A Posteriori Error Estimation in Mixed Finite Element Methods for Signorini's Problem

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Abstract This paper presents a posteriori error estimates for Signorini's problem which is discretized via a mixed finite element approach. The error control relies on the estimation of the discretization error of an auxiliary problem given as a variational equation. The resulting error estimates capture the discretization error of the auxiliary problem, the geometrical error and the error given by the complementary condition. The estimates are applied within adaptive finite element schemes. Numerical results confirm the applicability of the theoretical findings.

1 Introduction

The aim in this paper is to derive error estimates for mixed finite element discretization schemes for Signorini's problem, which plays an import role in mechanical engineering, cf. [6, 7, 14]. The mixed discretization is based on an approach introduced by Haslinger et al. in [8, 9, 10, 11, 13]. A saddle point formulation is used where the geometrical contact condition is captured by a Lagrange multiplier. The constraint for the Lagrange multiplier is a sign condition and, therefore, simpler than the original contact condition. However, the multiplier is an additional variable which also has to be discretized. The discretization approach is originally developed for lower-order finite elements. However, it can be extended to higher-order finite elements, cf. [17].

Modern discretization schemes usually include a posteriori error control and adaptivity. To derive an error estimation, we seize a suggestion in [4] for the obstacle problem, where a certain auxiliary problem is considered. We will extend this approach to Signorini's problem and, in particular, to the discretization schemes given by the mixed variational formulation. We obtain error bounds which capture the dis-

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cretization error of the auxiliary problem, the geometrical error and the error given by the complementary condition. Furthermore, we apply the estimates within adaptive schemes.
A posteriori error estimates based on the primal, non-mixed formulation are proposed in [2, 5, 18] for the obstacle problem and in [12] for Signorini's problem. In [20, 21], estimates for mixed formulations are introduced for the mortar approach. In [16], similar techniques of this work are applied to a simplified Signorini problem. In particular higher-order finite elements are discussed. Furthermore, the results can be applied to time-dependent problems, cf. [3].

2 Signorini's problem

Signorini's problem describes the deformation of a material body which gets in contact with a rigid foundation. The body is represented by a domain $\Omega \subset \mathbb{R}^k$, $k \in \{2,3\}$, with a sufficiently smooth boundary $\Gamma := \partial\Omega$ and is clamped at a boundary part which is represented by a closed set $\Gamma_D \subset \Gamma$ with positive measure. The boundary part of the body which possibly gets in contact with the foundation is described by an open set Γ_C. We assume that $\overline{\Gamma}_C \subsetneq \Gamma\backslash\Gamma_D$ and $\Gamma_N := \Gamma\backslash(\Gamma_D \cup \overline{\Gamma}_C)$. Volume and surface forces act on the body. They are described by functions $f \in L^2(\Omega;\mathbb{R}^k)$ and $q \in L^2(\Gamma_N;\mathbb{R}^k)$. The resulting deformation is described by a displacement field $v \in H^1(\Omega;\mathbb{R}^k)$ with linearized strain tensor $\varepsilon(v) := \frac{1}{2}(\nabla v + (\nabla v))^\top$. The stress tensor describing a linear-elastic material law is defined as $\sigma(v)_{ij} := \mathscr{C}_{ijkl}\varepsilon(v)_{kl}$, where $\mathscr{C}_{ijkl} \in L^\infty(\Omega)$ with $\mathscr{C}_{ijkl} = \mathscr{C}_{jilk} = \mathscr{C}_{klij}$ and $\mathscr{C}_{ijkl}\tau_{ij}\tau_{kl} \geq \kappa\tau_{ij}^2$ for all $\tau \in L^2(\Omega;\mathbb{R}^{k\times k})$ with $\tau_{ij} = \tau_{ji}$ and a constant $\kappa > 0$. We set $H_D^1(\Omega) := \{v \in H^1(\Omega;\mathbb{R}^k) \mid \gamma_{|\Gamma_D}(v_i) = 0,\ i = 1,\dots,k\}$ for the trace operator $\gamma \in L(H^1(\Omega), L^2(\Gamma))$ and define $(\sigma_n(u))_i := \sigma_{ij}(u)n_j$, $u_n := u_i n_i$, $\sigma_{nn}(u) := \sigma_{ij}(u)n_i n_j$, $\sigma_{nt}(u) := \sigma_n(u) - \sigma_{nn}(u)n$ with outer normal n. Signorini's problem is thus to find a displacement field u such that

$$\begin{aligned} -\operatorname{div}\sigma(u) &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \Gamma_D, \\ \sigma_n(u) &= q \text{ on } \Gamma_N, \\ u_n - g \leq 0,\ \sigma_{nn}(u) \leq 0,\ \sigma_{nn}(u)(u_n - g) &= 0,\ \sigma_{nt}(u) = 0 \text{ on } \Gamma_C. \end{aligned}$$

Here, the function $g \in H^{1/2}(\Gamma_C)$ is the usual linearized gap function describing the surface of the rigid foundation, cf. [14].
In this paper, the following notational conventions are used. The space $H^{-1/2}(\Gamma_C)$ denotes the topological dual space of $H^{1/2}(\Gamma_C)$ with norms $\|\cdot\|_{-1/2,\Gamma_C}$ and $\|\cdot\|_{1/2,\Gamma_C}$, respectively. Let $(\cdot,\cdot)_{0,\omega}$, $(\cdot,\cdot)_{0,\Gamma'}$ be the usual L^2-scalar products on $\omega \subset \Omega$ and $\Gamma' \subset \Gamma$, respectively, for vector and matrix-valued functions. We define $\|v\|_{0,\omega}^2 := (v,v)_{0,\omega}$ and omit the subscript ω whenever $\omega = \Omega$. Moreover, we state the energy norm $\|v\|^2 := (\sigma(v),\varepsilon(v))_0$, which is equivalent to the usual norm $\|\cdot\|_1$

in $H^1(\Omega;\mathbb{R}^k)$ due to Korn's inequality. We define $\gamma_N \in L(H^1_D(\Omega), L^2(\Gamma_N,\mathbb{R}^k))$ as $\gamma_N(v)_i = \gamma_{|\Gamma_N}(v_i)$ and $\gamma_{Cn} \in L(H^1_D(\Omega), H^{1/2}(\Gamma_C))$ as $\gamma_{Cn}(v) := \gamma_{|\Gamma_C}(v_i)n_i$ which is surjective due to the assumptions on Γ_C, cf. [14]. Furthermore, we define the norm $\|\cdot\|'_{1/2,\Gamma_C}$ by $\|w\|'_{1/2,\Gamma_C} := \inf_{v\in H^1(\Omega,\Gamma_D),\,\gamma_{Cn}(v)=w}\|v\|$, which is equivalent to the $\|\cdot\|_{1/2,\Gamma_C}$-norm. The negative part v_- of a function v is defined as $v_-(x) := v(x)$ if $v(x) \le 0$, $v_- := 0$ otherwise.

3 Mixed variational formulation of Signorini's problem and its discretization

It is well-known, that the solution of Signorini's problem u is also a solution $u \in K := \{v \in H^1_D(\Omega) \mid \gamma_{Cn}(v) \le g\}$ of the variational inequality

$$(\sigma(u), \varepsilon(v-u))_0 \ge (f, v-u)_0 + (q, \gamma_N(v-u))_{0,\Gamma_N}$$

for all $v \in K$. The inequality above is fulfilled if and only if u is a minimizer of the functional $E(v) := \frac{1}{2}(\sigma(v),\varepsilon(v))_0 - (f,v)_0 - (q,\gamma_N(v))_{0,\Gamma_N}$ in K. The functional E is strictly convex, continuous and coercive due to Cauchy's and Korn's inequalities. This implies the existence of a unique minimizer u.
Given the Lagrange functional $\mathscr{L}(v,\mu) := E(v) + \langle \mu, \gamma_{Cn}(v) - g\rangle$ on $H^1_D(\Omega) \times H^{-1/2}_+(\Gamma_C)$, the Hahn-Banach theorem yields

$$E(u) = \inf_{v\in H^1_D(\Omega)} \sup_{\mu\in H^{-1/2}_+(\Gamma_C)} \mathscr{L}(v,\mu). \tag{1}$$

for $H^{1/2}_+(\Gamma_C) := \{w \in H^{1/2}(\Gamma_C) \mid w \ge 0\}$ and $H^{-1/2}_+(\Gamma_C) := \{\mu \in H^{-1/2}(\Gamma_C) \mid \forall w \in H^{1/2}_+(\Gamma_C) : \ \langle \mu, w\rangle \ge 0\}$. Thus, u is a minimizer of E, whenever $(u,\lambda) \in H^1_D(\Omega) \times H^{-1/2}_+(\Gamma_C)$ is a saddle point of $\mathscr{L}$. The existence of a unique saddle point is guaranteed, if there exists a constant $\alpha > 0$ such that the inf-sup condition $\alpha\|\mu\|_{-1/2,\Gamma_C} \le \sup_{v\in H^1_D(\Omega),\,\|v\|_1=1}\langle \mu, \gamma_{Cn}(v)\rangle$ for all $\mu \in H^{-1/2}(\Gamma_C)$, cf. [14]. In fact, it follows from the closed range theorem and the surjectivity of γ_{Cn}, that the inf-sup condition is valid. Due to the stationary condition, $(u,\lambda) \in H^1_D(\Omega) \times H^{-1/2}_+(\Gamma_C)$ is a saddle point of $\mathscr{L}$, if and only if it fulfills the mixed variational formulation

$$\begin{aligned} &(\sigma(u),\varepsilon(v))_0 = (f,v)_0 + (q,\gamma_N(v))_0 - \langle \lambda, \gamma_{Cn}(v)\rangle, \\ &\langle \mu - \lambda, \gamma_{Cn}(u) - g\rangle \le 0 \end{aligned} \tag{2}$$

for all $v \in H^1_D(\Omega)$ and $\mu \in H^{-1/2}_+(\Gamma_C)$.
A finite element discretization based on quadrangles or hexahedrons is given in the following way: Let $\mathscr{T}_h$ and $\mathscr{T}_{C,H}$ be finite element meshes of Ω and Γ_C with mesh sizes h and H, respectively. Furthermore, let $\Psi_T : [-1,1]^k \to T \in \mathscr{T}_h$, $\Psi_{C,T} : [-1,1]^{k-1} \to T \in \mathscr{T}_{C,H}$ be bijective transformations. The space of bilinear

or trilinear functions on the reference element $[-1,1]^k$ is denoted by Q_k^1. We set $V_h := \{v \in H_D^1(\Omega) \mid \forall T \in \mathscr{T}_h : \ v_{|T} \circ \Psi_T \in (Q_k^1)^k\}$ and $M_H := \{\mu \in L^2(\Gamma_C) \mid \forall T \in \mathscr{T}_{C,H} : \ \mu_{|T} \equiv \text{constant}\}$. For $M_H^+ := \{\mu_H \in M_H \mid \mu_H \geq 0\}$ the discrete problem is to find $(u_h, \lambda_H) \in V_h \times M_H^+$ such that

$$\begin{aligned} &(\sigma(u_h), \varepsilon(v_h))_0 = (f, v_h)_0 + (q, \gamma_N(v_h))_{0,\Gamma_N} - (\lambda_H, \gamma_{Cn}(v_h))_{0,\Gamma_C}, \\ &(\mu_H - \lambda_H, \gamma_{Cn}(u_h) - g)_{0,\Gamma_C} \leq 0 \end{aligned} \tag{3}$$

for all $v_h \in V_h$ and all $\mu_H \in M_H^+$. To ensure the existence of a unique solution of (3), we have to verify a discrete version of inf-sup condition. To guarantee the discretization scheme to be stable, the corresponding constant has to be independent of h and H. This can be achieved by using meshes $\mathscr{T}_h$ and $\mathscr{T}_{C,H}$ which imply sufficiently small quotients h/H for $T \in \mathscr{T}_h$, $T_C \in \mathscr{T}_{C,H}$ and $T \subset T_C$, cf. [13]. In our implementation, we ensure $h/H \leq 0.5$, using hierarchical meshes with $\mathscr{T}_{C,H}$ being sufficiently coarser than $\mathscr{T}_h$.

4 Reliable a posteriori error estimates

The basic idea for the estimation of $\|u - u_h\|$ is to consider the following auxiliary problem: Find $u_0 \in H_D^1(\Omega)$ such that

$$(\sigma(u_0), \varepsilon(v))_0 = (f, v)_0 + (q, \gamma_N(v))_{0,\Gamma_N} - (\lambda_H, \gamma_{Cn}(v))_{0,\Gamma_C} \tag{4}$$

for all $v \in H_D^1(\Omega)$. Obviously, the solution u_0 of (4) exists and is unique. Moreover, u_h is a finite element solution of (4). We will show that $\|u - u_h\| \lesssim \|u_0 - u_h\| + \mathscr{R}$ where $\mathscr{R}$ are some remainder terms given below. Here, $\lesssim$ abbreviates $\leq$ up to some constant which is independent of h and H. The idea is to use an arbitrary error estimator η_0 for problem (4) and to set $\eta := \eta_0 + \mathscr{R}$. We then obtain $\|u - u_h\| \lesssim \eta$. In principle, each error estimator known from the literature of variational equations can be used, see [1, 19] for an overview. In the following, we will make use of the inequalities,

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \qquad \text{for } a, b \in \mathbb{R},\ \varepsilon > 0, \tag{5}$$

$$(a+b)^2 \leq 2a^2 + 2b^2 \qquad \text{for } a, b \in \mathbb{R}, \tag{6}$$

$$x \leq a + b^{1/2} \qquad \text{for } x, a, b > 0,\ x^2 \leq ax + b. \tag{7}$$

Lemma 1. *There holds*

$$\|u - u_h\|^2 \leq \|u_0 - u_h\| \|u - u_h\| + \langle \lambda, \gamma_{Cn}(u_h) - g \rangle.$$

Proof. Since $0, 2\lambda \in H_+^{-1/2}(\Gamma_C)$ and $0, 2\lambda_H \in M_H^+$, we have $\langle \lambda, \gamma_{Cn}(u) - g \rangle = (\lambda_H, \gamma_{Cn}(u_h) - g)_{0,\Gamma_C} = 0$. Furthermore, there holds $(\lambda_H, \gamma_{Cn}(u) - g)_{0,\Gamma_C} \leq 0$. Using

Cauchy's inequality, we obtain

$$\begin{aligned}\|u-u_h\|^2 &= (\sigma(u-u_0),\varepsilon(u-u_h))_0 + (\sigma(u_0-u_h),\varepsilon(u-u_h))_0\\ &\le (\lambda_H,\gamma_{Cn}(u-u_h))_{0,\Gamma_C} - \langle\lambda,\gamma_{Cn}(u-u_h)\rangle + \|u_0-u_h\|\,\|u-u_h\|\\ &= (\lambda_H,\gamma_{Cn}(u)-g)_{0,\Gamma_C} - \langle\lambda,g-\gamma_{Cn}(u_h)\rangle + \|u_0-u_h\|\,\|u-u_h\|\\ &\le \langle\lambda,\gamma_{Cn}(u_h)-g\rangle + \|u_0-u_h\|\|u-u_h\|.\end{aligned}$$

Theorem 1. *Let $\varepsilon>0$, thus*

$$\|u-u_h\| \le (1+\varepsilon)\|u_0-u_h\| + (1+\frac{1}{4\varepsilon})\|(g-\gamma_{Cn}(u_h))_-\|'_{1/2,\Gamma_C} + |(\lambda_H,(g-\gamma_{Cn}(u_h))_-)_{0,\Gamma_C}|^{1/2}.$$

Proof. Let $d\in W := \{v\in H^1_D(\Omega)\mid \gamma_{Cn}(v)=(g-\gamma_{Cn}(u_h))_-\}$ with $\|d\|=\inf_{v\in W}\|v\|$. Thus, we have $\|d\|=\|(g-\gamma_{Cn}(u_h))_-\|'_{1/2,\Gamma_C}$. Moreover, there holds $g-\gamma_{Cn}(u_h)-\gamma_{Cn}(d) = g-\gamma_{Cn}(u_h)-(g-\gamma_{Cn}(u_h))_- \ge 0$ on Γ_C and therefore $g-\gamma_{Cn}(u_h)-\gamma_{Cn}(d)\in H^{1/2}_+(\Gamma_C)$. Hence, we obtain

$$\begin{aligned}\langle\lambda,\gamma_{Cn}(u_h)-g\rangle &= -\langle\lambda,g-\gamma_{Cn}(u_h)-\gamma_{Cn}(d)\rangle - \langle\lambda,\gamma_{Cn}(d)\rangle\\ &\le (\sigma(u),\varepsilon(d))_0 - (f,d)_0 - (q,\gamma_N(d))_{0,\Gamma_N}\\ &= (\sigma(u-u_h),\varepsilon(d))_0 + (\sigma(u_h),\varepsilon(d))_0 - (f,d)_0 - (q,\gamma_N(d))_{0,\Gamma_N}\\ &\le \|u-u_h\|\|d\| + (\sigma(u_h-u_0),\varepsilon(d))_0 - (\lambda_H,\gamma_{Cn}(d))_{0,\Gamma_C}\\ &\le \|u-u_h\|\|(g-\gamma_{Cn}(u_h))_-\|'_{1/2,\Gamma_C}\\ &\quad + \|u_0-u_h\|\|(g-\gamma_{Cn}(u_h))_-\|'_{1/2,\Gamma_C} + |(\lambda_H,(g-\gamma_{Cn}(u_h))_-)_{0,\Gamma_C}|.\end{aligned}$$

Consequently, Lemma 1 implies

$$\begin{aligned}\|u-u_h\|^2 &\le \|u_0-u_h\|\,\|u-u_h\| + \langle\lambda,\gamma_{Cn}(u_h)-g\rangle\\ &\le \|u-u_h\|(\|u_0-u_h\| + \|(g-\gamma_{Cn}(u_h))_-\|'_{1/2,\Gamma_C}) +\\ &\quad \|u_0-u_h\|\|(g-\gamma_{Cn}(u_h))_-\|'_{1/2,\Gamma_C} + |(\lambda_H,(g-\gamma_{Cn}(u_h))_-)_{0,\Gamma_C}|.\end{aligned}$$

The application of (5) and (7) yields

$$\begin{aligned}\|u-u_h\| &\le \|u_0-u_h\| + \|(g-\gamma_{Cn}(u_h))_-\|'_{1/2,\Gamma_C}\\ &\quad + (\|u_0-u_h\|\|(g-\gamma_{Cn}(u_h))_-\|'_{1/2,\Gamma_C} + |(\lambda_H,(g-\gamma_{Cn}(u_h))_-)_{0,\Gamma_C}|)^{1/2}\\ &\le (1+\varepsilon)\|u_0-u_h\| + (1+\frac{1}{4\varepsilon})\|(g-\gamma_{Cn}(u_h))_-\|'_{1/2,\Gamma_C}\\ &\quad + |(\lambda_H,(g-\gamma_{Cn}(u_h))_-)_{0,\Gamma_C}|^{1/2}.\end{aligned}$$

Corollary 1. *Let $\eta_0>0$ with $\|u-u_h\|\lesssim\eta_0$ and*

$$\eta^2 := \eta_0^2 + \|(g-\gamma_{Cn}(u_h))_-\|^2_{1/2,\Gamma_C} + |(\lambda_H,(g-\gamma_{Cn}(u_h))_-)_{0,\Gamma_C}|. \tag{8}$$

Thus, there holds $\|u-u_h\| \lesssim \eta$.

Proof. Theorem 1, (6) and the equivalence of $\|\cdot\|_{1/2,\Gamma_C}$ and $\|\cdot\|'_{1/2,\Gamma_C}$ yield the assertion.

Remark 1. The terms in the error estimate of Corollary 1 correspond to typical error sources in Signorini's problem: $\|(g-\gamma_{Cn}(u_h))_-\|_{1/2,\Gamma_C}$ measures the error in the geometrical contact condition and $|(\lambda_H,(g-\gamma_{Cn}(u_h))_-)_{0,\Gamma_C}|$ describes the error in the complementary condition.

Remark 2. To calculate η in (8) we have to determine $\|(g-\gamma_{Cn}(u_h))_-\|_{1/2,\Gamma_C}$. Since $\gamma_{Cn}(u_h)$ is piecewise polynomial, we have $(g-\gamma_{Cn}(u_h))_- \in H^1(\Gamma_C)$ for $g \in H^1(\Gamma_C)$. By interpolation results, we get $\|(g-\gamma_{Cn}(u_h))_-\|^2_{1/2,\Gamma_C} \lesssim \|(g-\gamma(u_h))_-\|_{0,\Gamma_C}\|(g-\gamma(u_h))_-\|_{1,\Gamma_C}$, cf. [15, Thm.7.7.].

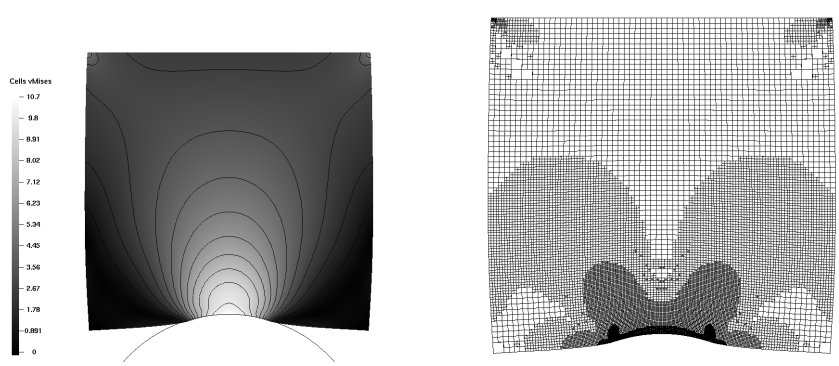
Corollary 2. *Let the assumptions of Corollary 1 be fulfilled. Hence, there holds*

$$\|u-u_h\| + \|\lambda-\lambda_H\|_{-1/2,\Gamma_C} \lesssim \eta.$$

Proof. Using $\|\lambda-\lambda_H\|_{-1/2,\Gamma_C} \lesssim \|u-u_0\|$, cf. [16], we obtain

$$\begin{aligned}\|u-u_h\| + \|\lambda-\lambda_H\|_{-1/2,\Gamma_C} &\lesssim \|u-u_h\| + \|u-u_0\| \\ &\lesssim 2\|u-u_h\| + \|u_0-u_h\| \lesssim \eta + \eta_0 \lesssim \eta.\end{aligned}$$

Remark 3. Since we do not use specific properties of quadrangles or hexahedrons, all results are also valid for discretizations based on triangles or tetrahedrons.



(a) (b)

Fig. 1 **(a)** Solution u of Signorini's problem with an obstacle function g, **(b)** adaptive mesh.

5 Numerical results

In our numerical experiments, we study Signorini's problem with $\Omega := (-1,1)^2$, $\Gamma_C := (-1,1)\times\{-1\}, \Gamma_D := [-1,1]\times\{1\}$, $f := 0$ and $q := 0$. The rigid foundation is given by $\{(x_1,(1-x_1^2)^{1/2}-1.85)\in\mathbb{R}^2 \mid x_1\in[-1,1]\}$. We use Hooke's law for plain stress with Young's modulus $E := 70kN/mm^2$ and Poisson's number $\nu := 0.33$. In Figure 1(a), the deformation caused by the contact with the rigid foundation is shown. Furthermore, the von-Mises-stress $\sigma_v := (\sigma_{11}+\sigma_{22}-\sigma_{11}\sigma_{22}+3\sigma_{12}^2)^{1/2}$ is depicted. We see high stress concentrations at the contact zone. An adaptive mesh is shown in Figure 1(b). We use a standard residual error estimator η_0, which is defined by $\eta_0^2 := \sum_{T\in\mathscr{T}_h}(h_T^2R_{0,T}^2+\sum_{e\in\mathscr{E}_T}h_eR_{0,e}^2)$ with

$$R_{0,T} := \|f+\operatorname{div}\sigma(u_h)\|_{0,T},\ T\in\mathscr{T}_h,$$

$$R_{0,e} := \begin{cases} \frac{1}{2}\|[\sigma_n(u_h)]\|_{0,e}, & e\in\mathscr{E}^\circ,\\ \|\sigma_n(u_h)-q\|_{0,e}, & e\in\mathscr{E}_N\\ \|\sigma_{nn}(u_h)+\lambda_H\|_{0,e}+\|\sigma_{nt}(u_h)\|_{0,e}, & e\in\mathscr{E}_C,\end{cases}$$

where $\mathscr{E}_T$ is the set of edges of $T\in\mathscr{T}_h$, $\mathscr{E}^0$ contains the internal edges and $\mathscr{E}_N$ and $\mathscr{E}_C$ the edges on Γ_N and Γ_C, respectively. As usual, $[\cdot]_e$ denotes the jump across an edge $e\in\mathscr{E}^\circ$.

In the adaptive mesh, we find local refinements towards both ends of the contact zone and towards two end points of the dirichlet boundary part Γ_D. Moreover, there are local refinements within the contact zone.

In Figure 2, the estimated error obtained by adaptive and uniform refinements are depicted. As the diagram shows, the estimated convergence rate is nearly $\mathscr{O}(h^{1/2})$ for uniform refinements which corresponds to a priori results, cf. [13]. For adaptive refinements, we obtain an optimal algebraic convergence rate $\mathscr{O}(h)$.

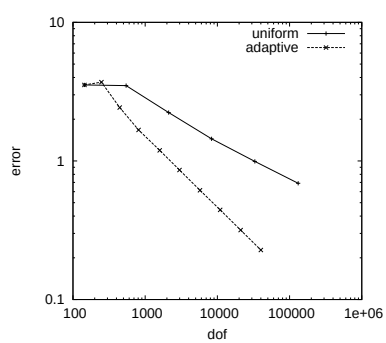


Fig. 2 Convergence rates.

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