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THE UNIVERSAL ABELIAN VARIETY OVER $\mathcal{A}_5$

BY GAVRIL FARKAS AND ALESSANDRO VERRA

ABSTRACT. – We establish a structure result for the universal abelian variety over $\mathcal{A}_5$. This implies that the boundary divisor of $\overline{\mathcal{A}}_6$ is unirational and leads to a lower bound on the slope of the cone of effective divisors on $\overline{\mathcal{A}}_6$.

RÉSUMÉ. – On établit un théorème de structure pour la variété abélienne universelle sur $\mathcal{A}_5$. Le résultat entraîne que le diviseur de la frontière de $\overline{\mathcal{A}}_6$ est unirationnel et ceci donne lieu à une borne inférieure pour la pente du cône des diviseurs effectifs en $\overline{\mathcal{A}}_6$.

The general principally polarized abelian variety $[A, \Theta] \in \mathcal{A}_g$ of dimension $g \leq 5$ can be realized as a Prym variety. Abelian varieties of small dimension can be studied in this way via the rich and concrete theory of curves, in particular, one can establish that $\mathcal{A}_g$ is unirational in this range. In the case $g = 5$, the Prym map $P : \mathcal{R}_6 \to \mathcal{A}_5$ is finite of degree 27, see [7]; three different proofs [6, 17], [22] of the unirationality of $\mathcal{R}_6$ are known. The moduli space $\mathcal{A}_g$ is of general type for $g \geq 7$, see [12, 18], [21]. Determining the Kodaira dimension of $\mathcal{A}_6$ is a notorious open problem.

The aim of this paper is to give a simple unirational parametrization of the universal abelian variety over $\mathcal{A}_5$ and hence of the boundary divisor of a compactification of $\mathcal{A}_6$. We denote by $\phi : \mathcal{X}_{g-1} \to \mathcal{A}_{g-1}$ the universal abelian variety of dimension $g-1$ (in the sense of stacks). The moduli space $\widetilde{\mathcal{A}}_g$ of principally polarized abelian varieties of dimension g and their rank 1 degenerations is a partial compactification of $\mathcal{A}_g$ obtained by blowing up $\mathcal{A}_{g-1}$ in the Satake compactification, cf. [18]. Its boundary $\partial\widetilde{\mathcal{A}}_g$ is isomorphic to the universal Kummer variety in dimension $g-1$ and there exists a surjective double covering $j : \mathcal{X}_{g-1} \to \partial\widetilde{\mathcal{A}}_g$. We establish a simple structure result for the boundary $\partial\widetilde{\mathcal{A}}_6$:

THEOREM 0.1. – *The universal abelian variety $\mathcal{X}_5$ is unirational.*

This immediately implies that the boundary divisor $\partial\widetilde{\mathcal{A}}_6$ is unirational as well. What we prove is actually stronger than Theorem 0.1. Over the moduli space $\mathcal{R}_g$ of smooth Prym curves of genus g, we consider the universal Prym variety $\varphi : \mathcal{Y}_g \to \mathcal{R}_g$ obtained by pulling back $\mathcal{X}_{g-1} \to \mathcal{A}_{g-1}$ via the Prym map $P : \mathcal{R}_g \to \mathcal{A}_{g-1}$. Let $\overline{\mathcal{R}}_g$ be the moduli space of stable

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Prym curves of genus g together with the universal Prym curve $\tilde{\pi} : \widetilde{\mathscr{C}} \to \overline{\mathscr{R}}_g$ of genus $2g-1$. If $\widetilde{\mathscr{C}}^{g-1} := \widetilde{\mathscr{C}} \times_{\overline{\mathscr{R}}_g} \cdots \times_{\overline{\mathscr{R}}_g} \widetilde{\mathscr{C}}$ is the $(g-1)$-fold product, one has a universal *Abel-Prym* rational map $\mathfrak{ap} : \widetilde{\mathscr{C}}^{g-1} \dashrightarrow \mathscr{Y}_g$, whose restriction on each individual Prym variety is the usual Abel-Prym map. The rational map $\mathfrak{ap}$ is dominant and generically finite (see Section 4 for details). We prove the following result:

THEOREM 0.2. – *The five-fold product $\widetilde{\mathscr{C}}^5$ of the universal Prym curve over $\overline{\mathscr{R}}_6$ is unirational.*

The key idea in the proof of Theorem 0.2 is to view smooth Prym curves of genus 6 as discriminants of conic bundles, via their representation as symmetric determinants of quadratic forms in three variables. We fix four general points $u_1, \ldots, u_4 \in \mathbf{P}^2$ and set $w_i := (u_i, u_i) \in \mathbf{P}^2 \times \mathbf{P}^2$. Since the action of the automorphism group $\mathrm{Aut}(\mathbf{P}^2 \times \mathbf{P}^2)$ on $\mathbf{P}^2 \times \mathbf{P}^2$ is 4-transitive, any set of four general points in $\mathbf{P}^2 \times \mathbf{P}^2$ can be brought to this form. We then consider the linear system

$$\mathbf{P}^{15} := \left|\mathscr{I}^2_{\{w_1,\ldots,w_4\}}(2,2)\right| \subset \left|\mathscr{O}_{\mathbf{P}^2\times\mathbf{P}^2}(2,2)\right|$$

of hypersurfaces $Q \subset \mathbf{P}^2 \times \mathbf{P}^2$ of bidegree $(2,2)$ which are nodal at $w_1, \ldots, w_4$. For a general threefold $Q \in \mathbf{P}^{15}$, the first projection $p : Q \to \mathbf{P}^2$ induces a conic bundle structure with a sextic discriminant curve $\Gamma \subset \mathbf{P}^2$ such that $p(\mathrm{Sing}(Q)) = \mathrm{Sing}(\Gamma)$. The discriminant curve Γ is nodal precisely at the points $u_1, \ldots, u_4$. Furthermore, Γ is equipped with an unramified double cover $p_\Gamma : \widetilde{\Gamma} \to \Gamma$, parametrizing the lines which are components of the singular fibres of $p : Q \to \mathbf{P}^2$. By normalizing, p_Γ lifts to an unramified double cover $f : \widetilde{C} \to C$ between the normalization $\widetilde{C}$ of $\widetilde{\Gamma}$ and the normalization C of Γ respectively. Note that there exists an exact sequence of generalized Prym varieties

$$0 \longrightarrow (\mathbf{C}^*)^4 \longrightarrow P(\widetilde{\Gamma}/\Gamma) \longrightarrow P(\widetilde{C}/C) \longrightarrow 0.$$

One can show without much effort that the assignment $\mathbf{P}^{15} \ni Q \mapsto [\widetilde{C} \xrightarrow{f} C] \in \mathscr{R}_6$ is dominant. This offers an alternative, much simpler, proof of the unirationality of $\mathscr{R}_6$. However, much more can be obtained with this construction.

Let $\mathbf{G} := \mathbf{P}^2 \times (\mathbf{P}^2)^\vee = \big\{(o,\ell) : o \in \mathbf{P}^2, \ell \in \{o\} \times (\mathbf{P}^2)^\vee\big\}$ be the Hilbert scheme of lines in the fibres of the first projection $p : \mathbf{P}^2 \times \mathbf{P}^2 \to \mathbf{P}^2$. Since containing a given line in a fibre of p imposes *three* linear conditions on the linear system $\mathbf{P}^{15}$ of threefolds $Q \subset \mathbf{P}^2 \times \mathbf{P}^2$ as above, it follows that imposing the condition $\{o_i\} \times \ell_i \subset Q$ for *five* general lines, singles out a *unique* conic bundle $Q \in \mathbf{P}^{15}$. This induces an étale double cover $f : \widetilde{C} \to C$, as above, over a smooth curve of genus 6. Moreover, f comes equipped with five marked points $\ell_1, \ldots, \ell_5 \in \widetilde{C}$. To summarize, we can define a rational map

$$\zeta : \mathbf{G}^5 \dashrightarrow \widetilde{\mathscr{C}}^5, \quad \zeta\Big((o_1,\ell_1), \ldots, (o_5,\ell_5)\Big) := \Big(f : \widetilde{C} \to C, \ell_1, \ldots, \ell_5\Big),$$

between two 20-dimensional varieties, where $\mathbf{G}^5$ denotes the 5-fold product of $\mathbf{G}$.

THEOREM 0.3. – *The morphism $\zeta : \mathbf{G}^5 \dashrightarrow \widetilde{\mathscr{C}}^5$ is dominant, so that $\widetilde{\mathscr{C}}^5$ is unirational.*

More precisely, we show that $\mathbf{G}^5$ is birationally isomorphic to the fibre product $\mathbf{P}^{15}\times_{\overline{\mathcal{R}}_6}\widetilde{\mathcal{C}}^5$. In order to set Theorem 0.3 on the right footing and in view of enumerative calculations, we introduce a $\mathbf{P}^2$-bundle $\pi : \mathbf{P}(\mathcal{M}) \to S$ over the quintic del Pezzo surface S obtained by blowing up $\mathbf{P}^2$ at the points $u_1, \ldots, u_4$. The rank 3 vector bundle $\mathcal{M}$ on S is obtained by making an elementary transformation along the four exceptional divisors $E_1, \ldots, E_4$ over $u_1, \ldots, u_4$. The nodal threefolds $Q \subset \mathbf{P}^2 \times \mathbf{P}^2$ considered above can be thought of as sections of a tautological linear system on $\mathbf{P}(\mathcal{M})$ and, via the identification

$$\left|\mathcal{I}^2_{\{w_1,\ldots,w_4\}}(2,2)\right| = \left|\mathcal{O}_{\mathbf{P}(\mathcal{M})}(2)\right|,$$

we can view 4-nodal conic bundles in $\mathbf{P}^2 \times \mathbf{P}^2$ as *smooth* conic bundles over S. In this way the numerical characters of a pencil of such conic bundles can be computed (see Sections 2 and 3 for details).

Theorem 0.3 is then used to give a lower bound for the slope of the effective cone of $\overline{\mathcal{A}}_6$ (though we stop short of determining the Kodaira dimension of $\overline{\mathcal{A}}_6$). Recall that if E is an effective divisor on the perfect cone compactification $\overline{\mathcal{A}}_g$ of $\mathcal{A}_g$ with no component supported on the boundary $D_g := \overline{\mathcal{A}}_g - \mathcal{A}_g$ and $[E] = a\lambda_1 - b[D_g]$, where $\lambda_1 \in CH^1(\widetilde{\mathcal{A}}_g)$ is the Hodge class, then the slope of E is defined as $s(E) := \frac{a}{b} \geq 0$. The slope $s(\overline{\mathcal{A}}_g)$ of the effective cone of divisors of $\overline{\mathcal{A}}_g$ is the infimum of the slopes of all effective divisors on $\overline{\mathcal{A}}_g$. This important invariant governs to a large extent the birational geometry of $\overline{\mathcal{A}}_g$; for instance, since $K_{\overline{\mathcal{A}}_g} = (g+1)\lambda_1 - [D_g]$, the variety $\overline{\mathcal{A}}_g$ is of general type if $s(\overline{\mathcal{A}}_g) < g+1$, and uniruled when $s(\overline{\mathcal{A}}_g) > g+1$. It is shown in the appendix of [14] that the slope of the moduli space $\overline{\mathcal{A}}_g$ is independent of the choice of a toroidal compactification.

It is known that $s(\overline{\mathcal{A}}_4) = 8$ and that the Jacobian locus $\overline{\mathcal{M}}_4 \subset \overline{\mathcal{A}}_4$ achieves the minimal slope [19]; one of the results of [9] is the calculation $s(\overline{\mathcal{A}}_5) = \frac{54}{7}$. Furthermore, the only irreducible effective divisor on $\overline{\mathcal{A}}_5$ of minimal slope is the closure of the Andreotti-Mayer divisor N_0' consisting of 5-dimensional ppav's $[A, \Theta]$ for which the theta divisor Θ is singular at a point which is not 2-torsion. Concerning $\overline{\mathcal{A}}_6$, we establish the following estimate:

THEOREM 0.4. – *The following lower bound holds:* $s(\overline{\mathcal{A}}_6) \geq \frac{53}{10}$.

Note that this is the first concrete lower bound on the slope of $\overline{\mathcal{A}}_6$. The idea of proof of Theorem 0.4 is very simple. Since $\widetilde{\mathcal{C}}^5$ is unirational, we choose a suitable sweeping rational curve $i : \mathbf{P}^1 \to \widetilde{\mathcal{C}}^5$, which we then push forward to $\overline{\mathcal{A}}_6$ as follows:

$$\begin{tikzcd} \mathbf{P}^1 \arrow[r, "i"] \arrow[rrrr, bend left, "h"] & \widetilde{\mathcal{C}}^5 \arrow[r, "\mathfrak{ap}"] & \widetilde{\mathcal{Y}}_6 \arrow[r] & \widetilde{\mathcal{X}}_5 \arrow[r, "j"] & D_6. \end{tikzcd}$$

Here $\widetilde{\mathcal{Y}}_6$ and $\widetilde{\mathcal{X}}_5$ are partial compactifications of $\mathcal{Y}_6$ and $\mathcal{X}_5$ respectively which are described in Section 4, whereas D_6 is the boundary divisor of $\overline{\mathcal{A}}_6$. The curve $h(\mathbf{P}^1)$ sweeps the boundary divisor of $\overline{\mathcal{A}}_6$ and intersects non-negatively any effective divisor on $\overline{\mathcal{A}}_6$ not containing D_6. In particular,

$$s(\overline{\mathcal{A}}_6) \geq \frac{h(\mathbf{P}^1) \cdot [D_6]}{h(\mathbf{P}^1) \cdot \lambda_1}.$$

To define $i:\mathbf{P}^1\to\widetilde{\mathscr{C}}^5$, we fix general points $(o_1,\ell_1),\ldots,(o_4,\ell_4)\in\mathbf{G}$ and a further general point $o\in\mathbf{P}^2$. Then we consider the image under ζ of the pencil of lines in $\mathbf{P}^2$ through o, that is, the sweeping curve i is defined as

$$\mathbf{P}(T_o(\mathbf{P}^2))\ni\ell\mapsto\zeta\Big((o_1,\ell_1),\ldots,(o_4,\ell_4),(o,\ell)\Big)\in\widetilde{\mathscr{C}}^5.$$

The calculation of the numerical characters of $h(R)\subset\overline{\mathscr{A}}_6$ is a consequence of the geometry of the map ζ and is completed in Section 4.

We close the Introduction by discussing the structure of the paper. Theorem 0.3 (and hence also Theorems 0.1 and 0.2) are proven rather quickly in Section 1. The bulk of the paper is devoted to the explicit description of the numerical characters of a curve that sweeps the boundary divisor of $\overline{\mathscr{A}}_6$ and to the proof of Theorem 0.4. Section 2 concerns enumerative properties of pencils of conic bundles over quintic del Pezzo surfaces. The sweeping curve for the boundary divisor of $\overline{\mathscr{A}}_6$ is constructed in Section 3. In the final Section 4, we prove Theorem 0.4.

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1. Determinantal nodal sextics and a parametrization of $\mathscr{X}_5$

In this section we prove Theorem 0.3. We begin by recalling basic facts about determinantal representation of nodal plane sextics, see [3, 5], [4]. Let $\Gamma\subset\mathbf{P}^2$ be an integral 4-nodal sextic and $\nu:C\to\Gamma$ the normalization map, thus C is a smooth curve of genus 6. One has an exact sequence at the level of 2-torsion groups

$$0\longrightarrow\mathbb{Z}_2^{\oplus 4}\longrightarrow\mathrm{Pic}^0(\Gamma)[2]\xrightarrow{\nu^*}\mathrm{Pic}^0(C)[2]\longrightarrow 0.$$

In particular, étale double covers $f:\Gamma'\to\Gamma$ with an irreducible source curve Γ' are induced by 2-torsion points $\eta\in\mathrm{Pic}^0(\Gamma)[2]$, such that $\eta_C:=\nu^*(\eta)\neq\mathscr{O}_C$.

Definition 1.1. – We denote by $\mathscr{P}_6$ the quasi-projective moduli space of pairs (Γ,η) as above, where $\Gamma\subset\mathbf{P}^2$ is an integral 4-nodal sextic and $\eta\in\mathrm{Pic}^0(\Gamma)[2]$ is a torsion point such that $\eta_C\neq\mathscr{O}_C$. Equivalently, the induced double cover $\Gamma'\to\Gamma$ is *unsplit*, that is, the curve Γ' is irreducible.

Starting with a general element $[C,\eta_C]\in\mathscr{R}_6$, since $|W^2_6(C)|=5$, there are five sextic nodal plane models $\nu:C\to\Gamma$. For each of them, there are 2^4 further ways of choosing $\eta\in(\nu^*)^{-1}(\eta_C)$. Thus there is a degree $80=5\cdot 2^4$ covering $\rho:\mathscr{P}_6\to\mathscr{R}_6$.

Suppose now that $(\Gamma,\eta)\in\mathscr{P}_6$ is a general point[(1)]. In particular $h^0(\Gamma,\eta(1))=0$, or equivalently, $h^0(\Gamma,\eta(2))=3$. Indeed, the condition $h^0(\Gamma,\eta(1))\geq 1$ implies that $\Gamma\subset\mathbf{P}^2$ possesses a totally tangent conic, that is, there exists a reduced conic $B\subset\mathbf{P}^2$ such that $\nu^*(B)=2b$, with b being an effective divisor of C. This condition is satisfied only if $\rho(\Gamma,\eta)$

(1) We shall soon establish that $\mathscr{P}_6$ is irreducible, but here we just require that $\rho(\Gamma,\eta)$ be a general point of the irreducible variety $\mathscr{R}_6$.

lies in the ramification divisor of the Prym map $P:\mathscr{R}_6\to\mathscr{A}_5$, see [9]. Thus we may assume that $h^0(\Gamma,\eta(2))=3$, for a general point $(\Gamma,\eta)\in\mathscr{P}_6$.

Following [3] Theorem B, it is known that such a sheaf η admits a resolution

$$(1)\qquad 0\longrightarrow \mathscr{O}_{\mathbf{P}^2}(-4)^{\oplus 3}\xrightarrow{A}\mathscr{O}_{\mathbf{P}^2}(-2)^{\oplus 3}\longrightarrow\eta\longrightarrow 0,$$

where the map A is given by a symmetric matrix $\Big(a_{ij}(x_1,x_2,x_3)\Big)_{i,j=1}^3$ of quadratic forms. More precisely, we can view the resolution (1) as a twist of the exact sequence

$$(2)\qquad 0\longrightarrow H^0(\Gamma,\eta(2))^\vee\otimes\mathscr{O}_{\mathbf{P}^2}(-2)\xrightarrow{A}H^0(\Gamma,\eta(2))\otimes\mathscr{O}_{\mathbf{P}^2}\xrightarrow{\mathrm{ev}}\eta(2)\longrightarrow 0,$$

where ev is the evaluation map on sections. Indeed, since the multiplication map $H^0(\Gamma,\eta(2))\otimes H^0(\mathscr{O}_{\mathbf{P}^2}(j))\to H^0(\Gamma,\eta(2+j))$ is surjective for all $j\in\mathbb{Z}$ (use again that $H^0(\Gamma,\eta(1))=0$, see also [23] Proposition 3.1 for a very similar situation), it follows that the kernel of the morphism ev splits as a sum of line bundles, which then necessarily must be $\mathscr{O}_{\mathbf{P}^2}(-2)^{\oplus 3}$.

Since η is invertible, for each point $x\in\Gamma$ one has

$$1=\dim_{\mathbb{C}}\eta(x)=3-\mathrm{rk}\,A(x),$$

where, as usual, $\eta(x):=\eta_x\otimes_{\mathscr{O}_{\Gamma,x}}\mathbb{C}(x)$ is the fibre of the sheaf η at the point x. Thus $\mathrm{rk}\,A(x)=2$, for each $x\in\Gamma$.

To the matrix $A\in M_3\big(H^0(\mathscr{O}_{\mathbf{P}^2}(2))\big)$ we can associate the following $(2,2)$ threefold in $\mathbf{P}^2_{[x_1:x_2:x_3]}\times\mathbf{P}^2_{[y_1:y_2:y_3]}=\mathbf{P}^2\times\mathbf{P}^2$

$$Q:\sum_{i,j=1}^3 a_{ij}(x_1,x_2,x_3)y_iy_j=0,$$

which is a conic bundle with respect to the two projections. Alternatively, if

$$A:H^0(\Gamma,\eta(2))^\vee\otimes H^0(\Gamma,\eta(2))^\vee\to H^0(\Gamma,\mathscr{O}_\Gamma(2))$$

is the symmetric map appearing in (2), then A induces the $(2,2)$ hypersurface

$$Q\subset\mathbf{P}\Big(H^0(\Gamma,\mathscr{O}_\Gamma(1))^\vee\Big)\times\mathbf{P}\Big(H^0(\Gamma,\eta(2))^\vee\Big)=\mathbf{P}^2\times\mathbf{P}^2.$$

We denote by $p:Q\to\mathbf{P}^2$ the first projection and then $\Gamma\subset\mathbf{P}^2$ is precisely the discriminant curve of Q given by determinantal equation $\Gamma:=\big\{\det A(x_1,x_2,x_3)=0\big\}$. Let Γ' denote the Fano scheme of lines $F_1\big(p^{-1}(\Gamma)/\Gamma\big)$ over the discriminant curve Γ. That means that Γ' parametrizes pairs (x,ℓ), where $x\in\Gamma$ and ℓ is an irreducible component of the fibre $p^{-1}(x)$. The map $f:\Gamma'\to\Gamma$ is given by $f(x,\ell):=x$. Since $\mathrm{rk}\,A(x)=2$ for all $x\in\Gamma$, it follows that f is an étale double cover.

Proposition 1.2. – *For a general point $(\Gamma,\eta)\in\mathscr{P}_6$, the restriction map $p|_{\mathrm{Sing}(Q)}$: $\mathrm{Sing}(Q)\to\mathrm{Sing}(\Gamma)$ is bijective.*

Proof. – Let $x \in \Gamma$ and $R := \mathcal{O}_{\mathbf{P}^2,x}$ be the local ring of $\mathbf{P}^2$ and $\mathfrak{m}$ its maximal ideal. After a linear change of coordinates, we may assume that the matrix A mod $\mathfrak{m} =: A(x)$ equals

$$A(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Suppose $(x, y = [y_1, y_2, y_3]) \in \mathrm{Sing}(Q)$. Then $A(x) \cdot^t y = 0$, hence $y_1 = y_2 = 0$. Imposing that the partials of the defining equation of Q with respect to x_1, x_2, x_3 vanish, we obtain that $a_{33} \in \mathfrak{m}^2$. Since $\det(a_{ij}) \equiv a_{33} \bmod \mathfrak{m}^2$, this implies that Γ is singular at x. Conversely, for $x \in \mathrm{Sing}(\Gamma)$, we obtain that $\mathrm{Sing}(Q) \cap p^{-1}(x) = \{(x, y)\}$, where $y \in \mathbf{P}^2$ is uniquely determined by the condition $A(x) \cdot^t y = 0$ (use once more that $\mathrm{rk}\, A(x) = 2$). □

Proposition 1.3. – *$f_*(\mathcal{O}_{\Gamma'}) = \mathcal{O}_\Gamma \oplus \eta$, that is, the double cover f is induced by η.*

Proof. – Essentially identical to [1] Lemme 6.14. □

To sum up, to a general point $(\Gamma, \eta) \in \mathcal{P}_6$ we have associated a 4-nodal conic bundle $p : Q \to \mathbf{P}^2$ as above. Conversely, as explained in the Introduction, the discriminant curve of a 4-nodal conic bundle in $Q \subset \mathbf{P}^2 \times \mathbf{P}^2$ gives rise to an element in $\mathcal{P}_6$.

Let $\mathbf{T} \subset \left|\mathcal{O}_{\mathbf{P}^2 \times \mathbf{P}^2}(2,2)\right|$ be the subvariety consisting of 4-nodal hypersurfaces of bidegree $(2,2)$. This is an irreducible 31-dimensional variety endowed with an action of $\mathrm{Aut}(\mathbf{P}^2 \times \mathbf{P}^2)$. The following result summarizes what has been achieved so far:

Theorem 1.4. – *A general Prym curve $(\Gamma, \eta) \in \mathcal{P}_6$ is the discriminant of a 4-nodal conic bundle $p : Q \to \boldsymbol{P}^2$, where $Q \subset \boldsymbol{P}^2 \times \boldsymbol{P}^2$ is a 4-nodal threefold of bidegree $(2,2)$. More precisely, we have a birational isomorphism $\boldsymbol{T} // \mathrm{Aut}(\boldsymbol{P}^2 \times \boldsymbol{P}^2) \overset{\cong}{\dashrightarrow} \mathcal{P}_6$.*

Remark 1.5. – A similar isomorphism between the moduli space of Prym curves over *smooth* plane sextics and the quotient $\left|\mathcal{O}_{\mathbf{P}^2 \times \mathbf{P}^2}(2,2)\right| // \mathrm{Aut}(\mathbf{P}^2 \times \mathbf{P}^2)$ has already been established and used in [1] and [23].

Remark 1.6. – Theorem 1.4 yields another (shorter) proof of the unirationality of $\mathcal{R}_6$.

The automorphism group of $\mathbf{P}^2 \times \mathbf{P}^2$ sits in an exact sequence

$$0 \longrightarrow PGL(3) \times PGL(3) \longrightarrow \mathrm{Aut}(\mathbf{P}^2 \times \mathbf{P}^2) \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

In particular, we can fix four general points $u_1, \ldots, u_4 \in \mathbf{P}^2$, as well as diagonal points $w_i := (u_i, u_i) \in \mathbf{P}^2 \times \mathbf{P}^2$, and consider the linear system $\mathbf{P}^{15} := \left|\mathcal{I}^2_{\{w_1, \ldots, w_4\}}(2,2)\right|$ of $(2,2)$ threefolds with assigned nodes at these points. Theorem 1.4 implies the existence of a dominant discriminant map $\mathfrak{d} : \mathbf{P}^{15} \dashrightarrow \mathcal{P}_6$ assigning $\mathfrak{d}(Q) := (\Gamma' \xrightarrow{f} \Gamma)$.

Proof of Theorem 0.3. – Using the notation introduced in this section and in the Introduction, setting $\mu := \varphi \circ \mathfrak{ap} : \widetilde{\mathscr{C}}_5 \dashrightarrow \overline{\mathscr{R}}_6$, one has the following commutative diagram:

$$\begin{tikzcd}
\mathbf{G}^5 \arrow[r] \arrow[d] & \mathscr{P}_6 \times_{\mathscr{R}_6} \widetilde{\mathscr{C}}^5 \arrow[r] \arrow[d] & \widetilde{\mathscr{C}}^5 \arrow[d, "\mu"'] \\
\mathbf{P}^{15} \arrow[r, "\mathfrak{d}"] & \mathscr{P}_6 \arrow[r, "\rho"] & \overline{\mathscr{R}}_6.
\end{tikzcd}$$

The dominance of the composite map $\zeta : \mathbf{G}^5 \dashrightarrow \widetilde{\mathscr{C}}^5$ follows once we observe, that the above diagram is birationally a fibre product, that is, $\mathbf{G}^5 \overset{\cong}{\dashrightarrow} \mathbf{P}^{15} \times_{\overline{\mathscr{R}}_6} \widetilde{\mathscr{C}}^5$. $\square$

2. Conic bundles over a del Pezzo surface

With view to further applications, we analyze the linear system of conic bundles of type $(2,2)$ in $\mathbf{P}^2 \times \mathbf{P}^2$ which are singular at four fixed general points and birationally, we reconstruct such a linear system as the complete linear system of smooth conic bundles in a certain $\mathbf{P}^2$-bundle over a smooth quintic del Pezzo surface.

We fix four general points $u_1, \ldots, u_4 \in \mathbf{P}^2$ and set $w_i := (u_i, u_i) \in \mathbf{P}^2 \times \mathbf{P}^2$. Let S be the Del Pezzo surface defined by the blow-up $\sigma : S \to \mathbf{P}^2$ of $u_1, \ldots, u_4$. For $i = 1, \ldots, 4$, we denote by $E_i := \sigma^{-1}(u_i)$ the exceptional line over u_i. Set $E := E_1 + \cdots + E_4$ and denote by $L \in |\sigma^* \mathcal{O}_{\mathbf{P}^2}(1)|$ the pull-back of a line in $\mathbf{P}^2$ under σ. An important role is played by the rank 3 vector bundle $\mathscr{M}$ on S defined by the following sequence

$$0 \longrightarrow \mathscr{M} \xrightarrow{j} H^0(S, \mathcal{O}_S(L)) \otimes \mathcal{O}_S(L) \xrightarrow{r} \bigoplus_{i=1}^{4} \mathcal{O}_{E_i}(2L) \longrightarrow 0. \tag{3}$$

Here $r_i : H^0(S, \mathcal{O}_S(L)) \otimes \mathcal{O}_S(L) \to \mathcal{O}_{E_i}(2L)$ is the evaluation map and $r := \oplus_{i=1}^4 r_i$. Since $\mathcal{O}_{E_i}(L)$ is trivial, it follows that $h^0(r)$ is surjective. Passing to cohomology, we write the exact sequence

$$0 \longrightarrow H^0(S, \mathscr{M}) \xrightarrow{h^0(j)} H^0(S, O_S(L)) \otimes H^0(S, \mathcal{O}_S(L)) \xrightarrow{h^0(r)} \bigoplus_{i=1}^{4} H^0(\mathcal{O}_{E_i}(2L)) \longrightarrow 0.$$

In particular, we obtain that $h^0(S, \mathscr{M}) = 5$. By direct calculation, we also find that

$$c_1(\mathscr{M}) = \mathcal{O}_S(-K_S) \quad \text{and} \quad c_2(\mathscr{M}) = 3. \tag{4}$$

Under the decomposition $H^0(\mathcal{O}_S(L)) \otimes H^0(\mathcal{O}_S(L)) = \wedge^2 H^0(\mathcal{O}_S(L)) \oplus H^0(\mathcal{O}_S(2L))$ into symmetric and anti-symmetric tensors, the space $H^0(S, \mathscr{M}) \subset H^0(\mathcal{O}_S(L)) \otimes H^0(\mathcal{O}_S(L))$ decomposes as

$$H^0(S, \mathscr{M}) = H^0(S, \mathscr{M})^- \oplus H^0(S, \mathscr{M})^+ = \bigwedge^2 H^0(S, \mathcal{O}_S(L)) \oplus H^0(S, \mathcal{O}_S(2L - E)).$$

LEMMA 2.1. – *The vector bundle $\mathscr{M}$ is globally generated.*

Proof. – Clearly, we only need to address the global generation of $\mathcal{M}$ along $\bigcup_{i=1}^4 E_i$ and to that end, we consider the restriction of the sequence (3) to E_i,

$$\mathcal{M}_{|_{E_i}} \xrightarrow{j|_{E_i}} H^0(S, \mathcal{O}_S(L)) \otimes \mathcal{O}_{E_i} \xrightarrow{r|_{E_i}} \mathcal{O}_{E_i} \longrightarrow 0,$$

where we recall that $\mathcal{O}_{E_i}(L)$ is trivial. The sheaf $H^0(\mathcal{O}_S(L-E_i)) \otimes \mathcal{O}_{E_i} = \mathcal{O}_{\mathbf{P}^1}^{\oplus 2}$ is the kernel of $r_{|_{E_i}}$. Since $\det(\mathcal{M}_{|_{E_i}}) = \mathcal{O}_{E_i}(1)$, it follows that $\mathcal{M}_{|_{E_i}}$ fits into an exact sequence of bundles on $\mathbf{P}^1$:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^1}(1) \longrightarrow \mathcal{M}_{|_{E_i}} \xrightarrow{j|E_i} \mathcal{O}_{\mathbf{P}^1}^{\oplus 2} \longrightarrow 0.$$

This sequence is split, so that $\mathcal{M}_{|_{E_i}} = \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}^{\oplus 2}$, which is globally generated. The same holds for $\mathcal{M}$ if the map $H^0(\mathcal{M}) \to H^0(\mathcal{M}_{|_{E_i}})$ is surjective; this is implied by the vanishing $H^1(S, \mathcal{M}(-E_i)) = 0$. We twist by $\mathcal{O}_S(-E_i)$ the sequence (3), and write

$$0 \longrightarrow \mathcal{M}(-E_i) \longrightarrow H^0(S, \mathcal{O}_S(L)) \otimes \mathcal{O}_S(L-E_i) \xrightarrow{r} \bigoplus_{i=1}^4 \mathcal{O}_{E_i}(2L-E_i) \longrightarrow 0.$$

Since $h^0(r)$ is surjective and $h^1(S, \mathcal{O}_S(L-E_i)) = 0$, it follows $h^1(S, \mathcal{M}(-E_i)) = 0$. □

From now on we set $\mathbf{P} := \mathbf{P}(\mathcal{M})$ and consider the $\mathbf{P}^2$-bundle $\pi : \mathbf{P} \to S$. The linear system $|\mathcal{O}_{\mathbf{P}}(1)|$ is base point free, for $\mathcal{M}$ is globally generated. We reserve the notation

$$h := \phi_{\mathcal{O}_{\mathbf{P}}(1)} : \mathbf{P} \to \mathbf{P}^4 := \mathbf{P}H^0(S, \mathcal{M})^\vee$$

for the induced morphism. A Chern classes count implies that $\deg(h) = 2$. The map j from the sequence (3) induces a birational morphism

$$\epsilon : S \times \mathbf{P}^2 \dashrightarrow \mathbf{P}.$$

We describe a factorization of ϵ. Since j is an isomorphism along $U := S - \bigcup_{i=1}^4 E_i$, it follows that $\epsilon : U \times \mathbf{P}^2 \to \pi^{-1}(U)$ is biregular. The behaviour of ϵ along $E_i \times \mathbf{P}^2$ can be understood in terms of the restriction of the sequence (3) to E_i. Following the proof of Lemma 2.1, one has the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^1}(1) \longrightarrow \mathcal{M}_{|_{E_i}} \xrightarrow{j|_{E_i}} H^0(S, \mathcal{O}_S(L)) \otimes \mathcal{O}_{E_i} \xrightarrow{r|_{E_i}} \mathcal{O}_{E_i} \longrightarrow 0,$$

where $\mathrm{Im}(j_{|_{E_i}}) = H^0(\mathcal{O}_S(L-E_i)) \otimes \mathcal{O}_{E_i}$. Now $j_{|_{E_i}}$ induces a rational map

$$\epsilon_{|_{E_i \times \mathbf{P}^2}} : E_i \times \mathbf{P}^2 \dashrightarrow \mathbf{P}(\mathcal{M}_{|_{E_i}}) \subset \mathbf{P}.$$

For a point $x \in E_i$, the restriction of ϵ to $\mathbf{P}^2 \times \{x\}$ is the projection $\{x\} \times \mathbf{P}^2 \to \mathbf{P}^1$ of center (x, u_i). This implies that:

Lemma 2.2. – *The birational map ϵ contracts $E_i \times \boldsymbol{P}^2$ to a surface which is a copy of $\mathbf{P}^1 \times \mathbf{P}^1$. Furthermore, the indeterminacy scheme of ϵ is equal to $\bigcup_{i=1}^4 E_i \times \{u_i\}$.*

Let $D_i := E_i \times \{u_i\} \subset S \times \mathbf{P}^2$ and $D := D_1 + \cdots + D_4$. We consider the blow-up

$$\alpha : \widetilde{S \times \mathbf{P}^2} \to S \times \mathbf{P}^2$$

of $S \times \mathbf{P}^2$ along D, and the birational map

$$\epsilon_2 := \epsilon \circ \alpha : \widetilde{S \times \mathbf{P}^2} \to \mathbf{P}.$$

The restriction of ϵ_2 to the strict transform $\widetilde{E_i \times \mathbf{P}^2}$ of $E_i \times \mathbf{P}^2$ is a regular morphism, for $\epsilon|_{E_i \times \mathbf{P}^2}$ is defined by the linear system $\left|\mathscr{I}_{E_i \times \{u_i\}/S \times \mathbf{P}^2}(1,1)\right|$. This implies that ϵ_2 itself is a regular morphism:

Proposition 2.3. – *The following commutative diagram resolves the indeterminacy locus of ϵ:*

$$\begin{tikzcd} & \widetilde{S \times \boldsymbol{P}^2} \arrow[dl, "\alpha"'] \arrow[dr, "\epsilon_2"] & \\ S \times \boldsymbol{P}^2 \arrow[rr, dotted, "\epsilon"] & & \boldsymbol{P}. \end{tikzcd}$$

In the sequel, it will be useful to consider the exact commutative diagram

$$\begin{tikzcd} H^0(S, \mathscr{M}) \arrow[r] \arrow[d] & H^0(\mathscr{O}_S(L)) \otimes H^0(\mathscr{O}_S(L)) \arrow[r] \arrow[d] & \bigoplus_{i=1}^4 H^0(\mathscr{O}_{E_i}(2L)) \arrow[d] \\ H^0\big(\mathscr{I}_{\{w_1 \dots w_4\}}(1,1)\big) \arrow[r] & H^0(\mathscr{O}_{\mathbf{P}^2}(1)) \otimes H^0(\mathscr{O}_{\mathbf{P}^2}(1)) \arrow[r] & \bigoplus_{i=1}^4 H^0(\mathscr{O}_{w_i}(2)) \end{tikzcd}$$

where the vertical arrows are isomorphisms induced by $\sigma : S \to \mathbf{P}^2$. Starting from the left arrow, one can construct the commutative diagram

$$\begin{tikzcd} H^0(S, \mathscr{M}) \otimes \mathscr{O}_S \arrow[rr] \arrow[d] & & \mathscr{M} \arrow[d, "j"] \\ H^0\big(\mathscr{I}_{\{w_1 \dots w_4\}}(1,1)\big) \otimes \mathscr{O}_S \arrow[r] & H^0(S, L) \otimes \mathscr{O}_S(L). & \end{tikzcd}$$

Passing to evaluation maps, we obtain the morphism $h : \mathbf{P} \to \mathbf{P}^4$ and the rational map $h_D : S \times \mathbf{P}^2 \dashrightarrow \mathbf{P}^4$ defined by the space $\left(\sigma \times \mathrm{id}_{\mathbf{P}^2}\right)^* H^0\Big(\mathbf{P}^2 \times \mathbf{P}^2, \mathscr{I}_{\{w_1, \dots, w_4\}}(1,1)\Big)$.

The discussion above is summarized in the following commutative diagram:

$$\begin{tikzcd} & \widetilde{S \times \mathbf{P}^2} \arrow[dl, "\alpha"'] \arrow[dr, "\epsilon_2"] & \\ S \times \mathbf{P}^2 \arrow[rr, dotted, "\epsilon"] \arrow[dr, dotted, "h_D"'] & & \mathbf{P} \arrow[dl, "h"] \\ & \mathbf{P}^4. & \end{tikzcd}$$

We derive a few consequences. Let $\pi_1 : S \times \mathbf{P}^2 \to S$ and $\pi_2 : S \times \mathbf{P}^2 \to \mathbf{P}^2$ be the two projections, then define the following effective divisors of $\widetilde{S \times \mathbf{P}^2}$:

$$\widetilde{H} \in \left|(\pi_1 \circ \alpha)^*(\mathscr{O}_S(-K_S))\right|, \ \widetilde{H}_1 \in \left|(\pi_1 \circ \alpha)^*(\mathscr{O}_S(L))\right|, \ \widetilde{H}_2 \in \left|(\pi_2 \circ \alpha)^*(\mathscr{O}_{\mathbf{P}^2}(1))\right|,$$

as well as

$$\widetilde{N}_i := \alpha^{-1}(D_i) \text{ and } \widetilde{N} = \sum_{i=1}^{4} \widetilde{N}_i.$$

Applying push-forward under ϵ_2, we obtain the following divisors on $\mathbf{P}$:

$$H := \epsilon_{2*}(\widetilde{H}), \;\; H_i := \epsilon_{2*}(\widetilde{H}_i), \;\; N_i := \epsilon_{2*}(\widetilde{N}_i), \text{ and } \;\; N := \sum_{i=1}^{4} N_i.$$

PROPOSITION 2.4. – $|\mathcal{O}_{\boldsymbol{P}}(1)| = |H_1 + H_2 - N|$.

Proof. – Using for instance [16] Theorem 1.4, we have $\epsilon_2^*(\mathcal{O}_{\mathbf{P}}(1)) = \mathcal{O}_{\widetilde{S\times \mathbf{P}^2}}(\widetilde{H}_1+\widetilde{H}_2-\widetilde{N})$. By pushing forward, we obtain the desired result. ☐

We have already remarked that $h : \mathbf{P} \to \mathbf{P}^4$ is a morphism of degree 2. The inverse image $E \subset \mathbf{P}$ under h of a general line in $\mathbf{P}^4$ is a smooth elliptic curve. The restriction h_E has 4 branch points and the branch locus of h is a quartic hypersurface $B \subset \mathbf{P}^4$.

PROPOSITION 2.5. – *For each $d \geq 0$, one has $h^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(d)) = \binom{d+4}{4} + \binom{2d}{4}$.*

Proof. – We pass to the Stein factorization $h := s \circ f$, where $f : \overline{\mathbf{P}} \to \mathbf{P}^4$ is a double cover and $s : \mathbf{P} \to \overline{\mathbf{P}}$ is birational. In particular, $h^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(d)) = h^0(f^*\mathcal{O}_{\mathbf{P}^4}(d))$. The involution $\iota : \overline{\mathbf{P}} \to \overline{\mathbf{P}}$ induced by f acts on $H^0(f^*\mathcal{O}_{\mathbf{P}^4}(d))$ and the eigenspaces are $f^*H^0(\mathcal{O}_{\mathbf{P}^4}(d))$ and $b \cdot f^*H^0(\mathcal{O}_{\mathbf{P}^4}(2d-4))$ respectively, where $b \in H^0(f^*\mathcal{O}_{\mathbf{P}^4}(2))$ and $\mathrm{div}(b) = f^{-1}(B)$. ☐

We can now relate the 15-dimensional linear system $|\mathcal{O}_{\mathbf{P}}(2)|$ of *smooth* conic bundles in $\mathbf{P}$ to the linear system of 4-*nodal* conic bundles of type $(2,2)$ in $\mathbf{P}^2 \times \mathbf{P}^2$. Let $\tilde{I}$ be the moving part of the total transform $\big((\sigma \times \mathrm{id}_{\mathbf{P}^2})\circ\alpha\big)^*\Big|\mathcal{I}^2_{\{w_1,\ldots,w_4\}}(2,2)\Big|$. Over $\mathbf{P}$, we consider the linear system $I' := (\epsilon_2)_*\tilde{I}$, and conclude that:

PROPOSITION 2.6. – *One has the equality $I' = |\mathcal{O}_{\boldsymbol{P}}(2)|$ of linear systems on $\boldsymbol{P}$.*

Proof. – Consider a general threefold $Y \in \big|\mathcal{I}_{\{w_1,\ldots,w_4\}}(1,1)\big|$. Its strict transform $\widetilde{Y}$ under the morphism $(\sigma \times \mathrm{id}_{\mathbf{P}^2}) \circ \alpha$ is smooth and has class $\widetilde{H}_1 + \widetilde{H}_2 - \widetilde{N}$. Therefore we obtain $(\epsilon_2)_*(\widetilde{Y}) \in |H_1 + H_2 - N| = |\mathcal{O}_{\mathbf{P}}(1)|$, and then $I' = |\mathcal{O}_{\mathbf{P}}(2)|$. ☐

To conclude this discussion, the identification

$$\big|\mathcal{O}_{\mathbf{P}}(2)\big| = \Big|\mathcal{I}^2_{\{w_1,\ldots w_2\}}(2,2)\Big| := \mathbf{P}^{15},$$

induced by the birational map ϵ, will be used throughout the rest of the paper.

REMARK 2.7. – One can describe $h : \mathbf{P} \to \mathbf{P}^4$ in geometric terms. Consider the rational map $h' := h_D \circ \big(\sigma \times \mathrm{id}_{\mathbf{P}^2}\big)^{-1} : \mathbf{P}^2 \times \mathbf{P}^2 \dashrightarrow \mathbf{P}^4$, where h_D appears in a previous diagram. Then h' is defined by the linear system $\big|\mathcal{I}_{\{w_1,\ldots,w_4\}}(1,1)\big|$. If $\mathbf{P}^2 \times \mathbf{P}^2 \subset \mathbf{P}^8$ is the Segre embedding and $\Lambda \subset \mathbf{P}^8$ the linear span of $w_1, \ldots, w_4$, then h' is the restriction to $\mathbf{P}^2 \times \mathbf{P}^2$ of the linear projection having center Λ.

One can also recover the tautological bundle $\mathcal{M}$ as follows. Consider the family of planes $\Big\{\mathbf{P}_x := h'_*\big(\{x\} \times \mathbf{P}^2\big)\Big\}_{x \in \mathbf{P}^2}$. Its closure in the Grassmannian $\mathbf{G}(2,4)$ of planes of $\mathbf{P}^4$ is equal to the image of S under the classifying map of $\mathcal{M}$. We omit further details.

PROPOSITION 2.8. – *The following relations hold in* $CH^4(\widetilde{S \times \boldsymbol{P}^2})$:

$\widetilde{N}^4 = -4,\ \widetilde{N}^3 \cdot \widetilde{H} = 4,\ \widetilde{N}^3 \cdot \widetilde{H}_1 = \widetilde{N}^3 \cdot \widetilde{H}_2 = 0,\ \widetilde{N}^2 \cdot \widetilde{H}^2 = \widetilde{N}^2 \cdot \widetilde{H}_1^2 = \widetilde{N}^2 \cdot \widetilde{H}_2^2 = 0.$

Proof. – These are standard calculations on blow-ups. We fix $i \in \{1, \ldots, 4\}$ and note that $\widetilde{N}_i = \mathbf{P}\big(\mathcal{O}_{D_i}^{\oplus 2} \oplus \mathcal{O}_{D_i}(1)\big)$. We denote by $\xi_i := c_1(\mathcal{O}_{\widetilde{N}_i}(1)) \in CH^1(\widetilde{N}_i)$ the class of the tautological bundle on the exceptional divisor, by $\alpha_i := \alpha|_{\widetilde{N}_i} : \widetilde{N}_i \to D_i$ the restriction of α, and by $j_i : \widetilde{N}_i \hookrightarrow \widetilde{S \times \mathbf{P}^2}$ the inclusion map. Then for $k = 1, \ldots, 4$, the formula $\widetilde{N}_i^k = (-1)^{k-1}(j_i)_*(\xi_i^{k-1})$ holds in $CH^k(\widetilde{S \times \mathbf{P}^2})$. In particular,

$$\widetilde{N}_i^4 = -(j_i)_*(\xi_i^3) = -c_1(\mathcal{O}_{D_i}^{\oplus 2} \oplus \mathcal{O}_{D_i}(1)) = -1,$$

which implies that $\widetilde{N}^4 = \widetilde{N}_1^4 + \cdots + \widetilde{N}_4^4 = -4$. Furthermore, based on dimension reasons, $\widetilde{N}_i^2 \cdot \alpha^*(\gamma) = -(j_i)_*\big(\xi_i \cdot \alpha_i^*(\gamma|_{D_i})\big) = 0$, for each class $\gamma \in CH^2(S \times \mathbf{P}^2)$. Finally, for a class $\gamma \in CH^1(S \times \mathbf{P}^2)$, we have that $\widetilde{N_i}^3 \cdot \alpha^*(\gamma) = (j_i)_*\big(\xi_i^2 \cdot \alpha_i^*(\gamma|_{D_i})\big) = (\alpha_i)_*(\xi_i^2) \cdot \gamma|_{D_i} = \gamma \cdot D_i$, where the last intersection product is computed on $S \times \mathbf{P}^2$. This determines all top intersection numbers involving $\widetilde{N}^3$, which finishes the proof. □

REMARK 2.9. – Since ϵ_2 contracts the divisors $\widetilde{E_i \times \mathbf{P}^2}$, clearly $H = 3H_1 - N$. An immediate consequence of Proposition 2.8 is that the degree of the morphism $h : \mathbf{P} \to \mathbf{P}^4$ equals $\deg(h) = (H_1 + H_2 - N)^4 = 6H_1^2 \cdot H_2^2 + N^4 = 2$.

2.1. Pencils of conic bundles in the projective bundle P

In this section we determine the numerical characters of a pencil of 4-nodal conic bundles of type $(2, 2)$. Let

$$P \subset \big|\mathcal{O}_{\mathbf{P}}(2)\big| = \big|\mathcal{O}_{\mathbf{P}}(2H_1 + 2H_2 - 2N)\big|$$

be a Lefschetz pencil in $\mathbf{P}$. We may assume that its base locus $B \subset \mathbf{P}$ is a smooth surface. We are primarily interested in the number of singular conic bundles and those having a double line respectively. We first describe B.

LEMMA 2.10. – *For the base surface $B \subset \boldsymbol{P}$ of a pencil of conic bundles, the following hold:*

1. $K_B = \mathcal{O}_B(H_1 + H_2 - N) \in \mathrm{Pic}(B)$.
2. $K_B^2 = 8$ *and* $c_2(B) = 64$.

Proof. – The surface B is a complete intersection in $\mathbf{P}$, hence by adjunction

$$K_B = K_{\mathbf{P}|B} \otimes \mathcal{O}_B(4H_1 + 4H_2 - 4N).$$

Furthermore, $K_{\widetilde{S \times \mathbf{P}^2}} = \alpha^*\big(\mathcal{O}_S(-H) \boxtimes \mathcal{O}_{\mathbf{P}^2}(-3)\big) \otimes \mathcal{O}_{\widetilde{S \times \mathbf{P}^2}}(2\widetilde{N})$, and by push-pull

$$K_{\mathbf{P}} = (\epsilon_2)_*\big(K_{\widetilde{S \times \mathbf{P}^2}}\big) = \mathcal{O}_{\mathbf{P}}(-H - 3H_2 + 2N) = \mathcal{O}_{\mathbf{P}}(-3H_1 - 3H_2 + 3N),$$

for $H = 3H_1 - N$. We find that $K_B = \mathcal{O}_B(H_1 + H_2 - N)$. From Lemma 2.8, we compute

$$K_B^2 = 4(H_1 + H_2 - N)^2 \cdot (H_1 + H_2 - N)^2 = 24H_1^2 \cdot H_2^2 + 4N^4 = 8.$$

Finally, from the Euler formula applied for B, we obtain $12\chi(B, \mathcal{O}_B) = K_B^2 + c_2(B)$. Since $\chi(B, \mathcal{O}_B) = 6$, this yields $c_2(B) = 64$. □

For a variety Z we denote as usual by $e(Z)$ its topological Euler characteristic.

LEMMA 2.11. – *For a general conic bundle $Q \in \left|\mathcal{O}_{\mathbf{P}}(2)\right|$, we have that $e(Q)=4$, whereas for conic bundle Q_0 with a single ordinary quadratic singularity, $e(Q_0)=5$.*

Proof. – We fix a conic bundle $\pi_1 : Q \to S$ with smooth discriminant curve $C \in |-2K_S|$. We then write the relation $e(Q-\pi_1^*(C)) = 2e(S-C)$. Since $e(\pi_1^*(C)) = 3e(C)$, we find that $e(Q) = 2e(S) + e(C) = 2\cdot 7 - 10 = 4$.

Similarly, if $\pi_1 : Q_0 \to \mathbf{P}^2$ is a conic bundle such that the discriminant curve $C_0 \subset S$ has a unique node, then $e(Q_0) = 2e(S) + e(C_0) = 14 - 9 = 5$. □

In the next statement we use the notation from [10] for divisors classes on $\overline{\mathcal{R}}_g$, see also the beginning of Section 3 for further details.

THEOREM 2.12. – *In a Lefschetz pencil of conic bundles $P \subset |\mathcal{O}_{\mathbf{P}}(2)|$ there are precisely 77 singular conic bundles and 32 conic bundles with a double line.*

Proof. – Retaining the notation from above, $B \subset \mathbf{P}$ is the base surface of the pencil. The number δ of nodal conic bundles in P is given by the formula:

$$\delta = e(\mathbf{P}) + e(B) - 2e(Q) = 3e(S) + 64 - 2\cdot 4 = 77,$$

where the relation $e(\mathbf{P}) = 3e(S)$ follows because $\pi : \mathbf{P} \to S$ is a $\mathbf{P}^2$-bundle.

The number of conic bundles in the pencil P having a double line equals the number of discriminant curves in the family induced by P in $\overline{\mathcal{R}}_6$, that lie in the ramification divisor Δ_0^{ram} of the projection map $\pi : \overline{\mathcal{R}}_6 \to \overline{\mathcal{M}}_6$. We choose general conic bundles $Q_1, Q_2 \in P$, and let $A = \big(a_{ij}(x_1,x_2,x_3)\big)_{i,j=1}^3$ and $B = \big(b_{ij}(x_1,x_2,x_3)\big)_{i,j=1}^3$ be the symmetric matrices of quadratic forms giving rise to Prym curves $(\Gamma_1,\eta_1) := \mathfrak{d}(Q_1)$ and $(\Gamma_2,\eta_2) := \mathfrak{d}(Q_2) \in \mathcal{P}_6$ respectively. Note that both curves Γ_1 and Γ_2 are nodal precisely at the points $u_1,\dots,u_4$. Let us consider the surface

$$Y := \Big\{\Big([x_1 : x_2 : x_3],[t_1 : t_2]\Big) \in \mathbf{P}^2 \times \mathbf{P}^1 : \det\big((t_1 a_{ij} + t_2 b_{ij})(x_1,x_2,x_3)\big) = 0\Big\},$$

together with the projection $\gamma : Y \to \mathbf{P}^1$. If $h_1, h_2 \in CH^1(\mathbf{P}^2 \times \mathbf{P}^1)$ are the pull-backs of the hyperplane classes under the two projections, then $Y \equiv 6h_1 + 3h_2$. Therefore $\omega_Y = \mathcal{O}_Y(3h_1 + h_2)$ and $h^0(Y,\omega_Y) = 20$. Observe that the surface Y is singular along the curves $L_j := \{u_j\} \times \mathbf{P}^1$ for $j = 1,\dots,4$, and let $\nu_Y : \mathcal{Y} \to Y$ be the normalization. From the exact sequence

$$0 \longrightarrow H^0(\mathcal{Y},\omega_{\mathcal{Y}}) \longrightarrow H^0(Y,\omega_Y) \longrightarrow \bigoplus_{j=1}^4 H^0(L_j,\omega_{Y|L_j}) \longrightarrow 0,$$

taking also into account that $\omega_{Y|L_j} = \mathcal{O}_{L_j}(1)$, we compute that $h^0(\mathcal{Y},\omega_{\mathcal{Y}}) = 12$, and hence $\chi(\mathcal{Y},\mathcal{O}_{\mathcal{Y}}) = 13$. The morphism $\tilde{\gamma} := \gamma \circ \nu_Y : \mathcal{Y} \to \mathbf{P}^1$ is a family of Prym curves of genus 6 and it induces a moduli map $m(\tilde{\gamma}) : \mathbf{P}^1 \to \overline{\mathcal{R}}_6$.

The points $u_1,\dots,u_4 \in \mathbf{P}^2$ being general, the curve $\mathfrak{e} := m(\tilde{\gamma})(\mathbf{P}^1) \subset \overline{\mathcal{R}}_6$ is disjoint from the pull-back $\pi^*(\overline{\mathcal{GP}}_6) \subset \overline{\mathcal{R}}_6$ of the Gieseker-Petri divisor consisting of curves of genus 6 lying on a singular quintic del Pezzo surface, see [9] for details on the geometry of $\pi^{-1}(\overline{\mathcal{GP}}_6)$. Since $\pi^*\big([\overline{\mathcal{GP}}_6]\big)_{|\widetilde{\mathcal{R}}_6} = 94\lambda - 12(\delta_0' + \delta_0'' + 2\delta_0^{\mathrm{ram}}) \in CH^1(\widetilde{\mathcal{R}}_6)$, and $\mathfrak{e}\cdot\delta_0' = 77$ (this being the already computed number of nodal conic bundles in P), whereas $\mathfrak{e}\cdot\delta_0''=0$, we obtain the following relation

$$47\mathfrak{e}\cdot\lambda-6\mathfrak{e}\cdot\delta_0'-12\mathfrak{e}\cdot\delta_0^{\mathrm{ram}}=0.$$

Finally, we observe that $\mathfrak{e}\cdot\lambda=\chi(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})+g-1=18$, which leads to $\mathfrak{e}\cdot\delta_0^{\mathrm{ram}}=32$. □

3. A sweeping rational curve in the boundary of $\overline{\mathcal{A}}_6$

In this section we construct an explicit sweeping rational curve in $\widetilde{\mathcal{C}}^5$, whose numerical properties we shall use in order to bound the slope of $\overline{\mathcal{A}}_6$. Before doing that, we quickly review basic facts concerning the moduli space $\overline{\mathcal{R}}_g$ of stable Prym curves of genus g, while referring to [10] for details.

Geometric points of $\overline{\mathcal{R}}_g$ correspond to triples (X,η,β), where X is a quasi-stable curve of arithmetic genus g, η is a line bundle on X of degree 0, such that $\eta_E=\mathcal{O}_E(1)$ for each smooth rational component $E\subset X$ with $|E\cap\overline{(X-E)}|=2$, and $\beta:\eta^{\otimes 2}\to\mathcal{O}_X$ is a sheaf homomorphism whose restriction to any non-exceptional component of X is an isomorphism. Denoting by $\pi:\overline{\mathcal{R}}_g\to\overline{\mathcal{M}}_g$ the forgetful map, one has the following formula [10] Example 1.4

$$\pi^*(\delta_0)=\delta_0'+\delta_0''+2\delta_0^{\mathrm{ram}}\in CH^1(\overline{\mathcal{R}}_g), \tag{5}$$

where $\delta_0':=[\Delta_0']$, $\delta_0'':=[\Delta_0'']$, and $\delta_0^{\mathrm{ram}}:=[\Delta_0^{\mathrm{ram}}]$ are boundary divisor classes on $\overline{\mathcal{R}}_g$ whose meaning we recall. Let us fix a general point $[C]\in\Delta_0$ corresponding to a smooth 2-pointed curve (N,x,y) of genus $g-1$ with normalization map $\nu:N\to C$, where $\nu(x)=\nu(y)$. A general point of Δ_0' (respectively of Δ_0'') corresponds to a stable Prym curve $[C,\eta]$, where $\eta\in\mathrm{Pic}^0(C)[2]$ and $\nu^*(\eta)\in\mathrm{Pic}^0(N)$ is non-trivial (respectively, $\nu^*(\eta)=\mathcal{O}_N$). A general point of Δ_0^{ram} is of the form (X,η), where $X:=N\cup_{\{x,y\}}\mathbf{P}^1$ is a quasi-stable curve of arithmetic genus g, whereas $\eta\in\mathrm{Pic}^0(X)$ is a line bundle characterized by $\eta_{\mathbf{P}^1}=\mathcal{O}_{\mathbf{P}^1}(1)$ and $\eta_N^{\otimes 2}=\mathcal{O}_N(-x-y)$. Throughout this paper, we only work on the partial compactification $\widetilde{\mathcal{R}}_g:=\pi^{-1}(\mathcal{M}_g\cup\Delta_0^0)$ of $\mathcal{R}_g$, where Δ_0^0 is the open subvariety of Δ_0 consisting of irreducible one-nodal curves. We denote by δ_0',δ_0'' and δ_0^{ram} the restrictions of the corresponding boundary classes to $\widetilde{\mathcal{R}}_g$. Note that $CH^1(\widetilde{\mathcal{R}}_g)=\mathbf{Q}\langle\lambda,\delta_0',\delta_0'',\delta_0^{\mathrm{ram}}\rangle$.

Recall that we use the identification $\mathbf{P}^{15}:=\left|\mathcal{I}^2_{\{w_1,\ldots,w_4\}}(2,2)\right|=|\mathcal{O}_{\mathbf{P}}(2)|$ for the linear system of $(2,2)$ threefolds in $\mathbf{P}^2\times\mathbf{P}^2$ which are nodal at $w_1,\ldots,w_4$. Recall also that $\mathbf{P}\to S$ is the $\mathbf{P}^2$-bundle constructed in Section 2.

We start constructing a sweeping curve $i:\mathbf{P}^1\to\widetilde{\mathcal{C}}^5$, by fixing general points $(o_1,\ell_1),\ldots,(o_4,\ell_4)\in\mathbf{P}^2\times(\mathbf{P}^2)^\vee$ and a general point $o\in\mathbf{P}^2$. We introduce the net

$$T:=\Big\{Q\in\mathbf{P}^{15}:(o,o)\in Q\ \text{ and } \{o_j\}\times\ell_i\subset Q \text{ for } j=1,\ldots,4\Big\},$$

consisting of conic bundles containing the lines $\{o_1\}\times\ell_1,\ldots,\{o_4\}\times\ell_4$ and passing through the point $(o,o)\in\mathbf{P}^2\times\mathbf{P}^2$. Because of the genericity of our choices, the restriction

$$\mathrm{res}|_{\{o\}\times\mathbf{P}^2}:T\to\left|\mathcal{O}_{\{o\}\times\mathbf{P}^2}(2)\right|$$

is an injective map and we can view T as a general net of conics in $\mathbf{P}^2$ passing through the fixed point $o\in\mathbf{P}^2$. The discriminant curve of the net is a nodal cubic curve $\Delta_T\subset T$; its singularity corresponds to the only conic of type $\ell_0 + m_0 \in T$, consisting of a pair of lines ℓ_0 and m_0 passing through o.

To ease notation, we identify $\{o\} \times \mathbf{P}^2$ with $\mathbf{P}^2$ in everything that follows. Denoting by $\mathbf{P}^1 := \mathbf{P}(T_o(\mathbf{P}^2))$ the pencil of lines through o, it is clear that the map

$$\tau : \mathbf{P}^1 \to \Delta_T, \quad \tau(\ell) := Q_\ell \in T, \text{ such that } Q_\ell \supset \{o\} \times \ell,$$

is the normalization map of Δ_T. In particular, we have $\tau(\ell_0) = \tau(m_0) = \ell_0 + m_0$, where, abusing notation, we identify Q_ℓ with its singular conic $\{o\} \times (\ell + m) = Q_\ell \cdot \big(\{o\} \times \mathbf{P}^2\big)$. For $\ell \in \mathbf{P}^1$, the double cover $f_\ell : \widetilde{\Gamma}_\ell \to \Gamma_\ell$ over the discriminant curve Γ_ℓ of Q_ℓ is an element of $\mathscr{P}_6$ (see Definition 1.1). Clearly $\widetilde{\Gamma}_\ell$ carries the marked points $\ell_1, \ell_2, \ell_3, \ell_4$ and ℓ. This procedure induces a moduli map into the universal symmetric product

$$i : \mathbf{P}^1 \to \widetilde{\mathscr{C}}^5, \quad i(\ell) := \big[\rho(\widetilde{\Gamma}_\ell/\Gamma_\ell),\ \ell_1,\ \ell_2,\ \ell_3,\ \ell_4,\ \ell\big].$$

We explicitly construct the family of discriminant curves Γ_ℓ of the conic bundles Q_ℓ, where $\tau(\ell) \in \Delta_T$. Setting coordinates $x := [x_1 : x_2 : x_3], y := [y_1 : y_2 : y_3]$ in $\mathbf{P}^2$, let

$$Z := \Big\{(x, y, t) \in \mathbf{P}^2 \times \mathbf{P}^2 \times \Delta_T : y \in \operatorname{Sing}\big(\pi_1^{-1}(x) \cap Q_t\big)\Big\} \subset \mathbf{P}^2 \times \mathbf{P}^2 \times T.$$

Concretely, if Q_1, Q_2, Q_3 is a basis of T, then the surface Z is given by the equations

$$\frac{\partial}{\partial y_i}\Big(t_1 Q_1(x, y) + t_2 Q_2(x, y) + t_3 Q_3(x, y)\Big) = 0, \quad \text{for } i = 1, 2, 3,$$

where $[t_1 : t_2 : t_3] \in \mathbf{P}^2$ gives rise to the point $t \in T$, once the basis Q_1, Q_2, Q_3 of T has been chosen. It follows immediately that Z is a complete intersection of three divisors of multidegree $(2, 1, 1)$, defined by the partial derivatives, and the divisor $\mathbf{P}^2 \times \mathbf{P}^2 \times \Delta_T$ of multidegree $(0, 0, 3)$.

Lemma 3.1. – *The first projection $\gamma_1 : Z \to \boldsymbol{P}^2$ is a map of degree* 9.

Proof. – Denoting by $h_1, h_2, h_3 \in \operatorname{Pic}(\mathbf{P}^2 \times \mathbf{P}^2 \times T)$ the pull-backs of the hyperplane bundles from the three factors, we find that $\deg(\gamma_1) = (2h_1 + h_2 + h_3)^3 \cdot (3h_3) \cdot h_1^2 = 9$. □

The third projection $\gamma_3 : Z \to \Delta_T$ is a birational model of the family $\{C_\ell\}_{\ell \in \mathbf{P}^1}$ of underlying genus 6 curves, induced by the map $i : \mathbf{P}^1 \to \widetilde{\mathscr{C}}^5$. However, the surface Z is not normal. It has singularities along the curves $\{w_j\} \times \Delta_T$ for $j = 1, \ldots, 4$, as well as along the fibre $\gamma_3^{-1}(\ell_0 + m_0)$ over the point $\tau(\ell_0) = \tau(m_0) = \ell_0 + m_0 \in T$. To construct a smooth model of Z, we pass instead to its natural birational model in the 5-fold $\mathbf{P} \times \mathbf{P}^1$.

Abusing notation, we still denote by $Q_\ell \subset \mathbf{P}$ the strict transform of the conic bundle Q_ℓ in $\mathbf{P}^2 \times \mathbf{P}^2$; its discriminant curve C_ℓ is viewed as an element of $|-K_S|$. We denote by $\pi_\ell : Q_\ell \to S$ the restriction of $\pi : \mathbf{P} \to S$, then consider the surface

$$\mathscr{Z} := \Big\{(z, \ell) \in \mathbf{P} \times \mathbf{P}^1 : z \in \operatorname{Sing}\, \pi_\ell^{-1}(C_\ell)\Big\}.$$

Clearly $\mathcal{Z}$ is endowed with the projection $q_{\mathbf{P}^1} : \mathcal{Z} \to \mathbf{P}^1$. We have the following commutative diagram, where $u := \big(\sigma \times \mathrm{id}_{\mathbf{P}^2}\big) \circ \epsilon^{-1} : \mathbf{P} \to \mathbf{P}^2 \times \mathbf{P}^2$ and the horizontal arrows are the natural inclusions or projections and $\nu_Z : \mathcal{Z} \to Z$ is the normalization map:

$$\begin{tikzcd}
\mathcal{Z} \arrow[r] \arrow[d, "\nu_Z"'] \arrow[rr, bend left, "q_{\mathbf{P}^1}"] & \mathbf{P} \times \mathbf{P}^1 \arrow[r] \arrow[d, "u \times \tau"] & \mathbf{P}^1 \arrow[d, "\tau"] \\
Z \arrow[r] \arrow[rr, bend right, "\gamma_3"'] & \mathbf{P}^2 \times \mathbf{P}^2 \times \Delta_T \arrow[r] & \Delta_T .
\end{tikzcd}$$

Since $u \times \tau$ is birational, it follows that $\deg(\mathcal{Z}/S) = \deg(Z/\mathbf{P}^2) = 9$. The fibration $q_{\mathbf{P}^1} : \mathcal{Z} \to \mathbf{P}^1$ admits sections

$$\tau_j : \mathbf{P}^1 \to \mathcal{Z} \;\text{ for } j = 1, \ldots, 5,$$

which we now define. For $1 \leq j \leq 4$ and each $\ell \in \mathbf{P}^1$, the fibre $Q_\ell \cdot \big(\{o_j\} \times \mathbf{P}^2\big)$ contains the line ℓ_j. Hence ℓ_j defines a point in the covering curve of $f_\ell : \widetilde{C}_\ell \to C_\ell$. By definition $\tau_j(\ell)$ is this point. Tautologically, $\tau_5(\ell)$ is the point corresponding to the line ℓ.

Finally, we consider the universal family $\mathcal{Q} \subset \mathbf{P} \times T$ defined by T. The pull-back of the projection $\mathcal{Q} \to T$ by the morphism $\mathrm{id}_{\mathbf{P}} \times \tau$ induces a flat family of conic bundles $\mathcal{Q}' \subset \mathbf{P} \times \mathbf{P}^1$ and a projection $q' : \mathcal{Q}' \to \mathbf{P}^1$. Clearly, $\mathcal{Z} \subset \mathcal{Q}'$ and $q_{\mathbf{P}^1} = q'|_{\mathcal{Z}}$.

Definition 3.2. – A conic bundle $Q \in |\mathcal{O}_{\mathbf{P}}(2)|$ is said to be *ordinary* if both Q and its discriminant curve C are nodal. A subvariety in $|\mathcal{O}_{\mathbf{P}}(2)|$ is said to be a *Lefschetz family*, if each of its members is an ordinary conic bundle.

Postponing the proof, we assume that the fibration $q' : \mathcal{Q} \to \mathbf{P}^1$ constructed above is a Lefschetz family of conic bundles, and we determine the properties of the Prym moduli map $m : \mathbf{P}^1 \to \overline{\mathcal{R}}_6$, where $m(\ell) := [f_\ell : \widetilde{C}_\ell \to C_\ell] = \rho\big(\widetilde{\Gamma}_\ell/\Gamma_\ell\big)$.

Proposition 3.3. – *The numerical features of* $m : \boldsymbol{P}^1 \to \widetilde{\mathcal{R}}_6 \subset \overline{\mathcal{R}}_6$ *are as follows:*

$$m(\boldsymbol{P}^1) \cdot \lambda = 9 \cdot 6,\; m(\boldsymbol{P}^1) \cdot \delta_0' = 3 \cdot 77,\; m(\boldsymbol{P}^1) \cdot \delta_0^{\mathrm{ram}} = 3 \cdot 32,\; m(\boldsymbol{P}^1) \cdot \delta_0'' = 0.$$

Proof. – We consider the composite map $\rho \circ \mathfrak{d}|_T : T \dashrightarrow \overline{\mathcal{R}}_6$, assigning to a conic bundle from the net $T \subset \mathbf{P}^{15}$ the double covering of its (normalized) discriminant curve. This map is well-defined outside the codimension two locus in T corresponding to conic bundles with non-nodal discriminant. Furthermore, $m = \rho \circ \mathfrak{d} \circ \tau : \mathbf{P}^1 \to \overline{\mathcal{R}}_6$, where we recall that $\tau(\mathbf{P}^1) = \Delta_T \subset T$ is a nodal cubic curve. It follows that the intersection number of $m(\mathbf{P}^1) \subset \overline{\mathcal{R}}_6$ with any divisor class on $\overline{\mathcal{R}}_6$ is three times the intersection number of the corresponding class in $CH^1(\overline{\mathcal{R}}_6)$ with the curve of discriminants induced by a *pencil* of conic bundles in $|\mathcal{O}_{\mathbf{P}}(2)|$. The latter numbers have been determined in Theorem 2.12. □

Definition 3.4. – An irreducible variety X is said to be swept by an irreducible curve Γ on X, if Γ flatly deforms in a family of curves $\{\Gamma_t\}_{t \in T}$ on X such that for a general point $x \in X$, there exists $t \in T$ with $x \in \Gamma_t$.

The composition of the map $i : \mathbf{P}^1 \to \widetilde{\mathcal{C}}^5$ with the projection $\widetilde{\mathcal{C}}^5 \to \overline{\mathcal{R}}_6$ is the map $m : \mathbf{P}^1 \to \overline{\mathcal{R}}_6$ discussed in Proposition 3.3. We discuss the numerical properties of i:

PROPOSITION 3.5. – *The moduli map* $i : \boldsymbol{P}^1 \to \widetilde{\mathcal{C}}^5$ *induced by the pointed family of Prym curves*

$$\big(q_{\boldsymbol{P}^1} : \mathcal{Z} \to \boldsymbol{P}^1, \tau_1, \ldots, \tau_5 : \boldsymbol{P}^1 \to \mathcal{Z}\big)$$

sweeps the five-fold product $\widetilde{\mathcal{C}}^5$. *Furthermore* $i(\boldsymbol{P}^1) \cdot \psi_{x_j} = 9$, *for* $j = 1, \ldots, 5$.

Proof. – For $1 \le j \le 4$, the image of the section $\tilde{\tau}_j := \nu_Z \circ \tau_j : \mathbf{P}^1 \to Z$ is the curve

$$L_j := \big\{(o_j,\ y_j(\ell),\ \nu(\ell)) \in \mathbf{P}^2 \times \mathbf{P}^2 \times T : \ell \in \mathbf{P}^1\big\},$$

where $y_j(\ell) = \ell_j \cap m_j(\ell)$, with $m_j(\ell)$ being the line in $\mathbf{P}^2$ defined by the equality of cycles $Q_\ell \cdot \big(\{o_j\} \times \mathbf{P}^2\big) = \{o_j\} \times (\ell_j + m_j(\ell))$. Here, recall that $\ell \in \mathbf{P}^1 = \mathbf{P}(T_o(\mathbf{P}^2))$, is a point corresponding to a line in $\mathbf{P}^2$ passing through o. In particular, noting that by the adjunction formula $\omega_Z = \mathcal{O}_Z(3h_1 + 3h_3)$, we compute that $L_j \cdot h_1 = 0$ and $L_j \cdot h_3 = 3$, hence $L_j \cdot \omega_Z = L_j \cdot (3h_1 + 3h_3) = 9$.

By definition $i(\mathbf{P}^1) \cdot \psi_{x_j} = \tau_j^*\big(c_1(\omega_{q_{\mathbf{P}^1}})\big)$. To evaluate the dualizing class, we note that $\omega_{q_{\mathbf{P}^1}} = \omega_{\mathcal{Z}} \otimes q_{\mathbf{P}^1}^*(T_{\mathbf{P}^1})$, therefore $\deg \tau_j^*\big(c_1(\omega_{q_{\mathbf{P}^1}})\big) = \deg \tau_j^*(\omega_{\mathcal{Z}}) + 2$. Furthermore,

$$\nu_Z^*(\omega_Z) = \omega_{\mathcal{Z}} \otimes \mathcal{O}_{\mathcal{Z}}\big(q_{\mathbf{P}^1}^{-1}(\ell_0) + q_{\mathbf{P}^1}^{-1}(m_0) + D\big),$$

where $D \subset \mathcal{Z}$ is a curve disjoint from $\nu_Z^{-1}(L_j)$. We compute that

$$\deg \tau_j^*(\omega_{\mathcal{Z}}) = \deg \tilde{\tau}_j^*(\omega_Z) - \deg \tau_j^* q_{\mathbf{P}^1}^*(\ell_0) - \deg \tau_j^* q_{\mathbf{P}^1}^*(m_0) = \omega_Z \cdot L_j - 2,$$

and finally, $i(\mathbf{P}_1) \cdot \psi_{x_j} = \omega_Z \cdot L_j = 9$. The calculation of $i(\mathbf{P}^1) \cdot \psi_{x_5}$ is largely similar and we skip it. □

Proof of the claim. – We show that $q' : \mathcal{Q} \to \mathbf{P}^1$ is a Lefschetz family, that is, it consists entirely of ordinary conic bundles. For $1 \le j \le 4$, let $\ell'_j \subset \mathbf{P}$ be the inverse image of the line $\{o_j\} \times \ell_j$ under the map $u : \mathbf{P} \dashrightarrow \mathbf{P}^2 \times \mathbf{P}^2$ and set $W := \big|\mathcal{I}_{\{\ell'_1, \ldots, \ell'_4\}}(2)\big| \subset |\mathcal{O}_{\mathbf{P}}(2)|$. The net $T := T_o$ of conic bundles passing through the point $(o, o) \in \mathbf{P}^2 \times \mathbf{P}^2$ is a plane in W. Let Δ_{no} denote the locus of non-ordinary conic bundles $Q \in W$. We aim to show that $\Delta_{\mathrm{no}} \cap \Delta_{T_o} = \varnothing$, for a general point $o \in \mathbf{P}^2$.

We consider the incidence correspondence

$$\Sigma := \Big\{\big(Q, (o, \ell)\big) \in \Delta_{\mathrm{no}} \times \mathbf{G} : \{o\} \times \ell \subset u(Q),\ \ o \in \ell\Big\}$$

together with the projection map $p_1 : \Sigma \to \Delta_{\mathrm{no}}$. Over a conic bundle $Q \in \Delta_{\mathrm{no}}$ for which the image $u(Q) \subset \mathbf{P}^2 \times \mathbf{P}^2$ is transversal to a general fibre $\{o\} \times \mathbf{P}^2$, the fibre $p_1^{-1}(Q)$ is finite. To account for the conic bundles not enjoying this property, we define Δ_{hr} to be the union of the irreducible components of Δ_{no} consisting of conic bundles $Q \in W$ such that the branch locus of $Q \to S$ is equal to S.

To conclude that $\Delta_{\mathrm{no}} \cap \Delta_{T_o} = \varnothing$ for a general $o \in \mathbf{P}^2$, it suffices to show that (1) Δ_{no} has codimension at least 2 in W, and (2) Δ_{hr} has codimension at least 3 in W. The next two lemmas are devoted to the proof of these assertions. □

LEMMA 3.6. – Δ_{no} *has codimension at least* 2 *in* W.

Proof. – We have established that $h : \mathbf{P} \to \mathbf{P}^4$ is a morphism of degree two. We claim that the 4 lines $l_i := h(\ell'_i) \subset \mathbf{P}^4$ are general, in the sense that $V := \left|\mathcal{I}_{\{l_1,\ldots,l_4\}}(2)\right|$ is a net of quadrics. Granting this and denoting by $L_{ij} \in H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(1))$ the linear form vanishing along $l_i \cup l_j$, the space V is generated by the quadrics $L_{12} \cdot L_{34}, L_{13} \cdot L_{24}$ and $L_{14} \cdot L_{23}$ respectively. The base locus bs $|V|$ of the net is a degenerate canonical curve of genus 5, which is a union of 8 lines, namely $l_1, \ldots, l_4$ and $b_1, \ldots, b_4$, where if $\{1,2,3,4\} = \{i,j,k,l\}$, then the line $b_l \subset \mathbf{P}^4$ is the common transversal to the lines l_i, l_j and l_k. Then by direct calculation, the pull-back P of a general pencil in V is a Lefschetz family of conic budles in $|\mathcal{O}_{\mathbf{P}}(2)|$. Since $P \cap \Delta_{\rm no} = \varnothing$, it follows that $\operatorname{codim}(\Delta_{\rm no}, W) \geq 2$. It remains to show that the lines $l_1, \ldots, l_4$ are general. To that end, we observe that the construction can be reversed. Four general lines $m_1, \ldots, m_4 \in \mathbf{G}(1,4) \subset \mathbf{P}^9$ define a codimension 4 linear section S' of $\mathbf{G}(1,4)$ which is isomorphic to S. The projectivized universal bundle $\mathbf{P}' \to S'$ is a copy of $\mathbf{P}$ and the projection $h' : \mathbf{P}' \to \mathbf{P}^4$ is the tautological map. This completes the proof. ☐

The second lemma follows from a direct analysis in $\mathbf{P}^2 \times \mathbf{P}^2$.

LEMMA 3.7. – *$\Delta_{\rm hr}$ has codimension at least 3 in W.*

Proof. – If Q is a general element of an irreducible component of $\Delta_{\rm hr}$, then the discriminant locus of the projection $p : Q \to S$ equals S, and necessarily $Q = D + D'$, where $p(D) = p(D') = S$. By a dimension count, it follows that W contains only *finitely many* elements $Q \in \Delta_{\rm hr}$, such that $D, D' \in |\mathcal{O}_{\mathbf{P}}(1)|$, and assume that we are not in this case.

Recall that $h' : \mathbf{P}^2 \times \mathbf{P}^2 \dashrightarrow \mathbf{P}^4$ is the map defined by $\left|\mathcal{I}_{\{w_1,\ldots,w_4\}}(1,1)\right|$. The case when both $u(D), u(D') \in |\mathcal{I}_{\{w_1,\ldots,w_4\}}(1,1)|$ having been excluded, we may assume that one of the components of $u(Q)$, say $u(D) \subset \mathbf{P}^2 \times \mathbf{P}^2$, has type $(0,1)$. In particular, $u(D) = \mathbf{P}^2 \times n$, where $n \subset \mathbf{P}^2$ is a line. Observe that $u(D)$ has degree three in the Segre embedding $\mathbf{P}^2 \times \mathbf{P}^2 \subset \mathbf{P}^8$ and the base scheme of $\left|\mathcal{I}_{\{w_1,\ldots,w_4\}}(1,1)\right|$ consists of the simple points $w_1, \ldots, w_4$. Since $h'(D)$ lies on a quadric, it follows $u(D) \cap \{w_1, \ldots, w_4\} \neq \varnothing$, therefore we have $u_i \in n$ for some i, say $i = 4$. Since the lines $\{o_i\} \times \ell_i$ are general, they do not lie on $u(D)$, for $\ell_i \neq n$. Hence $u(D') \subset \mathbf{P}^2 \times \mathbf{P}^2$ is a $(2,1)$ hypersurface which contains $\{o_1\} \times \ell_1, \ldots \{o_4\} \times \ell_4$, is singular at w_1, w_2, w_3 and such that $w_4 \in u(D')$. This contradicts the generality of the lines $\{o_i\} \times \ell_i$. ☐

4. The slope of $\overline{\mathcal{A}}_6$

For $g \geq 2$, let $\overline{\mathcal{A}}_g$ be the first Voronoi compactification of $\mathcal{A}_g$ — this is the toroidal compactification of $\mathcal{A}_g$ constructed using the perfect fan decomposition, see [20]. The rational Picard group of $\overline{\mathcal{A}}_g$ has rank 2 and it is generated by the first Chern class λ_1 of the Hodge bundle and the class of the *irreducible* boundary divisor $D = D_g := \overline{\mathcal{A}}_g - \mathcal{A}_g$. Following Mumford [18], we consider the moduli space $\widetilde{\mathcal{A}}_g$ of principally polarized abelian varieties of dimension g together with their rank 1 degenerations. Precisely, if $\xi : \overline{\mathcal{A}}_g \to \mathcal{A}_g^s = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \cdots \sqcup \mathcal{A}_1 \sqcup \mathcal{A}_0$ is the projection from the toroidal to the Satake compactification of $\mathcal{A}_g$, then

$$\widetilde{\mathcal{A}}_g := \overline{\mathcal{A}}_g - \xi^{-1}\Big(\bigcup_{j=2}^{g} \mathcal{A}_{g-j}\Big) := \mathcal{A}_g \sqcup \widetilde{D}_g,$$

where $\widetilde{D}_g$ is an open dense subvariety of D_g isomorphic to the universal Kummer variety $\mathrm{Kum}(\mathcal{X}_{g-1}) := \mathcal{X}_{g-1}/\pm$. Furthermore, if $\phi : \widetilde{\mathcal{X}}_{g-1} \to \widetilde{\mathcal{A}}_{g-1}$ is the extended universal abelian variety, there exists a degree two morphism $j : \widetilde{\mathcal{X}}_{g-1} \to \overline{\mathcal{A}}_g$, extending the Kummer map $\widetilde{\mathcal{X}}_{g-1} \stackrel{2:1}{\to} \widetilde{D}_g$. The geometry of the boundary divisor $\partial\widetilde{\mathcal{X}}_{g-1} = \phi^{-1}(\mathrm{Kum}(\mathcal{X}_{g-2}))$ is discussed in [13] and [8]. In particular, $\mathrm{codim}(j(\partial\widetilde{\mathcal{X}}_{g-1}), \overline{\mathcal{A}}_g) = 2$. As usual, let $\mathbb{E}_g$ denote the Hodge bundle on $\overline{\mathcal{A}}_g$.

Denoting by $\varphi : \widetilde{\mathcal{Y}}_g \to \widetilde{\mathcal{R}}_g$ the universal Prym variety restricted to the partial compactification $\widetilde{\mathcal{R}}_g$ of $\overline{\mathcal{R}}_g$ introduced in Section 3, we have the following commutative diagram summarizing the situation, where the lower horizontal arrow is the Prym map:

$$\begin{tikzcd}
\widetilde{\mathcal{Y}}_g \arrow[r, "\chi"] \arrow[d, "\varphi"'] & \widetilde{\mathcal{X}}_{g-1} \arrow[r, "j"] \arrow[d, "\phi"'] & \overline{\mathcal{A}}_g \\
\widetilde{\mathcal{R}}_g \arrow[r, "P"] & \widetilde{\mathcal{A}}_{g-1}. &
\end{tikzcd}$$

Furthermore, let us denote by $\theta \in CH^1(\widetilde{\mathcal{X}}_{g-1})$ the class of the universal theta divisor trivialized along the zero section and by $\theta_{\mathrm{pr}} := \chi^*(\theta) \in CH^1(\widetilde{\mathcal{Y}}_g)$ the Prym theta divisor. The following formulas have been pointed out to us by Sam Grushevsky:

Proposition 4.1. – *The following relations at the level of divisor classes hold:*

1. $j^*([D]) = -2\theta + \phi^*([D_{g-1}]) \in CH^1(\widetilde{\mathcal{X}}_{g-1})$.
2. $(j \circ \chi)^*(\lambda_1) = \varphi^*\big(\lambda - \frac{1}{4}\delta_0^{\mathrm{ram}}\big) \in CH^1(\widetilde{\mathcal{Y}}_g)$.
3. $(j \circ \chi)^*([D]) = -2\theta_{\mathrm{pr}} + \varphi^*(\delta_0') \in CH^1(\widetilde{\mathcal{Y}}_g)$.

Proof. – At the level of the restriction $j : \mathcal{X}_{g-1} \to \overline{\mathcal{A}}_g$, the formula

$$j^*(D) \equiv -2\theta \in CH^1(\mathcal{X}_{g-1})$$

is proven in [18] Proposition 1.8. To extend this calculation to $\widetilde{\mathcal{X}}_{g-1}$, it suffices to observe that the boundary divisor $\partial\widetilde{\mathcal{X}}_{g-1} = \phi^*(\widetilde{D}_{g-1})$ is mapped under j to the locus in $\overline{\mathcal{A}}_g$ parametrizing *rank* 2 degenerations and it will appear with multiplicity one in $j^*(D)$.

To establish relation (ii), we observe that $j^*(\lambda_1) = \phi^*(\lambda_1)$, where we use the same symbol to denote the Hodge class on $\mathcal{A}_g$ and that on $\mathcal{A}_{g-1}$. Indeed, there exists an exact sequence of vector bundles on $\widetilde{\mathcal{X}}_g$, see also [13] p.74:

$$0 \longrightarrow \phi^*(\mathbb{E}_{g-1}) \longrightarrow j^*(\mathbb{E}_g) \longrightarrow \mathcal{O}_{\widetilde{\mathcal{X}}_{g-1}} \longrightarrow 0.$$

It follows that $\chi^* j^*(\lambda_1) = \varphi^* P^*(\lambda_1) = \varphi^*(\lambda - \frac{1}{4}\delta_0^{\mathrm{ram}})$, where $P^*(\lambda_1) = \lambda - \frac{1}{4}\delta_0^{\mathrm{ram}}$, see [10, 14]. Finally, (iii) is a consequence of (i) and of the relation $P^*([\widetilde{D}_{g-1}]) = \delta_0'$, see [14]. $\square$

Assume now that g is an even integer and let $\tilde{\pi} : \widetilde{\mathcal{C}} \to \overline{\mathcal{R}}_g$ be the universal curve of genus $2g-1$, that is, $\widetilde{\mathcal{C}} = \overline{\mathcal{M}}_{2g-1,1} \times_{\overline{\mathcal{M}}_{2g-1}} \overline{\mathcal{R}}_g$, and $\pi : \overline{\mathcal{C}} \to \overline{\mathcal{R}}_g$ the universal curve of genus g, that is, $\overline{\mathcal{C}} = \overline{\mathcal{M}}_{g,1} \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{R}}_g$. There is a degree two map $f : \widetilde{\mathcal{C}} \to \overline{\mathcal{C}}$ unramified in codimension one and an involution $\iota : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}$, such that $f \circ \iota = f$. Note that $\omega_{\tilde{\pi}} = f^*(\omega_\pi)$.

We consider the global Abel-Prym map $\mathfrak{ap}:\widetilde{\mathscr{C}}^{g-1}\dashrightarrow\widetilde{\mathscr{Y}}_g$, defined by

$$\mathfrak{ap}\Big(\widetilde{C}/C,\ x_1,\dots,x_{g-1}\Big):=\Big(\widetilde{C}/C,\ \mathcal{O}_{\widetilde{C}}\big(x_1-\iota(x_1)+\cdots+x_{g-2}-\iota(x_{g-2})+2x_{g-1}-2\iota(x_{g-1})\big)\Big).$$

REMARK 4.2. – We recall that if $\widetilde{C}\to C$ is an étale double cover and $\iota:\widetilde{C}\to\widetilde{C}$ the induced involution, then the Prym variety $P(\widetilde{C}/C)\subset\mathrm{Pic}^0(\widetilde{C})$ can be realized as the locus of line bundles $\mathcal{O}_{\widetilde{C}}(E-\iota(E))$, where E is a divisor on $\widetilde{C}$ having *even* degree, see [2]. Furthermore, for a general point $[\widetilde{C}\to C]\in\overline{\mathscr{R}}_g$, where $g\geq 3$, and for an integer $1\leq n\leq g-1$, the difference map $\widetilde{C}_n\to\mathrm{Pic}^0(\widetilde{C})$ given by $E\mapsto\mathcal{O}_{\widetilde{C}}(E-\iota(E))$ is generically finite. In particular, for even g, the locus

$$Z_{g-2}(\widetilde{C}/C):=\Big\{\mathcal{O}_{\widetilde{C}}(E-\iota(E)):E\in\widetilde{C}_{g-2}\Big\}$$

is a divisor inside $P(\widetilde{C}/C)$. We refer to $Z_{g-2}(\widetilde{C}/C)$ as the *top difference Prym variety*.

One computes the pull-back of the universal theta divisor under the Abel-Prym map. Recall that $\psi_{x_1},\dots,\psi_{x_{g-1}}\in CH^1(\widetilde{\mathscr{C}}^{g-1})$ are the cotangent classes corresponding to the marked points on the curves of genus $2g-1$.

PROPOSITION 4.3. – *For even g, if $\mu=\varphi\circ\mathfrak{ap}:\widetilde{\mathscr{C}}^{g-1}\to\widetilde{\mathscr{R}}_g$ denotes the projection map, one has*

$$\mathfrak{ap}^*(\theta_{\mathrm{pr}})=\frac{1}{2}\sum_{j=1}^{g-2}\psi_{x_j}+2\psi_{x_{g-1}}+0\cdot\Big(\lambda+\mu^*(\delta_0'+\delta_0''+\delta_0^{\mathrm{ram}})\Big)-\cdots\in CH^1(\widetilde{\mathscr{C}}^{g-1}).$$

Proof. – We factor the map $\mathfrak{ap}:\widetilde{\mathscr{C}}^{g-1}\dashrightarrow\widetilde{\mathscr{Y}}_g$ as $\mathfrak{ap}=\mathfrak{aj}\circ\Delta$, where $\Delta:\widetilde{\mathscr{C}}^{g-1}\to\widetilde{\mathscr{C}}^{2g-2}$ is defined by $(x_1,\dots,x_{g-1})\mapsto(x_1,\dots,x_{g-1},\iota(x_1),\dots,\iota(x_{g-1}))$ and $\mathfrak{aj}:\widetilde{\mathscr{C}}^{2g-2}\dashrightarrow\overline{\mathfrak{Pic}}^0_{2g-1}$ is the difference Abel-Jacobi map between the first and the last $g-1$ marked points on each curve into the universal Jacobian of degree zero over $\overline{\mathscr{M}}_{2g-1}$ respectively . There is a generically injective rational map $\widetilde{\mathscr{Y}}_g\dashrightarrow\overline{\mathfrak{Pic}}^0_{2g-1}$, which globalizes the usual inclusion $P(\widetilde{C}/C)\subset\mathrm{Pic}^0(\widetilde{C})$ valid for each Prym curve $[\widetilde{C}\to C]\in\mathscr{R}_g$. Using [15] Theorem 6, one computes the pull-back $\mathfrak{aj}^*(\theta_{2g-1})\in CH^1(\widetilde{\mathscr{C}}^{2g-2})$ of the universal theta divisor θ_{2g-1} on $\overline{\mathfrak{Pic}}^0_{2g-1}$ trivialized along the zero section. Remarkably, the coefficient of λ, as well as that of the δ_0', δ_0'' and δ_0^{ram} classes in this expression, are all zero. This is then pulled back to $\widetilde{\mathscr{C}}^{g-1}$ keeping in mind that the pull-back of θ_{2g-1} to $\widetilde{\mathscr{Y}}_g$ is equal to $2\theta_{\mathrm{pr}}$. Using the formulas $\Delta^*(\psi_{x_j})=\Delta^*(\psi_{y_j})=\psi_{x_j}$, and $\Delta^*(\delta_{0:x_iy_j})=\delta_{0:x_ix_j}$, as well as $\Delta^*(\delta_{0:y_iy_j})=\delta_{0:x_ix_j}$, we conclude. $\square$

REMARK 4.4. – The other boundary coefficients of $\mathfrak{ap}^*(\theta_{\mathrm{pr}})\in CH^1(\widetilde{\mathscr{C}}^{g-1})$ can be determined explicitly, but play no role in our future considerations.

REMARK 4.5. – Restricting ourselves to even g, we consider the restricted (non-dominant) Abel-Prym map $\mathfrak{ap}_{g-2}:\widetilde{\mathscr{C}}^{g-2}\dashrightarrow\widetilde{\mathscr{Y}}_g$ given by

$$\mathfrak{ap}_{g-2}\big(\widetilde{C}/C,\ x_1,\dots,x_{g-2}\big):=\Big(\widetilde{C}/C,\ \mathcal{O}_{\widetilde{C}}\big((x_1-\iota(x_1)+\cdots+x_{g-2}-\iota(x_{g-2})\big)\Big),$$

and obtain the formula: $\mathfrak{ap}_{g-2}^*(\theta_{\mathrm{pr}}) = \frac{1}{2}\sum_{j=1}^{g-2}\psi_{x_j} + 0\cdot\big(\lambda + \mu^*(\delta_0' + \delta_0'' + \delta_0^{\mathrm{ram}})\big) - \cdots$.

The image of $\mathfrak{ap}_{g-2}$ is a divisor $\mathcal{Z}_{g-2}$ on $\widetilde{\mathcal{Y}}_g$ characterized by the property

$$(\mathcal{Z}_{g-2})|_{P(\widetilde{C}/C)} = Z_{g-2}(\widetilde{C}/C),$$

for each $[\widetilde{C}\to C]\in\mathcal{R}_g$. In other words, $\mathcal{Z}_{g-2}$ is the divisor cutting out on each Prym variety the top difference variety. A similar difference variety inside the universal Jacobian over $\overline{\mathcal{M}}_g$ has been studied in [11]. Specializing to the case $g=6$, the locus

$$\mathcal{U}_4 := \overline{(j\circ\chi)(\mathcal{Z}_4)} \subset \overline{\mathcal{A}}_6$$

is a codimension two cycle on $\overline{\mathcal{A}}_6$, which will appear as an obstruction for an effective divisor on $\overline{\mathcal{A}}_6$ to have small slope.

We use these considerations to bound from below the slope of $\overline{\mathcal{A}}_6$.

Proof of Theorem 0.4. – We have seen that the boundary divisor D_6 of $\overline{\mathcal{A}}_6$ is filled up by rational curves $h:\mathbf{P}^1\to D_6$ constructed in Theorem 3.5 by pushing forward the sweeping rational curve $i:\mathbf{P}^1\to\widetilde{\mathcal{C}}^5$ of discriminants of a pencil of conic bundles. In particular, $\gamma := h_*(\mathbf{P}^1)\in NE_1(\overline{\mathcal{A}}_6)$ is an effective class that intersects every non-boundary effective divisor on $\overline{\mathcal{A}}_6$ non-negatively. We compute using Propositions 4.1 and 4.3:

$$\gamma\cdot\lambda_1 = i_*(\mathbf{P}^1)\cdot\mu^*\Big(\lambda - \frac{1}{4}\delta_0^{\mathrm{ram}}\Big) = 6\cdot 9 - \frac{3\cdot 32}{4} = 30, \quad \text{and}$$

$$\gamma\cdot[D_6] = -i_*(\mathbf{P}^1)\cdot\Big(\sum_{j=1}^4\psi_{x_j} + 4\psi_{x_5}\Big) + i_*(\mathbf{P}^1)\cdot\mu^*(\delta_0') = -8\cdot 9 + 3\cdot 77 = 159.$$

We obtain the bound $s(\overline{\mathcal{A}}_6)\ge\frac{\gamma\cdot[D_6]}{\gamma\cdot\lambda_1} = \frac{53}{10}$. ☐

For effective divisors on $\overline{\mathcal{A}}_6$ transversal to $\mathcal{U}_4$, we obtain a better slope bound:

THEOREM 4.6. – *If E is an effective divisor on $\overline{\mathcal{A}}_6$ not containing the universal codimension two Prym difference variety $\mathcal{U}_4\subset\overline{\mathcal{A}}_6$, then $s(E)\ge\frac{13}{2}$.*

Proof. – We consider the family $(q_{\mathbf{P}^1}:\mathcal{Z}\to\mathbf{P}^1,\ \tau_1,\ldots,\tau_4:\mathbf{P}^1\to\mathcal{Z})$ obtained from the construction explained in Theorem 3.5, where we retain only the first four sections. We obtain an induced moduli map $i_4:\mathbf{P}^1\to\widetilde{\mathcal{C}}^4$. Pushing i_4 forward via the Abel-Prym map, we obtain a curve $h_4:\mathbf{P}^1\to\mathcal{U}_4\subset\overline{\mathcal{A}}_6$, which fills up the locus $\mathcal{U}_4$. Thus $\gamma_4 := (h_4)_*(\mathbf{P}^1)\in NE_1(\overline{\mathcal{A}}_6)$ is an effective class which intersects non-negatively any effective divisor on $\overline{\mathcal{A}}_6$ not containing $\mathcal{U}_4$. We compute using Theorems 3.3 and 3.5:

$$\gamma_4\cdot\lambda_1 = \gamma\cdot\lambda_1 = 30 \quad\text{and}\quad \gamma_4\cdot[D_6] = -4\cdot 9 + 3\cdot 77 = 195.$$ ☐

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