Tutorials

Exercise 1

Let $\mathcal{T}_1, \mathcal{T}_2 \subset \bigcup \mathbb{T}$ be two regular triangulations of Ω into admissible triangles and let $\mathcal{N}_j$ denote the set of nodes of the triangulation $\mathcal{T}_j$ for $j = 1, 2$. Given $\mathcal{N}_1 = \mathcal{N}_2$, show that $\mathcal{T}_1 = \mathcal{T}_2$.

Exercise 2

Show that the admissible triangulations $\mathbb{T}$ with respect to a regular triangulation $\mathcal{T}_0$ of the domain $\Omega \subset \mathbb{R}^2$ are exactly the regular triangulations of Ω into admissible triangles

$$\mathbb{T} = \left\{ \mathcal{T} \text{ regular triangulation } : \mathcal{T} \subseteq \bigcup \mathbb{T} \right\}.$$

Exercise 3

Given two admissible triangulations $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{T}$ with respect to a regular triangulation $\mathcal{T}_0$ of the domain $\Omega \subset \mathbb{R}^2$, prove that their overlay $\mathcal{T}_1 \oplus \mathcal{T}_2$ satisfies

$$|\mathcal{T}_1 \oplus \mathcal{T}_2| + |\mathcal{T}_0| \leqslant |\mathcal{T}_1| + |\mathcal{T}_2|.$$

Exercise 4

Write an essay on overhead control, explaining the proof of the theorem of Binev-Dahmen-DeVore and its meaning.

Exercise 5

Write an essay on the optimality of adaptive algorithms. Explain especially what adaptivity is and what optimality means for an adaptive algorithm.

Exercise 6

Given Theorem 2.11 (comparison lemma), Theorem 2.12 (optimality) and their proofs, find an upper bound $\Theta < 1$ for θ_0 from Theorem 2.11. Recall that θ_0 already satisfies $0 < \theta_0 < 1$ and $\theta_0 \eta^2 \leqslant \eta^2(\mathcal{T} \backslash \hat{\mathcal{T}})$ for $\hat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$.

Exercise 7

Let (V, a) a real Hilbert space, $V_0 \subset V_1 \subset V_2 \subset \ldots \subset V$ a nested sequence of finite-dimensional subspaces and $F \in V^*$. Prove that the Galerkin solutions $(u_\ell)_{\ell \in \mathbb{N}_0}$, i.e. $a(u_\ell, \bullet) = F$ in V_ℓ for all $\ell \in \mathbb{N}_0$, converge strongly to u_∞. When is u_∞ the solution to $a(u, \bullet) = F$ in V?

Exercise 8

Let (V, a) a real Hilbert space, $V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \ldots \subseteq V$ a strictly nested sequence of finite-dimensional subspaces and $(\delta_j)_{j\in\mathbb{N}_0}$ a monotone decreasing sequence of positive real numbers with $\lim_{j\to\infty} \delta_j = 0$.
Prove the existence of $F \in V^*$ such that the Galerkin solutions $(u_\ell)_{\ell\in\mathbb{N}_0}$ satisfy $\|u - u_\ell\| = \delta_\ell$ for all $\ell \in \mathbb{N}_0$ and the solution u to $a(u, \bullet) = F$ in V.

Exercise 9

Given the Crouzeix-Raviart-FEM solution u_{CR}, write $\left[\frac{\partial u_{\mathrm{CR}}}{\partial \nu_E}\right]_E$ as a function of the right-hand side F and the local geometry for any $E \in \mathcal{E}(\Omega)$.

Exercise 10

Given $T \in \mathcal{T} \in \mathbb{T}$, $w \in P_1(T)$ and $z \in \mathcal{N}(T)$, compute $C_1, C_2 \approx 1$ such that

$$C_1\|\nabla w\|_{L^2(T)} \leqslant h_T^{-1}\|w - w(z)\|_{L^2(T)} \leqslant C_2\|\nabla w\|_{L^2(T)}.$$

Exercise 11

Given $w \in H^1(\omega_z)$ and $T \in \mathcal{T}(z)$, compute $C_1, C_2 \approx 1$ with

$$\|w\|_{L^2(\omega_z)} \leqslant C_1\|w\|_{L^2(T)} + C_2 h_z\|\nabla w\|_{L^2(\omega_z)}$$

in terms of Poincaré constants.

Exercise 12

Compute all the pairs $(u, f) \in H_0^1(\Omega) \times L^2(\Omega)$ with vanishing seminorm, $|(u, f)|_{\mathcal{A}_\sigma} = 0$. Prove that the seminorm is a norm on the vector space of all pairs $(u, f) \in H_0^1(\Omega) \times L^2(\Omega)$ with $f + \Delta u = 0$.

Exercise 13

Given a polygonal bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ with $J + 1$ connection components $\Gamma_0, \Gamma_1, \ldots, \Gamma_J$ of the boundary $\partial\Omega$, define

$$\begin{aligned}\mathrm{CR}^1_* := \{v_{\mathrm{CR}} \in \mathrm{CR}^1(\mathcal{T}) | \exists c_1, c_2, \ldots, c_J \in \mathbb{R}, c_0 := 0 \\ \forall j = 0, \ldots, J \quad \forall E \in \mathcal{E}(\Gamma_j) \quad v_{\mathrm{CR}}(\mathrm{mid}(E)) = c_j\}.\end{aligned}$$

Prove that $P_0(\mathcal{T}, \mathbb{R}^2) = \nabla_{\mathrm{NC}}\mathrm{CR}^1_*(\mathcal{T}) \ominus \mathrm{Curl}(S^1(\mathcal{T})/\mathbb{R})$.

Exercise 14

Prove (A1)–(A2) for the error estimator of the CR-NCFEM.

Exercise 15

Conclude the proof of ($A4_\varepsilon$) for any $\varepsilon > 0$ for the MFEM from Lemma 5.11 in the spirit of the lectures on CR-NCFEM.

Exercise 16

Compute the constant $C_{\mathrm{qo}} \approx 1$ of the quasi-orthogonality for the CRFEM from Lemma 4.11.

Exercise 17

Prove that for any $\mathcal{T} \in \mathbb{T}$ and $E \in \mathcal{E}$, it holds

$$|E| \|[p_{RT}]_E \cdot \tau_E\|^2_{L^2(E)} \lesssim \|p - p_{RT}\|^2_{L^2(\omega_E)}.$$