EXPLICIT BRILL-NOETHER-PETRI GENERAL CURVES

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Abstract. Let $p_1, \dots, p_9$ be the points in $\mathbb{A}^2(\mathbb{Q}) \subset \mathbb{P}^2(\mathbb{Q})$ with coordinates

$$(-2,3), (-1,-4), (2,5), (4,9), (52,375), (5234,37866), (8,-23), (43,282), \left(\frac{1}{4}, -\frac{33}{8}\right)$$

respectively. We prove that, for any genus g, a plane curve of degree $3g$ having a g-tuple point at $p_1, \dots, p_8$, and a $(g-1)$-tuple point at p_9, and no other singularities, exists and is a Brill-Noether general curve of genus g, while a general curve in that g-dimensional linear system is a Brill-Noether-Petri general curve of genus g.

1. Introduction.

The Petri Theorem asserts that for a general curve C of genus $g > 1$, the multiplication map

$$\mu_{0,L} : H^0(C, L) \otimes H^0(C, \omega_C \otimes L^{-1}) \to H^0(C, \omega_C)$$

is injective for every line bundle L on C. While the result, which immediately implies the Brill-Noether Theorem, holds for almost every curve $[C] \in \mathcal{M}_g$, so far no explicitly computable examples of smooth curves of arbitrary genus satisfying this theorem have been known. Indeed, there are two types of known proofs of the Petri Theorem. These are: the proofs by degeneration due to Griffiths-Harris [12], Gieseker [11], and Eisenbud-Harris [8], or the recent proof using tropical geometry [5], which by their nature, shed little light on the explicit smooth curves which are Petri general; and the elegant proof by Lazarsfeld [15], asserting that every hyperplane section of a polarised $K3$ surface (X, H) of degree $2g-2$, such that the hyperplane class $[H]$ is indecomposable is a Brill-Noether general curve, while a general curve in the linear system $|H|$ is Petri general. However, there are no known concrete examples of polarised $K3$ surfaces of arbitrary degree satisfying the requirement above. It is a non-trivial instance of a theorem of André [1], [16], that there exists polarised $K3$ surfaces of degree $2g-2$ over a number field, having Picard number one. While the above mentioned results are all in characteristic zero, it has been observed by Welters [22] that a minor modification of the proof in [8], proves the Petri Theorem in positive characteristic as well.

This work originated from the paper [2], where a number of explicit families of curves lying on the projective plane or on a ruled elliptic surface were constructed. For these curves the question of whether they satisfy the Brill-Noether-Petri condition arises naturally. Among these families one, already studied by du Val [7], is particularly interesting. Curves in this family naturally sit on the blow-up of the projective plane in nine points.

The aim of this paper is to show that, by using the methods from [15] and [19], coupled with Nagata's classical results [17] on the effective cone of the blown-up projective plane, these curves provide *explicit* examples of Brill-Noether-Petri general curves of any genus. They also provide *computable* examples of Brill-Noether general curves of any genus.

In [21], Treibich sketches a construction of Brill-Noether (but not necessarily Petri) curves of any given genus.

We set the notation we are going to use throughout this note. We denote by S' the blow-up of $\mathbb{P}^2$ at nine points $p_1, \dots, p_9$ which are $3g$-*general* (see the Definition 2.2 below), and we let $E_1, \dots, E_9$ be the exceptional curves of this blow-up. We have that

$$-K_{S'} \sim 3\ell - E_1 - \cdots - E_9\,,$$

where ℓ is the proper transform of a line in $\mathbb{P}^2$. As the points p_i are general, there exists a unique curve

$$J' \in |-K_{S'}| \tag{1.1}$$

which corresponds to a smooth plane cubic passing through the p_i's. We next consider the linear system on S'

$$L_g := \big|3g\ell - gE_1 - \cdots - gE_8 - (g-1)E_9\big|.$$

This is a g-dimensional system whose general element is a smooth genus g curve. Since for each curve $C' \in L_g$, we have that $C' \cdot J' = 1$, the point $\{p\} := C' \cap J'$ is independent of C' and is thus a base point of the linear system L_g. Precisely, $p \in J'$ is determined by the equation $\mathcal{O}_{J'}\big(gp_1 + \ldots + gp_8 + (g-1)p_9 + p\big) = \mathcal{O}_{J'}(3g\ell_{|J'})$.

Let $\sigma : S \longrightarrow S'$ be the blow-up of S' at p, We denote again by $E_1, \dots, E_9$ the inverse images of the exceptional curves on S' and by E_{10} the exceptional curve of σ. We let J be the strict transform of J' and C the strict transform of C', so that we can write

$$\begin{aligned} &-K_S \sim J \sim 3\ell - E_1 - \cdots - E_{10}, \\ &C \sim 3g\ell - gE_1 - \cdots - gE_8 - (g-1)E_9 - E_{10}\,, \\ &C \cdot J = 0\,. \end{aligned} \tag{1.2}$$

The linear system $|C|$ is base-point-free and maps S to a surface $\overline{S} \subset \mathbb{P}^g$ having canonical sections and a single elliptic singularity resulting from the contraction of J. As we mentioned above, this linear system was first studied by Du Val in [7].

Definition 1.1. A curve in the linear system $|C|$ as in (1.2) is called a Du Val curve.

In [2] it is proved that Brill-Noether-Petri general curves whose Wahl map

$$\nu : \bigwedge^2 H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 3})$$

is not surjective, are hyperplane sections of a $K3$ surface, or limits of such, and it is shown that one such limit could be the surface $\overline{S}$ we just described. This is one of the reasons why it is interesting to determine whether Du Val curves are Brill-Noether-Petri general. In this note we answer this question in the affirmative.

Theorem 1.2. *A general Du Val curve $C \subset S$ satisfies the Brill-Noether-Petri Theorem.*

This, on the one hand, gives a strong indication that the result in [2] is the best possible. On the other hand, and more importantly, Theorem 1.2 provides a very concrete example of a Brill-Noether-Petri curve for every value of the genus. Since the locus of $3g$-general sets of 9 points is Zariski open in the symmetric product $(\mathbb{P}^2)^{(9)}$, we can choose $p_1, \dots, p_9$ to have rational coefficients. Then Theorem 1.2 implies the following result, which answers a question raised by Harris-Morrison in [14] p.343, in connection with the Lang-Mordell Conjecture:

Corollary 1.3. For every g, there exist smooth Brill-Noether-Petri general curves C of genus g defined over $\mathbb{Q}$.

In Section 5 we make Theorem 1.2 and Corollary 1.3 explicit by proving that the following set of 9 points in $\mathbb{A}^2(\mathbb{Q}) \subset \mathbb{P}^2(\mathbb{Q})$, lying on the elliptic curve $y^2 = x^3 + 17$, is $3g$-general, for every g, in particular they can be used to construct Petri general curves of any genus:

$$(-2,3),(-1,-4),(2,5),(4,9),(52,375),(5234,37866),(8,-23),(43,282),\Big(\frac{1}{4},-\frac{33}{8}\Big)$$

We give two proofs of Theorem 1.2. The first one, in Section 3, uses [17] and holds for every $3g$-general set of points $p_1,\ldots,p_9$ in $\mathbb{P}^2$. The second proof, presented in Section 4, works only for a general sets of points $p_1,\ldots,p_9$, and relies on the theory of limit linear series and the proof of the Gieseker-Petri theorem in [8].

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2. Preliminaries

As in the introduction, we denote by S' the blow-up of $\mathbb{P}^2$ at nine points $p_1,\ldots,p_9$ and let $E_1,\ldots,E_9$ be the corresponding exceptional curves on S'. We then consider the anticanonical elliptic curve $J' \subset S'$ as in (1.1).

Definition 2.1. The points $p_1,\ldots,p_9$ are said to be k-*Cremona general* for a positive integer k, if there exists a single cubic curve passing through them and the surface S' carries no effective (-2)-curve of degree at most k. The points $p_1,\ldots,p_9$ are *Cremona general*, if they are k-Cremona general for any $k>0$.

Nagata [17] has obtained an explicit characterization of the sets of Cremona special sets, which we now explain. A permutation $\sigma \in \mathfrak{S}_9$ gives rise to an isomorphism $\sigma : \mathrm{Pic}(S') \to \mathrm{Pic}(S')$ induced by permuting the curves $E_1,\ldots,E_9$. We define the following divisors on S':

$$\mathfrak{A}_1 := \ell - E_1 - E_2 - E_3, \quad \mathfrak{A}_2 := 2\ell - E_1 - \cdots - E_6,$$
$$\mathfrak{A}_3 := 3\ell - 2E_1 - E_2 - \cdots - E_8 \ \text{ and } \ \mathfrak{B} = 3\ell - \sum_{i=1}^{9} E_i.$$

It is shown in [17] Proposition 9 and Proposition 10, that a set $p_1,\ldots,p_9$ consisting of distinct points is k-Cremona general if and only if the following conditions are satisfied for all permutations $\sigma \in \mathfrak{S}_9$:

$$\big|\sigma(n\mathfrak{B}+\mathfrak{A}_{\mathrm{i}})\big| = \emptyset, \quad \text{for all } n \le \frac{k-i}{3} \ \text{ and } \ i=1,2,3. \tag{2.1}$$

Since the virtual dimension of each linear system $\big|n\mathfrak{B}+\mathfrak{A}_i\big|$ is negative, clearly a very general set of points $p_1,\ldots,p_9$ is Cremona general.

We now recall the following classical definition:

Definition 2.2. The points $p_1, \dots, p_9$ are said to be *k-Halphen special* if there exists a plane curve of degree $3d \leq k$ having points of multiplicity d at $p_1, \dots, p_9$ and no further singularities. We say that the set $p_1, \dots, p_9$ is *k-general* if it is simultaneously k-Cremona and k-Halphen general.

The locus of k-special points defines a proper Zariski closed subvariety of the symmetric product $(\mathbb{P}^2)^{(9)}$. If $p_1, \dots, p_9$ is a k-Halphen special set, then $\dim |dJ'| = 1$, thus $S' \to \mathbb{P}^1$ is an elliptic surface with a fibre of multiplicity $d \leq \frac{k}{3}$. If $\mathrm{Halph}(k) \subset (\mathbb{P}^2)^{(9)}$ denotes the locus of k-special Halphen sets, then the quotient $\mathrm{Halph}(k)//SL(3)$ is a variety of dimension 9, see [4] Remark 2.8.

The relevance of both Definitions 2.1 and 2.2 comes to the fore in the following result, which is essentially due to Nagata [17], see also [6].

Proposition 2.3. *The points $p_1, \dots, p_9$ are k-*general *if and only if, for every effective divisor D on S' such that*

$$D \sim d\ell - \sum_{i=1}^{9} \nu_i E_i\,, \quad \nu_i \geq 0\,, \qquad \textit{and} \quad D \cdot J' = 0, \tag{2.2}$$

where $d \leq k$, one has $D = mJ'$, for some m.

Proof. Clearly we may assume that D is irreducible. From the Hodge Index Theorem, it follows that $D^2 \leq 0$. If $D^2 < 0$, then by adjunction D is a smooth rational curve with $D^2 = -2$. But S' has no (-2)-curves of degree at most k, for $p_1, \dots, p_9$ are k-Cremona general. If $D^2 = 0$, then applying again the Hodge Index Theorem we obtain that $D^\perp = K_{S'}^\perp$, therefore $D \in |J'|$. Thus, for an arbitrary effective divisor D, with $D \cdot J' = 0$, we get that $D \in |mJ'|$, for some positive integer $m \leq \frac{k}{3}$. From the k-Halphen generality condition, we obtain $\dim |mJ'| = 0$, hence $D = mJ'$. The reverse implication follows directly from the definition of a k-general nine-tuple of points. □

Recall Definition 1.1.

Lemma 2.4. *If the points $p_1, \dots, p_9$ are 3-general, a general Du Val curve of genus g is smooth and irreducible.*

Proof. The linear system $|C|$ on S satisfies the hypothesis of Theorem 3.1 in [13] and it is then free of fixed divisors. In particular, since by hypothesis J is fixed, the general element of $|C|$ does not contain J. From Corollary 3.4 of [13] the linear system $|C|$ is also base point free. This property together with Bertini's theorem and the fact that $C^2 > 0$, implies that the general element of C is irreducible and hence smooth.

□

3. A general Du Val curve is a Petri general curve.

Let $|C|$ and S be as in the Introduction. By Lemma 2.4, a general element C of the linear system $|C|$ is smooth. Let L be a base-point-free line bundle on C with $h^0(C, L) = r + 1$ and consider the homomorphism $\mu_{0,L}$ given by multiplication of global sections

$$\mu_{0,L} : H^0(C, L) \otimes H^0(C, \omega_C \otimes L^{-1}) \longrightarrow H^0(C, \omega_C)$$

The curve C is said to be a *Brill-Noether-Petri general* curve, if the map $\mu_{0,L}$ is injective for every line bundle L on C. Consider the Lazarsfeld-Mukai bundle defined by the sequence

$$0 \longrightarrow F_L \longrightarrow H^0(C, L) \otimes \mathcal{O}_S \longrightarrow L \longrightarrow 0.$$

Note that $H^0(S, F_L) = 0$ and $H^1(S, F_L) = 0$. Setting, as usual, $E_L := F_L^\vee$, dually, we obtain the exact sequence

$$0 \longrightarrow H^0(C, L)^\vee \otimes \mathcal{O}_S \longrightarrow E_L \longrightarrow \omega_C \otimes L^{-1} \longrightarrow 0. \tag{3.1}$$

Here we have used that $\omega_{S|C} = \mathcal{O}_C$. Clearly $c_1(E_L) = \mathcal{O}_S(C)$, but unlike in the $K3$ situation, on S we have that $H^1(S, E_L) \cong H^0(C, L)^\vee$ is $(r + 1)$-dimensional (rather than trivial). Following closely Pareschi's proof of Lazarsfeld's Theorem, [19], [15], (see also Chapter XXI, section 7 of [3]), one proves the following lemma.

Lemma 3.1. *If* $h^0(S, F_L^\vee \otimes F_L) = 1$, *then* Ker $\mu_{0,L} = 0$.

Proof. For the benefit of the reader we outline the proof of this Lemma following very closely the treatment in [3]. By tensoring the exact sequence (3.1) by F_L and taking cohomology, since $H^0(S, F_L) = 0$ and $H^1(S, F_L) = 0$, we obtain

$$H^0(S, F_L^\vee \otimes F_L) \cong H^0\big(C, F_{L|C} \otimes \omega_C \otimes L^{-1}\big).$$

The twist by $\omega_C \otimes L^{-1}$ of the restriction $F_{L|C}$ of the Lazarsfeld-Mukai bundle to C sits in an exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow F_{L|C} \otimes \omega_C \otimes L^{-1} \longrightarrow M_L \otimes \omega_C \otimes L^{-1} \longrightarrow 0 \tag{3.2}$$

Moreover there is a canonical isomorphism Ker $\mu_{0,L} \cong H^0(C, M_L \otimes \omega_C \otimes L^{-1})$. Proposition 5.29 and diagrams (6.1) and (6.2) in [3] show that if $\eta : \mathcal{W}^r_d \to M_g$ is the family of $|L| = g^r_d$'s over moduli, then the image of $d\eta$ at a point $[C, L]$, is contained in

$$(\operatorname{Im} \mu_1)^\perp \subset H^1(C, T_C)$$

where

$$\mu_1 : \operatorname{Ker} \mu_0 \longrightarrow H^0(C, K_C^2) = H^1(C, T_C)^\vee$$

is the Gaussian map defined by diagram (6.1) in [3]. We must show that the coboundary map δ of the cohomology sequence (3.2) vanishes. Let $U \subset |C|$ be the open subscheme parametrising smooth Du Val curves in the linear system $|C|$ on S, and let $f : \mathcal{C} \to U \subset |C|$ be the family of smooth curves parametrised by U. Since S is regular, the characteristic map induces an isomorphism $T_{[C]}(U) \cong H^0(C, N_{C/S})$. Consider the relative family

$$p : \mathcal{W}^r_d(f) \to U$$

Since p is surjective and C is a general element of U, the differential

$$dp : T_{[C,L]}(\mathcal{W}^r_d(f)) \longrightarrow T_{[C]}(U) = H^0(C, N_{C/S})$$

is surjective. Since $K_{S|C} = \mathcal{O}_C$, we have $N_{C/S} = \omega_C$. Let $\rho : H^0(C, N_{C/S}) \to H^1(C, T_C)$ be the Kodaira-Spencer map of the family f. We then get

$$\operatorname{Im}(\rho \circ dp) \subset \operatorname{Im}(\rho) \cap \operatorname{Im}(\mu_1)^\perp,$$

hence

$$\operatorname{Im}(dp) \subset \operatorname{Im}(\rho^\vee \circ \mu_1)^\perp \subset H^0(C, N_{C/S}).$$

We set

$$\mu_{1,S} := \rho^\vee \circ \mu_1 : \operatorname{Ker}\mu_0 \to H^1(C, \omega_C \otimes N^\vee_{C/S}).$$

Since dp is surjective, we get

$$\mu_{1,S} = 0$$

In Lemma (7.9) of [3], using only the fact that $K_{S|C}$ is trivial on C, it is proved that $\mu_{1,S} = \delta$ up to multiplication by a nonzero scalar. Hence the coboundary map δ is zero. $\square$

Let us go back to the construction of S and S', and recall the role played by the points $p_1, \dots, p_9$. From the Riemann-Roch theorem on S', these points are $3g$-Halphen general if and only if

$$(3.3)\quad H^0(J', \mathcal{O}_{J'}(kJ')) = H^0\Big(J, \mathcal{O}_J(k(3\ell - E_1 - \cdots - E_9))\Big) = 0\,, \quad k = 1, \dots, g \qquad (J \cong J')$$

Theorem 3.2. *If $p_1, \dots, p_9$ is a $3g$-general set, then the general element of $|C|$ is a Brill-Noether-Petri general curve.*

Proof. We use the Lemma above. By contradiction, suppose there is a non-trivial endomorphism $\phi \in \operatorname{End}(F_L^\vee, F_L^\vee)$. As in Lazarsfeld's proof, we may assume that ϕ is not of maximal rank. Consider the blow-down $\sigma : S \to S'$. We have

$$\sigma(E_{10}) = p\,, \qquad \sigma : C \cong \sigma(C) = C'\,, \qquad \sigma : J \cong \sigma(J) = J'$$

Notice that

$$(J')^2 = 0\,, \qquad J' \cdot C' = 1$$

Let $U := S \setminus E_{10} \cong S' \setminus \{p\} =: V$. Let F be the sheaf defined on S' by the exact sequence

$$0 \longrightarrow F \longrightarrow H^0(C, L) \otimes \mathcal{O}_{S'} \longrightarrow L \longrightarrow 0.$$

Since

$$0 \longrightarrow H^0(C, L)^\vee \otimes \mathcal{O}_{S'} \longrightarrow F^\vee \longrightarrow \omega_C \otimes L^{-1}(p) \longrightarrow 0$$

is exact, and L is special, $F^\vee$ is generated by global sections away from a finite set of points. Consider the restriction

$$\phi : F^\vee_{|V} = F^\vee_{L|U} \longrightarrow F^\vee_{L|U} = F^\vee_{|V}$$

By Hartogs' Theorem, ϕ extends uniquely to a homomorphism

$$\phi' : F^\vee \longrightarrow F^\vee,$$

which is non trivial and not of maximal rank. Let

$$E := \operatorname{Im}\phi'\,, \qquad G := \operatorname{Coker}\phi'\,, \qquad \overline{G} := G/\mathcal{T}(G)\,,$$

Set

$$A = c_1(E)\,, \qquad B = c_1(\overline{G})\,, \qquad T = c_1(\mathcal{T}(G))\,,$$

therefore

$$[C'] = A + B + T.$$

Let us prove that A, B, and T are effective or trivial. The assertion for T is clear. As for A and B it suffices to notice that E and $\overline{G}$ are generated by global sections away from a finite set of points because they are positive rank torsion free quotients of $F^\vee$.

Since $(J')^2 = 0$, we have that

$$J' \cdot A \geq 0\,, \qquad J' \cdot B \geq 0\,, \qquad J' \cdot T \geq 0.$$

Since $C' \cdot J' = 1$, either $J' \cdot A = 0$ or $J' \cdot B = 0$. By Proposition 2.3, either

$$A = kJ' \text{ or } B = hJ',$$

with $k, h \geq 0$. Both cases lead to a contradiction. Suppose $A = kJ'$. This means that $\mathcal{O}_{J'}(A)$ is a degree-zero line bundle. Let us show that it is the trivial bundle. Since E is globally generated away from a finite set of points, the same holds for the restriction of its determinant to J'. Thus $h^0(J', \mathcal{O}_{J'}(A)) = h^0(J', \mathcal{O}_{J'}(kJ')) \neq 0$, which contradicts condition (3.3). To summarize, the non-trivial endomorphism ϕ cannot exist in the first place and C is a Brill-Noether-Petri general curve. $\square$

Remark 3.3. If the set $p_1, \ldots, p_9$ is $3d$-Halphen special, the linear system $|3d\ell - d\sum_{i=1}^{9} E_i|$ cuts out on C a g^1_d. In particular, one can realize curves of arbitrary gonality as special Du Val curves.

4. Lefschetz pencils of Du Val curves

In this section we determine the intersection numbers of a rational curve $j : \mathbb{P}^1 \to \overline{\mathcal{M}}_g$ induced by a pencil of Du Val curves on S with the generators of the Picard group of the moduli space $\overline{\mathcal{M}}_g$. Recall that λ denotes the Hodge class and $\delta_0, \ldots, \delta_{\lfloor \frac{g}{2} \rfloor} \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ are the classes corresponding to the boundary divisors of the moduli space. We denote by $\delta := \delta_0 + \cdots + \delta_{\lfloor \frac{g}{2} \rfloor}$ the total boundary. For integers $r, d \geq 1$, we denote by $\mathcal{M}^r_{g,d}$ the locus of curves $[C] \in \mathcal{M}_g$ such that $W^r_d(C) \neq \emptyset$. If $\rho(g, r, d) = -1$, in particular $g+1$ must be composite, $\mathcal{M}^r_{g,d}$ is an effective divisor. Eisenbud and Harris [9] famously computed the class of the closure of the Brill-Noether divisors:

$$[\overline{\mathcal{M}}^r_{g,d}] = c_{g.d,r}\Big((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} i(g-i)\delta_i\Big) \in \mathrm{Pic}(\overline{\mathcal{M}}_g). \tag{4.1}$$

We retain the notation of the introduction and observe that the linear system

$$\Lambda_{g-1} := \big|3(g-1)\ell - (g-1)E_1 - \cdots - (g-1)E_8 - (g-2)E_9\big|$$

appears as a hyperplane in the g-dimensional linear system L_g on the surface S. It consists precisely of the curves $D + J \in L_g$, where $D \in \Lambda_{g-1}$. We now choose a Lefschetz pencil in L_g, which has $2g-2 = C^2$ base points. Let $X := \mathrm{Bl}_{2g-2}(S)$ be the blow-up of S at those points and we denote by $f : X \to \mathbb{P}^1$ the induced fibration, which gives rise to a moduli map

$$j : \mathbb{P}^1 \to \overline{\mathcal{M}}_g.$$

We compute the numerical features of this Du Val pencil in the moduli space:

Theorem 4.1. *The intersection numbers of the Du Val pencil with the generators of the Picard group of $\overline{\mathcal{M}}_g$ are given as follows:*

$$j^*(\lambda) = g, \quad j^*(\delta_0) = 6(g+1), \quad j^*(\delta_1) = 1, \quad \text{and } j^*(\delta_i) = 0 \ \text{ for } i = 2, \ldots, \Big\lfloor \frac{g}{2} \Big\rfloor.$$

As a consequence: $j^*([\overline{\mathcal{M}}^r_{g,d}]) = 0$.

Proof. Using Grothendieck-Riemann-Roch, we have the following formulas valid for the moduli map j induced by $f : X \to \mathbb{P}^1$:

$$j^*(\lambda) = \chi(X, \mathcal{O}_X) + g - 1, \quad j^*(\delta) = c_2(X) + 4(g-1).$$

Clearly $\chi(X, \mathcal{O}_X) = 1$, therefore $j^*(\lambda) = g$. Furthermore, since X is $\mathbb{P}^2$ blown up at $2g+8$ points, $c_2(X) = 12\chi(X, \mathcal{O}_X) - K_X^2 = 2g+11$, and accordingly $j^*(\delta) = 6g+7$. Of these $6g+7$ singular curves in the pencil, there is precisely one of type $D+J$, where D is the proper transform of a curve in the linear system Λ_{g-1}. Note that $D \cdot J = 1$. Therefore $j^*(\delta_1) = 1$. A parameter count also shows that a general Du Val pencil contains no curves in the higher boundary divisors Δ_i, where $i \geq 2$, therefore $j^*(\delta_0) = 6g+6$. Using (4.1), we now compute $j^*([\overline{\mathcal{M}}^r_{g,d}]) = 0$, and finish the proof. $\square$

We record the following immediate consequence of Theorem 4.1

Corollary 4.2. For any choice of nine distinct points $p_1, \ldots, p_9 \in \mathbb{P}^2$, the Du Val pencil $j(\mathbb{P}^1)$ either lies entirely in or is disjoint from any Brill-Noether divisor $\overline{\mathcal{M}}^r_{g,d}$.

In particular, notice that when the points $p_1, \ldots, p_9$ belong to the Halphen stratum $\mathrm{Halp}(3d)$, then the elliptic pencil $|dJ'|$ on S' cut out a pencil of degree d on each curve C', in particular $\mathrm{gon}(C) \leq d$. Such Halphen surfaces S, appear as limits of polarised $K3$ surfaces (X, H), where X carries an elliptic pencil $|E|$ with $E \cdot H = k$. The enlargement of the Picard group on the side of $K3$ surfaces correspond on the Du Val side to the points $p_1, \ldots, p_9$ becoming Halphen special.

Remark 4.3. Du Val curves of genus g form a unirational subvariety of dimension

$$\min(g+10, 3g-3)$$

inside the moduli space $\mathcal{M}_g$. In particular, for $g = 7$, one has a divisor $\mathfrak{Dv}_7$ of Du Val curves of genus 7. It would be interesting to describe this divisor and compute the class $[\overline{\mathfrak{Dv}}_7] \in \mathrm{Pic}(\overline{\mathcal{M}}_7)$.

4.1. **Du Val curves are Petri general: a second proof.** We now describe an alternative approach, based on the theory of limit linear series, to prove a slightly weaker version of Theorem 1.2. We retain throughout the notation of the Introduction. We denote by $\mathcal{BN}$ (respectively $\mathcal{GP}$) the proper subvariety of $\mathcal{M}_g$ consisting of curves $[C]$ having a line bundle L which violates the Brill-Noether (respectively the Gieseker-Petri) condition. Clearly $\mathcal{BN} \subset \mathcal{GP}$.

Theorem 4.4. *Let S' be the blow-up of $\mathbb{P}^2$ at nine* general *points $p_1, \ldots, p_9$ and set as before*

$$L_g := \left|3g\ell - gE_1 - \cdots - gE_8 - (g-1)E_9\right|.$$

Then a general curve $C' \in L_g$ satisfies the Petri Theorem. Furthermore, an arbitrary irreducible nodal curve $C' \in L_g$ satisfies the Brill-Noether Theorem.

Proof. Assume by contradiction, that for a general choice of $p_1, \ldots, p_9 \in \mathbb{P}^2$, there exists a nodal curve $C' \in L_g$ that violates the Brill-Noether condition. We let the points $p_1, \ldots, p_9$ specialize to the base locus of a general pencil of plane cubics. Then S' becomes a rational elliptic surface $\pi : S' \to \mathbb{P}^1$ and $E := E_9$ can be viewed as a section of π.

By a standard calculation, since $\pi_* \mathcal{O}_{S'} = \mathcal{O}_{\mathbb{P}^1}$, we compute that

$$h^0(S', \mathcal{O}_{S'}(gJ')) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g)) = g+1.$$

Similarly, since $\pi_*(\mathcal{O}_{S'}(E)) = \mathcal{O}_{\mathbb{P}^1}$, we find that $h^0(S', \mathcal{O}_{S'}(gJ' + E)) = g + 1$. Therefore, every element of the linear system L_g is of the form $J_1 + \cdots + J_g + E$, where $J_i \in |\mathcal{O}_{S'}(J')|$ are elliptic curves on S' and $J_i \cdot E = 1$, for $i = 1, \ldots, g$.

Let $\varphi : \overline{\mathcal{M}}_{0,g} \times \overline{\mathcal{M}}_{1,1}^g \to \overline{\mathcal{M}}_g$ be the map obtained by attaching to each g-pointed stable rational curve $[R, x_1, \ldots, x_g] \in \overline{\mathcal{M}}_{0,g}$ elliptic tails $J_1, \ldots, J_g$ at the points $x_1, \ldots, x_g$ respectively. The symmetric group $\mathfrak{S}_g$ acts diagonally on the product $\overline{\mathcal{M}}_{0,g} \times \overline{\mathcal{M}}_{1,1}^g$, by simultaneously permuting the markings x_i and the tails J_i for $i = 1, \ldots, g$. The map φ is $\mathfrak{S}_g$-invariant. Observe that the moduli map $m : L_g \dashrightarrow \overline{\mathcal{M}}_g$ corresponding to the linear system L_g factors via $\big(\overline{\mathcal{M}}_{0,g} \times \overline{\mathcal{M}}_{1,1}^g\big)/\mathfrak{S}_g$. Since the morphism φ is regular, it follows that the variety of stable limits of L_g, defined as the image $\pi_2(\Sigma)$ of the graph $L_g \xleftarrow{\pi_1} \Sigma \xrightarrow{\pi_2} \overline{\mathcal{M}}_g$ of the rational map m, is actually contained in $\mathrm{Im}(\varphi)$.

Using [9] Theorem 1.1, no curve lying $\mathrm{Im}(\varphi)$ carries a limit linear series $\mathfrak{g}^r_d$ with negative Brill-Noether number (note that all the stable curves in $\mathrm{Im}(\varphi)$ are *tree-like* in the sense of [9], so the theory of limit linear series applies to them). It follows that $\mathrm{Im}(\varphi) \cap \overline{\mathfrak{BN}} = \emptyset$.

Our hypothesis implies that we can find a family of Du Val curve $f : \mathcal{C} \to (T, 0)$ over a 1-dimensional base, such that for the general fibre $[f^{-1}(t)] \in \mathcal{BN}$, whereas the central fibre $f^{-1}(0)$ is a (possibly non-reduced) curve from the linear system L_g. Applying stable reduction to f, we obtain a new family having in the central fibre a stable curve that lies simultaneously in $\mathrm{Im}(\varphi)$ and in $\overline{\mathcal{BN}}$, which is a contradiction.

Furthermore, the proof of the Gieseker-Petri Theorem in [8], implies that for any choice of elliptic tails $[J_1, x_1] \ldots, [J_g, x_g] \in \overline{\mathcal{M}}_{1,1}^g$, there exists $[R, x_1, \ldots, x_g] \in \overline{\mathcal{M}}_{0,g}$ such that $\varphi\Big([R, x_1, \ldots, x_g], [J_1, x_1], \ldots, [J_g, x_g]\Big) \notin \overline{\mathcal{GP}}$. This implies that for general $p_1, \ldots, p_9 \in \mathbb{P}^2$, a general curve $C' \in L_g$ satisfies Petri's condition. □

Remark 4.5. The conclusion of Theorems 1.2 and 4.4 cannot be improved, in the sense that it is *not true* that every smooth curve $C' \in L_g$ verifies the Petri condition. The classes of the closure of the divisorial components $\mathcal{GP}^r_{g,d}$ of $\mathcal{GP}$ corresponding to line bundles $L \in W^r_d(C)$ such that $g - (r+1)(g - d + r) = 0$, have been computed in [9] Section 5, when $r = 1$ and in [10] Theorem 1.6 in general. Taking the pencil $j : \mathbb{P}^1 \to \overline{\mathcal{M}}_g$ considered in Theorem 4.1, we immediately conclude that $j^*([\overline{\mathcal{GP}}^r_{g,d}]) \neq 0$.

5. An explicit system of nine general points

In this final section we show how, using standard techniques from the arithmetic of elliptic curves, we can exhibit an explicit system of nine points verifying the genericity assumption of Definition 2.2 for every k. Throughout this section we use the embedding $\mathbb{A}^2(\mathbb{Q}) \hookrightarrow \mathbb{P}^2(\mathbb{Q})$.

We start with the elliptic curve $E : y^2 = x^3 + 17$, and we denote by $p_\infty := [0, 1, 0] \in E$ its point at infinity and use the identification $\mathcal{O}_E(1) = \mathcal{O}_E(3p_\infty)$. If $q \in E$, we denote by $-q \in E$ its inverse element using the group law of E, having p_∞ as origin. Observe that the following points belong to $E(\mathbb{Q})$:

$$p_1 = (-2, 3),\ p_2 = (-1, -4),\ p_3 = (2, 5),\ p_4 = (4, 9),\ p_5 = (52, 375),$$

as well as,

$$p_6 = (5234, 37866),\ p_7 = (8, -23),\ p_8 = (43, 282),\ \text{ and } p_9 = \left(\frac{1}{4}, -\frac{33}{8}\right).$$

It is known that $\pm p_i$ for $i = 1, \ldots, 8$ are the only points in $E(\mathbb{Z}) - \{0\}$. Using the explicit formulas for the addition law on E, observe that $p_4 = p_1 - p_3$, $p_2 = 2p_1 - p_3$, $p_5 = 3p_1 - p_3$, $p_6 = 4p_1 - 3p_3$, $p_7 = 2p_1$, $p_8 = 2p_3 - p_1$ and $p_9 = p_1 + p_3$. The following facts are known to experts, we include an elementary proof for the sake of completeness.

Lemma 5.1. *1) One has* $E(\mathbb{Q})_{\text{tors}} = 0$.
2) One has an embedding $\mathbb{Z} \cdot p_1 \oplus \mathbb{Z} \cdot p_3 \hookrightarrow E(\mathbb{Q})$[1].

Proof. For the first part, we use that if p is a prime not dividing the discriminant of E, one has an embedding $E(\mathbb{Q})_{\text{tors}} \hookrightarrow E(\mathbb{F}_p)$, see for instance [20] Chapter 7. The curve E has good reduction at the primes 5 and 7 (in fact, at any prime different from $2, 3$ and 17). Therefore, the torsion subgroup $E(\mathbb{Q})_{\text{tors}}$ injects into both $E(\mathbb{F}_5)$ and $E(\mathbb{F}_7)$, which are of orders 6 and 13, respectively. It follows that $E(\mathbb{Q})_{\text{tors}}$ is trivial. We remark that the same conclusion can be obtained by applying the Nagell-Lutz Theorem.

We prove that the points p_1 and p_3 are independent in $E(\mathbb{Q})$. Since $E(\mathbb{Q})[2] = 0$, it will suffice to show that no linear combination $np_1 + mp_3$ of the points $p_1 = (-2, 3)$, $p_3 = (2, 5)$ can be zero, where at least one of $m, n \in \mathbb{Z}$ is odd. This follows once we show that p_1, p_3, as well as $p_4 = p_1 - p_3 = (4, 9)$ are non-zero in the quotient $E(\mathbb{Z})/2E(\mathbb{Z})$. Recall [20] page 58, that if $p = (a, b) \in E(\mathbb{Q})$, then the x-coordinate of the point $2p \in E$ is given by

$$x(2p) = \frac{a^4 - 136}{4a^3 + 68}.$$

Assuming $p_1 \in 2E(\mathbb{Z})$, we obtain that the equation $a^4 - 136a = 8(a^3 + 17)$ has an integral solution, which is a contradiction. The proof that $p_3 \notin 2E(\mathbb{Z})$ is identical. If $p_4 \in 2E(\mathbb{Z})$, then the equation $a^4 - 136a = 16(a^3 + 17)$ has an integral solution, again a contradiction. $\square$

Theorem 5.2. *The points* $p_1, \ldots, p_9$ *are* k*-general for every integer* k.

Proof. The condition that the nine points are k-Halphen special for some $k \geq 0$ is precisely that $p_1 + \cdots + p_9 \in E(\mathbb{Q})_{\text{tors}}$, that is, $p_1 + \cdots + p_9 = 13p_1 - p_3 = 0$, which contradicts Lemma 5.1.

To show that the points are Cremona general, we unwind the conditions appearing in (2.1) in terms of the group law on E. In turns out that if $p_1, \ldots, p_9$ are Cremona general, then there exists non-negative integers $n_1, \ldots, n_9$, not all equal to zero, such that the linear equivalence $n_1p_1 + \cdots + n_9p_9 \equiv (n_1 + \cdots + n_9)p_\infty$ holds, that is, $n_1p_1 + \cdots + n_9p_9 = 0 \in E$. Since with the exception of $p_4 = p_1 - p_3$, each of the points $p_1, \ldots, p_9$ are combinations of the type $mp_1 + np_3$, with $m + n > 0$, we obtain that such a combination of p_1 and p_3 is equal to zero, which contradicts the second part of Lemma 5.1. $\square$

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