\|_{2,\Omega}+t\|\boldsymbol{\gamma}\|_{1,\Omega}+\|\boldsymbol{\gamma}\|_{H(div)}\right)\left(\|e_{\boldsymbol{\theta}}\|_{\Omega}+\|e_w\|_{\Omega}\right),$$

from which the result follows. □

As a consequence of Theorem 6 and Theorem 7 we can prove the following error estimates for the displacement.

Corollary 8 *Under the assumption of Theorems 6 and 7 we have the following error estimate*

$$\|\nabla_h(w-w_h)\|_{0,h}\lesssim(h^2+ht)\left(\|\boldsymbol{\theta}\|_{2,\Omega}+t\|\boldsymbol{\gamma}\|_{1,\Omega}+\|\boldsymbol{\gamma}\|_{H(div)}\right).\tag{32}$$

Moreover, under the assumption of Theorem 6 we have

$$\|\nabla_h(w-w_h)\|_{0,h}\lesssim h\left(\|\boldsymbol{\theta}\|_{2,\Omega}+t\|\boldsymbol{\gamma}\|_{1,\Omega}+\|\boldsymbol{\gamma}\|_{H(div)}\right);\tag{33}$$

$$\|w-w_h\|_h\lesssim h\left(\|w\|_{2,\Omega}+\|\boldsymbol{\theta}\|_{2,\Omega}+t\|\boldsymbol{\gamma}\|_{1,\Omega}+\|\boldsymbol{\gamma}\|_{H(div)}\right).\tag{34}$$

Proof Since $\nabla_h(w-w_h)=\boldsymbol{\theta}-\boldsymbol{\theta}_h-t^2(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h)$ the error estimate (32) follows from triangle inequality and Theorems 6 and 7. For the same reason the error estimate (33) follows directly from the definition of $\|\!|\cdot\|\!|$ and Theorem 6.

For the last error estimate, since $\nabla_h \tilde{w} = \tilde{\boldsymbol{\theta}} - t^2\tilde{\boldsymbol{\gamma}}$ and based on triangle inequality and the definitions of $\tilde{w}$, $\tilde{\boldsymbol{\gamma}}$ and $\tilde{\boldsymbol{\theta}}$ given by (15), we have

$$\begin{aligned}
\|w - w_h\|_h &\lesssim \|\nabla_h(w - w^I)\|_{0,h}^2 + \|\nabla_h \tilde{w}\|_{0,h}^2 + \mathcal{J}(e_w, e_w) \\
&\lesssim \|\nabla_h(w - w^I)\|_{0,h}^2 + \|\tilde{\boldsymbol{\theta}}\|_{0,h}^2 + t^2\|\tilde{\boldsymbol{\gamma}}\|_{0,h}^2 + \mathcal{J}(e_w, e_w) \\
&\lesssim \|\nabla_h(w - w^I)\|_{0,h}^2 + \|\boldsymbol{\theta} - \boldsymbol{\theta}^I\|_{0,h}^2 + t^2\|\boldsymbol{\gamma} - \boldsymbol{\gamma}^I\|_{0,h}^2 + \|e_{\boldsymbol{\theta}}\|_{0,h}^2 \\
&\quad + t^2\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,h}^2 + \mathcal{J}(e_w, e_w).
\end{aligned}$$

Applying the interpolation estimates (2), (3) and (5) together with Theorem 6 we conclude the proof of (34). □

We note that the regularity required in Theorems 6 and 7 for the solution of (1) is that $\boldsymbol{\theta} \in H^2(\Omega; \mathbb{R}^2)$ and $w \in H^2(\Omega)$ and this regularity always holds if Ω is a convex polygon or a smooth bounded domain for $\boldsymbol{f} \in L^2(\Omega; \mathbb{R}^2)$ and $g \in L^2(\Omega)$. Furthermore, in this case ($k = 2$), the right-hand side $t\|\boldsymbol{\gamma}\|_{1,\Omega} + \|\boldsymbol{\gamma}\|_{H(div)} + \|\boldsymbol{\theta}\|_{2,\Omega}$ remains bounded as t tends to zero.

As this result was proved under the assumption that the Helmholtz decomposition holds for $\boldsymbol{\gamma}$ (and $\boldsymbol{\zeta}$ for Theorem 7) we highlight that the Helmholtz decomposition always holds if Ω is a convex domain.

5 *A priori* error analysis for high-order

In this section we will proceed with the analysis for high-order polynomials without the Helmholtz decomposition hypothesis and, as usual, we need to assume additional regularity for the solution $(\boldsymbol{\theta}, w)$ of (1). That is, regularity which is not theoretically established for this problem.

In Theorem 9 and Corollary 10 we extend the results of Theorem 6 and part of Corollary 8, respectively, combining the same arguments presented in [12] with the previous analysis. We emphasize that if we proceed in this way, when $k = 2$ the resulting estimates will not be optimal in relation to t.

Theorem 9 *Let $(\boldsymbol{\theta}, w)$ be the solution of (1), and let $(\boldsymbol{\theta}_h, w_h)$ be the solution of the dGWOPIP formulation (8) with $k > 2$. Assume that the solution $(\boldsymbol{\theta}, w) \in H^k(\Omega; \mathbb{R}^2) \times H^k(\Omega)$, $\boldsymbol{f} \in L^2(\Omega; \mathbb{R}^2)$ and $g \in L^2(\Omega)$. Then if $\rho = 2k - 1$ we have the following error estimate*

$$\begin{aligned}
&|||\boldsymbol{\theta} - \boldsymbol{\theta}_h||| + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,h} + \mathcal{J}(w - w_h, w - w_h) \\
&\quad \lesssim h^{k-1}\left(\|\boldsymbol{\gamma}\|_{k-1,\Omega} + t\|\boldsymbol{\gamma}\|_{k-1,\Omega} + \|\boldsymbol{\gamma}\|_{\Omega} + \|\boldsymbol{\theta}\|_{k,\Omega}\right).
\end{aligned}$$

Proof It is easy to check that inequality (17) is also valid for high order. Thus, we limited the last term of (17) using Cauchy–Schwarz inequality in the following way

$$\left|\sum_{T\in\mathcal{T}}\int_{\partial T}\boldsymbol{\gamma}\cdot\boldsymbol{\nu}\,\tilde{w}\,ds\right| \lesssim \left|\sum_{E\in\mathcal{E}}\left(\int_E\{\boldsymbol{\gamma}-\boldsymbol{\gamma}^I\}\cdot[\tilde{w}]\,ds+\int_E\{\boldsymbol{\gamma}^I\}\cdot[\tilde{w}]ds\right)\right|$$
$$\lesssim \left(\sum_{E\in\mathcal{E}}h_E\|\{\boldsymbol{\gamma}-\boldsymbol{\gamma}^I\}\|_E^2\right)^{1/2}\left(\sum_{E\in\mathcal{E}}h_E^{-1}\|[\tilde{w}]\|_E^2\right)^{1/2}+\left|\sum_{E\in\mathcal{E}}\int_E\{\boldsymbol{\gamma}^I\}\cdot[\tilde{w}]ds\right|.$$

Since $\boldsymbol{\gamma}^I{}_{|E}\in\mathcal{P}_{k-1}(E;\mathbb{R}^2)$ and applying Cauchy–Schwarz inequality it follows that

$$\left|\sum_{E\in\mathcal{E}}\int_E\{\boldsymbol{\gamma}^I\}\cdot[\tilde{w}]ds\right| = \left|\sum_{E\in\mathcal{E}}\int_E\{\boldsymbol{\gamma}^I\}\cdot\boldsymbol{\Pi}^{k-1}[\tilde{w}]ds\right|$$
$$\lesssim \left(\sum_{E\in\mathcal{E}}\frac{h_E^{\rho}}{\sigma_2}\|\{\boldsymbol{\gamma}^I\}\|_E^2\right)^{1/2}(\mathcal{J}(\tilde{w},\tilde{w}))^{1/2}.$$

Using Lemma 2.2 of [12] and the fact that $\nabla_h\tilde{w}=\tilde{\boldsymbol{\theta}}-t^2\tilde{\boldsymbol{\gamma}}$ we obtain

$$\left(\sum_{E\in\mathcal{E}}h_E^{-1}\|[\tilde{w}]\|_E^2\right)^{1/2} \lesssim \left(\|\nabla_h\tilde{w}\|_{0,h}^2+\mathcal{J}(\tilde{w},\tilde{w})\right)^{1/2}$$
$$\lesssim \left(\|\tilde{\boldsymbol{\theta}}\|_{0,h}^2+t^2\|\tilde{\boldsymbol{\gamma}}\|_{0,h}^2+\mathcal{J}(\tilde{w},\tilde{w})\right)^{1/2}.$$

Combining these inequalities with trace inequality, Cauchy–Schwarz inequality, inverse inequality for $\|\boldsymbol{\gamma}^I\|_E^2$ and triangular inequality gives

$$\left|\sum_{T\in\mathcal{T}}\int_{\partial T}\boldsymbol{\gamma}\cdot\boldsymbol{\nu}\,\tilde{w}\,ds\right| \lesssim \left(\|\tilde{\boldsymbol{\theta}}\|_{0,h}^2+t^2\|\tilde{\boldsymbol{\gamma}}\|_{0,h}^2+\mathcal{J}(\tilde{w},\tilde{w})\right)^{1/2}$$
$$\times\left(\sum_{T\in\mathcal{T}}\left(\|\boldsymbol{\gamma}-\boldsymbol{\gamma}^I\|_T^2+h_E^2|\boldsymbol{\gamma}-\boldsymbol{\gamma}^I|_{1,T}^2+h_E^{\rho-1}\left(\|\boldsymbol{\gamma}-\boldsymbol{\gamma}^I\|_T^2+\|\boldsymbol{\gamma}\|_T^2\right)\right)\right)^{1/2}.$$

Applying the estimate for the interpolant (5) we obtain that

$$\left|\sum_{T\in\mathcal{T}}\int_{\partial T}\boldsymbol{\gamma}\cdot\boldsymbol{\nu}\,\tilde{w}\,ds\right| \lesssim \left(\sum_{T\in\mathcal{T}}\left(h_T^{2k-2}\|\boldsymbol{\gamma}\|_{k-1,T}^2+h_T^{\rho-1}\|\boldsymbol{\gamma}\|_T^2\right)\right)^{1/2}$$
$$\times\left(\|\tilde{\boldsymbol{\theta}}\|_{0,h}^2+t^2\|\tilde{\boldsymbol{\gamma}}\|_{0,h}^2+\mathcal{J}(\tilde{w},\tilde{w})\right)^{1/2}.$$

Following as in the demonstration of Theorem 6 and choosing $\rho=2k-1$ we complete the proof. □

Proceeding as in the demonstration of Corollary 8, but now using Theorem 9, we can also establish the following estimates.

Corollary 10 *Under the assumption of Theorem 9 we have the following error estimate*

$$\|w - w_h\|_h \lesssim h^{k-1}\left(\|w\|_{k,\Omega} + \|\boldsymbol{\theta}\|_{k,\Omega} + t\|\boldsymbol{\gamma}\|_{k-1,\Omega} + \|\boldsymbol{\gamma}\|_{k-1,\Omega} + \|\boldsymbol{\gamma}\|_{\Omega}\right);$$
$$\|\nabla_h(w - w_h)\|_{0,h} \lesssim h^{k-1}\left(\|\boldsymbol{\gamma}\|_{k-1,\Omega} + t\|\boldsymbol{\gamma}\|_{k-1,\Omega} + \|\boldsymbol{\gamma}\|_{\Omega} + \|\boldsymbol{\theta}\|_{k,\Omega}\right).$$

It is well known that the norm $\|\boldsymbol{\gamma}\|_{k-1,\Omega}$ behaves like $t^{-(k-3/2)}$ as t tends to zero. For this reason multiplying this norm by the factor t when $k = 2$ keeps this term controlled, in the sense that it remains limited as t tends to zero. Unfortunately, when we are dealing with polynomials of higher degree the norm $\|\boldsymbol{\gamma}\|_{k-1,\Omega}$ (for $k > 2$) will appear on the right-hand side without the adequate factor multiplying it. This means that the error estimate can blow up as t tends to zero. However, if we keep t fixed, Theorem 9 and Corollary 10, as well as the Theorem 13 below, show that the method will maintain the rate of convergence in relation to h.

The remainder of this section will be dedicated to deriving another energy norm error estimate using a different technique. Here, enriching operators and residual estimates will be the main tools used to perform the analysis. We start by recalling Theorem 3 of [15] which gives the following residual estimates.

Theorem 11 *Let $g_h \in \mathcal{P}_k(\mathcal{T})$, $\boldsymbol{f}_h \in \mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)$ and $\boldsymbol{\phi}, \boldsymbol{\eta} \in \mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)$ be arbitrary. Then, it holds for all $T \in \mathcal{T}$ and for all $E \in \mathcal{E}(\Omega)$ that*

$$h_T\|\boldsymbol{f}_h + \mathbf{div}_h\mathcal{C}e_h(\boldsymbol{\eta}) - \boldsymbol{\phi}\|_T \lesssim \|e(\boldsymbol{\theta}) - e_h(\boldsymbol{\eta})\|_T + h_T\|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T + \|\boldsymbol{f}_T - \boldsymbol{f}_h\|_{H^{-1}(T)};$$

$$h_T\|g_h - \mathrm{div}_h(\boldsymbol{\phi})\|_T \lesssim \|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T + \|g_T - g_h\|_{H^{-1}(T)};$$
$$h_E^{1/2}\|[\![\mathcal{C}e_h(\boldsymbol{\eta})]\!]\|_E \lesssim \|e(\boldsymbol{\theta}) - e_h(\boldsymbol{\eta})\|_{\omega_E} + h_E\|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_{\omega_E} + \|\boldsymbol{f}_E - \boldsymbol{f}_h\|_{H^{-1}(\omega_E)};$$
$$h_E^{1/2}\|[\boldsymbol{\phi}]\|_E \lesssim \|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_{\omega_E} + \|g_E - g_h\|_{H^{-1}(\omega_E)}.$$

Here, and throughout this paper, $g_T = g_{|T}$, $g_E = g_{|\omega_E}$ (idem for $\boldsymbol{f}$) and ω_E is the patch of two triangles sharing the face E.

The following enriching operators use averaging techniques (see [9] and [10] for details):

$\mathsf{E}_h : \mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2) \to \mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2) \cap H_0^1(\Omega,\mathbb{R}^2)$ such that

$$\left(\sum_{T\in\mathcal{T}} h_E^{-2}\|\mathsf{E}_h\boldsymbol{\eta} - \boldsymbol{\eta}\|_T^2\right)^{1/2} + \|\nabla_h(\mathsf{E}_h\boldsymbol{\eta} - \boldsymbol{\eta})\|_{\Omega} \lesssim \|\boldsymbol{\eta}\|_h \tag{35}$$

and $\mathsf{E}_h : \mathcal{P}_k(\mathcal{T}) \to \mathcal{P}_k(\mathcal{T}) \cap H_0^1(\Omega)$ such that

$$\left(\sum_{T\in\mathcal{T}} h_E^{-2}\|\mathsf{E}_h v - v\|_T^2\right)^{1/2} + \|\nabla_h(\mathsf{E}_h v - v)\|_{\Omega} \lesssim \|v\|_h. \tag{36}$$

The previous inequality (35) follows from the enriching operator properties and from the discrete Korn's inequality (see [11] and [2]), while (36) follows from the enriching operator properties and from [12, Lemma 2.2] (recall that $\rho > 1$).

We recall now the following definitions of oscillation for a scalar function and for a vector function

$$Osc(g) = \left(\sum_{E \in \mathcal{E}} \|g_E - Pg\|^2_{H^{-1}(\omega_E)} \right)^{1/2}$$

and

$$Osc(\boldsymbol{f}) = \left(\sum_{E \in \mathcal{E}} \|\boldsymbol{f}_E - \mathbf{P}\boldsymbol{f}\|^2_{H^{-1}(\omega_E)} \right)^{1/2},$$

where $P : L^2(\Omega) \to \mathcal{P}_k(\mathcal{T})$ is the L^2 orthogonal projection onto $\mathcal{P}_k(\mathcal{T})$ and $\mathbf{P} : L^2(\Omega; \mathbb{R}^2) \to \mathcal{P}_{k-1}(\mathcal{T}; \mathbb{R}^2)$ is the L^2 orthogonal projection onto $\mathcal{P}_{k-1}(\mathcal{T}; \mathbb{R}^2)$. That is,

$$\int_\Omega (Pg - g)v\, dx = 0 \ \ \forall\ v \in \mathcal{P}_k(\mathcal{T}) \ \text{(analogous to } \mathbf{P}\boldsymbol{f}).$$

As proved in [6], if $\boldsymbol{f} \in L^p(\Omega; \mathbb{R}^2)$ for $p > 1$ we have

$$Osc(\boldsymbol{f}) \lesssim h^{1-2(1/2-1/q)} \|\boldsymbol{f} - \mathbf{P}\boldsymbol{f}\|_{L^p(\Omega)}, \tag{37}$$

where p and q are such that $1/p + 1/q = 1$. In the same way it is possible to obtain

$$Osc(g) \lesssim h^{1-2(1/2-1/q)} \|g - Pg\|_{L^p(\Omega)} \tag{38}$$

if $g \in L^p(\Omega)$ for $p > 1$.

The main steps of the proof for the next theorem are analogous to those performed in Theorem 7 of [15]. However, as we are dealing with a different method we will write almost the complete proof for clarity reasons. We observe that under conditions N2 and N3 of [26, Lemma 2.1] the term Υ_a defined below is related to the interpolation error part while the term Υ_b is related with consistency/nonconforming error part. Unfortunately, the analysis here is more complex because condition N3 for our energy norm was not established. However, as in [26] we perform the analysis using residual estimates and enriching operator properties.

Theorem 12 *Let $(\boldsymbol{\theta}, w)$ be the solution of (1) and let $(\boldsymbol{\theta}_h, w_h)$ be the solution of the dGWOPIP formulation (8) with $k > 2$. Then we have*

$$\begin{aligned} |||\boldsymbol{\theta} - \boldsymbol{\theta}_h, w - w_h|||^2 \lesssim \inf_{\substack{\boldsymbol{\eta} \in \mathcal{P}_{k-1}(\mathcal{T}; \mathbb{R}^2) \\ v \in \mathcal{P}_k(\mathcal{T})}} \Bigg\{ & \left(h^{\rho-1} + t^2 + 1\right) \|\boldsymbol{\gamma} - \boldsymbol{\phi}\|^2_{0,h} \\ & + \mathcal{J}(w - v,\ w - v) + \|e(\boldsymbol{\theta}) - e_h(\boldsymbol{\eta})\|^2_{0,h} + \boldsymbol{J}(\boldsymbol{\theta} - \boldsymbol{\eta}, \boldsymbol{\theta} - \boldsymbol{\eta}) \\ & + \sum_{E \in \mathcal{E}} \frac{h_E}{\sigma_1} \|\{\mathcal{C} e_h(\boldsymbol{\theta} - \boldsymbol{\eta})\}\|^2_E \Bigg\} + Osc^2(g) + Osc^2(\boldsymbol{f}) + h^{\rho-1} \|\boldsymbol{\gamma}\|^2_\Omega \end{aligned}$$

where $\boldsymbol{\phi} = t^{-2}(\boldsymbol{\eta} - \nabla_h v)$.

Proof **Step 0:** Let $\tilde{\boldsymbol{\eta}} = \boldsymbol{\theta}_h - \boldsymbol{\eta}$ and $\tilde{v} = w_h - v$ where $\boldsymbol{\eta}$ and v are arbitrary in $\mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)$ and $\mathcal{P}_k(\mathcal{T})$, respectively, and $\boldsymbol{\theta}_h$ and w_h are the solution of the dGWOPIP formulation (8). The coercivity of the bilinear form given by Lemma 3 and (8) implies that

$$\begin{aligned}\|\!|\tilde{\boldsymbol{\eta}}, \tilde{v}|\!\|^2 &\lesssim \mathcal{A}_h(\tilde{\boldsymbol{\eta}}, \tilde{v}; \tilde{\boldsymbol{\eta}}, \tilde{v}) = \mathcal{A}_h(\boldsymbol{\theta}_h, w_h; \tilde{\boldsymbol{\eta}}, \tilde{v}) - \mathcal{A}_h(\boldsymbol{\eta}, v; \tilde{\boldsymbol{\eta}}, \tilde{v})\\ &= (\boldsymbol{f}, \tilde{\boldsymbol{\eta}} - \mathsf{E}_h\tilde{\boldsymbol{\eta}})_\Omega + (g, \tilde{v} - \mathsf{E}_h\tilde{v})_\Omega - \mathcal{A}_h(\boldsymbol{\eta}, v; \tilde{\boldsymbol{\eta}} - \mathsf{E}_h\tilde{\boldsymbol{\eta}}, \tilde{v} - \mathsf{E}_h\tilde{v})\\ &\quad + (\boldsymbol{f}, \mathsf{E}_h\tilde{\boldsymbol{\eta}})_\Omega + (g, \mathsf{E}_h\tilde{v})_\Omega - \mathcal{A}_h(\boldsymbol{\eta}, v; \mathsf{E}_h\tilde{\boldsymbol{\eta}}, \mathsf{E}_h\tilde{v}).\end{aligned} \tag{39}$$

Step 1: Proof of

$$\begin{aligned}\Upsilon_a &\lesssim \left(\sum_{T\in\mathcal{T}} \left(\|e(\boldsymbol{\theta}) - e_h(\boldsymbol{\eta})\|_T^2 + \|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T^2\right)\right)^{1/2} \|\tilde{\boldsymbol{\eta}}\|_h\\ &\quad + \left(\sum_{T\in\mathcal{T}} \|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T^2\right)^{1/2} \|\tilde{v}\|_h + (\boldsymbol{J}(\boldsymbol{\theta} - \boldsymbol{\eta}, \boldsymbol{\theta} - \boldsymbol{\eta}))^{1/2} \|\tilde{\boldsymbol{\eta}}\|_h,\end{aligned} \tag{40}$$

where $\boldsymbol{\phi} = t^{-2}(\boldsymbol{\eta} - \nabla_h v)$ and

$$\Upsilon_a := (\boldsymbol{f}, \mathsf{E}_h\tilde{\boldsymbol{\eta}})_\Omega + (g, \mathsf{E}_h\tilde{v})_\Omega - \mathcal{A}_h(\boldsymbol{\eta}, v; \mathsf{E}_h\tilde{\boldsymbol{\eta}}, \mathsf{E}_h\tilde{v}).$$

For the analysis of Υ_a observe that $\mathsf{E}_h\tilde{\boldsymbol{\eta}} \in H_0^1(\Omega;\mathbb{R}^2) \cap \mathcal{P}_{k-1}(\mathcal{T};\mathbb{R}^2)$ and $\mathsf{E}_h\tilde{v} \in H_0^1(\Omega) \cap \mathcal{P}_k(\mathcal{T})$. Hence, (1) and (6) lead to

$$\begin{aligned}\Upsilon_a &= \sum_{T\in\mathcal{T}} ((\mathcal{C}e(\boldsymbol{\theta}) - \mathcal{C}e_h(\boldsymbol{\eta}), e(\mathsf{E}_h\tilde{\boldsymbol{\eta}}))_T + \mu(\boldsymbol{\gamma} - \boldsymbol{\phi}, \mathsf{E}_h\tilde{\boldsymbol{\eta}})_T - \mu(\boldsymbol{\gamma} - \boldsymbol{\phi}, \nabla\mathsf{E}_h\tilde{v})_T)\\ &\quad + \sum_{E\in\mathcal{E}} \delta\langle[\![\boldsymbol{\eta}]\!], \{\mathcal{C}e(\mathsf{E}_h\tilde{\boldsymbol{\eta}})\}\rangle_E =: \Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4.\end{aligned}$$

Applying Cauchy–Schwarz inequality for each term we obtain

$$\begin{aligned}\Upsilon_1 &= \sum_{T\in\mathcal{T}} (\mathcal{C}e(\boldsymbol{\theta}) - \mathcal{C}e_h(\boldsymbol{\eta}), e(\mathsf{E}_h\tilde{\boldsymbol{\eta}}))_T \lesssim \sum_{T\in\mathcal{T}} \|e(\boldsymbol{\theta}) - e_h(\boldsymbol{\eta})\|_T \|e(\mathsf{E}_h\tilde{\boldsymbol{\eta}})\|_T;\\ \Upsilon_2 + \Upsilon_3 &\lesssim \sum_{T\in\mathcal{T}} \|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T \|\mathsf{E}_h\tilde{\boldsymbol{\eta}}\|_T + \sum_{T\in\mathcal{T}} \|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T \|\nabla\mathsf{E}_h\tilde{v}\|_T;\\ \Upsilon_4 &\lesssim \sum_{E\in\mathcal{E}} \delta \left\|\sqrt{\frac{\sigma_1}{h_E}}[\![\boldsymbol{\eta}]\!]\right\|_E \left\|\sqrt{\frac{h_E}{\sigma_1}}\{\mathcal{C}e(\mathsf{E}_h\tilde{\boldsymbol{\eta}})\}\right\|_E\\ &\lesssim (\boldsymbol{J}(\boldsymbol{\eta}, \boldsymbol{\eta}))^{1/2} \left(\sum_{T\in\mathcal{T}} \|e(\mathsf{E}_h\tilde{\boldsymbol{\eta}})\|_T^2\right)^{1/2}\end{aligned}$$

where we also used inverse inequality for the fourth term. The combination of these bounds shows

$$\Upsilon_a \lesssim \sum_{T\in\mathcal{T}} (\|e(\boldsymbol{\theta}) - e_h(\boldsymbol{\eta})\|_T \|e(\mathsf{E}_h\tilde{\boldsymbol{\eta}})\|_T + \|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T \|\mathsf{E}_h\tilde{\boldsymbol{\eta}}\|_T)$$
$$+ \sum_{T\in\mathcal{T}} \|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T \|\nabla \mathsf{E}_h\tilde{v}\|_T + (\boldsymbol{J}(\boldsymbol{\eta},\boldsymbol{\eta}))^{1/2}\left(\sum_{T\in\mathcal{T}} \|e(\mathsf{E}_h\tilde{\boldsymbol{\eta}})\|_T^2\right)^{1/2}.$$

Applying the properties of the enriching operators (35) and (36) we obtain (40).
Step 2: Proof of

$$\Upsilon_b \lesssim \left(\sum_{T\in\mathcal{T}} \left(\|e(\boldsymbol{\theta}) - e_h(\boldsymbol{\eta})\|_T^2 + \|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T\right)\right)^{1/2} \|\tilde{\boldsymbol{\eta}}\|_h$$
$$+ \left(\mathcal{J}(w - v,\ w - v) + Osc^2(g)\right)^{1/2} \|\tilde{v}\|_h + \left(\boldsymbol{J}(\boldsymbol{\theta} - \boldsymbol{\eta},\ \boldsymbol{\theta} - \boldsymbol{\eta}) + Osc^2(\boldsymbol{f})\right)^{1/2}$$
$$\|\tilde{\boldsymbol{\eta}}\|_h$$
$$+ \left(\sum_{T\in\mathcal{T}} (\|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T^2 + h_T^{\rho-1}(\|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T^2 + \|\boldsymbol{\gamma}\|_T^2))\right)^{1/2} \|\tilde{v}\|_h. \tag{41}$$

Where, as before, $\boldsymbol{\phi} = t^{-2}(\boldsymbol{\eta} - \nabla_h v)$ and

$$\Upsilon_b := (\boldsymbol{f}, \tilde{\boldsymbol{\eta}} - \mathsf{E}_h\tilde{\boldsymbol{\eta}})_\Omega + (g, \tilde{v} - \mathsf{E}_h\tilde{v})_\Omega - \mathcal{A}_h(\boldsymbol{\eta}, v; \tilde{\boldsymbol{\eta}} - \mathsf{E}_h\tilde{\boldsymbol{\eta}}, \tilde{v} - \mathsf{E}_h\tilde{v}).$$

To facilitate the handling, we use the definition of the bilinear form $\mathcal{A}_h(\cdot,\cdot;\cdot,\cdot)$ to write all terms of Υ_b, that is,

$$\Upsilon_b = (\boldsymbol{f}, \tilde{\boldsymbol{\eta}} - \mathsf{E}_h\tilde{\boldsymbol{\eta}})_\Omega + (g, \tilde{v} - \mathsf{E}_h\tilde{v})_\Omega - \sum_{T\in\mathcal{T}} (\mathcal{C}e_h(\boldsymbol{\eta}), e_h(\tilde{\boldsymbol{\eta}} - \mathsf{E}_h\tilde{\boldsymbol{\eta}}))_T$$
$$- \sum_{T\in\mathcal{T}} (\boldsymbol{\phi}, \tilde{\boldsymbol{\eta}} - \mathsf{E}_h\tilde{\boldsymbol{\eta}})_T + \sum_{T\in\mathcal{T}} (\boldsymbol{\phi}, \nabla_h(\tilde{v} - \mathsf{E}_h\tilde{v}))_T$$
$$+ \sum_{E\in\mathcal{E}} \langle\{\mathcal{C}e_h(\boldsymbol{\eta})\}, [\![\tilde{\boldsymbol{\eta}} - \mathsf{E}_h\tilde{\boldsymbol{\eta}}]\!]\rangle_E + \delta \sum_{E\in\mathcal{E}} \langle\{\mathcal{C}e_h(\tilde{\boldsymbol{\eta}} - \mathsf{E}_h\tilde{\boldsymbol{\eta}})\}, [\![\boldsymbol{\eta}]\!]\rangle_E$$
$$- \boldsymbol{J}(\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}}) - \mathcal{J}(v, \tilde{v}) =: \Upsilon_1 + \ldots + \Upsilon_9.$$

Proceeding as in the proof of Theorem 7 of [15] (Step 2) we obtain the following limitations

$$\Upsilon_1 + \Upsilon_3 + \Upsilon_4 + \Upsilon_6 \lesssim + \left(\sum_{T\in\mathcal{T}} h_E^{-2}\|\tilde{\boldsymbol{\eta}} - \mathsf{E}_h\tilde{\boldsymbol{\eta}}\|_T^2\right)^{1/2} \left(\sum_{E\in\mathcal{E}} h_E \|[\![\mathcal{C}e_h(\boldsymbol{\eta})]\!]\|_E^2\right)^{1/2}$$
$$+ \left(\sum_{T\in\mathcal{T}} h_T^2 \|\mathbf{P}\boldsymbol{f} + \mathbf{div}_h(\mathcal{C}e_h(\boldsymbol{\eta})) - \boldsymbol{\phi}\|_T^2\right)^{1/2} \left(\sum_{T\in\mathcal{T}} h_T^{-2}\|\tilde{\boldsymbol{\eta}} - \mathsf{E}_h\tilde{\boldsymbol{\eta}}\|_T^2\right)^{1/2};$$

$$\Upsilon_2 + \Upsilon_5 \lesssim \left(\sum_{T\in\mathcal{T}} h_E^2 \|Pg - \mathrm{div}_h(\boldsymbol{\phi})\|_T^2 \right)^{1/2} \left(\sum_{T\in\mathcal{T}} h_E^{-2} \|\tilde{v} - \mathsf{E}_h\tilde{v}\|_T^2 \right)^{1/2}$$
$$+ \left(\sum_{T\in\mathcal{T}} \frac{h_E^{\rho-1}}{\sigma_2} \left(\|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T^2 + \|\boldsymbol{\gamma}\|_T^2 \right) \right)^{1/2} \mathcal{J}(\tilde{v}, \tilde{v})^{1/2}$$
$$+ \left(\sum_{E\in\mathcal{E}} h_E \|[\boldsymbol{\phi}]\|_E^2 \right)^{1/2} \left(\sum_{T\in\mathcal{T}} h_E^{-2} \|\tilde{v} - \mathsf{E}_h\tilde{v}\|_T^2 \right)^{1/2};$$

and

$$\Upsilon_7 + \Upsilon_8 + \Upsilon_9 \lesssim \left(\left(\sum_{T\in\mathcal{T}} \|e_h(\tilde{\boldsymbol{\eta}} - \mathsf{E}_h\tilde{\boldsymbol{\eta}})\|_T^2 \right)^{1/2} + (\boldsymbol{J}(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\eta}}))^{1/2} \right)$$
$$\times (\boldsymbol{J}(\boldsymbol{\eta} - \boldsymbol{\theta}, \boldsymbol{\eta} - \boldsymbol{\theta}))^{1/2} + (\mathcal{J}(w - v, w - v))^{1/2} (\mathcal{J}(\tilde{v}, \tilde{v}))^{1/2}.$$

Combining all of these inequalities and using enriching operator properties (35) and (36) together with the Theorem 11 we prove (41).

Step 3: We combine the previous steps to finish the proof. Firstly observe that $\|\boldsymbol{\eta}\|_h \leq \|\boldsymbol{\eta}, v\|_h$, $\|v\|_h \leq \|\boldsymbol{\eta}, v\|_h$ and that there exists positive constants $\tilde{c}_1$ and $\tilde{c}_2$ such that (finite dimension)

$$\tilde{c}_1 \|\!|\boldsymbol{\eta}, v|\!\| \leq \|\boldsymbol{\eta}, v\|_h \leq \tilde{c}_2 \|\!|\boldsymbol{\eta}, v|\!\|. \tag{42}$$

Finally from (39)–(41) and (42) we have

$$\|\!|\tilde{\boldsymbol{\eta}}, \tilde{v}|\!\|^2 \lesssim \sum_{T\in\mathcal{T}} \left(\|e(\boldsymbol{\theta}) - e_h(\boldsymbol{\eta})\|_T^2 + \|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T^2 \right)$$
$$+ \sum_{T\in\mathcal{T}} (\|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T^2 + h_T^{\rho-1}(\|\boldsymbol{\gamma} - \boldsymbol{\phi}\|_T^2 + \|\boldsymbol{\gamma}\|_T^2))$$
$$+ \mathcal{J}(w - v, w - v) + \boldsymbol{J}(\boldsymbol{\theta} - \boldsymbol{\eta}, \boldsymbol{\theta} - \boldsymbol{\eta}) + Osc^2(\boldsymbol{f}) + Osc^2(g).$$

From triangle inequality we complete the proof. □

Theorem 13 *Let $(\boldsymbol{\theta}, w)$ be the solution of (1), and let $(\boldsymbol{\theta}_h, w_h)$ be the solution of the dGWOPIP formulation (8) with $k > 2$. Assume that the solution $(\boldsymbol{\theta}, w) \in H^k(\Omega; \mathbb{R}^2) \times H^k(\Omega)$, $\boldsymbol{f} \in H^{k-2}(\Omega; \mathbb{R}^2)$ and $g \in H^{k-2}(\Omega)$. Then, if $\rho = 2k - 1$ we have the following error estimate*

$$\|\!|\boldsymbol{\theta} - \boldsymbol{\theta}_h, w - w_h|\!\| \lesssim h^{k-1} \left(\|\boldsymbol{\gamma}\|_{k-1,\Omega} + t\|\boldsymbol{\gamma}\|_{k-1,\Omega} + \|\boldsymbol{\gamma}\|_\Omega + \|\boldsymbol{\theta}\|_{k,\Omega} \right)$$
$$+ h^{k-1} \left(\|\boldsymbol{f}\|_{k-2,\Omega} + \|g\|_{k-2,\Omega} \right).$$

Proof We prove this result exploring the infimum on the right-hand side of Theorem 12. Choosing $\boldsymbol{\eta} = \boldsymbol{\theta}^I$ and $v = w^I$ we have from (4) that $\boldsymbol{\phi} = t^{-2}(\boldsymbol{\eta} - \nabla_h v) = \boldsymbol{\gamma}^I$. This allows us to use the interpolation estimates (3) and (5).

Using the trace inequality and interpolation estimate (3) we obtain

$$\|e(\boldsymbol{\theta}) - e_h(\boldsymbol{\theta}^I)\|_{0,h}^2 + \boldsymbol{J}(\boldsymbol{\theta} - \boldsymbol{\theta}^I, \boldsymbol{\theta} - \boldsymbol{\theta}^I) + \sum_{E \in \mathcal{E}} \frac{h_E}{\sigma_1} \left\| \{\mathcal{C} e_h(\boldsymbol{\theta} - \boldsymbol{\theta}^I)\} \right\|_E^2$$
$$\lesssim \|e(\boldsymbol{\theta}) - e_h(\boldsymbol{\theta}^I)\|_{0,h}^2 + \sum_{T \in \mathcal{T}} h_T \left(h_T^{-1} \|e_h(\boldsymbol{\theta} - \boldsymbol{\theta}^I)\|_T^2 + h_T |e_h(\boldsymbol{\theta} - \boldsymbol{\theta}^I)|_{1,T}^2 \right)$$
$$+ \sum_{E \in \mathcal{E}} \frac{\sigma_1}{h_E} \|[\![\boldsymbol{\theta} - \boldsymbol{\theta}^I]\!]\|_E^2 \lesssim h^{2k-2} \|\boldsymbol{\theta}\|_{k,\Omega}^2.$$

For $v = w^I$ since $w^I \in H^1(\Omega) \cap \mathcal{P}_k(\mathcal{T})$ and $w \in H^1(\Omega)$, we have $[w - v] = 0$. Therefore, $\mathcal{J}(w - v, w - v) = 0$. Applying the interpolation estimate (5) we find that

$$\Upsilon := \left(h^{\rho-1} + t^2 + 1 \right) \|\boldsymbol{\gamma} - \boldsymbol{\gamma}^I\|_{0,h}^2 \lesssim h^{2k-2} \left(h^{\rho-1} + t^2 + 1 \right) \|\boldsymbol{\gamma}\|_{k-1,\Omega}^2$$

as $\rho > 1$ it follows that

$$\Upsilon \lesssim h^{2k-2} \left(\|\boldsymbol{\gamma}\|_{k-1,\Omega}^2 + t^2 \|\boldsymbol{\gamma}\|_{k-1,\Omega}^2 \right).$$

Combining this result and choosing $\rho = 2k - 1$ we obtain from Theorem 12 that,

$$|\!|\!|\boldsymbol{\theta} - \boldsymbol{\theta}_h, w - w_h|\!|\!| \lesssim h^{k-1} \left(\|\boldsymbol{\gamma}\|_{k-1,\Omega} + t \|\boldsymbol{\gamma}\|_{k-1,\Omega} + \|\boldsymbol{\gamma}\|_\Omega + \|\boldsymbol{\theta}\|_{k,\Omega} \right)$$
$$+ Osc(\boldsymbol{f}) + Osc(g).$$

The result follows from (37) and (38). □

6 Numerical results

In this section we will show some numerical results that illustrate the performance of the dGWOPIP formulation. Aiming at the calculation of the error and the numerical order of convergence we will consider the following slight modification of the solution given in [20]. If

$$w_1(x, y) = \frac{1}{3} x^3 (x-1)^3 y^3 (y-1)^3,$$
$$w_2(x, y) = y^3 (y-1)^3 x (x-1)(5x^2 - 5x + 1)$$
$$+ x^3 (x-1)^3 y (y-1)(5y^2 - 5y + 1),$$

it follows that

$$w(x, y) = \check{c} w_1(x, y) - t^2 \frac{8(\mu + \lambda)}{3(2\mu + \lambda)} \check{c} w_2(x, y),$$
$$\theta_1(x, y) = \check{c} y^3 (y-1)^3 x^2 (x-1)^2 (2x - 1),$$
$$\theta_2(x, y) = \check{c} x^3 (x-1)^3 y^2 (y-1)^2 (2y - 1),$$

solve the Reissner–Mindlin equation in $\Omega = (0, 1) \times (0, 1)$ with $\boldsymbol{f} = 0$ and

$$g = \frac{4(\mu+\lambda)\mu}{3(2\mu+\lambda)}\breve{c}\{12y(y-1)(5x^2-5x+1)[2y^2(y-1)^2+x(x-1)(5y^2-5y+1)] \\ +12x(x-1)(5y^2-5y+1)[2x^2(x-1)^2+y(y-1)(5x^2-5x+1)]\}.$$

Here, we introduce the constant $\breve{c}$ simply to stretch out the solution. In the numerical result we set $\breve{c} = 10^4$.

The dGWOPIP described above was implemented in the PZ environment [22]. We proceeded to check the convergence for both the symmetric and nonsymmetric versions and for lower-order polynomials (that is, $k = 2$) and higher-order polynomials. In our numerical simulations we set the Lamé coefficients $\lambda = \mu = 1$ and after some numerical tests we selected $\sigma_1 = 10$ for all cases, along with $\sigma_2 = 2 \times 10^3$ for lower-order and $\sigma_2 = 4$ for higher-order polynomials.

Unfortunately, due to the over-penalization, parameter σ_2 is a more complex choice than parameter σ_1. However, as the solution $(\boldsymbol{\theta}, w)$ of the Reissner–Mindlin equation converges to $(\nabla\Phi, \Phi)$ as t tends to zero, where Φ is the solution of the biharmonic problem, and considering the penalization parameters of the dG formulation for biharmonic equation [31–33], we see that here there is no power of k in these parameters. On the other hand, the power of h here may be greater. This suggests that in some way the penalization parameter σ_2 needs to "compensate" this lack/excess indicating that, for example, it should be large for $k = 2$ and small for $k = 4$.

We successively divide the domain using 2^{2L+1} triangles. Thus, if e_L denotes the error at the level of refinement L, the rate of convergence for this level is given by $r_L = \log\left(\frac{e_L}{e_{L-1}}\right) / \log(0.5)$.

In Tables 1 and 2 we investigate the convergence rates for the rotations and vertical displacements for $k = 2$. Table 1 shows the results for the symmetric formulation and Table 2 for the nonsymmetric formulation.

Table 1 Numerical convergence with the symmetric formulation for $k = 2$ with $t_1 = 10^{-1}$, $t_2 = 10^{-3}$ and $t_3 = 10^{-6}$

t	L	$L^2(\mathcal{T})$		$H^1(\mathcal{T})$		$L^2(\mathcal{T};\mathbb{R}^2)$		$H^1(\mathcal{T};\mathbb{R}^2)$	
		e_ω	r_L	e_ω	r_L	$e_{\boldsymbol{\theta}}$	r_L	$e_{\boldsymbol{\theta}}$	r_L
	3	1.249e−1	2.93	7.441e+0	1.86	7.439e−2	1.83	3.672e+0	0.724
t_1	4	1.546e−2	3.01	1.890e+0	1.98	1.882e−2	1.98	1.936e+0	0.924
	5	1.956e−3	2.98	4.724e−1	2.00	4.691e−3	2.00	9.810e−1	0.980
	3	1.140e−2	2.05	1.059e−1	1.71	1.053e−1	1.70	3.439e+0	0.779
t_2	4	2.442e−3	2.22	2.760e−2	1.94	2.748e−2	1.94	1.794e+0	0.939
	5	5.782e−4	2.08	6.616e−3	2.06	6.571e−3	2.06	9.068e−1	0.984
	3	1.144e−2	2.04	1.061e−1	1.71	1.055e−1	1.70	3.439e+0	0.779
t_3	4	2.478e−3	2.21	2.782e−2	1.93	2.771e−2	1.93	1.794e+0	0.939
	5	5.953e−4	2.06	6.804e−3	2.03	6.778e−3	2.03	9.064e−1	0.985

Table 2 Numerical convergence with the nonsymmetric formulation for $k = 2$ with $t_1 = 10^{-1}$, $t_2 = 10^{-3}$ and $t_3 = 10^{-6}$

t	L	$L^2(\mathcal{T})$		$H^1(\mathcal{T})$		$L^2(\mathcal{T};\mathbb{R}^2)$		$H^1(\mathcal{T};\mathbb{R}^2)$	
		e_ω	r_L	e_ω	r_L	$e_{\boldsymbol{\theta}}$	r_L	$e_{\boldsymbol{\theta}}$	r_L
	3	1.255e−1	2.92	7.441e+0	1.86	9.816e−2	1.58	3.738e+0	0.730
t_1	4	1.565e−2	3.00	1.890e+0	1.98	2.771e−2	1.82	1.956e+0	0.935
	5	1.987e−3	2.98	4.724e−1	2.00	7.262e−3	1.93	9.860e−1	0.988
	3	1.533e−2	1.74	1.314e−1	1.49	1.305e−1	1.49	3.536e+0	0.771
t_2	4	3.913e−3	1.97	3.860e−2	1.77	3.839e−2	1.77	1.825e+0	0.954
	5	8.996e−4	2.12	1.020e−2	1.92	1.014e−2	1.92	9.146e−1	0.997
	3	1.535e−2	1.74	1.315e−1	1.49	1.306e−1	1.49	3.536e+0	0.771
t_3	4	3.944e−3	1.96	3.878e−2	1.76	3.858e−2	1.76	1.825e+0	0.954
	5	9.313e−4	2.08	1.040e−2	1.90	1.035e−2	1.90	9.142e−1	0.998

Table 3 Numerical convergence with the symmetric formulation for $k = 4$ with $t_1 = 10^{-1}$, $t_2 = 10^{-3}$ and $t_3 = 10^{-6}$

t	L	$L^2(\mathcal{T})$		$H^1(\mathcal{T})$		$L^2(\mathcal{T};\mathbb{R}^2)$		$H^1(\mathcal{T};\mathbb{R}^2)$	
		e_ω	r_L	e_ω	r_L	$e_{\boldsymbol{\theta}}$	r_L	$e_{\boldsymbol{\theta}}$	r_L
	2	2.872e−2	4.19	1.235e+0	3.21	2.250e−2	3.24	7.108e−1	2.23
t_1	3	1.019e−3	4.82	8.953e−2	3.79	1.666e−3	3.76	1.064e−1	2.74
	4	3.398e−5	4.91	5.893e−3	3.93	1.094e−4	3.93	1.387e−2	2.94
	2	1.297e−2	2.99	4.797e−2	2.45	4.617e−2	2.39	8.074e−1	2.11
t_2	3	3.794e−4	5.10	2.344e−3	4.36	2.308e−3	4.32	1.177e−1	2.78
	4	1.261e−5	4.91	1.161e−4	4.34	1.141e−4	4.34	1.469e−2	3.00
	2	1.299e−2	2.99	4.800e−2	2.45	4.621e−2	2.39	8.081e−1	2.11
t_3	3	3.818e−4	5.09	2.349e−3	4.35	2.317e−3	4.32	1.182e−1	2.77
	4	1.418e−5	4.75	1.252e−4	4.23	1.244e−4	4.22	1.499e−2	2.98

According to Theorem 6 it follows that the convergence rate for the rotation in the $H^1(\mathcal{T};\mathbb{R}^2)$ norm needs to be equal to one for both formulations (symmetric and nonsymmetric). The numerical results clearly show this in the last columns of Tables 1 and 2.

For the symmetric version, we observe from Theorem 7 that the convergence rates for the rotation and vertical displacement in the L^2 norm must be quadratic. We can see from Table 1 that the numerical convergence rate for the rotation in L^2 is quadratic for all thickness values considered and that this value is attained for the vertical displacement for t_2 and t_3, while, for t_1 it is one order better. We note that a similar result was also obtained for the nonsymmetric version.

From the tables we can observe that the numerical rates of convergence in $H^1(\mathcal{T})$ for the vertical displacement for both formulations are very similar to those obtained for the rotation in $L^2(\mathcal{T};\mathbb{R}^2)$. To be precise, this rate tends to be quadratic, which is better (one order) than the theoretical result proved in Corollary 8.

In Table 3 we report the results for the symmetric formulation with $k = 4$. The last column shows that the rates of convergence for the rotation in $H^1(\mathcal{T};\mathbb{R}^2)$ are equal to $k-1$ for all thickness values considered. This is in agreement with the theoretical result present in Theorem 9. Once again the numerical rates for the displacement in $H^1(\mathcal{T})$ are better (by at least one order) than the theoretical result present in Corollary 10. Once again, similar results were obtained for the nonsymmetric version of the dGWOPIP.

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