

THE GEOMETRY OF THE MODULI SPACE OF ODD SPIN CURVES

GAVRIL FARKAS AND ALESSANDRO VERRA

The set of odd theta-characteristics on a general curve C of genus g is in bijection with the set $\theta(C)$ of *theta hyperplanes* $H \in (\mathbf{P}^{g-1})^\vee$ everywhere tangent to the canonically embedded curve $C \xrightarrow{|K_C|} \mathbf{P}^{g-1}$. Even though the geometry and the intricate combinatorics of $\theta(C)$ have been studied classically, see [Dol], [DK] for a modern account, it has only been recently proved in [CS] that one can reconstruct a general curve $[C] \in \mathcal{M}_g$ from the hyperplane configuration $\theta(C)$.

Odd theta-characteristics form a moduli space $\pi : \mathcal{S}_g^- \rightarrow \mathcal{M}_g$. At the level of stacks, π is an étale cover of degree $2^{g-1}(2^g-1)$. The normalization of $\overline{\mathcal{M}}_g$ in the function field of \mathcal{S}_g^- gives rise to a finite covering $\pi : \overline{\mathcal{S}}_g^- \rightarrow \overline{\mathcal{M}}_g$. Furthermore, $\overline{\mathcal{S}}_g^-$ has a modular meaning being isomorphic to the coarse moduli space of the Deligne-Mumford stack of odd stable spin curves, cf. [C], [CCC], [AJ]. The map π is branched along the boundary of $\overline{\mathcal{M}}_g$ and one expects $K_{\overline{\mathcal{S}}_g^-}$ to enjoy better positivity properties than $K_{\overline{\mathcal{M}}_g}$.

The aim of this paper is to describe the birational geometry of $\overline{\mathcal{S}}_g^-$ for all g . Our goals are (1) to understand the transition from rationality to maximal Kodaira dimension for $\overline{\mathcal{S}}_g^-$ as g increases, and (2) to use the existence of *Mukai models* of $\overline{\mathcal{M}}_g$ in order to construct explicit unirational parameterizations of $\overline{\mathcal{S}}_g^-$ for small genus. Remarkably, we end up having no gaps in the classification of $\overline{\mathcal{S}}_g^-$. First, we show that in the range where the general curve $[C] \in \mathcal{M}_g$ lies on a $K3$ surface, the existence of special *theta pencils* on $K3$ surfaces, provides an *explicit* uniruled parameterization of $\overline{\mathcal{S}}_g^-$:

Theorem 0.1. *The odd spin moduli space $\overline{\mathcal{S}}_g^-$ is uniruled for $g \leq 11$.*

We fix a general spin curve $[C, \eta] \in \mathcal{S}_g^-$, therefore $h^0(C, \eta) = 1$. When $g \leq 9$ or $g = 11$, the underlying curve C is the hyperplane section of a $K3$ surface $X \subset \mathbf{P}^g$, such that if $d \in C_{g-1}$ is the (unique) effective divisor with $\eta = \mathcal{O}_C(d)$, then the linear span $\langle d \rangle \subset \mathbf{P}^g$ is a codimension two linear subspace. A rational curve $P \subset \overline{\mathcal{S}}_g^-$ is induced by the pencil $\mathbf{P}H^0(X, \mathcal{I}_{d/X}(C))$ of hyperplanes containing $\langle d \rangle$. We show in Section 3 that $P \subset \overline{\mathcal{S}}_g^-$ is a *covering* rational curve, satisfying

$$P \cdot K_{\overline{\mathcal{S}}_g^-} = 2g - 24 < 0.$$

Thus $P \cdot K_{\overline{\mathcal{S}}_g^-} < 0$ precisely when $g \leq 11$, which highlights the fact that the nature of $\overline{\mathcal{S}}_g^-$ is expected to change exactly when $g \geq 12$. This is something we shall achieve in the course of proving Theorem 0.4.

The previous argument no longer works for $\overline{\mathcal{S}}_{10}^-$, when the condition that a curve $[C] \in \overline{\mathcal{M}}_{10}$ lie on a $K3$ surface is divisorial in moduli [FP]. This case is a specialization of the genus 11 case. A general one-nodal irreducible curve $[C] \in \Delta_0 \subset \overline{\mathcal{M}}_{11}$ of arithmetic

genus 11 lies on a $K3$ surface $X \subset \mathbf{P}^{11}$. By a degeneration argument, we show that this construction can also be carried out in such a way, that if $\nu : C' \rightarrow C$ denotes the normalization of C , then the points $x, y \in C'$ with $\nu(x) = \nu(y)$ (that is, mapping to the node of C), lie in the support of the zero locus of one of the odd-theta characteristics of $[C'] \in \mathcal{M}_{10}$. Ultimately, this produces a rational curve $P \subset \overline{\mathcal{S}}_{10}^-$ through a general point, which shows that $\overline{\mathcal{S}}_{10}^-$ is uniruled as well.

In the range in which a *Mukai model* of $\overline{\mathcal{M}}_g$ exists, our results are more precise:

Theorem 0.2. $\overline{\mathcal{S}}_g^-$ is unirational for $g \leq 8$.

The proof relies on the existence, in this range, of *Mukai varieties* $V_g \subset \mathbf{P}^{n_g+g-2}$, where $n_g = \dim(V_g)$, which have the property that general 1-dimensional linear sections of V_g are canonical curves $[C] \in \mathcal{M}_g$ with general moduli. We fix an integer $1 \leq \delta \leq g - 1$ and consider the correspondence

$$\mathcal{P}_{g,\delta}^o := \left\{ (C, \Gamma, Z) : Z \subset C \cap \Gamma \subset V_g, \quad |\text{sing}(\Gamma)| = \delta, \text{sing}(\Gamma) \subset Z \right\},$$

where $Z \subset V_g$ is a *cluster*, that is, a 0-dimensional subscheme of V_g of length $2g - 2$, supported at $g - 1$ points and such that $\dim\langle Z \rangle = g - 2$ (see Section 3 for a precise definition), $\Gamma \subset V_g$ is an irreducible δ -nodal curve section of V_g whose nodes are among the points in the support of Z , and $C \subset V_g$ is an arbitrary curve linear section of V_g containing Z as a subscheme. Thus if C is smooth, then $Z \subset C$ is a divisor of even degree at each point in its support, and $\mathcal{O}_C(Z/2)$ can be viewed as an odd theta-characteristic. The quotient variety $\overline{\mathbb{P}}_{g,\delta} := \mathcal{P}_{g,\delta}^o // \text{Aut}(V_g)$ comes equipped with two projections

$$\overline{\mathcal{S}}_g^- \xleftarrow{\overline{\alpha}} \overline{\mathbb{P}}_{g,\delta} \xrightarrow{\overline{\beta}} B_{g,\delta}^-,$$

where $B_{g,\delta}^- \subset \overline{\mathcal{S}}_g^-$ denotes the moduli space of irreducible δ -nodal curves of arithmetic genus g together with an odd theta-characteristic on the normalization. It is easy to see that $\overline{\mathbb{P}}_{g,\delta}$ is birational to a projective bundle over the irreducible variety $B_{g,\delta}^-$. Thus the unirationality of $\overline{\mathcal{S}}_g^-$ follows once we prove that (i) $\overline{\alpha}$ is dominant, and (ii) $B_{g,\delta}^-$ itself is unirational. We carry out this program when $g \leq 8$. When $\delta = n_g - 1$, we show in Section 3 that the map $\overline{\alpha}$ is birational, hence in this case $\overline{\beta}$ realizes a birational isomorphism between $\overline{\mathcal{S}}_g^-$ and a (Zariski trivial) projective bundle over B_{g,n_g-1}^- . Very interesting is the case $g = 8$, when $n_g = 8$, see [M1], and $B_{8,7}^-$ is isomorphic to the moduli space $\overline{\mathcal{M}}_{1,14}/\mathbb{Z}_2^{\oplus 7}$ of elliptic curves with seven pairs of points; here each copy of \mathbb{Z}_2 identifies a pair of points.

Theorem 0.3. $\overline{\mathcal{S}}_8^-$ is birational to $\mathbf{P}^7 \times (\overline{\mathcal{M}}_{1,14}/\mathbb{Z}_2^{\oplus 7})$.

In the process of proving Theorem 0.2, we establish some facts of independent interest concerning the Mukai models

$$\mathfrak{M}_g := \mathbf{G}(g - 1, n_g + g - 2)^{\text{ss}} // \text{Aut}(V_g).$$

These are birational models of $\overline{\mathcal{M}}_g$ having $\text{Pic}(\mathfrak{M}_g) = \mathbb{Z}$ and appearing as GIT quotients of Grassmannians; they can be viewed as log-minimal models of $\overline{\mathcal{M}}_g$ emerging from the constructions carried out in [M1], [M2], [M3].

Theorem 0.1 is sharp and the remaining moduli spaces $\overline{\mathcal{S}}_g^-$ are of general type:

Theorem 0.4. *The space $\overline{\mathcal{S}}_g^-$ is a variety of general type for $g > 11$.*

The border case of $\overline{\mathcal{S}}_{12}^-$ is particularly challenging and takes up the entire Section 6. We remark that in the range $11 < g < 17$, of the two moduli spaces $\overline{\mathcal{S}}_g^-$ and $\overline{\mathcal{M}}_g$, one is of general type whereas the other has negative Kodaira dimension. More strikingly, Theorems 0.4 and 0.1 coupled with results from [F3], show that for $9 \leq g \leq 11$, the space $\overline{\mathcal{S}}_g^-$ is uniruled while $\overline{\mathcal{S}}_g^+$ is of general type! Finally, we note that $\overline{\mathcal{S}}_8^-$ is unirational whereas $\overline{\mathcal{S}}_8^+$ is of Calabi-Yau type [FV].

We describe the main steps in the proof of Theorem 0.4. First, we use that for all $g \geq 4$ and $\ell \geq 0$, if $\epsilon : \widehat{\mathcal{S}}_g \rightarrow \overline{\mathcal{S}}_g^-$ denotes a resolution of singularities, then there is an induced isomorphism at the level of global sections, see [Lud]

$$\epsilon^* : H^0\left(\overline{\mathcal{S}}_{g,\text{reg}}^-, K_{\overline{\mathcal{S}}_g^-}^{\otimes \ell}\right) \xrightarrow{\sim} H^0\left(\widehat{\mathcal{S}}_g, K_{\widehat{\mathcal{S}}_g}^{\otimes \ell}\right).$$

Thus to conclude that $\overline{\mathcal{S}}_g^-$ is of general type, it suffices to exhibit an effective divisor D on $\overline{\mathcal{S}}_g^-$ such that for appropriately chosen rational constants $\alpha, \beta > 0$, a relation of the type $K_{\overline{\mathcal{S}}_g^-} \equiv \alpha \lambda + \beta D + E \in \text{Pic}(\overline{\mathcal{S}}_g^-)$ holds, where $\lambda \in \text{Pic}(\overline{\mathcal{S}}_g^-)$ is the pull-back to $\overline{\mathcal{S}}_g^-$ of the Hodge class, and E is an effective \mathbb{Q} -class which is typically a combination of boundary divisors. It is essential to pick D so that (i) its class can be explicitly computed, that is, points in D have good geometric characterization, and (ii) $[D] \in \text{Pic}(\overline{\mathcal{S}}_g^-)$ is in some way an extremal point of the effective cone of divisors so that the coefficients α, β stand a chance of being positive. In the case of $\overline{\mathcal{S}}_g^+$, the role of D is played by the divisor $\overline{\Theta}_{\text{null}}$ of vanishing theta-nulls, see [F3]. In the case of $\overline{\mathcal{S}}_g^-$ we compute the class of *degenerate theta-characteristics*, that is, curves carrying a non-reduced odd theta-characteristic.

Theorem 0.5. *We fix $g \geq 3$. The locus consisting of odd spin curves*

$$\mathcal{Z}_g := \left\{ [C, \eta] \in \mathcal{S}_g^- : \eta = \mathcal{O}_C(2x_1 + x_2 + \cdots + x_{g-2}) \text{ where } x_i \in C \text{ for } i = 1, \dots, g-2 \right\}$$

is a divisor on \mathcal{S}_g^- . The class of its compactification inside $\overline{\mathcal{S}}_g^-$ equals

$$\overline{\mathcal{Z}}_g \equiv (g+8)\lambda - \frac{g+2}{4}\alpha_0 - 2\beta_0 - \sum_{i=1}^{[g/2]} 2(g-i)\alpha_i - \sum_{i=1}^{[g/2]} 2i\beta_i \in \text{Pic}(\overline{\mathcal{S}}_g^-),$$

where $\lambda, \alpha_0, \beta_0, \dots, \alpha_{[g/2]}, \beta_{[g/2]}$ are the standard generators of $\text{Pic}(\overline{\mathcal{S}}_g^-)$.

For low genus, \mathcal{Z}_g specializes to well-known geometric loci. For instance \mathcal{Z}_3 is the divisor of hyperflexes on plane quartics. In particular, Theorem 0.5 yields the formula

$$\pi_*(\overline{\mathcal{Z}}_3) \equiv 308\lambda - 32\delta_0 - 76\delta_1 \in \text{Pic}(\overline{\mathcal{M}}_3),$$

for the class of quartic curves having a hyperflex. This matches [Cu] formula (5.5). Moreover, one has the following relation in $\text{Pic}(\overline{\mathcal{M}}_3)$

$$\left[\{ [C] \in \mathcal{M}_3 : \exists x \in C \text{ with } 4x \equiv K_C \} \right]^- \equiv 8 \cdot \overline{\mathcal{M}}_{3,2}^1 + \pi_*(\overline{\mathcal{Z}}_3),$$

where $\overline{\mathcal{M}}_{3,2}^1 \equiv 9\lambda - \delta_0 - 3\delta_1$ is the hyperelliptic class and the multiplicity 8 accounts for the number of Weierstrass points.

We briefly explain how Theorem 0.5 implies that $\overline{\mathcal{S}}_g^-$ is of general type for $g > 11$. We choose an effective divisor $D \in \text{Eff}(\overline{\mathcal{M}}_g)$ of small slope; for composite $g + 1$ one can take $D = \overline{\mathcal{M}}_{g,d}^r$ the closure of the Brill-Noether divisor of curves with a \mathfrak{g}_d^r , where $\rho(g, r, d) = -1$; there exists a constant $c_{g,d,r} > 0$ such that [EH2],

$$\overline{\mathcal{M}}_{g,d}^r \equiv c_{g,d,r} \left((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{\lfloor g/2 \rfloor} i(g-i)\delta_i \right) \in \text{Pic}(\overline{\mathcal{M}}_g).$$

We form the linear combination of divisors on $\overline{\mathcal{S}}_g^-$

$$\frac{2}{g-2}\overline{\mathcal{Z}}_g + \frac{3(3g-10)}{c_{g,d,r}(g-2)(g+1)}\pi^*(\overline{\mathcal{M}}_{g,d}^r) \equiv \frac{11g+37}{g+1}\lambda - 2\alpha_0 - 3\beta_0 - \sum_{i=1}^{\lfloor g/2 \rfloor} (a_i \cdot \alpha_i + b_i \cdot \beta_i),$$

where $a_i, b_i \geq 2$ for $i \neq 1$ and $a_1, b_1 > 3$ are explicitly known rational constants. The canonical class of $\overline{\mathcal{S}}_g^-$ is given by the Riemann-Hurwitz formula

$$K_{\overline{\mathcal{S}}_g^-} \equiv \pi^*(K_{\overline{\mathcal{M}}_g}) + \beta_0 \equiv 13\lambda - 2\alpha_0 - 3\beta_0 - 2 \sum_{i=1}^{\lfloor g/2 \rfloor} (\alpha_i + \beta_i) - (\alpha_1 + \beta_1),$$

and by comparison, it follows that for $g > 12$ one can find a constant $\mu_g \in \mathbb{Q}_{>0}$ such that

$$K_{\overline{\mathcal{S}}_g^-} - \mu_g \cdot \lambda \in \mathbb{Q}_{\geq 0} \langle [\overline{\mathcal{Z}}_g], \alpha_1, \beta_1, \dots, \alpha_{\lfloor g/2 \rfloor}, \beta_{\lfloor g/2 \rfloor} \rangle,$$

which shows that $K_{\overline{\mathcal{S}}_g^-}$ is big and thus proves Theorem 0.4.

For $g = 12$, there is no Brill-Noether divisor, and the reasoning above shows that in order to conclude that $\overline{\mathcal{S}}_{12}^-$ is of general type, one needs an effective divisor $\overline{\mathcal{D}}_{12}$ of slope $s(\overline{\mathcal{D}}_{12}) < 6 + \frac{12}{13}$, that is, a counterexample to the Slope Conjecture on effective divisors on $\overline{\mathcal{M}}_{12}$, see [FP]. We define the locus

$$\mathfrak{D}_{12} := \left\{ [C] \in \mathcal{M}_{12} : \exists L \in W_{14}^4(C) \text{ with } \text{Sym}^2 H^0(C, L) \xrightarrow{\mu_0(L)} H^0(C, L^{\otimes 2}) \text{ not injective} \right\},$$

that is, points in \mathfrak{D}_{12} correspond to curves that admit an embedding $C \subset \mathbf{P}^4$ with $\deg(C) = 14$, such that $H^0(\mathbf{P}^4, \mathcal{I}_{C/\mathbf{P}^4}(2)) \neq 0$. The computation of the class of the closure $\overline{\mathfrak{D}}_{12} \subset \overline{\mathcal{M}}_{12}$ is carried out in Section 6 and it turns out that $s(\overline{\mathfrak{D}}_{12}) = \frac{4415}{642} < 6 + \frac{12}{13}$. In particular $\overline{\mathfrak{D}}_{12}$ violates the Slope Conjecture on $\overline{\mathcal{M}}_{12}$, and as such, it contains the locus $\mathcal{K}_{12} := \{[C] \in \mathcal{M}_{12} : C \text{ lies on a } K3 \text{ surface}\}$.

We discuss the structure of the paper. Section 1 is of preliminary nature and establishes basic facts about the moduli space $\overline{\mathcal{S}}_g^-$ which will be used both in Section 3 in the course of proving Theorem 0.2, as well as in Section 5, when calculating the class $[\overline{\mathcal{Z}}_g]$. In Section 2, we prove Theorem 0.1, whereas Section 3 is devoted to the construction of Mukai models for $\overline{\mathcal{S}}_g^-$ and to establishing Theorem 0.2. The proof of Theorems 0.4 for $g > 12$ is completed in Section 5. Finally, in Section 6, we construct two counterexamples to the Slope Conjecture on $\overline{\mathcal{M}}_{12}$, which implies that $\overline{\mathcal{S}}_{12}^-$ is of general type.

1. FAMILIES OF STABLE SPIN CURVES

We briefly review some relevant facts about the moduli space $\overline{\mathcal{S}}_g^-$ that will be used throughout the paper, see also [C], [F3], [Lud] for details. As a matter of notation, we follow the convention set in [FL]; if \mathbf{M} is a Deligne-Mumford stack, then we denote by \mathcal{M} its associated coarse moduli space. Slightly abusing notation, if C is a smooth curve of genus g and $\eta \in \text{Pic}^{g-1}(C)$ an isolated odd theta-characteristic, that is, satisfying $h^0(C, \eta) = 1$, we define the *support* $\text{supp}(\eta) := \text{supp}(D)$, where $D \in C_{g-1}$ is the unique effective divisor with $\eta = \mathcal{O}_C(D)$. An isolated theta-characteristic η is said to be non-reduced if $\text{supp}(\eta)$ is a non-reduced divisor on C .

A connected, nodal curve X is called *quasi-stable*, if for any component $E \subset X$ which is isomorphic to \mathbf{P}^1 , one has that (i) $k_E := |E \cap (\overline{X - E})| \geq 2$, and (ii) any two rational components $E, E' \subset X$ with $k_E = k_{E'} \geq 2$ are disjoint. Such irreducible components are called *exceptional*. We recall the following definition from [C]:

Definition 1.1. A stable *spin curve* of genus g consists of a triple (X, η, β) , where X is a genus g quasi-stable curve, $\eta \in \text{Pic}^{g-1}(X)$ is a line bundle of total degree $g - 1$ with $\eta_E = \mathcal{O}_E(1)$ for all exceptional components $E \subset X$, and $\beta : \eta^{\otimes 2} \rightarrow \omega_X$ is a homomorphism of sheaves which is generically non-zero along each non-exceptional component of X .

Sometimes the morphism $\beta \in \mathbf{P}H^0(X, \omega_X \otimes \eta^{\otimes(-2)})$ appearing in the Definition 1.1 is uniquely determined by X and η and is accordingly dropped from the notation. In such a case, to ease notation, we view spin curves as pairs $[X, \eta] \in \overline{\mathcal{S}}_g$. It follows from the definition that if (X, η, β) is a spin curve with exceptional components E_1, \dots, E_r and $\{p_i, q_i\} = E_i \cap (\overline{X - E_i})$ for $i = 1, \dots, r$, then $\beta_{E_i} = 0$. Moreover, if $\tilde{X} := X - \bigcup_{i=1}^r E_i$ (viewed as a subcurve of X), then we have an isomorphism of sheaves $\eta_{\tilde{X}}^{\otimes 2} \xrightarrow{\sim} \omega_{\tilde{X}}$.

We denote by $\overline{\mathcal{S}}_g$ the non-singular Deligne-Mumford stack of spin curves of genus g . Because the parity $h^0(X, \eta) \bmod 2$ of a spin curve is invariant under deformations [Mu], the stack $\overline{\mathcal{S}}_g$ splits into two connected components $\overline{\mathcal{S}}_g^+$ and $\overline{\mathcal{S}}_g^-$ of relative degree $2^{g-1}(2^g + 1)$ and $2^{g-1}(2^g - 1)$ respectively. It is proved in [C] that the coarse moduli space of $\overline{\mathcal{S}}_g$ is isomorphic to the normalization of $\overline{\mathcal{M}}_g$ in the function field of \mathcal{S}_g . There is a proper morphism $\pi : \overline{\mathcal{S}}_g \rightarrow \overline{\mathcal{M}}_g$ given by $\pi([X, \eta, \beta]) := [\text{st}(X)]$, where $\text{st}(X)$ denotes the stable model of X obtained by contracting all exceptional components.

1.1. Spin curves of compact type. We recall the description of the pull-back divisors $\pi^*(\Delta_i)$ for $1 \leq i \leq [g/2]$. We choose a spin curve $[X, \eta, \beta] \in \pi^{-1}([C \cup_y D])$, where $[C, y] \in \mathcal{M}_{i,1}$ and $[D, y] \in \mathcal{M}_{g-i,1}$. Then necessarily $X := C \cup_{y_1} E \cup_{y_2} D$, where E is an exceptional component such that $C \cap E = \{y_1\}$ and $D \cap E = \{y_2\}$. Moreover $\eta = (\eta_C, \eta_D, \eta_E = \mathcal{O}_E(1)) \in \text{Pic}^{g-1}(X)$. Since $\beta_E = 0$, it follows that $\eta_C^{\otimes 2} = K_C$ and $\eta_D^{\otimes 2} = K_D$, that is, η_C and η_D are "honest" theta-characteristics on C and D respectively. The condition $h^0(X, \eta) \equiv 1 \bmod 2$ implies that η_C and η_D must have opposite parities. We denote by $A_i \subset \overline{\mathcal{S}}_g^-$ the closure in $\overline{\mathcal{S}}_g^-$ of the locus corresponding to pairs

$$([C, \eta_C, y], [D, \eta_D, y]) \in \mathcal{S}_{i,1}^- \times \mathcal{S}_{g-i,1}^+,$$

and by $B_i \subset \overline{\mathcal{S}}_g^-$ the closure in $\overline{\mathcal{S}}_g^-$ of the locus corresponding to pairs

$$([C, \eta_C, y], [D, \eta_D, y]) \in \mathcal{S}_{i,1}^+ \times \mathcal{S}_{g-i,1}^-.$$

One has the relation $\pi^*(\Delta_i) = A_i + B_i$ and clearly $\deg(A_i/\Delta_i) = 2^{g-2}(2^i - 1)(2^{g-i} + 1)$ and $\deg(B_i/\Delta_i) = 2^{g-2}(2^i + 1)(2^{g-i} - 1)$. One denotes $\alpha_i := [A_i], \beta_i := [B_i] \in \text{Pic}(\overline{\mathcal{S}}_g^-)$.

1.2. Spin curves with an irreducible stable model. In order to describe $\pi^*(\Delta_0)$ we pick a point $[X, \eta, \beta]$ such that $\text{st}(X) = C_{yq} := C/y \sim q$, where $[C, y, q] \in \mathcal{M}_{g-1,2}$ is a general point of Δ_0 . Unlike the case of curves of compact type, here there are two possibilities depending on whether X possesses an exceptional component or not. If $X = C_{yq}$ and $\eta_C := \nu^*(\eta)$ where $\nu : C \rightarrow X$ denotes the normalization map, then $\eta_C^{\otimes 2} = K_C(y + q)$. For each choice of $\eta_C \in \text{Pic}^{g-1}(C)$ as above, there is precisely one choice of gluing the fibres $\eta_C(y)$ and $\eta_C(q)$ such that $h^0(X, \eta) \equiv 1 \pmod{2}$. We denote by A_0 the closure in $\overline{\mathcal{S}}_g^-$ of the locus of those points $[C_{yq}, \eta_C \in \sqrt{K_C(y+q)}]$ with $\eta_C(y)$ and $\eta_C(q)$ glued as above. One has that $\deg(A_0/\Delta_0) = 2^{2g-2}$.

If $X = C \cup_{\{y,q\}} E$ where E is an exceptional component, then since $\beta_E = 0$ it follows that $\beta_C \in H^0(C, \omega_{X|C} \otimes \eta_C^{\otimes (-2)})$ must vanish at both y and q and then for degree reasons $\eta_C := \eta \otimes \mathcal{O}_C$ is a theta-characteristic on C . The condition $H^0(X, \omega) \cong H^0(C, \omega_C) \equiv 1 \pmod{2}$ implies that $[C, \eta_C] \in \mathcal{S}_{g-1}^-$. In an étale neighborhood of a point $[X, \eta, \beta]$, the covering π is given by

$$(\tau_1, \tau_2, \dots, \tau_{3g-3}) \mapsto (\tau_1^2, \tau_2, \dots, \tau_{3g-3}),$$

where one identifies \mathbb{C}_τ^{3g-3} with the versal deformation space of (X, η, β) and the hyperplane $(\tau_1 = 0) \subset \mathbb{C}_\tau^{3g-3}$ denotes the locus of spin curves where the exceptional component E persists. This discussion shows that π is simply branched over Δ_0 and we denote the ramification divisor by $B_0 \subset \overline{\mathcal{S}}_g^-$, that is, the closure of the locus of spin curves $[C \cup_{\{y,q\}} E, (C, \eta_C) \in \mathcal{S}_{g-1}^-, \eta_E = \mathcal{O}_E(1)]$. If $\alpha_0 = [A_0] \in \text{Pic}(\overline{\mathcal{S}}_g^-)$ and $\beta_0 = [B_0] \in \text{Pic}(\overline{\mathcal{S}}_g^-)$, we then have the relation

$$(1) \quad \pi^*(\delta_0) = \alpha_0 + 2\beta_0.$$

We define several test curves in the boundary of $\overline{\mathcal{S}}_g^-$ which will be later used to compute divisor classes on the moduli space.

1.3. The family F_i . We fix $1 \leq i \leq [g/2]$ and construct a covering family for the boundary divisor A_i . We fix general curves $[C] \in \mathcal{M}_i$ and $[D, q] \in \mathcal{M}_{g-i,1}$ as well as an odd theta-characteristic η_C^- on C and an even theta-characteristic η_D^+ on D . If $E \cong \mathbf{P}^1$ is a fixed exceptional component, we define the family of spin curves

$$F_i := \left\{ [C \cup_y \cup E \cup_q D, \eta] : \eta_C = \eta_C^-, \eta_E = \mathcal{O}_E(1), \eta_D = \eta_D^+, E \cap C = \{y\}, E \cap D = \{q\} \right\}_{y \in C}.$$

One has that $F_i \cdot \beta_i = 0$ and then $F_i \cdot \alpha_i = -2i + 2$; furthermore F_i has intersection number zero with the remaining generators of $\text{Pic}(\overline{\mathcal{S}}_g^-)$.

1.4. The family G_i . As above, we fix an integer $1 \leq i \leq [g/2]$ and curves $[C] \in \mathcal{M}_i$ and $[D, q] \in \mathcal{M}_{g-i,1}$. This time we choose an even theta-characteristic η_C^+ on C and an odd theta-characteristic η_D^- on D . The following family covers the divisor B_i :

$$G_i := \left\{ [C \cup_y E \cup_q D, \eta] : \eta_C = \eta_C^+, \eta_E = \mathcal{O}_E(1), \eta_D = \eta_D^-, E \cap C = \{y\}, E \cap D = \{q\} \right\}_{y \in C}.$$

Clearly $G_i \cdot \alpha_i = 0$, $G_i \cdot \beta_i = 2 - 2i$ and $G_i \cdot \lambda = G_i \cdot \alpha_j = G_i \cdot \beta_j = 0$ for $j \neq i$.

1.5. Two elliptic pencils. The boundary divisor $\Delta_1 \subset \overline{\mathcal{M}}_g$ is covered by a standard elliptic pencil R obtained by attaching to a fixed general pointed curve $[C, y] \in \mathcal{M}_{g-1,1}$ a pencil of plane cubic curves $\{E_\lambda = f^{-1}(\lambda)\}_{\lambda \in \mathbf{P}^1}$ where $f : \text{Bl}_g(\mathbf{P}^2) \rightarrow \mathbf{P}^1$. The points of attachment on the elliptic pencil are given by a section $\sigma : \mathbf{P}^1 \rightarrow \text{Bl}_g(\mathbf{P}^2)$ given by one of the base points of the pencil of cubics. We lift this pencil in two possible ways to the space $\overline{\mathcal{S}}_g$, depending on the parity of the theta-characteristic on the varying elliptic tail. We fix an even theta-characteristic $\eta_C^+ \in \text{Pic}^{g-2}(C)$ and $E \cong \mathbf{P}^1$ will again denote an exceptional component. We define the family

$$F_0 := \left\{ [C \cup_q E \cup_{\sigma(\lambda)} f^{-1}(\lambda), \eta_C = \eta_C^+, \eta_E = \mathcal{O}_E(1), \eta_{f^{-1}(\lambda)} = \mathcal{O}_{f^{-1}(\lambda)}] : \lambda \in \mathbf{P}^1 \right\} \subset \overline{\mathcal{S}}_g^-.$$

Since $F_0 \cap B_1 = \emptyset$, we find that $F_0 \cdot \alpha_1 = \pi_*(F_0) \cdot \delta_1 = -1$. Similarly, $F_0 \cdot \lambda = \pi_*(F_0) \cdot \lambda = 1$ and obviously $F_0 \cdot \alpha_i = F_0 \cdot \beta_i = 0$ for $2 \leq i \leq [g/2]$. For each of the 12 points $\lambda_\infty \in \mathbf{P}^1$ corresponding to singular fibres of R , the associated $\eta_{\lambda_\infty} \in \overline{\text{Pic}}^{g-1}(C \cup E \cup f^{-1}(\lambda_\infty))$ are actual line bundles on $C \cup E \cup f^{-1}(\lambda_\infty)$, that is, we do not have to blow-up the extra node. Thus we obtain that $F_0 \cdot \beta_0 = 0$ and then $F_0 \cdot \alpha_0 = \pi_*(F_0) \cdot \delta_0 = 12$.

A second lift of the elliptic pencil to $\overline{\mathcal{S}}_g^-$ is obtained by choosing an odd theta-characteristic $\eta_C^- \in \text{Pic}^{g-2}(C)$ whereas on E_λ one takes each of the 3 possible even theta-characteristics, that is,

$$G_0 := \left\{ [C \cup_q E \cup_{\sigma(\lambda)} f^{-1}(\lambda), \eta_C = \eta_C^-, \eta_E = \mathcal{O}_E(1), \eta_{f^{-1}(\lambda)} \in \gamma^{-1}[f^{-1}(\lambda)]] : \lambda \in \mathbf{P}^1 \right\},$$

where $\gamma : \overline{\mathcal{S}}_{1,1}^+ \rightarrow \overline{\mathcal{M}}_{1,1}$ is the projection of degree 3. Since $\pi_*(G_0) = 3R \subset \Delta_1$, we obtain that $G_0 \cdot \lambda = 3$. Obviously $G_0 \cdot \alpha_1 = 0$, hence $G_0 \cdot \beta_1 = \pi_*(G_0) \cdot \delta_1 = -3$. The map $\gamma : \overline{\mathcal{S}}_{1,1}^+ \rightarrow \overline{\mathcal{M}}_{1,1}$ is simply ramified over the point corresponding to j -invariant ∞ . Hence, $G_0 \cdot \alpha_0 = 12$ and $G_0 \cdot \beta_0 = 12$.

1.6. A covering family in B_0 . We fix a general pointed spin curve $[C, q, \eta_C^-] \in \mathcal{S}_{g-1,1}^-$ and as usual $E \cong \mathbf{P}^1$ denotes an exceptional component. We construct a family of spin curves $H_0 \subset B_0$ with general member

$$[C \cup_{\{y,q\}} E, \eta_C = \eta_C^-, \eta_E = \mathcal{O}_E(1)]_{y \in C} \subset \overline{\mathcal{S}}_g^-$$

and with special fibre corresponding to $y = q$ being the odd spin curve with support

$$C \cup_q E' \cup_{q'} E_2 \cup_{\{y_2, q_2\}} E,$$

where E' and E_2 are both smooth rational curves and $y_2, q_2 \in E$, $E_2 \cap E = \{y_2, q_2\}$, while $E_2 \cap E' = \{q'\}$. The stable model of this curve is $C \cup_q \left(\frac{E_2}{y_2 \sim q_2} \right)$, having an elliptic tail of j -invariant ∞ . The underlying line bundle $\eta \in \text{Pic}^{g-1}(C \cup E' \cup E_2 \cup E)$ satisfies

$\eta_C = \eta_C^-, \eta_{E'} = \mathcal{O}_{E'}(1), \eta_E = \mathcal{O}_E(1)$ and, for degree reasons, $\eta_{E_2} = \mathcal{O}_{E_2}(-1)$. We have the following relations for the numerical parameters of H_0 :

$$H_0 \cdot \lambda = 0, H_0 \cdot \beta_0 = 1 - g, H_0 \cdot \alpha_0 = 0, H_0 \cdot \beta_1 = 1, H_0 \cdot \alpha_1 = 0.$$

(The only non-trivial calculation here uses that $H_0 \cdot \beta_0 = \pi_*(H_0) \cdot \delta_0/2 = 1 - g$, cf. [HM]).

2. THETA PENCILS ON K3 SURFACES.

In this section we prove Theorem 0.1. As usual, we denote by \mathcal{F}_g the moduli space of polarized $K3$ surfaces $[X, H]$, where X is a $K3$ surface and $H \in \text{Pic}(X)$ is a (primitive) polarization of degree $H^2 = 2g - 2$, see [M4]. For an integer $0 \leq \delta \leq g$, we introduce the universal *Severi variety* consisting of pairs

$$\mathcal{V}_{g,\delta} := \left\{ ([X, H], C) : [X, H] \in \mathcal{F}_g \text{ and } C \in |\mathcal{O}_X(H)| \text{ is an integral } \delta\text{-nodal curve} \right\}.$$

If $\sigma : \mathcal{V}_{g,\delta} \rightarrow \mathcal{F}_g$ is the obvious projection, we set $V_{g,\delta}(|H|) := \sigma^{-1}([X, H])$. It is known that every irreducible component of $\mathcal{V}_{g,\delta}$ has dimension $19 + g - \delta$ and maps dominantly onto \mathcal{F}_g . It is conjectured that $\mathcal{V}_{g,\delta}$ is irreducible. This is established in [CD] in the range $g \leq 9$ and $g = 11$.

For a point $[X, H] \in \mathcal{F}_g$, we consider a pencil of curves $P \subset |H|$, and denote by Z the base locus of P . We assume that a general member $C \in P$ is a nodal integral curve. It follows that $C - Z$ is smooth and that $S := \text{sing}(C)$ is a, possibly empty, subset of Z . Let $\epsilon : X' := \text{Bl}_S(X) \rightarrow X$ be the blow-up of X along the locus S of nodes, and denote by E the exceptional divisor of ϵ . Let

$$P' \subset |\epsilon^*H \otimes \mathcal{O}_{X'}(-2E)|$$

be the strict transform of P by ϵ , and Z' its base locus. Since a general member $C \in P$ is nodal precisely along S , a general curve $C' \in P'$ is smooth. We view $h' := Z' + E \cdot C'$ as a divisor on the smooth curve C' . By the adjunction formula, $h' \in |\omega_{C'}|$.

Definition 2.1. We say that P is a *theta pencil*, if h' has even multiplicity at each of its points, that is, $\mathcal{O}_{C'}(\frac{1}{2}h')$ is an odd theta-characteristic for every smooth curve $C' \in P'$.

The definition implies that the intersection multiplicity of two curves in P is even at each point $p \in \text{supp}(Z)$. For every pair $[X, H] \in \mathcal{F}_g$ we have that:

Proposition 2.2. *Every smooth curve $C \in |H|$ belongs to a theta pencil.*

Proof. Let η be an odd theta-characteristic with $h^0(C, \eta) = 1$ and write $\eta = \mathcal{O}_C(d)$, with $d \in C_{g-1}$. Then $\mathbf{P}H^0(X, \mathcal{I}_{d/X}(H))$ is a theta pencil. \square

We can reverse the construction of a theta pencil, starting instead with the normalization of a nodal section of a $K3$ surface. Suppose

$$t := [C', x_1, y_1, \dots, x_\delta, y_\delta, \eta] \in \mathcal{M}_{g-\delta, 2\delta} \times_{\mathcal{M}_{g-\delta}} \mathcal{S}_{g-\delta}^-$$

is a 2δ -pointed curve C' together with an isolated odd theta-characteristic η , such that:

- (i) $h^0\left(C', \eta\left(-\sum_{i=1}^{\delta} (x_i + y_i)\right)\right) \geq 1$; we write $\eta = \mathcal{O}_{C'}\left(\sum_{i=1}^{\delta} (x_i + y_i) + d\right)$, where $d \in C'_{g-3\delta-1}$ is the residual divisor.
- (ii) There exists a polarized $K3$ surface $[X, H] \in \mathcal{F}_g$ and a map $f : C' \rightarrow X$, such that $f(x_i) = f(y_i) = p_i$ for all $i = 1, \dots, \delta$, $f_*(C') \in |H|$, and moreover $f : C' \rightarrow C$ is the normalization map of the δ -nodal curve $C := f(C')$.

If $\epsilon : X' \rightarrow X$ is the blow-up of X at the points p_1, \dots, p_δ and $E \subset X'$ denotes the exceptional divisor, we may view $C' \subset X'$, as a smooth curve in the linear system $|\epsilon^*H \otimes \mathcal{O}_{X'}(-2E)|$. Note that

$$\mathcal{O}_{C'}(C') = K_{C'} \left(- \sum_{i=1}^{\delta} (x_i + y_i) \right) = \eta \otimes \mathcal{O}_{C'}(d).$$

We pass to cohomology in the following short exact sequences

$$0 \longrightarrow \mathcal{O}_{X'} \longrightarrow \mathcal{I}_{d/X'}(C') \longrightarrow \mathcal{O}_{C'}(C')(-d) \longrightarrow 0,$$

and

$$0 \longrightarrow \mathcal{O}_{X'} \longrightarrow \mathcal{I}_{2d+\sum_{i=1}^{\delta}(x_i+y_i)/X'}(C') \longrightarrow \mathcal{O}_{C'} \longrightarrow 0$$

respectively, in order to obtain that

$$|\mathcal{I}_{d/X'}(C')| = |\mathcal{I}_{2d/X'}(C')| = |\mathcal{I}_{2d+\sum_{i=1}^{\delta}(x_i+y_i)/X'}(C')| = \mathbf{P}^1$$

is a theta pencil of δ -nodal curves on X . The link between this description of a theta pencil and the one provided by Definition 2.1 is given by the relation $h' = 2E \cdot C' + 2d$.

If $\mathcal{K}_{g-\delta,\delta}^- \subset \mathcal{M}_{g-\delta,2\delta} \times_{\mathcal{M}_{g-\delta}} \mathcal{S}_{g-\delta}^-$ is the locus of elements $[C, (x_i, y_i)_{i=1,\dots,\delta}, \eta]$ satisfying conditions (i) and (ii), the previous discussion proves the following:

Proposition 2.3. *Every irreducible component of $\mathcal{K}_{g-\delta,\delta}^-$ is uniruled.*

This implies the following consequence of Proposition 3.4 to be established in the next section:

Theorem 2.4. *We set $g \leq 9$ and $0 \leq \delta \leq (g+1)/3$. Then the variety $\mathcal{K}_{g-\delta,\delta}^-$ is non-empty, uniruled and dominates the spin moduli space $\mathcal{S}_{g-\delta}^-$.*

Definition 2.5. We say that a theta pencil P is δ -nodal if its general member is a δ -nodal curve, that is, $|S| = \delta$. We say that P is *regular* if the support $\text{supp}(Z)$ of its base locus consists of $g-1$ distinct points.

A δ -nodal theta pencil P on a $K3$ surface X , induces a map

$$m' : P' \cong \mathbf{P}^1 \rightarrow \overline{\mathcal{S}}_{g-\delta}^-,$$

obtained by sending a general $C' \in P'$ to the moduli point $[C', \mathcal{O}_{C'}(\frac{1}{2}h')] \in \overline{\mathcal{S}}_{g-\delta}^-$.

We note in passing that a theta pencil also induces a map $m : P' \rightarrow \overline{\mathcal{S}}_g^-$ defined as follows. Consider the pencil $E + P'$ having fixed component E . The general member is a quasi-stable curve $D \in (E + P')$ of arithmetic genus g , with exceptional components $\{E_i\}_{i=1,\dots,\delta}$ corresponding to the exceptional divisors of the blow-up $\epsilon : X' \rightarrow X$. Then

$$m(C) := \left[C \cup \left(\bigcup_{i=1}^{\delta} E_i \right), \eta_{E_i} = \mathcal{O}_{E_i}(1), \eta_{C'} = \mathcal{O}_{C'}\left(\frac{1}{2}h'\right) \right] \in \overline{\mathcal{S}}_g^-.$$

These pencils will be used extensively in the proof of Theorem 0.2.

Assume that $[X, H] \in \mathcal{F}_g$ is a general point, in particular $\text{Pic}(X) = \mathbb{Z} \cdot H$. Then every smooth curve $C \in |H|$ is Brill-Noether general, see [La], which implies that $h^0(C, \eta) = 1$, for every odd theta-characteristic η on C . Theta pencils with smooth general member define a locally closed subset in the Grassmannian $G(2, H^0(S, \mathcal{O}_S(H)))$ of lines in $|H|$. Let $\Theta^-(X, H)$ be its Zariski closure in $G(2, H^0(S, \mathcal{O}_S(H)))$.

Proposition 2.6. $\Theta^-(X, H)$ is pure of dimension $g - 1$.

Proof. Let $f : P^-(X, H) \rightarrow |H|$ be the projection map from the projectivized universal bundle over $\Theta^-(X, H)$, and $V_{g,0}(|H|) \subset |H|$ be the open locus of smooth curves. Under our assumptions f has finite fibres over $V_{g,0}(|H|)$. Thus $P^-(X, H)$ has pure dimension g , and $\Theta^-(X, H)$ has pure dimension $g - 1$. \square

For a general (thus necessarily regular) theta pencil $P \in \Theta^-(X, H)$, we study in more detail the map $m : P' \rightarrow \overline{\mathcal{S}}_g^-$. Let $\Delta(X, H) \subset |H|$ be the discriminant locus. Since $[X, H] \in \mathcal{F}_g$ is general, $\Delta(X, H)$ is an integral hypersurface parameterizing the singular elements of $|H|$. It is well-known that $\deg \Delta(X, H) = 6g + 18$.

Proposition 2.7. Let $P \in \Theta^-(X, H)$ be a general theta pencil with base locus Z . Then every singular curve $C \in P$ is nodal. Furthermore,

$$P \cdot \Delta(X, H) = 2(a_1 + \cdots + a_{g-1}) + b_1 + \cdots + b_{4g+20},$$

where a_i is the parameter point of a curve $A_i \in P$ having a point of Z as its only singularity, and b_j is the parameter point of a curve $B_j \in P$ such that $\text{sing}(B_j) \subset X - Z$. Accordingly,

$$P \cdot \alpha_0 = 4g + 20 \text{ and } P \cdot \beta_0 = g - 1.$$

Proof. We set $\text{supp}(Z) = \{p_1, \dots, p_{g-1}\}$. Since P is regular, for $i = 1, \dots, g - 1$, there exists a unique curve $A_i \in P$ singular at p_i . Moreover, for degree reasons, p_i is the unique double point of A_i . Each pencil $T \subset |H|$ having p_i in its base locus is a tangent line to $\Delta(X, H)$ at A_i . Hence the intersection multiplicity $(P \cdot \Delta(X, H))_{A_i}$ is at least 2. It follows that the assertion to prove is open on any family of pairs $(P, [X, H])$ such that $P \in \Theta^-(X, H)$. Since \mathcal{F}_g is irreducible, it suffices to produce one polarized K3 surface (X, H) satisfying this condition.

For this purpose, we use *hyperelliptic* polarized K3 surfaces (X, H) . Consider a rational normal scroll $\mathbb{F} := \mathbb{F}_a \subset \mathbf{P}^g$, where $a \in \{0, 1\}$ and $g = 2n + 1 - a$. A general section $R \in |\mathcal{O}_{\mathbb{F}}(1)|$ is a rational normal curve of degree $g - 1$. From the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{F}}(-2K_{\mathbb{F}} - R) \rightarrow \mathcal{O}_{\mathbb{F}}(-2K_{\mathbb{F}}) \rightarrow \mathcal{O}_R(-2K_{\mathbb{F}}) \rightarrow 0,$$

one finds that there exist a smooth curve $B \in |-2K_{\mathbb{F}}|$ and distinct points $o_1, \dots, o_{g-1} \in B$ such that the pencil $Q \subset |\mathcal{O}_{\mathbb{F}}(R)|$ of hyperplane sections through o_1, \dots, o_{g-1} cuts out a pencil with simple ramification on B .

Let $\rho : X \rightarrow \mathbb{F}$ be the double covering of \mathbb{F} branched along B . Then X is a K3 surface and $|H| := |\mathcal{O}_X(\rho^*R)|$ is a hyperelliptic linear system on X of genus g . Then $\rho^*(Q)$ is a regular theta pencil on X with the required properties. \square

Since theta pencils cover $\overline{\mathcal{S}}_g^-$ when $g \leq 11$ and $g \neq 10$, the following consequence of Proposition 2.7 is very suggestive concerning the variation of $\kappa(\overline{\mathcal{S}}_g^-)$ as g increases, in particular, in highlighting the significance of the case $g = 12$.

Corollary 2.8. With the same notation as above, we have that $P \cdot K_{\overline{\mathcal{S}}_g^-} = 2g - 24$. In particular general theta pencils of genus $g < 12$ are $K_{\overline{\mathcal{S}}_g^-}$ -negative.

Proof. Use that $(P \cdot \lambda)_{\overline{\mathcal{S}}_g^-} = (\pi_*(P) \cdot \lambda)_{\overline{\mathcal{M}}_g} = g + 1$, $P \cdot \alpha_0 = 4g + 20$ and $P \cdot \beta_0 = g - 1$. \square

Proposition 2.9. The locally closed set of nodal theta pencils in $\Theta^-(X, H)$ is non empty. If P is a general nodal theta pencil, then a general curve $C \in P$ has one node as its only singularity.

Proof. We keep the notation from the previous proof and construct a smooth curve $B \in |-2K_{\mathbb{F}}|$ and choose general points $o, o_1, \dots, o_{g-3} \in B$, such that the pencil $Q \subset |\mathcal{O}_{\mathbb{F}}(R)|$ of the hyperplane sections through $o_1 + \dots + o_{g-3} + 2o$ cuts out a pencil with simple ramification on B . Then $\rho^*(Q)$ is a nodal theta pencil with the required properties. \square

Theorem 2.10. $\overline{\mathcal{S}}_g^-$ is uniruled for $g \leq 11$.

Proof. By [M1-4], a general curve $[C] \in \overline{\mathcal{M}}_g$ is embedded in a K3 surface X precisely when $g \leq 9$ or $g = 11$. By Proposition 2.7, C belongs to a theta pencil $P \subset |\mathcal{O}_X(C)|$ (which moreover, is $K_{\overline{\mathcal{S}}_g^-}$ -negative). Thus the statement follows for $g \leq 9$ and $g = 11$.

To settle the case of $\overline{\mathcal{S}}_{10}^-$, we show that $\mathcal{K}_{10,1}^-$ is non-empty and irreducible. Indeed, then by Proposition 2.3 it follows that $\mathcal{K}_{10,1}^-$ is uniruled, and since the projection map $\mathcal{K}_{10,1}^- \rightarrow \mathcal{S}_{10}^-$ is finite, $\mathcal{K}_{10,1}^-$ dominates \mathcal{S}_{10}^- . This implies that $\overline{\mathcal{S}}_{10}^-$ is uniruled.

The variety $\mathcal{K}_{10,1}^-$ is an open subvariety of the irreducible locus

$$\mathcal{U} := \left\{ ([C, x, y], \eta) \in \mathcal{M}_{10,2} \times_{\mathcal{M}_{10}} \mathcal{S}_{10}^- : h^0(C, \eta \otimes \mathcal{O}_C(-x-y)) \geq 1 \right\},$$

hence it is irreducible as well. To establish its non-emptiness, it suffices to produce an example of an element $([C, x, y], \eta) \in \mathcal{U}$, such that the curve C_{xy} can be embedded in a K3 surface. We specialize to the case when C is hyperelliptic and $x, y \in C$ are distinct Weierstrass points, in which case one can choose $\eta = \mathcal{O}_C(x+y+w_1+\dots+w_7)$, where w_i are distinct Weierstrass points in $C - \{x, y\}$. Again we let $\rho : X \rightarrow \mathbb{F} \subset \mathbf{P}^{11}$ be a hyperelliptic K3 surface branched along $B \in |-2K_{\mathbb{F}}|$, with polarization $H := \rho^*\mathcal{O}_{\mathbb{F}}(1)$, so that $[X, H] \in \mathcal{F}_{11}$. We set $C := \rho^*(R)$, where $R \in |\mathcal{O}_{\mathbb{F}}(1)|$ is a rational normal curve of degree 10. We need to ensure that C is 1-nodal, with its node $p \in C$ such that if $f : C' \rightarrow C$ denotes the normalization map, then both points in $f^{-1}(p)$ are Weierstrass points. This is satisfied once we choose R in such a way that $B \cdot R \geq 2\rho(p)$. \square

3. UNIRATIONALITY OF $\overline{\mathcal{S}}_g^-$ FOR $g \leq 8$

To prove the claimed unirationality results, we use that a general curve $[C] \in \overline{\mathcal{M}}_g$ has a sextic plane model when $g \leq 6$, or is a linear section of a Mukai variety, when $7 \leq g \leq 9$. We start with the easy case of small genus, before moving on to the more substantial study of Mukai models.

Theorem 3.1. $\overline{\mathcal{S}}_g^-$ is unirational for $g \leq 6$.

Proof. We fix $3 \leq g \leq 6$ and a general odd spin curve $[C, \eta] \in \mathcal{S}_g^-$. Write $\eta = \mathcal{O}_C(d)$, where $d \in C_{g-1}$, then choose a general linear system $A \in G_6^2(C)$. The induced morphism $\phi_A : C \rightarrow \Gamma \subset \mathbf{P}^2$ realizes C as a sextic with $\delta = 10 - g$ nodes. By choosing $[C, \eta]$ and A generically, we may assume that $\text{supp}(d)$ consists of $g - 1$ points and is disjoint from $\phi_A^{-1}(\text{sing}(\Gamma))$. Accordingly, we identify d with its image $\phi_A(d)$ on Γ . By adjunction

$$\mathcal{O}_C(2d) = \omega_C = \mathcal{O}_C(3) \left(-\phi_A^{-1}(\text{sing}(\Gamma)) \right),$$

therefore the unique plane cubic $E \in |\mathcal{O}_{\mathbf{P}^2}(3)|$ passing through the $10 - g$ nodes of Γ as well as through the $g - 1$ points of $\text{supp}(d)$ is actually tangent to Γ along $\text{supp}(d)$.

We denote by $\mathcal{U} \subset (\mathbf{P}^2)^9$ the open set parameterizing general 9-tuples of points $(\bar{x}, \bar{y}) := (x_1, \dots, x_\delta, y_1, \dots, y_{g-1})$, where $g = 10 - \delta$. Over \mathcal{U} lies a projective bundle

\mathcal{P} whose fibre at (\bar{x}, \bar{y}) is the linear system of plane sextics Γ which are singular along \bar{x} and totally tangent to $E_{\bar{x}, \bar{y}}$ along \bar{y} . Here $E_{\bar{x}, \bar{y}} \in |\mathcal{O}_{\mathbf{P}^2}(3)|$ denotes the unique plane cubic through the points $x_1, \dots, x_\delta, y_1, \dots, y_{g-1}$. Then \mathcal{P} is a rational variety, and by the previous remark, it dominates $\overline{\mathcal{S}}_g$. Thus $\overline{\mathcal{S}}_g$ is unirational. \square

We assume now that $7 \leq g \leq 10$ and denote by $V_g \subset \mathbf{P}^{N_g}$ the rational homogeneous space defined as follows, see [M1], [M2], [M3]:

- V_{10} : the 5-dimensional variety $G_2/P \subset \mathbf{P}^{13}$ corresponding to the Lie group G_2 ,
- V_9 : the Plücker embedding of the symplectic Grassmannian $SG(3, 6) \subset \mathbf{P}^{13}$,
- V_8 : the Plücker embedding of the Grassmannian $G(2, 6) \subset \mathbf{P}^{14}$,
- V_7 : the Plücker embedding of the orthogonal Grassmannian $OG(5, 10) \subset \mathbf{P}^{15}$.

Note that $N_g = g + \dim(V_g) - 2$. Inside the Hilbert scheme $\text{Hilb}(V_g)$ of curvilinear sections of V_g , we consider the open set \mathcal{U}_g classifying curves $C \subset V_g$ such that

- C is a nodal integral section of V_g by a linear space of dimension $g - 1$,
- the residue map $\rho : H^0(C, \omega_C) \rightarrow H^0(C, \omega_C \otimes \mathcal{O}_{\text{sing}(C)})$ is surjective.

A general point $[C \hookrightarrow \mathbf{P}^{g-1}] \in \mathcal{U}_g$ is a smooth, canonical curve of genus g . Mukai's results [M1], [M2], [M3] imply that C has general moduli if $g \leq 9$. For each $0 \leq \delta \leq g-1$, we define the locally closed sets of δ -nodal curvilinear sections of V_g

$$\mathcal{U}_{g,\delta} := \{[C \hookrightarrow \mathbf{P}^{g-1}] \in \mathcal{U}_g : |\text{sing}(C)| = \delta\}.$$

Proposition 3.2. *For $g \leq 9$, the variety $\mathcal{U}_{g,\delta}$ is smooth of pure codimension δ in \mathcal{U}_g .*

Proof. A general 2-dimensional linear section of V_g is a polarized K3 surface $[X, H] \in \mathcal{F}_g$ with general moduli. It is known [Ta], that δ -nodal hyperplane sections of S form a pure $(g - \delta)$ -dimensional family $V_{g,\delta}(|H|) \subset |H|$. Thus, $\mathcal{U}_{g,\delta} \neq \emptyset$ and $\text{codim}(\mathcal{U}_{g,\delta}, \mathcal{U}_g) \leq \delta$. We fix a curve $[C] \in \mathcal{U}_{g,\delta}$, then consider the normal bundle N_C of C in V_g and the map $r : H^0(C, N_C) \rightarrow \mathcal{O}_{\text{sing}(C)}$ induced by the exact sequence

$$(2) \quad 0 \rightarrow T_C \rightarrow T_{V_g} \otimes \mathcal{O}_C \rightarrow N_C \xrightarrow{r} T_C^1 \rightarrow 0,$$

where $T_C^1 = \mathcal{O}_{\text{sing}(C)}$ is the Lichtenbaum-Schlessinger sheaf of C classifying the deformations of $\text{sing}(C)$. Using the identification $T_{[C]}(\mathcal{U}_g) = H^0(C, N_C)$, it is known that $\text{Ker}(r)$ is isomorphic to $T_{[C]}(\mathcal{U}_{g,\delta})$, see e.g. [HH]. Furthermore, $N_C \cong \omega_C^{\oplus(N_g - g + 1)}$ and $r = \rho^{\oplus(N_g - g + 1)}$, where $\rho : H^0(C, \omega_C) \rightarrow H^0(C, \mathcal{O}_{\text{sing}(C)})$ is the map given by the residues at the nodes. Since ρ is surjective, $\text{Ker}(r)$ has codimension δ inside $T_{[C]}(\mathcal{U}_g)$ and the statement follows. \square

The automorphism group $\text{Aut}(V_g)$ acts in the natural way on $\text{Hilb}(V_g)$. The locus of singular curvilinear sections $[C] \in \mathcal{U}_g$ is an $\text{Aut}(V_g)$ -invariant divisor which misses a general point of \mathcal{U}_g , therefore $\mathcal{U}_g^{\text{ss}} := \mathcal{U}_g \cap \text{Hilb}(V_g)^{\text{ss}} \neq \emptyset$. Since $\rho(V_g) = 1$, the notion of stability is independent of the polarization. The (quasi-projective) GIT-quotient

$$\mathfrak{M}_g := \mathcal{U}_g^{\text{ss}} // \text{Aut}(V_g)$$

is said to be the *Mukai model* of $\overline{\mathcal{M}}_g$. We have the following commutative diagram

$$\begin{array}{ccc} \mathcal{U}_g^{\text{ss}} & \longrightarrow & \mathcal{U}_g \\ u_g \downarrow & & m_g \downarrow \\ \mathfrak{M}_g & \xrightarrow{\phi_g} & \overline{\mathcal{M}}_g \end{array}$$

where $u_g : \mathcal{U}_g^{\text{ss}} \rightarrow \mathfrak{M}_g$ is the quotient map and $m_g : \mathcal{U}_g \rightarrow \overline{\mathcal{M}}_g$ is the moduli map. The general fibre of m_g is an $\text{Aut}(V_g)$ -orbit. Summarizing results from [M1], [M2], [M3], we state the following:

Theorem 3.3. *For $7 \leq g \leq 9$, the map $\phi_g : \mathfrak{M}_g \dashrightarrow \overline{\mathcal{M}}_g$ is a birational isomorphism. The inverse map ϕ_g^{-1} contracts the (unique) Brill-Noether divisor $\overline{\mathcal{M}}_{g,d}^r \subset \overline{\mathcal{M}}_g$ of curves with a \mathfrak{g}_d^r when $\rho(g, r, d) = -1$, as well as the boundary divisors Δ_i with $1 \leq i \leq [g/2]$.*

Next, let $\Delta_g^\delta \subset \Delta_0 \subset \overline{\mathcal{M}}_g$ be the locus of integral stable curves of arithmetic genus g with δ nodes. Then Δ_g^δ is irreducible of codimension δ in $\overline{\mathcal{M}}_g$.

Lemma 3.4. *Set $7 \leq g \leq 9$ and let D be any irreducible component of $\mathcal{U}_{g,\delta}$. Then the restriction morphism $m_{g|D} : D \rightarrow \Delta_g^\delta$ is dominant. In particular, a general δ -nodal curve $[C] \in \Delta_g^\delta$ lies on a smooth K3 surface.*

Proof. Since $\mathcal{U}_{g,\delta}$ is smooth, D is a connected component of $\mathcal{U}_{g,\delta}$, that is, for $[C] \in D$, the tangent spaces to D and to $\mathcal{U}_{g,\delta}$ coincide. We consider again the sequence (2):

$$0 \rightarrow T_C \rightarrow T_{V_g} \otimes \mathcal{O}_C \rightarrow N'_C \rightarrow 0,$$

where $N'_C := \text{Im} \{T_{V_g} \otimes \mathcal{O}_C \rightarrow N_C\}$ is the *equisingular sheaf* of C . We have that $H^0(C, N'_C) = \text{Ker}(r)$. As remarked in the proof of Proposition 3.2, $H^0(C, N'_C)$ is the tangent space $T_{[C]}(\mathcal{U}_{g,\delta})$ and its codimension in $H^0(C, N_C)$ equals δ . Consider the coboundary map $\partial : H^0(C, N'_C) \rightarrow H^1(C, T_C)$. Since $H^1(C, T_C)$ classifies topologically trivial deformations of the nodal curve C , the image $\text{Im}(\partial)$ is isomorphic to the image of the tangent map $dm_{g|_{\mathcal{U}_{g,\delta}}}$ at $[C]$. On the other hand $H^0(C, T_{V_g} \otimes \mathcal{O}_C)$ is the tangent space to the orbit of C under the action of $\text{Aut}(V_g)$. This is reduced and the stabilizer of C , being a subgroup of $\text{Aut}(C)$, is finite, hence we obtain:

$$\dim \text{Im}(\partial) = h^0(C, N_C) - \delta - \dim \text{Aut}(V_g) = 3g - 3 - \delta.$$

Since Δ_g^δ has codimension δ in $\overline{\mathcal{M}}_g$, it follows that $m_{g|D}$ is dominant. \square

Proposition 3.5. *Fix $0 \leq \delta \leq g - 1$ and D an irreducible component of $\mathcal{U}_{g,\delta}$. Then $D^{\text{ss}} \neq \emptyset$.*

Proof. It suffices to construct an $\text{Aut}(V_g)$ -invariant divisor which does not contain D . We carry out the construction when $g = 8$, the remaining cases being largely similar.

We fix a complex vector space $V \cong \mathbb{C}^6$, and then $V_8 := G(2, V) \subset \mathbf{P}(\wedge^2 V)$ and $\mathcal{U}_8 \subset G(8, \wedge^2 V)$. For a projective 7-plane $\Lambda \in G(8, \wedge^2 V)$, we denote the set of containing hyperplanes $F_\Lambda := \{H \in \mathbf{P}(\wedge^2 V)^\vee : H \supset \Lambda\}$, and define the $\text{Aut}(V_8)$ -invariant divisor

$$Z := \{\Lambda \in \mathcal{U}_8 : F_\Lambda \cap G(2, V^\vee) \subset \mathbf{P}(\wedge^2 V)^\vee \text{ is not a transverse intersection}\}.$$

We claim that $D \not\subset Z$. Indeed, let us fix a general point $[C \hookrightarrow \Lambda] \in D$, where $\Lambda = \langle C \rangle$, corresponding to a general curve $[C] \in \Delta_g^\delta$. In particular, we may assume that C lies outside the closure in $\overline{\mathcal{M}}_g$ of curves violating the Petri theorem. Thus C possesses no

generalized g_7^2 's, that is, $\overline{W}_7^2(C) = \emptyset$, whereas $\overline{W}_5^1(C) \subset \text{Pic}(C)$ consists of locally free pencils satisfying the Petri condition. We recall from [M2] the construction of $\phi_g^{-1}[C]$, which generalizes to irreducible Petri general nodal curves: There exists a unique rank two vector bundle E on C with $\det(E) = \omega_C$ and $h^0(C, E) = 6$. This appears as an extension

$$0 \rightarrow A \rightarrow E \rightarrow \omega_C \otimes A^\vee \rightarrow 0,$$

for every $A \in \overline{W}_5^1(C)$. Then one sets $\phi_g^{-1}([C]) := [C \hookrightarrow G(2, H^0(C, E)^\vee)]$. Moreover,

$$F_\Lambda = \mathbf{P}(\text{Ker}\{\wedge^2 H^0(C, E) \rightarrow H^0(C, \omega_C)\}).$$

In particular, the intersection $F_\Lambda \cap G(2, H^0(C, E))$ corresponds to the pencils $A \in \overline{W}_5^1(C)$. Since C is Petri general, $\overline{W}_5^1(C)$ is a smooth scheme, thus $[C \hookrightarrow \Lambda] \notin Z$. \square

We consider the quotient $\mathfrak{M}_{g,\delta} := \mathcal{U}_{g,\delta}^{\text{ss}} // \text{Aut}(V_g)$ and the induced map

$$\phi_{g,\delta} : \mathfrak{M}_{g,\delta} \rightarrow \Delta_g^\delta.$$

Theorem 3.6. *The variety $\mathfrak{M}_{g,\delta}$ is irreducible and $\phi_{g,\delta}$ is a birational isomorphism.*

Proof. By Lemma 3.4, any irreducible component Y of $\mathfrak{M}_{g,\delta}$ dominates Δ_g^δ . On the other hand, $\phi_g : \mathfrak{M}_g \rightarrow \overline{\mathcal{M}}_g$ is a birational morphism and $\phi_{g,\delta} = \phi_g|_{\mathfrak{M}_{g,\delta}}$. Since $\overline{\mathcal{M}}_g$ is normal, each fibre of ϕ_g is connected, thus $\mathfrak{M}_{g,\delta}$ is irreducible and $\deg(\phi_{g,\delta}) = 1$. \square

We lift our construction to the space of odd spin curves. Keeping $7 \leq g \leq 9$, we consider the Hilbert scheme $\text{Hilb}_{2g-2}(V_g)$ of 0-dimensional subschemes of V_g having length $2g - 2$.

Definition 3.7. Let $\mathfrak{Z}_{g-1} \subset \text{Hilb}_{2g-2}(V_g)$ be the parameter space of those 0-dimensional schemes $Z \subset V_g$ such that:

- (1) Z is a hyperplane section of a smooth curve section $[C] \in \mathcal{U}_g$,
- (2) Z has multiplicity two at each point of its support,
- (3) $\text{supp}(Z)$ consists of $g - 1$ linearly independent points.

The space \mathfrak{Z}_{g-1} classifies *clusters* of length $2g - 2$ on V_g . The cycle associated under the Hilbert-Chow morphism to a general point of \mathfrak{Z}_{g-1} corresponds to a 0-cycle of the form $2p_1 + \dots + 2p_{g-1} \in \text{Sym}^{2g-2}(V_g)$ satisfying the condition

$$\dim \langle p_1, \dots, p_{g-1} \rangle \cap \mathbb{T}_{p_i}(V_g) \geq 1, \text{ for } i = 1, \dots, g - 1.$$

Clearly $\dim(\mathfrak{Z}_{g-1}) = \dim \mathbf{G}(g - 1, N_g) - (N_g - g + 1) = (g - 1)(N_g - g + 1)$. We consider the incidence correspondence between clusters and curvilinear sections of V_g

$$\mathcal{U}_g^- := \{(C, Z) \in \mathcal{U}_g \times \mathfrak{Z}_{g-1} : Z \subset C\}.$$

The first projection map $\pi_1 : \mathcal{U}_g^- \rightarrow \mathcal{U}_g$ is finite of degree $2^{g-1}(2g - 1)$; its fibre at a general point $[C] \in \mathcal{U}_g$ is in bijective correspondence with the set of odd theta-characteristics of C . In particular, both \mathcal{U}_g^- and \mathfrak{Z}_{g-1} are irreducible varieties. The spin moduli map

$$m_g^- : \mathcal{U}_g^- \dashrightarrow \overline{\mathcal{S}}_g^-$$

is defined by $m_g^-(C, Z) := [C, \mathcal{O}_C(Z/2)]$, for each point $(C, Z) \in \mathcal{U}_g^-$ corresponding to a smooth curve C . Later we shall extend the rational map m_g^- to a regular map over \mathcal{U}_g^- . It is clear that m_g^- induces a map $\phi_g^- : Q_g^- \dashrightarrow \overline{\mathcal{S}}_g^-$ from the quotient

$$Q_g^- := \pi_1^{-1}(\mathcal{U}_g^{\text{ss}}) // \text{Aut}(V_g).$$

We may think of Q_g^- as being the *Mukai model* of $\overline{\mathcal{S}}_g^-$. If $\pi^- : Q_g^- \rightarrow \mathfrak{M}_g$ is the map induced by π at the level of Mukai models, we have a commutative diagram:

$$\begin{array}{ccc} Q_g^- & \xrightarrow{\phi_g^-} & \overline{\mathcal{S}}_g^- \\ \pi^- \downarrow & & \downarrow \pi \\ \mathfrak{M}_g & \xrightarrow{\phi_g} & \overline{\mathcal{M}}_g \end{array}$$

Proposition 3.8. *The spin Mukai model Q_g^- is irreducible and $\phi_g^- : Q_g^- \rightarrow \overline{\mathcal{S}}_g^-$ is a birational isomorphism.*

One extends the rational map m_g^- (therefore ϕ_g^- as well) to a regular morphism over the locus of points with nodal underlying curve section of V_g as follows. Let $(C, Z) \in \mathcal{U}_g^-$ be an arbitrary point and set $\text{supp}(Z) := \{p_1, \dots, p_{g-1}\}$. Assume that $\text{sing}(C) \cap \text{supp}(Z) = \{p_1, \dots, p_\delta\}$, where $\delta \leq g-1$. Consider the partial normalization $\nu : N \rightarrow C$ at the points p_1, \dots, p_δ . In particular, there exists an effective Cartier divisor e on C of degree $g-\delta-1$, such that $2e = Z \cap (C - \text{sing}(C))$, and set $\epsilon := \mathcal{O}_N(\nu^*e)$. Then $m_g^-(C, Z)$ is the spin curve $[X, \eta] \in \overline{\mathcal{S}}_g^-$ defined as follows:

Definition 3.9. We describe the following stable spin curve:

- (1) $X := N \cup E_1 \cup \dots \cup E_\delta$, where $E_i = \mathbf{P}^1$ for $i = 1, \dots, \delta$.
- (2) $E_i \cap N = \nu^{-1}(p_i)$, for every node $p_i \in \text{sing}(C) \cap \text{supp}(Z)$.
- (3) $\eta \otimes \mathcal{O}_N \cong \epsilon$ and $\eta \otimes \mathcal{O}_{E_i} \cong \mathcal{O}_{\mathbf{P}^1}(1)$.

We note that N is smooth of genus $g-\delta$, precisely when $\text{sing}(C) \subset \text{supp}(Z)$. In this case $\epsilon \in \text{Pic}^{g-1-\delta}(N)$ is a theta characteristic and $h^0(N, \epsilon) = 1$. Observe also that there is an isomorphism $H^0(X, \omega_X \otimes \eta^{\otimes(-2)}) \cong H^0(N, \omega_N \otimes \epsilon^{\otimes(-2)}) = \mathbb{C}$, so the spin curve in Definition 3.9 is uniquely determined by specifying X and η .

For $1 \leq \delta \leq g-1$, we refine our incidence correspondence and consider

$$\mathcal{U}_{g,\delta}^- := \left\{ (C, Z) \in \mathcal{U}_g^- : \text{sing}(C) \subset \text{supp}(Z), \quad |\text{sing}(C)| = \delta \right\}.$$

We denote by $B_{g,\delta}^-$ the closure of $m_g^-(\mathcal{U}_{g,\delta}^-)$ inside $\overline{\mathcal{S}}_g^-$; this is the closure in $\overline{\mathcal{S}}_g^-$ of the locus of δ -nodal spin curves having δ exceptional components. Clearly $B_{g,\delta}^-$ is an irreducible component of $\pi^{-1}(\Delta_g^\delta)$ and it is birationally isomorphic to $\overline{\mathcal{S}}_{g-\delta, 2\delta} / \mathbb{Z}_2^\delta$. We set

$$Q_{g,\delta}^- := \mathcal{U}_{g,\delta}^- \cap \pi_1^{-1}(\mathcal{U}_g^{\text{ss}}) // \text{Aut}(V_g),$$

and let $u_g^- : \mathcal{U}_{g,\delta}^- \dashrightarrow Q_{g,\delta}^-$ denote the quotient map. Keeping all previous notation, we have a further commutative diagram

$$\begin{array}{ccccc} \mathcal{U}_{g,\delta}^- & \xrightarrow{u_g^-} & Q_{g,\delta}^- & \xrightarrow{\phi_{g,\delta}^-} & B_{g,\delta}^- \\ \downarrow & & \pi^- \downarrow & & \pi \downarrow \\ \mathcal{U}_{g,\delta} & \xrightarrow{u_g} & \mathfrak{M}_{g,\delta} & \xrightarrow{\phi_{g,\delta}} & \Delta_g^\delta \end{array}$$

where $\phi_{g,\delta}^-$ is the morphism induced on $Q_{g,\delta}^-$ by m_g^- .

Theorem 3.10. *We fix $7 \leq g \leq 9$ and $1 \leq \delta \leq g - 1$. Then $\phi_{g,\delta}^- : Q_{g,\delta}^- \rightarrow B_{g,\delta}^-$ is a birational isomorphism.*

Proof. It suffices to note that $\phi_{g,\delta}$ is birational, and the vertical arrows of the diagram are finite morphisms of the same degree, namely the number of odd theta-characteristics on a curve of genus $g - \delta$. \square

We construct a projective bundle over $B_{g,\delta}^-$, then show that for certain values $\delta \leq g - 1$, the locus $B_{g,\delta}^-$ itself is unirational, whereas the above mentioned bundle dominates \mathcal{S}_g^- . Let $\mathcal{C}_{g,\delta} \subset \mathcal{U}_{g,\delta}^- \times V_g$ be the universal curve, endowed with its two projection maps

$$\mathcal{U}_{g,\delta}^- \xleftarrow{p} \mathcal{C}_{g,\delta} \xrightarrow{q} V_g.$$

We fix a point $(\Gamma, Z) \in \mathcal{U}_{g,\delta}^-$ and let $\nu : N \rightarrow \Gamma$ be the normalization map. Recall that $\text{sing}(\Gamma)$ consists of δ linearly independent points and that $h^0(N, \mathcal{O}_N(\nu^*e)) = 1$, where e is the effective divisor on Γ characterized by $Z|_{\Gamma_{\text{reg}}} = 2e$. Thus the restriction map $H^0(\Gamma, \omega_\Gamma) \rightarrow H^0(\omega_\Gamma \otimes \mathcal{O}_Z)$ has 1-dimensional kernel. In particular the relative cotangent sheaf ω_p admits a global section s inducing an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{C}_{g,\delta}} \rightarrow \omega_p \rightarrow \mathcal{O}_W \otimes \omega_p \rightarrow 0,$$

which defines a subscheme $W \subset \mathcal{C}_{g,\delta}$, whose fibre at the point $(\Gamma, Z) \in \mathcal{U}_{g,\delta}^-$ is Z itself. We set

$$\mathcal{A} := p_*(\mathcal{I}_{W/\mathcal{C}_{g,\delta}} \otimes q^*\mathcal{O}_{V_g}(1)),$$

which is a vector bundle on $\mathcal{U}_{g,\delta}^-$ of rank $N_g - g + 2$. The fibre of $\mathcal{A}(\Gamma, Z)$ is identified with $H^0(V_g, \mathcal{I}_{Z/V_g}(1))$. One has a natural identification

$$\mathbf{P}H^0(\mathcal{I}_{Z/V_g}(1))^\vee = \{1\text{-dimensional linear sections of } V_g \text{ containing } Z\}.$$

Definition 3.11. $\mathcal{P}_{g,\delta}$ is the projectivized dual of \mathcal{A} .

From the definitions and the previous remark it follows:

Proposition 3.12. $\mathcal{P}_{g,\delta}$ is the Zariski closure of the incidence correspondence

$$\mathcal{P}_{g,\delta}^o := \left\{ (C, (\Gamma, Z)) \in \mathcal{U}_g \times \mathcal{U}_{g,\delta}^- : Z \subset C \right\}.$$

Consider the projection maps

$$\mathcal{U}_g^- \xleftarrow{\alpha} \mathcal{P}_{g,\delta}^o \xrightarrow{\beta} \mathcal{U}_{g,\delta}^-.$$

We wish to know when is α a dominant map. For $1 \leq \delta < g \leq 9$, we have the following:

Proposition 3.13. *The morphism α is dominant if and only if $\delta \leq N_g + 1 - g = \dim(V_g) - 1$.*

Proof. By definition, the morphism β is surjective. Let $(\Gamma, Z) \in \mathcal{U}_{g,\delta}^-$ be an arbitrary point, and set $\text{sing}(\Gamma) := \{p_1, \dots, p_\delta\} \subset Z$. We define \mathbf{P}_Z to be the locus of one-dimensional linear sections of V_g containing Z . Inside \mathbf{P}_Z we consider the space

$$\mathbf{P}_{\Gamma,Z} := \{\Gamma' \in \mathbf{P}_Z : \text{sing}(\Gamma') \cap Z \supseteq \text{sing}(\Gamma) \cap Z\},$$

For $p \in \text{sing}(\Gamma)$, the locus $H_p := \{\Gamma' \in \mathbf{P}_Z : p \in \text{sing}(\Gamma')\}$ is a hyperplane in \mathbf{P}_Z . Indeed, we identify \mathbf{P}_Z with the family of linear spaces $L \in \mathbf{G}(g-1, N_g)$ such that $\langle Z \rangle \subset L$. By the definition of the cluster Z , it follows that $\mathbb{T}_p(V_g) \cap \langle Z \rangle$ is a line. For $L \in \mathbf{P}_Z$, the intersection $L \cap V_g$ is singular at p if and only if $\dim L \cap \mathbb{T}_p(V_g) \geq 2$. This is obviously a codimension 1 condition in \mathbf{P}_Z . Therefore, if for $1 \leq i \leq \delta$ we define the hyperplane $H_i := \{L = \langle \Gamma' \rangle \in \mathbf{P}_Z : \dim L \cap \mathbb{T}_{p_i}(V_g) \geq 2\}$, then

$$\mathbf{P}_{\Gamma,Z} = H_1 \cap \dots \cap H_\delta.$$

This shows that the general point in $\beta^{-1}(C, Z)$ corresponds to a smooth curve $C \supset Z$. We now fix a general point $(\Gamma, Z) \in \mathcal{U}_{g,\delta}^-$, corresponding to a general cluster $Z \in \mathfrak{Z}_{g-1}$.

Claim: $\mathbf{P}_{\Gamma,Z}$ has codimension δ in \mathbf{P}_Z ; its general element is a nodal curve with δ nodes.

Proof of the claim: Indeed \mathbf{P}_Z is a general fibre of the projective bundle $\mathcal{U}_g^- \rightarrow \mathfrak{Z}_{g-1}$. The claim follows since $\text{codim}(\mathcal{U}_{g,\delta}^-, \mathcal{U}_g^-) = \delta$.

The fibre $\alpha^{-1}((C, Z))$ over a general point $(C, Z) \in \mathcal{U}_g^-$, is the union of $\binom{g-1}{\delta}$ linear spaces $H_1 \cap \dots \cap H_\delta \subset \mathbf{P}_Z$ as above. By the claim above, when $Z \in \mathfrak{Z}_{g-1}$ is a general cluster, this is a union of linear spaces $\mathbf{P}_{\Gamma,Z}$ as before, having codimension δ in \mathbf{P}_Z . Hence $\alpha^{-1}((C, Z))$ is not empty if and only if $\delta \leq \dim \mathbf{P}_Z$, that is, $\delta \leq N_g - g + 1$. \square

Let us fix the following notation:

Definition 3.14.

(1) $\overline{\mathbb{P}}_{g,\delta} := (\mathcal{P}_{g,\delta}^o)^{\text{ss}} // \text{Aut}(V_g)$.

(2) $\overline{\alpha} : \overline{\mathbb{P}}_{g,\delta} \rightarrow \overline{\mathcal{S}}_g^-$ is the morphism induced by α at the level of quotients.

Note that $\beta : \mathcal{P}_{g,\delta} \rightarrow \mathcal{U}_{g,\delta}^-$ is a projective bundle and $\text{Aut}(V_g)$ acts linearly on its fibres, therefore β descends to a projective bundle on $B_{g,\delta}^-$. Then it follows from the previous remark that $\mathcal{P}_{g,\delta}$ is birationally isomorphic to $\mathbf{P}^{N_g - g + 1} \times B_{g,\delta}^-$. To finish the proof of the unirationality of \mathcal{S}_g^- , we proceed as follows:

Theorem 3.15. *Let $7 \leq g \leq 9$ and assume that (i) $B_{g,\delta}^-$ is unirational and (ii) $\delta \leq N_g - g + 1$. Then $\overline{\mathcal{S}}_g^-$ is unirational.*

Proof. By assumption (ii), the map $\alpha : \mathcal{P}_{g,\delta}^o \rightarrow \mathcal{U}_g^-$ is dominant, Hence the same is true for the induced morphism $\overline{\alpha} : \overline{\mathbb{P}}_{g,\delta} \rightarrow \overline{\mathcal{S}}_g^-$. By (i) and the above remark, $\overline{\mathbb{P}}_{g,\delta}$ is unirational. Therefore $\overline{\mathcal{S}}_g^-$ is unirational as well. \square

Theorem 3.15 has some straightforward applications. The case $\delta = g - 1$ is particularly convenient, since $B_{g,g-1}^-$ is isomorphic to the moduli space of integral curves of geometric genus 1 with $g - 1$ nodes. For $\delta = g - 1$, the assumptions of Theorem 3.15 hold when $g \leq 8$. In this range, the unirationality of \mathcal{S}_g^- follows from that of $B_{g,g-1}^-$.

Theorem 3.16. $B_{g,g-1}^-$ is unirational for $g \leq 10$.

Proof. Let $I \subset \mathbf{P}^2 \times (\mathbf{P}^2)^\vee$ be the natural incidence correspondence consisting of pairs (x, ℓ) such that x is a point on the line ℓ . For $\delta \leq 9$, we define

$$\Pi_\delta := \{(x_1, \ell_1, \dots, x_\delta, \ell_\delta, E) \in I^\delta \times \mathbf{P}H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(3)) : x_1, \dots, x_\delta \in E\}.$$

Then there exists a rational map $f_\delta : \Pi_\delta \dashrightarrow B_{\delta+1,\delta}^-$ sending $(x_1, \ell_1, \dots, x_\delta, \ell_\delta, E)$ to the moduli point of the δ -nodal, integral curve C obtained from the elliptic curve E , by identifying the pairs of points in $E \cap \ell_i - \{x_i\}$ for $1 \leq i \leq \delta$. It is easy to see that Π_δ is rational if $\delta \leq 9$. Clearly f_δ is dominant, just because every elliptic curve can be realized as a plane cubic. It follows that $B_{\delta+1,\delta}^-$ is unirational when $\delta \leq 9$. \square

Unfortunately one cannot apply Theorem 3.16 to the case $g = 9$, since the assumptions of Theorem 3.15 are satisfied only if $\delta \leq 5$.

4. THE SCORZA CURVE

This section serves as a preparation for the proof of Theorem 0.5 and we discuss in detail a correspondence $T_\eta \subset C \times C$ associated to each (non-vanishing) theta-characteristic $[C, \eta] \in \mathcal{S}_g^+ - \Theta_{\text{null}}$. This correspondence was used by G. Scorza [Sc] to provide a birational isomorphism between \mathcal{M}_3 and \mathcal{S}_3^+ (see also [DK]), and recently in [TZ], where several conditional statements of Scorza's have been rigorously established.

For a fixed theta-characteristic $[C, \eta] \in \mathcal{S}_g^+ - \Theta_{\text{null}}$, we consider the curve

$$T_\eta := \{(x, y) \in C \times C : H^0(C, \eta \otimes \mathcal{O}_C(x - y)) \neq 0\}.$$

By Riemann-Roch, it follows that T_η is a symmetric correspondence which misses the diagonal $\Delta \subset C \times C$. The curve T_η has a natural fixed point free involution and we denote by $f : T_\eta \rightarrow \Gamma_\eta$ the associated étale double covering. Under the assumption that T_η is a reduced curve, its class is computed in [DK] Proposition 7.1.5:

$$T_\eta \equiv (g-1)F_1 + (g-1)F_2 + \Delta.$$

Here $F_i \in H^2(C \times C, \mathbb{Q})$ denotes the class of the fibre of the i -th projection $C \times C \rightarrow C$.

Theorem 4.1. For a general theta-characteristic $[C, \eta] \in \mathcal{S}_g^+$, the Scorza curve T_η is a smooth curve of genus $g(T_\eta) = 3g(g-1) + 1$.

Proof. It is straightforward to show that a point $(x, y) \in T_\eta$ is singular if and only if

$$(3) \quad H^0(C, \eta \otimes \mathcal{O}_C(x - 2y)) \neq 0 \text{ and } H^0(C, \eta \otimes \mathcal{O}_C(y - 2x)) \neq 0.$$

By induction on g , we show that for a general even spin curve such a pair (x, y) cannot exist. We assume the result holds for a general $[C, \eta_C] \in \mathcal{S}_{g-1}^+$. We fix a general point $q \in C$, an elliptic curve D together with $\eta_D \in \text{Pic}^0(D) - \{\mathcal{O}_D\}$ with $\eta_D^{\otimes 2} = \mathcal{O}_D$ and consider the spin curve $t := [C \cup_q E \cup D, \eta|_C = \eta_C, \eta|_E = \mathcal{O}_E(1), \eta|_D = \eta_D] \in \overline{\mathcal{S}}_g^+$, obtained from $C \cup_q D$ by inserting an exceptional component E . Since the exceptional component plays no further role in the proof, we are going to suppress it.

We assume by contradiction that $t \in \overline{\mathcal{S}}_g^+$ lies in the closure of the locus of spin curves with singular Scorza curve. Then there exists a nodal curve $C \cup_q D'$ semistably equivalent to $C \cup_q D$ obtained by inserting a possibly empty chain on \mathbf{P}^1 's at the node

q (therefore, $p_a(D') = 1$ and we may regard D as a subcurve of D'), as well as smooth points $x, y \in C \cup D'$ together with two limit linear series $\sigma = \{\sigma_C, \sigma_{D'}\}$ and $\tau = \{\tau_C, \tau_{D'}\}$ of type \mathfrak{g}_{g-2}^0 on $C \cup D'$ such that the underlying line bundles corresponding to σ (resp. τ) are uniquely determined twists at the nodes of the line bundle $\eta \otimes \mathcal{O}_{C \cup D'}(x - 2y)$ (resp. $\eta \otimes \mathcal{O}_{C \cup D'}(y - 2x)$). The precise twists are determined by the limit linear series condition that each aspect of a limit \mathfrak{g}_{g-2}^0 have degree $g - 2$. We distinguish three cases depending on which components of $C \cup D'$ the points x and y specialize.

(i) $x, y \in C$. Then $\sigma_C \in H^0(C, \eta_C \otimes \mathcal{O}_C(x - 2y + q))$, $\tau_C \in H^0(C, \eta_C \otimes \mathcal{O}_C(y - 2x + q))$, while $\sigma_D, \tau_D \in H^0(D, \eta_D \otimes \mathcal{O}_D((g - 2)q))$. Denoting by $\{q'\} \in D \cap \overline{(C \cup D')} - D$ the point where D meets the rest of the curve, one has the compatibility conditions

$$\text{ord}_q(\sigma_C) + \text{ord}_{q'}(\sigma_D) \geq g - 2 \quad \text{and} \quad \text{ord}_q(\tau_C) + \text{ord}_{q'}(\tau_D) \geq g - 2,$$

which leads to $\text{ord}_q(\sigma_C) \geq 1$ and $\text{ord}_q(\tau_C) \geq 1$, that is, we have found two points $x, y \in C$ such that $H^0(C, \eta_C(x - 2y)) \neq 0$ and $H^0(C, \eta_C(y - 2x)) \neq 0$, which contradicts the inductive assumption on C .

(ii) $x, y \in D'$. This case does not appear if we choose η_C such that $H^0(C, \eta_C) = 0$. Indeed, for degree reason, both non-zero sections σ_C, τ_C must lie in the space $H^0(C, \eta_C)$.

(iii) $x \in C, y \in D'$. For simplicity, we assume first that $y \in D$. We find that

$$\begin{aligned} \sigma_C &\in H^0(C, \eta_C \otimes \mathcal{O}_C(x - q)), \quad \sigma_D \in H^0(D, \eta_D \otimes \mathcal{O}_D(g \cdot q' - 2y)) \quad \text{and} \\ \tau_C &\in H^0(C, \eta_C \otimes \mathcal{O}_C(2q - 2x)), \quad \tau_D \in H^0(D, \eta_D \otimes \mathcal{O}_D(y + (g - 3) \cdot q')). \end{aligned}$$

We claim that $\text{ord}_q(\sigma_C) = \text{ord}_q(\tau_C) = 0$ which can be achieved by a generic choice of $q \in C$. Then $\text{ord}_{q'}(\sigma_D) \geq g - 2$, which implies that $\eta_D = \mathcal{O}_D(2y - 2q)$. Similarly, $\text{ord}_q(\tau_D) \geq g - 2$ which yields that $\eta_D = \mathcal{O}_D(q - y)$, that is, $\eta_D^{\otimes 3} = \mathcal{O}_D$. Since η_D was assumed to be a non-trivial point of order 2 this leads to a contradiction. Finally, the case $y \in D' - D$, that is, when y lies on an exceptional subcurve $E' \subset D'$ is dealt with similarly: Since $\text{ord}_q(\sigma_C) = \text{ord}_q(\tau_C) = 0$, by compatibility, after passing through the component E' , one obtains that $\text{ord}_{q'}(\sigma_D) \geq g - 2$. Since $\sigma_D \in H^0(D, \eta_D \otimes \mathcal{O}_D((g - 2)q'))$ and $\eta_D \neq \mathcal{O}_D$, we obtain a contradiction \square

5. THE STACK OF DEGENERATE ODD THETA-CHARACTERISTICS

In this section we define a Deligne-Mumford stack $\mathbf{X}_g \rightarrow \overline{\mathcal{S}}_g^-$ parameterizing limit linear series \mathfrak{g}_{g-1}^0 which appear as limits of degenerate theta-characteristics on smooth curves. The push-forward of $[\mathbf{X}_g]$ is going to be precisely our divisor $\overline{\mathcal{Z}}_g$. Having a good description of \mathbf{X}_g over the boundary will enable us to determine all the coefficients in the expression of $[\overline{\mathcal{Z}}_g]$ in $\text{Pic}(\overline{\mathcal{S}}_g^-)$ and thus prove Theorem 0.5. We will use throughout the test curves in $\overline{\mathcal{S}}_g^-$ constructed in Section 1.

We first define a partial compactification $\tilde{\mathbf{M}}_g := \mathbf{M}_g \cup \tilde{\Delta}_0 \cup \dots \cup \tilde{\Delta}_{[g/2]}$ of $\overline{\mathbf{M}}_g$, obtaining by adding to \mathbf{M}_g the open sub-stack $\tilde{\Delta}_0 \subset \Delta_0$ of one-nodal irreducible curves $[C_{yq} := C/y \sim q]$, where $[C, y, q] \in \mathcal{M}_{g-1,2}$ is a Brill-Noether general curve together with their degenerations $[C \cup D_\infty]$ where D_∞ is an elliptic curve with $j(D_\infty) = \infty$, as well as the open substacks $\tilde{\Delta}_j \subset \Delta_j$ for $1 \leq j \leq [g/2]$ classifying curves $[C \cup_y D]$, where $[C] \in \mathcal{M}_j$ and $[D] \in \mathcal{M}_{g-j}$ are Brill-Noether general curves in the respective moduli spaces. Let $p : \tilde{\mathbf{M}}_{g,1} \rightarrow \tilde{\mathbf{M}}_g$ be the universal curve. We denote $\tilde{\mathcal{S}}_g^- := \pi^{-1}(\tilde{\mathbf{M}}_g) \subset \overline{\mathcal{S}}_g^-$ and note that for all $0 \leq j \leq [g/2]$ the boundary divisors $A'_j := A_j \cap \tilde{\mathcal{S}}_g^-$, $B'_j := B_j \cap \tilde{\mathcal{S}}_g^-$

are mutually disjoint inside $\tilde{\mathcal{S}}_g^-$. Finally, we consider $\mathcal{Z} := \tilde{\mathcal{S}}_g^- \times_{\tilde{\mathcal{M}}_g} \tilde{\mathcal{M}}_{g,1}$ and denote by $p_1 : \mathcal{Z} \rightarrow \tilde{\mathcal{S}}_g^-$ the projection.

Following the local description of the projection $\tilde{\mathcal{S}}_g^- \rightarrow \tilde{\mathcal{M}}_g$ carried out in [C], in order to obtain the universal spin curve over $\tilde{\mathcal{S}}_g^-$ one has first to blow-up the codimension 2 locus $V \subset \mathcal{Z}$ corresponding to points

$$v = \left([C \cup_{\{y,q\}} E, \eta_C^{\otimes 2} = K_C, h^0(\eta_C) \equiv 1 \pmod{2}, \eta_E = \mathcal{O}_E(1)], \nu(y) = \nu(q) \right) \in B'_0 \times_{\tilde{\mathcal{M}}_g} \tilde{\mathcal{M}}_{g,1}$$

(recall that $\nu : C \rightarrow C_{yq}$ denotes the normalization map, so v corresponds to the marked point specializing to the node of the curve C_{yq}). Suppose that $(\tau_1, \dots, \tau_{3g-3})$ are local coordinates in an étale neighbourhood of $[C \cup_{\{y,q\}} E, \eta_C, \eta_E] \in \tilde{\mathcal{S}}_g^-$, such that the local equation of the divisor B'_0 is $(\tau_1 = 0)$. Then \mathcal{Z} around v admits local coordinates $(x, y, \tau_1, \dots, \tau_{3g-3})$ verifying the equation $xy = \tau_1^2$, in particular, \mathcal{Z} is singular along V . Next, for $1 \leq j \leq [g/2]$ one blows-up the codimension 2 loci $V_j \subset \mathcal{Z}$ consisting of points

$$\left([C \cup_q D, \eta_C, \eta_D], q \in C \cap D \right) \in (A'_j \cup B'_j) \times_{\tilde{\mathcal{M}}_g} \tilde{\mathcal{M}}_{g,1}.$$

This corresponds to inserting an exceptional component in each spin curve in $\pi^*(\tilde{\Delta}_j)$. We denote by

$$\mathcal{C} := \text{Bl}_{V \cup V_1 \cup \dots \cup V_{[g/2]}}(\mathcal{Z})$$

and by $f : \mathcal{C} \rightarrow \tilde{\mathcal{S}}_g^-$ the induced family of spin curves. Then for every $[X, \eta, \beta] \in \tilde{\mathcal{S}}_g^-$ we have an isomorphism between $f^{-1}([X, \eta, \beta])$ and the quasi-stable curve X .

There exists a spin line bundle $\mathcal{P} \in \text{Pic}(\mathcal{C})$ of relative degree $g - 1$ as well as a morphism of $\mathcal{O}_{\mathcal{C}}$ -modules $B : \mathcal{P}^{\otimes 2} \rightarrow \omega_f$ having the property that $\mathcal{P}|_{f^{-1}([X, \eta, \beta])} = \eta$ and $B|_{f^{-1}([X, \eta, \beta])} = \beta : \eta^{\otimes 2} \rightarrow \omega_X$, for all spin curves $[X, \eta, \beta] \in \tilde{\mathcal{S}}_g^-$. We note that for the even moduli space $\tilde{\mathcal{S}}_g^+$ one has an analogous construction of the universal spin curve.

Next we define the stack $\tau : \mathbf{X}_g \rightarrow \tilde{\mathcal{S}}_g^-$ classifying limit \mathfrak{g}_{g-1}^0 which are twists of degenerate odd-spin curves. For a tree-like curve X we denote by $\overline{G}_d^r(X)$ the scheme of limit linear series \mathfrak{g}_d^r . The fibres of the morphism τ have the following description:

- $\tau^{-1}(\mathbf{S}_g^-)$ parameterizes triples $([C, \eta], \sigma, x)$, where $[C, \eta] \in \mathcal{S}_g^-$, $x \in C$ is a point and $\sigma \in \mathbf{PH}^0(C, \eta)$ is a section such that $\text{div}(\sigma) \geq 2x$.
- For $1 \leq j \leq [g/2]$ the inverse image $\tau^{-1}(A'_j \cup B'_j)$ parameterizes elements of the form

$$\left(X, \sigma \in \overline{G}_{g-1}^0(X), x \in X_{\text{reg}} \right),$$

where (X, x) is a 1-pointed quasi-stable curve semistably equivalent to the underlying curve of a spin curve $[C \cup_q E \cup_{q'} D, \eta_C, \eta_E, \eta_D] \in A'_j \cup B'_j$, with E denoting the exceptional component, $g(C) = j$, $g(D) = g - j$, $\{q\} = C \cap E$, $\{q'\} = E \cap D$ and

$$\sigma_C \in \mathbf{PH}^0(C, \eta_C \otimes \mathcal{O}_C((g-j)q)), \sigma_D \in \mathbf{PH}^0(D, \eta_D \otimes \mathcal{O}_D(jq')), \sigma_E \in \mathbf{PH}^0(E, \mathcal{O}_E(g-1))$$

are aspects of the limit linear series σ on X . Moreover, we require that $\text{ord}_x(\sigma) \geq 2$.

- $\tau^{-1}(B'_0)$ parameterizes elements $(X, \eta \in \text{Pic}^{g-1}(X), \sigma \in \mathbf{PH}^0(X, \eta), x \in X_{\text{reg}})$, where (X, x) is a 1-pointed quasi-stable curve equivalent to the curve underlying a point $[C \cup_{\{y,q\}} E, \eta_C, \eta_E] \in B'_0$, the line bundle η on X satisfies $\eta|_C = \eta_C$ and $\eta|_E = \eta_E$ and $\eta|_Z = \mathcal{O}_Z$ for the remaining components of X . Finally, we require $\text{ord}_x(\sigma) \geq 2$.

• $\tau^{-1}(A'_0)$ corresponds to points $(X, \eta \in \text{Pic}^{g-1}(X), \sigma \in \mathbf{PH}^0(X, \eta), x \in X_{\text{reg}})$, where (X, x) is a 1-pointed quasi-stable curve equivalent to the curve underlying a point $[C_{yq}, \eta_{C_{yq}}] \in A'_0$, and if $\mu : X \rightarrow C_{yq}$ is the map contracting all exceptional components, then $\mu^*(\eta_{C_{yq}}) = \eta$ (in particular η is trivial along exceptional components), and finally $\text{ord}_x(\sigma) \geq 2$.

Using general constructions of stacks of limit linear series cf. [EH1], [F2], it is clear that \mathbf{X}_g is a Deligne-Mumford stack. There exists a proper morphism

$$\tau : \mathbf{X}_g \rightarrow \tilde{\mathbf{S}}_g^-$$

that factors through the universal curve and we denote by $\chi : \mathbf{X}_g \rightarrow \mathcal{C}$ the induced morphism, hence $\tau = f \circ \chi$. The push-forward of the coarse moduli space $\tau_*([\mathcal{X}_g])$ equals scheme-theoretically $\overline{\mathcal{Z}}_g \cap \tilde{\mathcal{S}}_g^-$. It appears possible to extend \mathbf{X}_g over the entire $\overline{\mathcal{S}}_g^-$ but this is not necessary in order to prove Theorem 0.4 and we skip the details.

We are now in a position to calculate the class of the divisor $\overline{\mathcal{Z}}_g$ and we expand its class in the Picard group of $\overline{\mathcal{S}}_g^-$

$$(4) \quad \overline{\mathcal{Z}}_g \equiv \bar{\lambda} \cdot \lambda - \bar{\alpha}_0 \cdot \alpha_0 - \bar{\beta}_0 \cdot \beta_0 - \sum_{i=1}^{[g/2]} \bar{\alpha}_i \cdot \alpha_i - \sum_{i=1}^{[g/2]} \bar{\beta}_i \cdot \beta_i \in \text{Pic}(\overline{\mathcal{S}}_g^-),$$

where $\bar{\lambda}, \bar{\alpha}_i, \bar{\beta}_i \in \mathbb{Q}$ for $i = 0, \dots, [g/2]$. We start by determining the coefficients of the divisors α_i and β_i for $1 \leq i \leq [g/2]$.

Proposition 5.1. *For $1 \leq i \leq [g/2]$ we have that $F_i \cdot \overline{\mathcal{Z}}_g = 4(g-i)(i-1)$ and the intersection is everywhere transverse. It follows that $\bar{\alpha}_i = 2(g-i)$.*

Proof. We recall from the definition of F_i that we have fixed theta-characteristics of opposite parity $\eta_C^- \in \text{Pic}^{i-1}(C)$ and $\eta_D^+ \in \text{Pic}^{g-i-1}(D)$. Choose a point $t = (X, \eta, \sigma, x) \in \tau^{-1}(F_i)$. It is a simple exercise to show that the "double" point x of $\sigma \in \overline{\mathcal{G}}_{g-1}^0(X)$ cannot specialize to the exceptional component, therefore one has only two cases to consider depending on whether x lies on C or on D . Assume first that $x \in C$ and then $\sigma_C \in \mathbf{PH}^0(C, \eta_C^- \otimes \mathcal{O}_C((g-i)q))$ and $\sigma_D \in \mathbf{PH}^0(D, \eta_D^+ \otimes \mathcal{O}_D(iq))$, where $\{q\} = C \cap D$ is a point which moves on C but is fixed on D . Then $\text{ord}_q(\sigma_D) \leq i-1$, therefore $\text{ord}_q(\sigma_C) \geq g-i$ and then $\sigma_C(-(g-i)q) \in \mathbf{PH}^0(C, \eta_C^-)$. In particular, if we choose $[C, \eta_C^-] \in \mathcal{S}_i - \mathcal{Z}_i$, then the section $\sigma_C(-(g-i)q)$ has only simple zeros, which shows that x cannot lie on C , so this case does not occur.

We are left with the possibility $x \in D - \{q\}$. One observes that $\text{ord}_q(\sigma_C) = g-i+1$ and $\text{ord}_q(\sigma_D) = i-2$. In particular, $q \in \text{supp}(\eta_C^-)$ which gives $i-1$ choices for the moving point $q \in C$. Furthermore $\sigma_D(-(i-2)q) \in H^0(D, \eta_D^+ \otimes \mathcal{O}_D(2q-2x))$, that is, x specializes to one of the ramification points of the pencil $\eta_D^+ \otimes \mathcal{O}_D(2q) \in W_{g-i+1}^1(D)$. We note that because of the generality of $[D, \eta_D^+] \in \mathcal{S}_{g-i}^+$ as well as that of $q \in D$, the pencil is base point free and complete. From the Hurwitz-Zeuthen formula one finds $4(g-i)$ ramification points of $|\eta_D^+ \otimes \mathcal{O}_D(2q)|$, which leads to the formula $F_i \cdot \overline{\mathcal{Z}}_g = 4(g-i)(i-1)$. The fact that $\tau_*(\mathbf{X}_g)$ is transverse to F_i follows because the formation of \mathbf{X}_g commutes with restriction to B'_0 and then one can easily show in a way similar to [EH2] Lemma 3.4, or by direct calculation that $\mathbf{X}_g \times_{\tilde{\mathcal{S}}_g^-} B'_0$ is smooth at any of the points in $\tau^{-1}(F_i)$. \square

Proposition 5.2. *For $1 \leq i \leq [g/2]$ we have that $G_i \cdot \overline{\mathcal{Z}}_g = 4i(i-1)$ and the intersection is transversal. In particular $\overline{\beta}_i = 2i$.*

Proof. This time we fix general points $[C, \eta_C^+] \in \mathcal{S}_i^+$ and $[D, \eta_D^-] \in \mathcal{S}_{g-i}^-$ and $q \in C \cap D$ which is a fixed general point on D but an arbitrary point on C . Again, it is easy to see that if $t = (X, \sigma, x) \in \tau^{-1}(G_i)$ then x must lie either on C or on D . Assume first that $x \in C - \{q\}$. Then the aspects of σ are described as follows

$$\sigma_C \in \mathbf{P}H^0(C, \eta_C^+ \otimes \mathcal{O}_C((g-i)q)), \quad \sigma_D \in \mathbf{P}H^0(D, \eta_D^- \otimes \mathcal{O}_D(iq))$$

and moreover $\text{ord}_x(\sigma_C) \geq 2$. The point $q \in D$ can be chosen so that it does not lie in $\text{supp}(\eta_D^-)$, hence $\text{ord}_q(\sigma_D) \leq i$ and then $\text{ord}_q(\sigma_C) \geq g-i-1$. This leads to the conclusion $H^0(C, \eta_C^+ \otimes \mathcal{O}_C(y-2x)) \neq 0$, or equivalently $(x, y) \in C \times C$ is a ramification point of the degree i covering $p_1 : T_{\eta_C^+} \rightarrow C$ from the associated Scorza curve. We have shown that $T_{\eta_C^+}$ is smooth of genus $1 + 3i(i-1)$ (cf. Theorem 4.1) and moreover all the ramification points of p_1 are ordinary, therefore we find

$$\deg \text{Ram}(p_1) = 2g(T_{\eta_C^+}) - 2 - \deg(p_1)(2i-2) = 4i(i-1)$$

choices when $x \in C$. Next possibility is $x \in D - \{q\}$. The same reasoning as above shows that $\text{ord}_q(\sigma_C) \leq g-i-1$, therefore $\text{ord}_q(\sigma_D) \geq i$ as well as $\text{ord}_x(\sigma_D) \geq 2$. Since $\sigma_D(-iq) \in \mathbf{P}H^0(D, \eta_D^-)$, this case does not occur if $[D, \eta_D^-] \in \mathcal{S}_{g-i}^- - \mathcal{Z}_{g-i}$. \square

Next we prove that $\overline{\mathcal{Z}}_g$ is disjoint from both elliptic pencils F_0 and G_0 :

Proposition 5.3. *We have that $F_0 \cdot \overline{\mathcal{Z}}_g = 0$ and $G_0 \cdot \overline{\mathcal{Z}}_g = 0$. The equalities $\overline{\alpha} - 12\overline{\alpha}_0 + \overline{\alpha}_1 = 0$ and $3\overline{\alpha} - 12\overline{\alpha}_0 - 12\overline{\beta}_0 + 3\overline{\beta}_1 = 0$ follow.*

Proof. We first show that $F_0 \cap \overline{\mathcal{Z}}_g = \emptyset$ and we assume by contradiction that there exists $t = (X, \sigma, x) \in \tau^{-1}(F_0)$. Let us deal first with the case when $st(X) = C \cap E_\lambda$, with E_λ being a smooth curve of genus 1. The key point is that the point of attachment $q \in C \cap E_\lambda$ being general, we can assume that $(x, q) \notin \text{Ram}\{p_1 : T_{\eta_C^+} \rightarrow C\}$, for all $x \in C$. This implies that $H^0(C, \eta_C^+ \otimes \mathcal{O}_C(q-2x)) = 0$ for all $x \in C$, therefore a section $\sigma_C \in \mathbf{P}H^0(C, \eta_C^+ \otimes \mathcal{O}_C(q))$ cannot vanish twice anywhere. Thus either $x \in E_\lambda - \{q\}$ or x lies on some exceptional component of X . In the former case, since $\text{ord}_q(\sigma_C) = 0$, it follows that $\text{ord}_q(\sigma_{E_\lambda}) \geq g-1$, that is, σ_{E_λ} has no zeroes other than q (simple or otherwise). In the latter case, when $x \in E$, with E being an exceptional component, we denote by $q' \in E$ the point of intersection of E with the connected subcurve of X containing C as a subcomponent. Since as above, $\text{ord}_q(\sigma_C) = 0$, by compatibility it follows that $\text{ord}_{q'}(\sigma_E) = g-1$. But $\sigma_E \in \mathbf{P}H^0(E, \mathcal{O}_E(g-1))$, that is, σ_E does not vanish at x , a contradiction. The proof that $G_0 \cap \overline{\mathcal{Z}}_g = \emptyset$ is similar and we omit the details. \square

The trickiest part in the calculation of $[\overline{\mathcal{Z}}_f]$ is the computation of the following intersection number:

Proposition 5.4. *If $H_0 \subset B_0$ is the covering family lying in the ramification divisor of $\overline{\mathcal{S}}_g^-$, then one has that $H_0 \cdot \overline{\mathcal{Z}}_g = 2(g-2)$ and the intersection consists of $g-2$ points each counted with multiplicity 2. Therefore the relation $(g-1)\overline{\beta}_0 - \overline{\beta}_1 = 2(g-2)$ holds.*

Proof. We first describe the set-theoretic intersection $\tau_*(\mathcal{X}_g) \cap H_0$. We recall that we have fixed $[C, q, \eta_C^-] \in \mathcal{S}_{g-1,1}^-$ and start by choosing a point $t = (X, \eta, \sigma, x) \in \tau^{-1}(H_0)$.

Assume first that $X = C \cup_{\{y,q\}} E$, where $y \in C$, that is, x does not specialize to one of the nodes of $C \cup E$. Suppose first that $x \in C - \{y, q\}$. From the Mayer-Vietoris sequence on X we write

$$0 \neq \sigma \in H^0(X, \eta \otimes \mathcal{O}_X(-2x)) = \text{Ker}\{H^0(C, \eta_C^- \otimes \mathcal{O}_C(-2x)) \oplus H^0(E, \mathcal{O}_E(1)) \xrightarrow{ev_{y,q}} \mathbb{C}_{\{y,q\}}^2\},$$

we obtain that $H^0(C, \eta_C^- \otimes \mathcal{O}_C(-2x)) \neq 0$. This case can be avoided by choosing $[C, \eta_C^-] \in \mathcal{S}_{g-1}^- - \mathcal{Z}_{g-1}$.

Next we consider the possibility $x \in E - \{y, q\}$. The same Mayer-Vietoris argument reads in this case $0 \neq \text{Ker}\{H^0(C, \eta_C^-) \oplus H^0(E, \mathcal{O}_E(-1)) \xrightarrow{ev_{y,q}} \mathbb{C}_{\{y,q\}}^2\}$, that is, $y + q \in \text{supp}(\eta_C^-)$. This case can be avoided as well by starting with a general point $q \in C - \text{supp}(\eta_C^-)$. Thus the only possibility is that x specializes to one of the nodes y or q .

We deal first with the case when x and q coalesce and there is no loss of generality in assuming that $X = C \cup E \cup E'$, where both components E and E' are copies of \mathbf{P}^1 and $C \cap E = \{y\}$, $C \cap E' = \{q\}$, $E \cap E' = \{y'\}$ and moreover $x \in E' - \{y', q\}$. The restrictions of the line bundle $\eta \in \text{Pic}^{g-1}(X)$ are such that $\eta|_C = \eta_C^-$, $\eta_E = \mathcal{O}_E(1)$ and $\eta_{E'} = \mathcal{O}_{E'}$. We write

$$0 \neq \sigma = (\sigma_C, \sigma_E, \sigma_{E'}) \in \text{Ker}\{H^0(C, \eta_C^-) \oplus H^0(E, \mathcal{O}_E(1)) \oplus H^0(E', \mathcal{O}_{E'}(1)) \xrightarrow{ev_{y,y',q}} \mathbb{C}_{y,y',q}\},$$

hence $\sigma_{E'} = 0$, and then by compatibility $\sigma_C(q) = 0$, that is, $q \in \text{supp}(\eta_C^-)$ and again this case can be ruled out by a suitable choice of q . The last possible situation is when x and the moving point $y \in C$ coalesce, in which case $X = C \cup E \cup E'$, where this time $C \cap E = \{q\}$, $C \cap E' = \{y\}$, $E \cap E' = \{y'\}$ and again $x \in E' - \{y', q\}$. Writing one last time the Mayer-Vietoris sequence we find that $\sigma_{E'} = 0$ and then $\sigma_E(y') = 0$ and $\sigma_C(y) = 0$, that is, $y \in \text{supp}(\eta_C^-)$ and then σ_C is uniquely determined up to a constant. Finally $\sigma_E \in H^0(E, \mathcal{O}_E(1)(-y'))$ is uniquely specified by the gluing condition $\sigma_E(q) = \sigma_C(q)$. All in all, $H_0 \cap \tilde{\mathcal{Z}}_g = |\text{supp}(\eta_C^-)| = g - 2$.

This discussion singles out an irreducible component $\Xi \subset \chi_*(\mathcal{X}_g) \subset \mathcal{C}$ of the intersection $\chi(\mathcal{X}_g) \cap f^{-1}(B'_0)$, namely

$$\Xi = \left\{ ([C \cup_{\{y,q\}} E, \eta_C, \eta_E], x) : y \in \text{supp}(\eta_C^-) \text{ and } x = y \in X_{\text{sing}} \right\},$$

where recall that $f : \mathcal{C} \rightarrow \tilde{\mathbf{S}}_g^-$ is the universal spin curve. Since $\Xi \subset \text{Sing}(\chi_*(\mathcal{X}_g))$, after a simple local analysis, it follows that each point in $\tau^{-1}(H_0)$ occurs counted with multiplicity 2. \square

Remark 5.5. A partial independent check of Theorem 0.5 is obtained by using the Porteous formula to determine the coefficient $\bar{\lambda}$ in the expression of $[\tilde{\mathcal{Z}}_g]$. By abuse of notation we still denote by $f : \mathcal{C} \rightarrow \mathbf{S}_g^-$ the restriction of the universal spin curve to the locus of smooth curves and $\eta \in \text{Pic}(\mathcal{C})$ the spin bundle of relative degree $g - 1$. Then \mathcal{Z}_g is the push-forward via $f : \mathcal{C} \rightarrow \mathbf{S}_g^-$ of the degeneration locus of the sheaf morphism $\phi : f_*(\eta) \rightarrow J_1(\eta)$ (both these sheaves are locally free away a subset of codimension 3 in \mathbf{S}_g^- and throwing away this locus has no influence on divisor class calculations). Since $\det(f_*\eta) = (f_*\eta)^{\otimes 2}$, it follows that $c_1(f_*(\eta)) = -\lambda/4$, whereas the Chern classes of the first jet bundle $J_1(\eta)$ are calculated using the standard exact sequence on \mathcal{C}

$$0 \longrightarrow \eta \otimes \omega_f \longrightarrow J_1(\eta) \longrightarrow \eta \longrightarrow 0.$$

Remembering Mumford's formula $f_*(c_1^2(\omega_f)) = 12\lambda$, one finally writes that

$$[\mathcal{Z}_g] = f_*c_2\left(J_1(\eta) - f_*(\eta)\right) = f_*\left(\frac{3}{4}c_1(\omega_f)^2 - 2c_1(\omega_f) \cdot c_1(f_*(\eta))\right) = (g+8)\lambda \in \text{Pic}(\mathbf{S}_g^-).$$

6. A DIVISOR OF SMALL SLOPE ON $\overline{\mathcal{M}}_{12}$

The aim of this section is to construct an effective divisor $D \in \text{Eff}(\overline{\mathcal{M}}_{12})$ of slope $s(D) < 6 + 12/13$, that is, violating the Slope Conjecture. As pointed out in the proof of Theorem 0.4, this is precisely what is required in order to show that $\overline{\mathcal{S}}_{12}$ is a variety of general type.

Theorem 6.1. *The following locus consisting of curves of genus 12*

$\mathfrak{D}_{12} := \{[C] \in \mathcal{M}_{12} : \exists L \in W_{14}^4(C) \text{ with } \text{Sym}^2 H^0(C, L) \xrightarrow{\mu_0(L)} H^0(C, L^{\otimes 2}) \text{ not injective}\}$
is a divisor on \mathcal{M}_{12} . The class of its compactification inside $\overline{\mathcal{M}}_{12}$ equals

$$\overline{\mathfrak{D}}_{12} \equiv 13245 \lambda - 1926 \delta_0 - 9867 \delta_1 - \sum_{j=2}^6 b_j \delta_j \in \text{Pic}(\overline{\mathcal{M}}_{12}),$$

where $b_j \geq b_1$ for $j \geq 2$. In particular, $s(\overline{\mathfrak{D}}_{12}) = \frac{4415}{642} < 6 + \frac{12}{13}$.

This implies the following upper bound for the slope $s(\overline{\mathcal{M}}_{12})$ of the moduli space:

Corollary 6.2.

$$6 + \frac{10}{12} \leq s(\overline{\mathcal{M}}_{12}) := \inf_{D \in \text{Eff}(\overline{\mathcal{M}}_{12})} s(D) \leq \frac{4415}{642} \left(= 6 + \frac{10}{12} + \frac{14}{321} \right).$$

Another immediate application, via [Log], [F1], concerns the birational type of the moduli space $\overline{\mathcal{M}}_{g,n}$ of n -pointed stable curves of genus g :

Theorem 6.3. *The moduli space of n -pointed curves $\overline{\mathcal{M}}_{12,n}$ is of general type for $n \geq 11$.*

The divisor \mathfrak{D}_{12} is constructed as the push-forward of a codimension 3 cycle in the stack $\mathfrak{G}_{14}^4 \rightarrow \mathbf{M}_{12}$ classifying linear series \mathfrak{g}_{14}^4 . We describe the construction of this cycle, then extend this determinantal structure over a partial compactification of \mathcal{M}_{12} . This will be essential to understand the intersection of $\overline{\mathfrak{D}}_{12}$ with the boundary divisors Δ_0 and Δ_1 of $\overline{\mathcal{M}}_{12}$. We denote by \mathbf{M}_{12}^p the open substack of \mathbf{M}_{12} consisting of curves $[C] \in \mathcal{M}_{12}$ such that $W_{13}^4(C) = \emptyset$ and $W_{14}^5(C) = \emptyset$. Results in Brill-Noether theory guarantee that $\text{codim}(\mathcal{M}_{12} - \mathcal{M}_{12}^p, \mathcal{M}_{12}) \geq 3$. If \mathfrak{Pic}_{12}^{14} denotes the Picard stack of degree 14 over \mathbf{M}_{12}^p , then we consider the smooth Deligne-Mumford substack $\mathfrak{G}_{14}^4 \subset \mathfrak{Pic}_{12}^{14}$ parameterizing pairs $[C, L]$, where $[C] \in \mathcal{M}_{12}^p$ and $L \in W_{14}^4(C)$ is a (necessarily complete and base point free) linear series. We denote by $\sigma : \mathfrak{G}_{14}^4 \rightarrow \mathbf{M}_{12}^p$ the forgetful morphism. For a general $[C] \in \mathcal{M}_{12}^p$, the fibre $\sigma^{-1}([C]) = W_{14}^4(C)$ is a smooth surface.

Let $\pi : \mathbf{M}_{12,1}^p \rightarrow \mathbf{M}_{12}^p$ be the universal curve and then $p_2 : \mathbf{M}_{12,1}^p \times_{\mathbf{M}_{12}^p} \mathfrak{G}_{14}^4 \rightarrow \mathfrak{G}_{14}^4$ denotes the natural projection. If \mathcal{L} is a Poincaré bundle over $\mathbf{M}_{12,1}^p \times_{\mathbf{M}_{12}^p} \mathfrak{G}_{14}^4$ (or over an étale cover of it), then by Grauert's Theorem, both

$$\mathcal{E} := (p_2)_*(\mathcal{L}) \text{ and } \mathcal{F} := (p_2)_*(\mathcal{L}^{\otimes 2})$$

are vector bundles over \mathfrak{G}_{14}^4 , with $\text{rank}(\mathcal{E}) = 5$ and $\text{rank}(\mathcal{F}) = h^0(C, L^{\otimes 2}) = 17$ respectively. There is a natural vector bundle morphism over \mathfrak{G}_{14}^4 given by multiplication of sections,

$$\phi : \text{Sym}^2(\mathcal{E}) \rightarrow \mathcal{F},$$

and we denote by $\mathcal{U}_{12} \subset \mathfrak{G}_{14}^4$ its first degeneracy locus. We set $\mathfrak{D}_{12} := \sigma_*(\mathcal{U}_{12})$. Since the degeneracy locus \mathcal{U}_{12} has expected codimension 3 inside \mathfrak{G}_{14}^4 , the locus \mathfrak{D}_{12} is a virtual divisor on \mathcal{M}_{12}^p .

We extend the vector bundles \mathcal{E} and \mathcal{F} over a partial compactification of \mathfrak{G}_{14}^4 given by limit \mathfrak{g}_{14}^4 . We denote by $\Delta_1^p \subset \Delta_1 \subset \overline{\mathcal{M}}_{12}$ the locus of curves $[C \cup_y E]$, where E is an arbitrary elliptic curve, $[C] \in \mathcal{M}_{11}$ is a Brill-Noether general curve and $y \in C$ is an arbitrary point. We then denote by $\Delta_0^p \subset \Delta_0 \subset \overline{\mathcal{M}}_{12}$ the locus consisting of curves $[C_{yq}] \in \Delta_0$, where $[C, q] \in \mathcal{M}_{11,1}$ is Brill-Noether general and $y \in C$ is arbitrary, as well as their degenerations $[C \cup_q E_\infty]$ where E_∞ is a rational nodal curve. Once we set

$$\overline{\mathbf{M}}_{12}^p := \mathbf{M}_{12}^p \cup \Delta_0^p \cup \Delta_1^p \subset \overline{\mathbf{M}}_{12},$$

we can extend the morphism σ to a proper morphism

$$\sigma : \tilde{\mathfrak{G}}_{14}^4 \rightarrow \overline{\mathbf{M}}_{12}^p,$$

from the stack $\tilde{\mathfrak{G}}_{14}^4$ of limit linear series \mathfrak{g}_{14}^4 over the partial compactification $\overline{\mathbf{M}}_{12}^p$ of \mathbf{M}_{12} .

We extend the vector bundles \mathcal{E} and \mathcal{F} over the stack $\tilde{\mathfrak{G}}_{14}^4$. The proof of the following result proceeds along the lines of the proof of Proposition 3.9 in [F1]:

Proposition 6.4. *There exist two vector bundles \mathcal{E} and \mathcal{F} defined over $\tilde{\mathfrak{G}}_{14}^4$ with $\text{rank}(\mathcal{E}) = 5$ and $\text{rank}(\mathcal{F}) = 17$, together with a vector bundle morphism $\phi : \text{Sym}^2(\mathcal{E}) \rightarrow \mathcal{F}$, such that the following statements hold:*

- For $[C, L] \in \mathfrak{G}_{14}^4$, with $[C] \in \mathcal{M}_{12}^p$, we have that

$$\mathcal{E}(C, L) = H^0(C, L) \text{ and } \mathcal{F}(C, L) = H^0(C, L^{\otimes 2}).$$

- For $t = (C \cup_y E, l_C, l_E) \in \sigma^{-1}(\Delta_1^p)$, where $g(C) = 11, g(E) = 1$ and $l_C = |L_C|$ is such that $L_C \in W_{14}^4(C)$ has a cusp at $y \in C$, then $\mathcal{E}(t) = H^0(C, L_C)$ and

$$\mathcal{F}(t) = H^0(C, L_C^{\otimes 2}(-2y)) \oplus \mathbb{C} \cdot u^2,$$

where $u \in H^0(C, L_C)$ is any section such that $\text{ord}_y(u) = 0$. If L_C has a base point at y , then $\mathcal{E}(t) = H^0(C, L_C) = H^0(C, L_C \otimes \mathcal{O}_C(-y))$ and the image of a natural map $\mathcal{F}(t) \rightarrow H^0(C, L_C^{\otimes 2})$ is the subspace $H^0(C, L_C^{\otimes 2} \otimes \mathcal{O}_C(-2y))$.

- Fix $t = [C_{yq} := C/y \sim q, L] \in \sigma^{-1}(\Delta_0^p)$, with $q, y \in C$ and $L \in \overline{W}_{14}^4(C_{yq})$ such that $h^0(C, \nu^*L \otimes \mathcal{O}_C(-y-q)) = 4$, where $\nu : C \rightarrow C_{yq}$ is the normalization map. In the case when L is locally free we have that

$$\mathcal{E}(t) = H^0(C, \nu^*L) \text{ and } \mathcal{F}(t) = H^0(C, \nu^*L^{\otimes 2} \otimes \mathcal{O}_C(-y-q)) \oplus \mathbb{C} \cdot u^2,$$

where $u \in H^0(C, \nu^*L)$ is any section not vanishing at y and q . In the case when L is not locally free, that is, $L \in \overline{W}_{14}^4(C_{yq}) - W_{14}^4(C_{yq})$, then $L = \nu_*(A)$, where $A \in W_{13}^4(C)$ and the image of the natural map $\mathcal{F}(t) \rightarrow H^0(C, \nu^*L^{\otimes 2})$ is the subspace $H^0(C, A^{\otimes 2})$.

To determine the push-forward $[\overline{\mathcal{D}}_{12}]^{virt} = \sigma_*(c_3(\mathcal{F} - \text{Sym}^2(\mathcal{E})) \in A^1(\mathcal{M}_{12}^p)$, we study the restriction of the morphism ϕ along the pull-backs of two curves sitting in the boundary of $\overline{\mathcal{M}}_{12}$ and which are defined as follows: We fix a general pointed curve $[C, q] \in \mathcal{M}_{11,1}$ and a general elliptic curve $[E, y] \in \mathcal{M}_{1,1}$. Then we consider the families

$$C_0 := \{C/y \sim q : y \in C\} \subset \Delta_0^p \subset \overline{\mathcal{M}}_{12} \text{ and } C_1 := \{C \cup_y E : y \in C\} \subset \Delta_1^p \subset \overline{\mathcal{M}}_{12}.$$

These curves intersect the generators of $\text{Pic}(\overline{\mathcal{M}}_{12})$ as follows:

$$C_0 \cdot \lambda = 0, C_0 \cdot \delta_0 = \deg(\omega_{C_{yq}}) = -22, C_0 \cdot \delta_1 = 1 \text{ and } C_0 \cdot \delta_j = 0 \text{ for } 2 \leq j \leq 6, \text{ and}$$

$$C_1 \cdot \lambda = 0, C_1 \cdot \delta_0 = 0, C_1 \cdot \delta_1 = -\deg(K_C) = -20 \text{ and } C_1 \cdot \delta_j = 0 \text{ for } 2 \leq j \leq 6.$$

Next, we fix a general pointed curve $[C, q] \in \mathcal{M}_{11,1}$ and describe the geometry of the pull-back $\sigma^*(C_0) \subset \tilde{\mathfrak{S}}_{14}^4$. We consider the determinantal 3-fold

$$Y := \{(y, L) \in C \times W_{14}^4(C) : h^0(C, L \otimes \mathcal{O}_C(-y - q)) = 4\}$$

together with the projection $\pi_1 : Y \rightarrow C$. Inside Y we consider the following divisors

$$\Gamma_1 := \{(y, A \otimes \mathcal{O}_C(y)) : y \in C, A \in W_{13}^4(C)\} \text{ and}$$

$$\Gamma_2 := \{(y, A \otimes \mathcal{O}_C(q)) : y \in C, A \in W_{13}^4(C)\}$$

intersecting transversally along the curve $\Gamma := \{(q, A \otimes \mathcal{O}_C(q)) : A \in W_{13}^4(C)\} \cong W_{13}^4(C)$. We introduce the blow-up $Y' \rightarrow Y$ of Y along Γ and denote by $E_\Gamma \subset Y'$ the exceptional divisor and by $\tilde{\Gamma}_1, \tilde{\Gamma}_2 \subset Y'$ the strict transforms of Γ_1 and Γ_2 respectively. We then define $\tilde{Y} := Y'/\tilde{\Gamma}_1 \cong \tilde{\Gamma}_2$, to be the variety obtained from Y' by identifying the divisors $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ over each $(y, A) \in C \times W_{13}^4(C)$. Let $\epsilon : \tilde{Y} \rightarrow Y$ be the projection map.

Proposition 6.5. *With notation as above, one has a birational morphism of 3-folds*

$$f : \sigma^*(C_0) \rightarrow \tilde{Y},$$

which is an isomorphism outside a curve contained in $\epsilon^{-1}(\pi_1^{-1}(q))$. The map $f|_{(\pi_1 \circ f)^{-1}(q)}$ corresponds to forgetting the E_∞ -aspect of each limit linear series. Accordingly, the vector bundles $\mathcal{E}|_{\sigma^*(C_0)}$ and $\mathcal{F}|_{\sigma^*(C_0)}$ are pull-backs under $\epsilon \circ f$ of vector bundles on Y .

Proof. We fix a point $y \in C - \{q\}$ and denote by $\nu : C \rightarrow C_{yq}$ the normalization map, with $\nu(y) = \nu(q)$. We investigate the variety $\overline{W}_{14}^4(C_{yq}) \subset \overline{\text{Pic}}^{14}(C_{yq})$ of torsion-free sheaves L on C_{yq} with $\deg(L) = 14$ and $h^0(C_{yq}, L) \geq 5$. A locally free $L \in \overline{W}_{14}^4(C_{yq})$ is determined by $\nu^*(L) \in W_{14}^4(C)$, which has the property $h^0(C, \nu^*L \otimes \mathcal{O}_C(-y - q)) = 4$ (since $W_{12}^4(C) = \emptyset$, there exists a section of L that does not vanish simultaneously at both y and q). However, the bundles of type $A \otimes \mathcal{O}_C(y)$ or $A \otimes \mathcal{O}_C(q)$ with $A \in W_{13}^4(C)$, do not appear in this association, though $(y, A \otimes \mathcal{O}_C(y)), (y, A \otimes \mathcal{O}_C(q)) \in Y$. In fact, they correspond to the situation when $L \in \overline{W}_{14}^4(C_{yq})$ is not locally free, in which case necessarily $L = \nu_*(A)$ for some $A \in W_{13}^4(C)$. Thus, for a point $y \in C - \{q\}$, there is a birational morphism $\pi_1^{-1}(y) \rightarrow \overline{W}_{14}^4(C_{yq})$ which is an isomorphism over the locus of locally free sheaves. More precisely, $\overline{W}_{14}^4(C_{yq})$ is obtained from $\pi_1^{-1}(y)$ by identifying the disjoint divisors $\Gamma_1 \cap \pi_1^{-1}(y)$ and $\Gamma_2 \cap \pi_1^{-1}(y)$.

A special analysis is required when $y = q$, when C_{yq} degenerates to $C \cup_q E_\infty$, where E_∞ is a rational nodal cubic. If $\{l_C, l_{E_\infty}\} \in \sigma^{-1}([C \cup_q E_\infty])$, then the corresponding Brill-Noether numbers with respect to q satisfy $\rho(l_C, q) \geq 0$ and $\rho(l_{E_\infty}, q) \leq 2$. The

statement about the restrictions $\mathcal{E}|_{\sigma^*(C_0)}$ and $\mathcal{F}|_{\sigma^*(C_0)}$ follows, because both restrictions are defined by dropping the information coming from the elliptic tail. \square

To describe $\sigma^*(C_1) \subset \tilde{\mathfrak{G}}_{14}^4$, where $[C] \in \mathcal{M}_{11}$, we define the determinantal 3-fold

$$X := \{(y, L) \in C \times W_{14}^4(C) : h^0(L \otimes \mathcal{O}_C(-2y)) = 4\}.$$

In what follows we use notation from [EH1], to denote vanishing sequences of limit linear series:

Proposition 6.6. *With notation as above, the 3-fold X is an irreducible component of $\sigma^*(C_1)$. Moreover one has that $c_3((\mathcal{F} - \text{Sym}^2 \mathcal{E})|_{\sigma^*(C_1)}) = c_3((\mathcal{F} - \text{Sym}^2 \mathcal{E})|_X)$.*

Proof. By the additivity of the Brill-Noether number, if $\{l_C, l_E\} \in \sigma^{-1}([C \cup_y E])$, we have that $2 = \rho(12, 4, 14) \geq \rho(l_C, y) + \rho(l_E, y)$. Since $\rho(l_E, y) \geq 0$, we obtain that $\rho(l_C, y) \leq 2$. If $\rho(l_E, y) = 0$, then $l_E = 9y + |\mathcal{O}_E(5y)|$, that is, l_E is uniquely determined, while the aspect $l_C \in G_{14}^4(C)$ is a complete \mathfrak{g}_{14}^4 with a cusp at the variable point $y \in C$. This gives rise to an element from X . The remaining components of $\sigma^*(C_1)$ are indexed by Schubert indices $\bar{\alpha} := (0 \leq \alpha_0 \leq \dots \leq \alpha_4 \leq 10)$ such that $\bar{\alpha} > (0, 1, 1, 1, 1)$ and $5 \leq \sum_{j=0}^4 \alpha_j \leq 7$. For such $\bar{\alpha}$, we set $\bar{\alpha}^c := (10 - \alpha_4, \dots, 10 - \alpha_0)$ to be the complementary Schubert index, then define

$$X_{\bar{\alpha}} := \{(y, l_C) \in C \times G_{14}^4(C) : \alpha^{l_C}(y) \geq \bar{\alpha}\} \text{ and } Z_{\bar{\alpha}} := \{l_E \in G_{14}^4(E) : \alpha^{l_E}(y) \geq \bar{\alpha}^c\}.$$

Then $\sigma^*(C_1) = X + \sum_{\bar{\alpha}} X_{\bar{\alpha}} \times Z_{\bar{\alpha}}$. The last claim follows by dimension reasons. Since $\dim X_{\bar{\alpha}} = 1 + \rho(11, 4, 14) - \sum_{j=0}^4 \alpha_j < 3$, for every $\bar{\alpha} > (0, 1, 1, 1, 1)$ and the restrictions of both \mathcal{E} and \mathcal{F} are pulled-back from $X_{\bar{\alpha}}$, one obtains that $c_3(\mathcal{F} - \text{Sym}^2 \mathcal{E})|_{X_{\bar{\alpha}} \times Z_{\bar{\alpha}}} = 0$. \square

We also recall standard facts about intersection theory on Jacobians. For a Brill-Noether general curve $[C] \in \mathcal{M}_g$, we denote by \mathcal{P} a Poincaré bundle on $C \times \text{Pic}^d(C)$ and by $\pi_1 : C \times \text{Pic}^d(C) \rightarrow C$ and $\pi_2 : C \times \text{Pic}^d(C) \rightarrow \text{Pic}^d(C)$ the projections. We define the cohomology class $\eta = \pi_1^*([\text{point}]) \in H^2(C \times \text{Pic}^d(C))$, and if $\delta_1, \dots, \delta_{2g} \in H^1(C, \mathbb{Z}) \cong H^1(\text{Pic}^d(C), \mathbb{Z})$ is a symplectic basis, then we set

$$\gamma := - \sum_{\alpha=1}^g \left(\pi_1^*(\delta_\alpha) \pi_2^*(\delta_{g+\alpha}) - \pi_1^*(\delta_{g+\alpha}) \pi_2^*(\delta_\alpha) \right) \in H^2(C \times \text{Pic}^d(C)).$$

One has the formula $c_1(\mathcal{P}) = d\eta + \gamma$, corresponding to the Hodge decomposition of $c_1(\mathcal{P})$, as well as the relations $\gamma^3 = 0$, $\gamma\eta = 0$, $\eta^2 = 0$ and $\gamma^2 = -2\eta\pi_2^*(\theta)$. On $W_d^r(C)$ there is a tautological rank $r+1$ vector bundle $\mathcal{M} := (\pi_2)_*(\mathcal{P}|_{C \times W_d^r(C)})$. To compute the Chern numbers of \mathcal{M} we employ the Harris-Tu formula [HT]. We write

$$\sum_{i=0}^r c_i(\mathcal{M}^\vee) = (1 + x_1) \cdots (1 + x_{r+1}),$$

and then for every class $\zeta \in H^*(\text{Pic}^d(C), \mathbb{Z})$ one has the following formula:

$$(5) \quad x_1^{i_1} \cdots x_{r+1}^{i_{r+1}} \zeta = \det \left(\frac{\theta^{g+r-d+i_j-j+l}}{(g+r-d+i_j-j+l)!} \right)_{1 \leq j, l \leq r+1} \zeta.$$

We compute the classes of the 3-folds that appear in Propositions 6.5 and 6.6:

Proposition 6.7. *Let $[C, q] \in \mathcal{M}_{11,1}$ be a Brill-Noether general pointed curve. If \mathcal{M} denotes the tautological rank 5 vector bundle over $W_{14}^4(C)$ and $c_i := c_i(\mathcal{M}^\vee) \in H^{2i}(W_{14}^4(C), \mathbb{C})$, then one has the following relations:*

- (i) $[X] = \pi_2^*(c_4) - 6\eta\theta\pi_2^*(c_2) + (48\eta + 2\gamma)\pi_2^*(c_3) \in H^8(C \times W_{14}^4(C), \mathbb{C})$.
- (ii) $[Y] = \pi_2^*(c_4) - 2\eta\theta\pi_2^*(c_2) + (13\eta + \gamma)\pi_2^*(c_3) \in H^8(C \times W_{14}^4(C), \mathbb{C})$.

Proof. We start by noting that $W_{14}^4(C)$ is a smooth 6-fold isomorphic to the symmetric product C_6 . We realize X as the degeneracy locus of a vector bundle morphism defined over $C \times W_{14}^4(C)$. For each pair $(y, L) \in C \times W_{14}^4(C)$, there is a natural map

$$H^0(C, L \otimes \mathcal{O}_{2y})^\vee \rightarrow H^0(C, L)^\vee$$

which globalizes to a vector bundle morphism $\zeta : J_1(\mathcal{P})^\vee \rightarrow \pi_2^*(\mathcal{M})^\vee$ over $C \times W_{14}^4(C)$. Then we have the identification $X = Z_1(\zeta)$ and the Thom-Porteous formula gives that $[X] = c_4(\pi_2^*(\mathcal{M}) - J_1(\mathcal{P}^\vee))$. From the usual exact sequence over $C \times \text{Pic}^{14}(C)$

$$0 \rightarrow \pi_1^*(K_C) \otimes \mathcal{P} \rightarrow J_1(\mathcal{P}) \rightarrow \mathcal{P} \rightarrow 0,$$

we can compute the total Chern class of the jet bundle

$$c_t(J_1(\mathcal{P})^\vee)^{-1} = \left(\sum_{j \geq 0} (d(L)\eta + \gamma)^j \right) \cdot \left(\sum_{j \geq 0} ((2g(C) - 2 + d(L))\eta + \gamma)^j \right) = 1 - 6\eta\theta + 48\eta + 2\gamma,$$

which quickly leads to the formula for $[X]$. To compute $[Y]$ we proceed in a similar way. We denote by $\mu, \nu : C \times C \times \text{Pic}^{14}(C) \rightarrow C \times \text{Pic}^{14}(C)$ the two projections, by $\Delta \subset C \times C \times \text{Pic}^{14}(C)$ the diagonal and we set $\Gamma_q := \{q\} \times \text{Pic}^{14}(C)$. We introduce the rank 2 vector bundle $\mathcal{B} := (\mu)_*(\nu^*(\mathcal{P}) \otimes \mathcal{O}_{\Delta + \nu^*(\Gamma_q)})$ defined over $C \times W_{14}^4(C)$. We note that there is a bundle morphism $\chi : \mathcal{B}^\vee \rightarrow (\pi_2)^*(\mathcal{M})^\vee$, such that $Y = Z_1(\chi)$. Since we also have that

$$c_t(\mathcal{B}^\vee)^{-1} = (1 + (d(L)\eta + \gamma) + (d(L)\eta + \gamma)^2 + \dots)(1 - \eta),$$

we immediately obtained the stated expression for $[Y]$. \square

Proposition 6.8. *Let $[C] \in \mathcal{M}_{11}$ and denote by $\mu, \nu : C \times C \times \text{Pic}^{14}(C) \rightarrow C \times \text{Pic}^{14}(C)$ the natural projections. We define the vector bundles \mathcal{A}_2 and \mathcal{B}_2 on $C \times \text{Pic}^{14}(C)$ having fibres*

$$\mathcal{A}_2(y, L) = H^0(C, L^{\otimes 2} \otimes \mathcal{O}_C(-2y)) \text{ and } \mathcal{B}_2(y, L) = H^0(C, L^{\otimes 2} \otimes \mathcal{O}_C(-y - q)),$$

respectively. One has the following formulas:

$$\begin{aligned} c_1(\mathcal{A}_2) &= -4\theta - 4\gamma - 76\eta & c_1(\mathcal{B}_2) &= -4\theta - 2\gamma - 27\eta, \\ c_2(\mathcal{A}_2) &= 8\theta^2 + 280\eta\theta + 16\gamma\theta, & c_2(\mathcal{B}_2) &= 8\theta^2 + 100\eta\theta + 8\theta\gamma, \\ c_3(\mathcal{A}_2) &= -\frac{32}{3}\theta^3 - 512\eta\theta^2 - 32\theta^2\gamma & \text{and } c_3(\mathcal{B}_2) &= -\frac{32}{3}\theta^3 - 184\eta\theta^2 - 16\theta^2\gamma. \end{aligned}$$

Proof. Immediate application of Grothendieck-Riemann-Roch with respect to ν . \square

Before our next result, we recall that if \mathcal{V} is a vector bundle of rank $r + 1$ on a variety X , we have the formulas:

- (i) $c_1(\text{Sym}^2(\mathcal{V})) = (r + 2)c_1(\mathcal{V})$.
- (ii) $c_2(\text{Sym}^2(\mathcal{V})) = \frac{r(r+3)}{2}c_1^2(\mathcal{V}) + (r + 3)c_2(\mathcal{V})$.
- (iii) $c_3(\text{Sym}^2(\mathcal{V})) = \frac{r(r+4)(r-1)}{6}c_1^3(\mathcal{V}) + (r + 5)c_3(\mathcal{V}) + (r^2 + 4r - 1)c_1(\mathcal{V})c_2(\mathcal{V})$.

We expand $\sigma_*(c_3(\mathcal{F} - \text{Sym}^2\mathcal{E})) \equiv a\lambda - b_0\delta_0 - b_1\delta_1 \in A^1(\mathcal{M}_{12}^p)$ and determine the coefficients a, b_0 and b_1 . This will suffice in order to compute $s(\overline{\mathcal{D}}_{12})$.

Theorem 6.9. *Let $[C] \in \mathcal{M}_{11}$ be a Brill-Noether general curve and denote by $C_1 \subset \Delta_1 \subset \overline{\mathcal{M}}_{12}$ the associated test curve. Then the coefficient of δ_1 in the expansion of $\overline{\mathcal{D}}_{22}$ is equal to*

$$b_1 = \frac{1}{2g(C) - 2} \sigma^*(C_1) \cdot c_3(\mathcal{F} - \text{Sym}^2\mathcal{E}) = 9867.$$

Proof. We intersect the degeneracy locus of the map $\phi : \text{Sym}^2(\mathcal{E}) \rightarrow \mathcal{F}$ with the 3-fold $\sigma^*(C_1) = X + \sum_{\bar{\alpha}} X_{\bar{\alpha}} \times Z_{\bar{\alpha}}$. As already explained in Proposition 6.6, it is enough to estimate the contribution coming from X and we can write

$$\begin{aligned} \sigma^*(C_1) \cdot c_3(\mathcal{F} - \text{Sym}^2\mathcal{E}) &= c_3(\mathcal{F}|_X) - c_3(\text{Sym}^2\mathcal{E}|_X) - c_1(\mathcal{F}|_X)c_2(\text{Sym}^2\mathcal{E}|_X) + \\ &+ 2c_1(\text{Sym}^2\mathcal{E}|_X)c_2(\text{Sym}^2\mathcal{E}|_X) - c_1(\text{Sym}^2\mathcal{E}|_X)c_2(\mathcal{F}|_X) + c_1^2(\text{Sym}^2\mathcal{E}|_X)c_1(\mathcal{F}|_X) - c_1^3(\text{Sym}^2\mathcal{E}|_X). \end{aligned}$$

We are going to compute each term in the right-hand-side of this expression.

Recall that we have constructed in Proposition 6.7 a vector bundle morphism $\zeta : J_1(\mathcal{P})^\vee \rightarrow \pi_2^*(\mathcal{M})^\vee$. We consider the kernel line bundle $\text{Ker}(\zeta)$. If U is the line bundle on X with fibre

$$U(y, L) = \frac{H^0(C, L)}{H^0(C, L \otimes \mathcal{O}_C(-2y))} \hookrightarrow H^0(C, L \otimes \mathcal{O}_{2y})$$

over a point $(y, L) \in X$, then one has an exact sequence over X

$$0 \rightarrow U \rightarrow J_1(\mathcal{P}) \rightarrow \text{Ker}(\zeta)^\vee \rightarrow 0.$$

In particular, $c_1(U) = 2\gamma + 48\eta - c_1(\text{Ker}(\zeta)^\vee)$. The products of the Chern class of $\text{Ker}(\zeta)^\vee$ with other classes on $C \times W_{14}^4(C)$ can be computed from the Harris-Tu formula [HT]:

$$(6) \quad c_1(\text{Ker}(\zeta)^\vee) \cdot \xi|_X = -c_5(\pi_2^*(\mathcal{M})^\vee - J_1(\mathcal{P})^\vee) \cdot \xi|_X = -(\pi_2^*(c_5) - 6\eta\theta\pi_2^*(c_3) + (48\eta + 2\gamma)\pi_2^*(c_4)) \cdot \xi|_X,$$

for any class $\xi \in H^2(C \times W_{14}^4(C), \mathbb{C})$.

If \mathcal{A}_3 denotes the rank 18 vector bundle on X having fibres $\mathcal{A}_3(y, L) = H^0(C, L^{\otimes 2})$, then there is an injective morphism $U^{\otimes 2} \hookrightarrow \mathcal{A}_3/\mathcal{A}_2$, and we consider the quotient sheaf

$$\mathcal{G} := \frac{\mathcal{A}_3/\mathcal{A}_2}{U^{\otimes 2}}.$$

Since the morphism $U^{\otimes 2} \rightarrow \mathcal{A}_3/\mathcal{A}_2$ vanishes along the locus of pairs (y, L) where L has a base point, \mathcal{G} has torsion along $\Gamma \subset X$. A straightforward local analysis now shows that $\mathcal{F}|_X$ can be identified as a subsheaf of \mathcal{A}_3 with the kernel of the map $\mathcal{A}_3 \rightarrow \mathcal{G}$. Therefore, there is an exact sequence of vector bundles on X

$$0 \rightarrow \mathcal{A}_{2|X} \rightarrow \mathcal{F}|_X \rightarrow U^{\otimes 2} \rightarrow 0,$$

which over a general point of X corresponds to the decomposition

$$\mathcal{F}(y, L) = H^0(C, L^{\otimes 2} \otimes \mathcal{O}_C(-2y)) \oplus \mathbb{C} \cdot u^2,$$

where $u \in H^0(C, L)$ is such that $\text{ord}_y(u) = 1$. The analysis above, shows that the sequence stays exact over the curve Γ as well. Hence

$$\begin{aligned} c_1(\mathcal{F}|_X) &= c_1(\mathcal{A}_{2|X}) + 2c_1(U), \quad c_2(\mathcal{F}|_X) = c_2(\mathcal{A}_{2|X}) + 2c_1(\mathcal{A}_{2|X})c_1(U) \quad \text{and} \\ c_3(\mathcal{F}|_X) &= c_3(\mathcal{A}_2) + 2c_2(\mathcal{A}_{2|X})c_1(U). \end{aligned}$$

Furthermore, since $\mathcal{E}|_X = \pi_2^*(\mathcal{M})|_X$, we obtain that:

$$\begin{aligned} \sigma^*(C_1) \cdot c_3(\mathcal{F} - \text{Sym}^2 \mathcal{E}) &= c_3(\mathcal{A}_{2|X}) + c_2(\mathcal{A}_{2|X})c_1(U^{\otimes 2}) - c_3(\text{Sym}^2 \pi_2^* \mathcal{M}|_X) - \\ &- \left(\frac{r(r+3)}{2} c_1(\pi_2^* \mathcal{M}|_X) + (r+3)c_2(\pi_2^* \mathcal{M}|_X) \right) \cdot \left(c_1(\mathcal{A}_{2|X}) + c_1(U^{\otimes 2}) - 2(r+2)c_1(\pi_2^* \mathcal{M}|_X) \right) - \\ &- (r+2)c_1(\pi_2^* \mathcal{M}|_X)c_2(\mathcal{A}_{2|X}) - (r+2)c_1(\pi_2^* \mathcal{M}|_X)c_1(\mathcal{A}_{2|X})c_1(U^{\otimes 2}) + \\ &+ (r+2)^2 c_1^2(\pi_2^* \mathcal{M}|_X)c_1(\mathcal{A}_{2|X}) + (r+2)^2 c_1^2(\pi_2^* \mathcal{M}|_X)c_1(U^{\otimes 2}) - (r+2)^3 c_1^3(\pi_2^* \mathcal{M}|_X). \end{aligned}$$

As before, $c_i(\pi_2^* \mathcal{M}|_X) = \pi_2^*(c_i) \in H^{2i}(X, \mathbb{C})$. The coefficient of $c_1(\text{Ker}(\zeta)^\vee)$ in the product $\sigma^*(C_1) \cdot c_3(\mathcal{F} - \text{Sym}^2 \mathcal{E})$ is evaluated via (6). The part of this product that does not contain $c_1(\text{Ker}(\zeta)^\vee)$ equals

$$\begin{aligned} 28\pi_2^*(c_2)\theta - 88\pi_2^*(c_1^2)\theta + 440\eta\pi_2^*(c_1^2) - 53\pi_2^*(c_1c_2) - \frac{32}{3}\theta^3 + 128\eta\theta^2 - 432\eta\theta\pi_2^*(c_1) \\ + 64\pi_2^*(c_1^3) - 140\eta\pi_2^*(c_2) + 48\theta^2\pi_2^*(c_1) + 9\pi_2^*(c_3) \in H^6(C \times W_{14}^4(C), \mathbb{C}). \end{aligned}$$

Multiplying this quantity by the class $[X]$ obtained in Proposition 6.7 and then adding to it the contribution coming from $c_1(\text{Ker}(\zeta)^\vee)$, one obtains a homogeneous polynomial of degree 7 in η, θ and $\pi_2^*(c_i)$ for $1 \leq i \leq 4$. The only non-zero monomials are those containing η . After retaining only these monomials, the resulting degree 6 polynomial in $\theta, c_i \in H^*(W_{14}^4(C), \mathbb{Z})$ can be brought to a manageable form, by noting that, since $h^1(C, L) = 1$, the classes c_i are not independent. Precisely, if one fixes a divisor $D \in C_e$ of large degree, there is an exact sequence

$$0 \rightarrow \mathcal{M} \rightarrow (\pi_2)_*(\mathcal{P} \otimes \mathcal{O}(\pi^* D)) \rightarrow (\pi_2)_*(\mathcal{P} \otimes \mathcal{O}(\pi_1^* D)|_{\pi_1^* D}) \rightarrow R^1 \pi_{2*}(\mathcal{P}|_{C \times W_{14}^4(C)}) \rightarrow 0,$$

from which, via the well-known fact $c_t((\pi_2)_*(\mathcal{P} \otimes \mathcal{O}(\pi_1^* D))) = e^{-\theta}$, it follows that

$$c_t R^1 \pi_{2*}(\mathcal{P}|_{C \times W_{14}^4(C)}) \cdot e^{-\theta} = \sum_{i=0}^4 (-1)^i c_i.$$

Hence $c_{i+1} = \theta^i c_i / i! - i\theta^{i+1} / (i+1)!$, for all $i \geq 2$. After routine manipulations, one finds that $b_1 = \sigma^*(C_1) \cdot c_3(\mathcal{F} - \text{Sym}^2(\mathcal{E})) / 20 = 9867$. \square

Theorem 6.10. *Let $[C, q] \in \mathcal{M}_{11,1}$ be a Brill-Noether general pointed curve and we denote by $C_0 \subset \Delta_0 \subset \overline{\mathcal{M}}_{12}$ the associated test curve. Then $\sigma^*(C_0) \cdot c_3(\mathcal{F} - \text{Sym}^2 \mathcal{E}) = 22b_0 - b_1 = 32505$. It follows that $b_0 = 1926$.*

Proof. As already noted in Proposition 6.5, the vector bundles $\mathcal{E}|_{\sigma^*(C_0)}$ and $\mathcal{F}|_{\sigma^*(C_0)}$ are both pull-backs of vector bundles on Y and we denote these vector bundles \mathcal{E} and \mathcal{F} as well, that is, $\mathcal{E}|_{\sigma^*(C_0)} = (\epsilon \circ f)^*(\mathcal{E}|_Y)$ and $\mathcal{F}|_{\sigma^*(C_0)} = (\epsilon \circ f)^*(\mathcal{F}|_Y)$. Like in the proof of Theorem 6.9, we evaluate each term appearing in $\sigma^*(C_0) \cdot c_3(\mathcal{F} - \text{Sym}^2(\mathcal{E}))$.

Let V be the line bundle on Y with fibre

$$V(y, L) = \frac{H^0(C, L)}{H^0(C, L \otimes \mathcal{O}_C(-y-q))} \hookrightarrow H^0(C, L \otimes \mathcal{O}_{y+q})$$

over a point $(y, L) \in Y$. There is an exact sequence of vector bundles over Y

$$0 \rightarrow V \rightarrow \mathcal{B} \rightarrow \text{Ker}(\chi)^\vee \rightarrow 0,$$

where $\chi : \mathcal{B}^\vee \rightarrow \pi_2^*(\mathcal{M})^\vee$ is the bundle morphism defined in the second part of Proposition 6.7. In particular, $c_1(V) = 13\eta + \gamma - c_1(\text{Ker}(\chi^\vee))$. By using again [HT], we find the following formulas for the Chern numbers of $\text{Ker}(\chi)^\vee$:

$$c_1(\text{Ker}(\chi)^\vee) \cdot \xi|_Y = -c_5(\pi_2^*(\mathcal{M})^\vee - \mathcal{B}^\vee) \cdot \xi|_Y = -(\pi_2^*(c_5) + \pi_2^*(c_4)(13\eta + \gamma) - 2\pi_2^*(c_3)\eta\theta) \cdot \xi|_Y,$$

for any class $\xi \in H^2(C \times W_{14}^4(C), \mathbb{C})$. Recall that we introduced the vector bundle \mathcal{B}_2 over $C \times W_{14}^4(C)$ with fibre $\mathcal{B}_2(y, L) = H^0(C, L^{\otimes 2} \otimes \mathcal{O}_C(-y - q))$. We claim that one has an exact sequence of bundles over Y

$$(7) \quad 0 \rightarrow \mathcal{B}_{2|Y} \rightarrow \mathcal{F}|_Y \rightarrow V^{\otimes 2} \rightarrow 0.$$

If \mathcal{B}_3 is the vector bundle on Y with fibres $\mathcal{B}_3(y, L) = H^0(C, L^{\otimes 2})$, we have an injective morphism of sheaves $V^{\otimes 2} \hookrightarrow \mathcal{B}_3/\mathcal{B}_2$ locally given by

$$v^{\otimes 2} \mapsto v^2 \bmod H^0(C, L^{\otimes 2} \otimes \mathcal{O}_C(-y - q)),$$

where $v \in H^0(C, L)$ is any section not vanishing at q and y . Then $\mathcal{F}|_Y$ is canonically identified with the kernel of the projection morphism

$$\mathcal{B}_3 \rightarrow \frac{\mathcal{B}_3/\mathcal{B}_2}{V^{\otimes 2}}$$

and the exact sequence (7) now becomes clear. Therefore $c_1(\mathcal{F}|_Y) = c_1(\mathcal{B}_{2|Y}) + 2c_1(V)$, $c_2(\mathcal{F}|_Y) = c_2(\mathcal{B}_{2|Y}) + 2c_1(\mathcal{B}_{2|Y})c_1(V)$ and $c_3(\mathcal{F}|_Y) = c_3(\mathcal{B}_{2|Y}) + 2c_2(\mathcal{B}_{2|Y})c_1(V)$. The part of the total intersection number $\sigma^*(C_0) \cdot c_3(\mathcal{F} - \text{Sym}^2(\mathcal{E}))$ that does not contain $c_1(\text{Ker}(\chi^\vee))$ equals

$$28\pi_2^*(c_2)\theta - 88\pi_2^*(c_1^2)\theta - 22\eta\pi_2^*(c_1^2) - 53\pi_2^*(c_1c_2) - \frac{32}{3}\theta^3 +$$

$$-8\eta\theta^2 + 24\eta\theta\pi_2^*(c_1) + 64\pi_2^*(c_1^3) + 7\eta\pi_2^*(c_2) + 48\theta^2\pi_2^*(c_1) + 9\pi_2^*(c_3) \in H^6(C \times W_{14}^4(C), \mathbb{C})$$

and this gets multiplied with the class $[Y]$ from Proposition 6.7. The coefficient of $c_1(\text{Ker}(\zeta)^\vee)$ in $\sigma^*(C_0) \cdot c_3(\mathcal{F} - \text{Sym}^2\mathcal{E})$ equals

$$-2c_2(\mathcal{B}_{2|Y}) - 2(r+2)^2\pi_2^*(c_1^2) - 2(r+2)c_1(\mathcal{B}_{2|Y})\pi_2^*(c_1) + r(r+3)\pi_2^*(c_1^2) + 2(r+3)\pi_2^*(c_2).$$

All in all, $22b_0 - b_1 = \sigma^*(C_0) \cdot c_3(\mathcal{F} - \text{Sym}^2\mathcal{E})$ and we evaluate this using (6). \square

The following result follows from the definition of the vector bundles \mathcal{E} and \mathcal{F} given in Proposition 6.4:

Theorem 6.11. *Let $[C, q] \in \mathcal{M}_{11,1}$ be a Brill-Noether general pointed curve and $R \subset \overline{\mathcal{M}}_{12}$ the pencil obtained by attaching at the fixed point $q \in C$ a pencil of plane cubics. Then*

$$a - 12b_0 + b_1 = \sigma_*c_3(\mathcal{F} - \text{Sym}^2\mathcal{E}) \cdot R = 0.$$

End of the proof of Theorem 6.1. First we note that the virtual divisor \mathcal{D}_{12} is a genuine divisor on \mathcal{M}_{12} . Assuming by contradiction that for every curve $[C] \in \mathcal{M}_{12}$, there exists $L \in W_{14}^4(C)$ such that $\mu_0(L)$ is not-injective, one can construct a stable vector bundle E of rank 2 sitting in an extension

$$0 \rightarrow K_C \otimes L^\vee \rightarrow E \rightarrow L \rightarrow 0,$$

such that $h^0(C, E) = h^0(C, L) + h^1(C, L) = 7$, and for which the Mukai-Petri map $\text{Sym}^2 H^0(C, E) \rightarrow H^0(C, \text{Sym}^2 E)$ is not injective. This contradicts the main result from [T]. To determine the slope of $\overline{\mathcal{D}}_{12}$, we write $\overline{\mathcal{D}}_{12} \equiv a\lambda - \sum_{j=0}^6 b_j\delta_j \in \text{Pic}(\overline{\mathcal{M}}_{12})$. Since

$\frac{a}{b_0} = \frac{4415}{642} \leq \frac{71}{10}$, we can apply Corollary 1.2 from [FP], which gives the inequalities $b_j \geq b_0$ for $1 \leq j \leq 6$. Therefore $s(\overline{\mathcal{D}}_{12}) = \frac{a}{b_0} < 6 + \frac{12}{13}$. \square

We close by discussing a second counterexample to the Slope Conjecture on $\overline{\mathcal{M}}_{12}$.

Definition 6.12. Let V be a vector space. A pencil of quadrics $\ell \subset \mathbf{P}(\mathrm{Sym}^2(V))$ is said to be *degenerate* if the intersection of ℓ with the discriminant divisor $\mathbb{D}(V) \subset \mathbf{P}(\mathrm{Sym}^2(V))$ is non-reduced.

A general curve $[C] \in \mathcal{M}_{12}$ has finitely many linear systems $A \in W_{15}^5(C)$. As a consequence of the maximal rank conjecture [Vo], the multiplication map

$$\mu_0(A) : \mathrm{Sym}^2 H^0(C, A) \rightarrow H^0(C, A^{\otimes 2})$$

is surjective for each $A \in W_{15}^5(C)$, in particular $\mathbf{P}_{C,A} := \mathbf{P}(\mathrm{Ker} \mu_0(A))$ is a pencil of quadrics in \mathbf{P}^5 containing the image of the map $C \xrightarrow{|A|} \mathbf{P}^5$. One expects that the pencil $\mathbf{P}_{C,A}$ to be non-degenerate. By imposing the condition that it be degenerate, we produce a divisor on \mathcal{M}_{12} , whose class we compute.

We shall make essential use of the following result [FR]. Let X be a smooth projective variety, \mathcal{E} and \mathcal{F} vector bundles on X with $\mathrm{rk}(\mathcal{E}) = e$ and $\mathrm{rk}(\mathcal{F}) = \binom{e+1}{2} - 2$, and $\varphi : \mathrm{Sym}^2(\mathcal{E}) \rightarrow \mathcal{F}$ a surjective vector bundle morphism. Then the class of the locus

$$\mathcal{H} := \left\{ x \in X : \mathbf{P}(\mathrm{Ker} \varphi(x)) \subset \mathbf{P}(\mathrm{Sym}^2 \mathcal{E}(x)) \text{ is a degenerate pencil} \right\},$$

assuming it is of codimension one in X , is equal to

$$(8) \quad [\mathcal{H}] = (e-1)(e c_1(\mathcal{F}) - (e^2 + e - 4)c_1(\mathcal{E})) \in A^1(X).$$

Theorem 6.13. *The locus consisting of smooth curves of genus 12*

$$\mathcal{H}_{12} := \{ [C] \in \mathcal{M}_{12} : \mathbf{P}_{C,A} \text{ is degenerate for a } A \in W_{15}^5(C) \}$$

is an effective divisor. The slope of its closure $\overline{\mathcal{H}}_{12}$ inside $\overline{\mathcal{M}}_{12}$ equals $s(\overline{\mathcal{H}}_{12}) = \frac{373}{54} < 6 + \frac{12}{13}$.

Proof. We only sketch the main steps. We retain the notation in the proof of Theorem 6.1 and consider the stack $\sigma : \tilde{\mathfrak{G}}_{15}^5 \rightarrow \overline{\mathbf{M}}_{12}^p$ of limit linear series of type \mathfrak{g}_{15}^5 . Using [F2] Proposition 2.8, there exist two vector bundles \mathcal{E} and \mathcal{F} over $\tilde{\mathfrak{G}}_{15}^5$ together with a morphism $\varphi : \mathrm{Sym}^2(\mathcal{E}) \rightarrow \mathcal{F}$, such that over a point $[C, A] \in \sigma^{-1}(\mathcal{M}_{12}^p)$ corresponding to a smooth underlying curve one has the description of its fibres $\mathcal{E}(C, A) = H^0(C, A)$ and $\mathcal{F}(C, A) = H^0(C, A^{\otimes 2})$. Moreover $\varphi(C, A)$ is the multiplication map $\mu_0(A)$. The extension of \mathcal{E} and \mathcal{F} over the boundary of $\tilde{\mathfrak{G}}_{15}^5$ is identical to the one appearing in Proposition 6.4. Applying (8), the class of the restriction $\tilde{\mathcal{H}}_{12} := \overline{\mathcal{H}}_{12} \cap \mathcal{M}_{12}^p$ is equal to

$$[\tilde{\mathcal{H}}_{12}]^{\mathrm{virt}} = 10\sigma_*(6c_1(\mathcal{F}) - 38c_1(\mathcal{E})) \in A^1(\overline{\mathbf{M}}_{12}^p).$$

The pushforward classes $\sigma_*(c_1(\mathcal{E}))$ and $\sigma_*(c_1(\mathcal{F}))$ can be determined following [F2] Propositions 2.12 and 2.13, which after manipulations leads to the claimed slope.

To prove that \mathcal{H}_{12} is indeed a divisor, note first that \mathfrak{G}_{15}^5 being isomorphic to the Hurwitz space \mathfrak{G}_7^1 is irreducible. To establish that for a general curve $[C] \in \mathcal{M}_{12}$, the pencil $\mathbf{P}_{C,A}$ is non-degenerate for all linear systems $A \in W_{15}^5(C)$, it suffices to produce *one example* of a smooth curve $C \subset \mathbf{P}^5$ with $g(C) = 12$ and $\mathrm{deg}(C) = 15$, with $\mathbf{P}_{C, \mathcal{O}_C(1)}$ non-degenerate. This is carried out via the use of *Macaulay* in a way similar to the proof of Theorem 2.7 in [F1] for a curve C lying on a particular rational surface in \mathbf{P}^5 . \square

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HUMBOLDT-UNIVERSITÄT ZU BERLIN, INSTITUT FÜR MATHEMATIK, UNTER DEN LINDEN 6
10099 BERLIN, GERMANY
E-mail address: farkas@math.hu-berlin.de

UNIVERSITÀ ROMA TRE, DIPARTIMENTO DI MATEMATICA, LARGO SAN LEONARDO MURIALDO
1-00146 ROMA, ITALY
E-mail address: verra@mat.uniroma3.it