Sparse Grids for High Dimensional Problems

Michael Griebel
griebel@ins.uni-bonn.de
Abteilung Wissenschaftliches Rechnen und Numerische Simulation
Institut für Angewandte Mathematik
Universität Bonn
Wegelerstr. 6
D-53115 Bonn (Germany)

Abstract

The numerical treatment of high(er) dimensional problems suffers in general from the so-called curse of dimensionality. In special cases, i.e. for special function classes, this exponential dependence of $O(n^{-r/d})$ of the achieved accuracy on the invested work n can be substantially reduced. Here, r denotes smoothness and d dimensionality. This is e.g. the case for spaces of functions with bounded mixed derivatives. The associated numerical schemes involve a series expansion in a multiscale basis for the one-dimensional problem. Then, a product construction and a proper truncation of the resulting d-dimensional expansion result in a so-called sparse grid approximation which is closely related to hyperbolic crosses. Here, depending on the respective problem and the one-dimensional multiscale basis used, a variety of algorithms for higher dimensional problems result which allow to break the curse of dimensionality, at least to some extent, and result in complexities of the order $O(n^{-r} \log(n)^{\alpha(d)})$. In special cases even $\alpha(d) = 0$ can be achieved. This is for example possible if the error is measured in the H_1 seminorm or if the different dimensions as well as their interactions are not equally important and dimension-adaptive strategies are used. The constant in these order estimates, however, is still dependent on d. It also reflects subtile details of the implementation of the respective numerical scheme. In general, the order constant grows exponentially with d. In some cases, however, it can be shown that it decays exponentially with d.This allows to treat quite high dimensional problems in moderate computing time.

We discuss such sparse grid algorithms for the numerical treatment of partial differential equations and related problems in higher dimensions for various applications.