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The universal difference variety over $\overline{\mathcal{M}}_g$

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Abstract We determine the Kodaira dimension of the Deligne–Mumford compactification $\overline{\mathfrak{Diff}}_g$ of the universal difference variety over the moduli space of curves.

Keywords Moduli space of curves · Kodaira dimension · Difference variety · Jacobian variety

Mathematics Subject Classification 14H10 · 14H40 · 14H42

1 Introduction

For an algebraic curve C of genus $g \geq 2$, setting $i := \lfloor \frac{g+1}{2} \rfloor$, the *difference map*

$$\varphi_C : C_i \times C_i \to \mathrm{Pic}^0(C) \quad \text{defined by} \quad \varphi_C(D, E) := \mathcal{O}_C(D - E),$$

is surjective, that is, $C_i - C_i = \mathrm{Pic}^0(C)$. For even genus, φ_C is generically finite of degree $\binom{2i}{i}$, see [2, Chapter V]. For odd genus $g = 2i - 1$, when the curve C is non-hyperelliptic, the one degree lower difference variety $C_{i-1} - C_{i-1}$ is a divisor of class $\binom{2i-2}{i-1}\theta$, where $\theta \in H^2(\mathrm{Pic}^0(C), \mathbb{Q})$ is the class of the theta divisor. This divisorial difference variety has another incarnation via the results of [7] as the generalized theta divisor of the middle exterior power of the normal bundle of C inside its Jacobian variety $\mathrm{Pic}^0(C)$. If Q_C is the rank $g-1$ vector bundle on C defined via the exact sequence

$$0 \longrightarrow Q_C^{\vee} \longrightarrow H^0(C, K_C) \otimes \mathcal{O}_C \longrightarrow K_C \longrightarrow 0, \tag{1}$$

then the following equality of cycles holds for any non-hyperelliptic curve C, see [7]:

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$$C_{i-1} - C_{i-1} = \Theta_{\bigwedge^{i-1} Q_C} := \left\{ \xi \in \mathrm{Pic}^0(C) : h^0\left(C, \bigwedge^{i-1} Q_C \otimes \xi\right) \geq 1 \right\}.$$

The difference map is thus a canonical resolution of singularities of $\Theta_{\bigwedge^{i-1} Q_C}$, in the same way that the Abel–Jacobi map $C_{g-1} \to \mathrm{Pic}^{g-1}(C)$ provides a resolution of singularities of the classical theta divisor of C. This suggests an alternative approach to Green's Conjecture [19] on the syzygies of a canonical curve of odd genus and maximal Clifford index. Green's Conjecture holds for such a curve C, if and only if the map

$$\bigwedge^{i-1} H^0(C, K_C)^\vee \to H^0\left(C, \bigwedge^{i-1} Q_C\right)$$

induced by taking exterior powers and cohomology in the sequence (1) is an isomorphism, that is, $h^0(C, \bigwedge^{i-1} Q_C) = \binom{g}{i-1}$. On the other hand, results from [15] link $h^0(C, \bigwedge^{i-1} Q_C)$ to the singularities of the theta divisor $\Theta_{\bigwedge^{i-1} Q_C}$, which could open the way towards understanding the syzygies of C in terms of the singularities of φ_C.

Motivated by this connection, our aim is to pose this problem variationally and the birational properties of the universal difference variety $\mathfrak{Diff}_g := \mathcal{M}_{g,2i}/\mathfrak{S}_i \times \mathfrak{S}_i$; the two copies of the symmetric group $\mathfrak{S}_i$ act by permuting the first and the last i marked points of each $2i$-pointed curve $[C, x_1, \ldots, x_i, y_1, \ldots, y_i]$ respectively. The difference variety $\mathfrak{Diff}_g$ is equipped with a surjective difference map $\mathrm{diff} : \mathfrak{Diff}_g \to \mathfrak{Pic}_g^0$ to the universal degree zero Jacobian variety over $\mathcal{M}_g$, as well as with an Abel–Jacobi map $\mathrm{aj} : \mathfrak{Diff}_g \to \mathfrak{Pic}_g^{2i}$. Note than the universal Jacobian $\mathfrak{Pic}_g^d$, being a fibration in abelian varieties over $\mathcal{M}_g$, is never of maximal Kodaira dimension, precisely $\mathrm{kod}(\mathfrak{Pic}_g^d) \leq 3g-3$. Our main result concerns the birational classification of Deligne–Mumford compactification $\overline{\mathfrak{Diff}}_g := \overline{\mathcal{M}}_{g,2i}/\mathfrak{S}_i \times \mathfrak{S}_i$:

Theorem 1.1 *The universal difference variety $\overline{\mathfrak{Diff}}_g$ is a variety of general type for $g \geq 13$. The Kodaira dimension of $\overline{\mathfrak{Diff}}_{10}$ is equal to zero and kod$(\overline{\mathfrak{Diff}}_{12}) \geq 33$.*

It is known that the Kodaira dimension of $\overline{\mathcal{M}}_{g,2i}$ is negative for $g \leq 9$, so the same conclusion holds for $\overline{\mathfrak{Diff}}_g$ in this range. Theorem 1.1 fits into a pattern of recent classification results. It is shown in [9] that the Kodaira dimension of the degree g universal Jacobian variety $\mathfrak{Pic}_g^g$ is equal to $3g-3$ for $g \geq 12$, to 19 for $g = 11$, whereas $\mathrm{kod}(\mathfrak{Pic}_{10}^{10}) = 0$. This result was extended to any degree d satisfying the condition $\gcd(2g-2, g+1-d) = 1$ in [3], where it is shown that the Kodaira dimension of the degree d universal Jacobian $\mathfrak{Pic}_g^d$ is independent of d. In the paper [10], we proved that the universal theta divisor $\mathfrak{Th}_g$ over the moduli space of curves has general type for $g \geq 12$ and is uniruled for $g \leq 11$.

The proof of Theorem 1.1 distinguishes between the cases when g is odd or even, and uses in an essential way results from [7,9] in order to produce a big, effective representative of the canonical class $K_{\overline{\mathfrak{Diff}}_g}$. Precisely, if $\pi : \overline{\mathcal{M}}_{g,2i} \to \overline{\mathfrak{Diff}}_g$ is the quotient map, then the Hurwitz formula implies that

$$\pi^*\left(K_{\overline{\mathfrak{Diff}}_g}\right) = K_{\overline{\mathcal{M}}_{g,2i}} - \delta_{0:xx} - \delta_{0:yy} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,2i}), \tag{2}$$

where $\delta_{0:xx}$ (respectively $\delta_{0:yy}$) is the boundary divisor class on $\overline{\mathcal{M}}_{g,2i}$ corresponding to curves having a rational tail containing precisely two marked points, both of the type x_a and x_b (respectively y_a and y_b), where $1 \leq a < b \leq i$. Since on one hand, at the level of rational Picard groups, the map $\pi^* : \mathrm{Pic}(\overline{\mathfrak{Diff}}_g) \to \mathrm{Pic}(\overline{\mathcal{M}}_{g,2i})$ is injective and, on the other hand, the sum of cotangent lines $\sum_{j=1}^{2i} \psi_j \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,2i})$ descends to a big and nef line bundle on

$\overline{\mathfrak{Diff}}_g$ (essentially the same proof like that of Proposition 1.2 in [10]), in order to conclude that $\overline{\mathfrak{Diff}}_g$ is of general type for a given g, it suffices to find a $\mathfrak{S}_i \times \mathfrak{S}_i$-invariant effective divisor $\overline{\mathcal{D}}$ on $\overline{\mathcal{M}}_{g,2i}$ such that $\pi^*(K_{\overline{\mathfrak{Diff}}_g})$ is an effective $\mathbb{Q}$-combination of the class $[\overline{\mathcal{D}}]$, the Hodge class λ, boundary divisors invariant under the action of $\mathfrak{S}_i \times \mathfrak{S}_i$ and a *positive* multiple of $\sum_{j=1}^{2i} \psi_j$. In carrying this out, the choice of the divisor $\overline{\mathcal{D}}$ is crucial. In odd genus, this role is played by the fibrewise pull-back of the generalized theta divisor $\Theta_{\bigwedge^{i-1} Q_C} = C_{i-1} - C_{i-1}$ considered in Sect. 1 under the difference map diff : $\mathfrak{Diff}_g \to \mathfrak{Pic}^0_g$. Precisely, we define the locus

$$\mathcal{U}_g := \left\{ [C, x_1, \ldots, x_i, y_1, \ldots, y_i] \in \mathcal{M}_{g,2i} : \mathcal{O}_C\left(\sum_{j=1}^{i} (x_j - y_j)\right) \in C_{i-1} - C_{i-1} \right\},$$

and refer to Sect. 2 for more details.

In [10], we constructed $\mathfrak{S}_n$-invariant effective divisors on $\overline{\mathcal{M}}_{g,n}$ for $n \in \{g-1, g-2\}$ which determine an extremal ray of the respective cones of effective divisors. For instance, on $\overline{\mathcal{M}}_{g,g-2}$, the closure of the locus of pointed curves $[C, x_1, \ldots, x_{g-2}] \in \mathcal{M}_{g,g-2}$ such that there exists a pencil $A \in W^1_{g-1}(C)$ containing all the marked points in one of its fibres is an extremal divisor on $\overline{\mathcal{M}}_{g,g-2}$, see [10, Theorem 0.7]. We carry out a somewhat similar construction on $\overline{\mathcal{M}}_{g,g-3}$. If $D \in C_{g-3}$ is a general effective divisor of degree $g-3$ on a curve $[C] \in \mathcal{M}_g$, we observe that $K_C(-D) \in W^2_{g+1}(C)$. A natural codimension one condition on $\overline{\mathcal{M}}_{g,g-3}$ is that this plane model have a triple point [a similar construction requiring instead that $K_C(-D)$ have a cusp, produces a "less extremal" divisor]:

Theorem 1.2 *The closure inside $\overline{\mathcal{M}}_{g,g-3}$ of the locus*

$$\mathcal{D}_g := \left\{ [C, x_1, \ldots, x_{g-3}] \in \mathcal{M}_{g,g-3} : \exists L \in W^2_g(C) \text{ with } H^0\left(C, L\left(-\sum_{j=1}^{g-3} x_j\right)\right) \neq 0 \right\}$$

is an effective divisor. Its class in Pic$(\overline{\mathcal{M}}_{g,g-3})$ is equal to

$$[\overline{\mathcal{D}}_g] = -\frac{2(g-17)}{3}\binom{g-3}{2}\lambda + \frac{2g-3}{3}\binom{g-4}{2}\sum_{i=1}^{g-3}\psi_i - \binom{g-3}{2}\delta_{\mathrm{irr}}$$
$$-(g^2-5g+5)(g-5)\delta_{0:2} - \cdots .$$

One immediate consequence of Theorem 1.2 is that it gives a bound on the effective cone of the symmetric product C_{g-3} of any curve $[C] \in \mathcal{M}_g$. Restricting $\overline{\mathcal{D}}_g$ under the rational map $C_{g-3} \dashrightarrow \overline{\mathcal{M}}_{g,g-3}/\mathfrak{S}_{g-3}$ obtained by fixing the moduli of C, we obtain that the class $\theta - \frac{g}{g-3}x \in H^2(C_{g-3}, \mathbb{Q})$ is effective. Here θ and $x \in H^2(C_{g-3}, \mathbb{Z})$ denote the class of the pullback of the theta divisor and that of the locus of effective divisors of degree $g-3$ having a fixed point in their support respectively. We conjecture that this class spans an extremal ray of the effective cone of C_{g-3}, see Sect. 2 for details.

We close by thanking the referee for insightful comments that clearly improved this paper.

2 Effective divisors on universal difference varieties

The aim of this section is to prove Theorem 1.1 and we begin by reviewing the notation for boundary divisors and tautological classes on $\overline{\mathcal{M}}_{g,n}$. All the Picard groups of moduli spaces of curves considered in this paper are with rational coefficients, in particular, we identify the Picard group of the moduli stack with that of the corresponding coarse moduli space. A standard reference is [1]. For an integer $0 \leq j \leq \lfloor \frac{g}{2} \rfloor$ and a subset $T \subset \{1, \ldots, n\}$, we denote by $\Delta_{j:T}$ the closure in $\overline{\mathcal{M}}_{g,n}$ of the locus of n-pointed curves $[C_1 \cup C_2, x_1, \ldots, x_n]$, where C_1 and C_2 are smooth curves of genera j and $g - j$ respectively, meeting transversally in one point, and with the marked points lying on C_1 being exactly those indexed by T. We define $\delta_{j:T} := [\Delta_{j:T}]_{\mathbb{Q}} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$. For $0 \leq j \leq \lfloor \frac{g}{2} \rfloor$ and $0 \leq s \leq n$, we set

$$\Delta_{j:s} := \sum_{|T|=s} \delta_{j:T}, \quad \delta_{j:s} := [\Delta_{j:s}]_{\mathbb{Q}} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,n})^{\mathfrak{S}_n}.$$

By convention, $\delta_{0:s} := \emptyset$, for $s < 2$, and $\delta_{j:s} := \delta_{g-j:n-s}$. For $j = 1, \ldots, n$, we denote, as usual, by $\psi_j \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ the cotangent class corresponding to the marked point labeled by j, then set $\psi := \sum_{j=1}^{n} \psi_j \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,n})^{\mathfrak{S}_n}$.

If $\phi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_g$ is the morphism forgetting the marked points, we introduce the Hodge class $\lambda := \phi^*(\lambda)$ and $\delta_{\mathrm{irr}} := \phi^*(\delta_{\mathrm{irr}})$, where $\delta_{\mathrm{irr}} := [\Delta_{\mathrm{irr}}] \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ denotes the class of the locus of irreducible nodal curves. The canonical class of $\overline{\mathcal{M}}_{g,n}$ is computed via Kodaira–Spencer theory, see [13, Theorem 2] and [16, Theorem 2.6] respectively:

$$K_{\overline{\mathcal{M}}_{g,n}} = 13\lambda - 2\delta_{\mathrm{irr}} + \psi - 2 \sum_{\substack{T \subset \{1,\ldots,n\} \\ j \geq 0}} \delta_{j:T} - \delta_{1:\emptyset} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,n}). \tag{3}$$

Assume from now on that $i := \lfloor \frac{g+1}{2} \rfloor$ and $n := 2i$. We divide the $2i$-marked points into two equal groups and denote a general element of $\overline{\mathcal{M}}_{g,2i}$ by $[C, x_1, \ldots, x_i, y_1, \ldots, y_i]$. The group $\mathfrak{S}_i \times \mathfrak{S}_i$ acts on the marked points as follows

$$(\sigma, \tau) \cdot (x_1, \ldots, x_i, y_1, \ldots, y_i) = (x_{\sigma(1)}, \ldots, x_{\sigma(i)}, y_{\tau(1)}, \ldots, y_{\tau(i)}).$$

With respect to this action, there are invariant cotangent divisor classes

$$\psi_x := \sum_{j=1}^{i} \psi_{x_j} \quad \text{and} \quad \psi_y := \sum_{j=1}^{i} \psi_{y_j} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,2i})^{\mathfrak{S}_i \times \mathfrak{S}_i}.$$

For non-negative integers $0 \leq j \leq i$, $0 \leq s \leq 2i$ and $0 \leq \ell \leq s$, we define the $\mathfrak{S}_i \times \mathfrak{S}_i$-invariant boundary divisor class on $\overline{\mathcal{M}}_{g,2i}$

$$\delta_{j:s}^{\ell,s-\ell} := \sum_{T} \left\{\delta_{j:T} : |T \cap \{x_1, \ldots, x_i\}| = \ell, |T \cap \{y_1, \ldots, y_i\}| = s - \ell\right\} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,2i})^{\mathfrak{S}_i \times \mathfrak{S}_i}.$$

To ease notation, we write suggestively $\delta_{0:xx} = \delta_{0:2}^{2,0}$, $\delta_{0:yy} := \delta_{0:2}^{0,2}$ and $\delta_{0:xy} := \delta_{0,2}^{1,1}$, indicating how many marked points labeled by either x_j or y_j lie on a rational tail of the corresponding stable pointed curve. The classes $\lambda, \psi_x, \psi_y, \delta_{\mathrm{irr}}$ together with all the boundaries $\{\delta_{j:s}^{\ell,s-\ell}\}_{j,s,\ell \geq 0}$ generate the group $\mathrm{Pic}(\overline{\mathcal{M}}_{g,2i})^{\mathfrak{S}_i \times \mathfrak{S}_i}$. For $g \geq 3$, there are no relations between these classes.

We also introduce the forgetful morphisms

$$\overline{\mathcal{M}}_{g,i} \xleftarrow{\pi_x} \overline{\mathcal{M}}_{g,2i} \xrightarrow{\pi_y} \overline{\mathcal{M}}_{g,i},$$

where π_x drops all the marked points $y_1, \ldots, y_i$, whereas π_y drops the marked points $x_1, \ldots, x_i$ respectively. We summarize the pullback properties of the generators of $\mathrm{Pic}(\overline{\mathcal{M}}_{g,i})^{\mathfrak{S}_i}$ under the morphisms π_x and π_y respectively.

Proposition 2.1 *For $0 \leq j \leq i$ and $0 \leq s \leq 2i$, the following relations hold in $Pic(\overline{\mathcal{M}}_{g,2i})^{\mathfrak{S}_i \times \mathfrak{S}_i}$:*

$$\pi_x^*(\psi) = \psi_x - \sum_{s=2}^{i-1} \delta_{0:s}^{1,s-1}, \quad \pi_y^*(\psi) = \psi_y - \sum_{s=2}^{i-1} \delta_{0:s}^{s-1,1},$$
$$\pi_x^*(\delta_{j:s}) = \sum_{\ell \geq 0} \delta_{j:s+\ell}^{s,\ell}, \quad \pi_y^*(\delta_{j:s}) = \sum_{\ell \geq 0} \delta_{j:s+\ell}^{\ell,s}, \quad \text{and}$$

$$\pi_x^*(\lambda) = \pi_y^*(\lambda) = \lambda, \quad \pi_x^*(\delta_{\mathrm{irr}}) = \pi_y^*(\delta_{\mathrm{irr}}) = \delta_{\mathrm{irr}}.$$

Proof The morphism $\pi_x : \overline{\mathcal{M}}_{g,2i} \to \overline{\mathcal{M}}_{g,i}$ can be expressed as a composition of i forgetful morphisms, by dropping successively the marked points $y_1, \ldots, y_i$ respectively. This reduces the problem to that of understanding the pull-back of divisor classes under a forgetful map $f : \overline{\mathcal{M}}_{g,P\cup\{y\}} \to \overline{\mathcal{M}}_{g,P}$, where P is an arbitrary set of labels and $y \notin P$ is another label. The following formulas hold, cf. [16, Theorem 2.3]:

$$f^*(\lambda) = \lambda, \quad f^*(\delta_{\mathrm{irr}}) = \delta_{\mathrm{irr}}, \quad f^*(\psi_x) = \psi_x - \delta_{0:xy}, \quad \text{for each } x \in P,$$

as well as $f^*(\delta_{j:T}) = \delta_{j:T} + \delta_{j:T\cup\{y\}}$, for each $0 \leq j \leq g$ and $T \subset P$. Applying these formulas i times, the conclusion follows. □

As already pointed out, $\overline{\mathfrak{Diff}}_g := \overline{\mathcal{M}}_{g,2i}/\mathfrak{S}_i \times \mathfrak{S}_i$ serves as a birational model for the universal difference variety. Let us denote by $\pi : \overline{\mathcal{M}}_{g,2i} \to \overline{\mathfrak{Diff}}_g$ the projection morphism, which is simply ramified precisely along the divisor $\Delta_{0:xx} + \Delta_{0:yy}$. Indeed, by studying the action of $\mathfrak{S}_i \times \mathfrak{S}_i$ at the general point of each of the boundary divisors $\Delta_{j:s}^{\ell,s-\ell} \subset \overline{\mathcal{M}}_{g,2i}$, the only fixed divisors are those corresponding to a 1-nodal curve with a genus 0 tail containing two marked points, both either of type $\{x_a, x_b\}$, or $\{y_a, y_b\}$. These correspond either to the boundary divisor $\Delta_{0:xx}$ or to $\Delta_{0:yy}$.

Proposition 2.2 *The singularities of $\overline{\mathfrak{Diff}}_g$ do not impose adjunction conditions, that is, if $\epsilon : \widetilde{\mathfrak{Diff}}_g \to \overline{\mathfrak{Diff}}_g$ is a resolution of singularities, then for any $\ell \geq 0$, there is an isomorphism*

$$\epsilon^* : H^0\left((\overline{\mathfrak{Diff}}_g)reg, K_{\overline{\mathfrak{Diff}}_g}^{\otimes \ell}\right) \xrightarrow{\cong} H^0\left(\widetilde{\mathfrak{Diff}}_g, K_{\widetilde{\mathfrak{Diff}}_g}^{\otimes \ell}\right).$$

Proof Follows entirely along the lines of [9] Theorem 1.1. □

As a consequence, just like in the case of a smooth projective variety, the Kodaira dimension of $\overline{\mathfrak{Diff}}_g$ is equal to the Kodaira–Iitaka dimension of the canonical class $K_{\overline{\mathfrak{Diff}}_g}$. In particular, the space $\overline{\mathfrak{Diff}}_g$ has general type if and only if the $\mathbb{Q}$-divisor class $K_{\overline{\mathfrak{Diff}}_g}$ is a linear combination of an ample and an effective $\mathbb{Q}$-class on $\overline{\mathfrak{Diff}}_g$.

In our proof of Theorem 1.1 we consider $\mathfrak{S}_i$-invariant effective divisors on $\overline{\mathcal{M}}_{g,i}$, having, preferably, negative λ-coefficient. If $g = 2i$, let $\overline{\mathcal{D}}_2$ be the closure in $\overline{\mathcal{M}}_{g,i}$ of the locus $\mathcal{D}_2$ of smooth curves $[C, x_1, \ldots, x_i] \in \mathcal{M}_{g,i}$ such that $h^0(C, \mathcal{O}_C(2x_1 + \cdots + 2x_i)) \geq 2$. It follows from [16, Theorem 5.4], that $\overline{\mathcal{D}}_2$ is an effective divisor on $\overline{\mathcal{M}}_{g,i}$ and its class is equal to

$$[\overline{\mathcal{D}}_2] = -\lambda + 3\psi - 10\delta_{0:2} - \sum_{\substack{j,s \geq 0 \\ (j,s) \neq (0,2)}} b_{j:s}\delta_{j:s} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,i})^{\mathfrak{S}_i}, \quad (4)$$

where $b_{j:s} \geq 0$ for $(j, s) \neq (0, 2)$.

We shall also need an $\mathfrak{S}_{2i}$-invariant effective divisor on $\overline{\mathcal{M}}_{g,2i}$ having, if possible, negative λ-coefficient. We consider the locus $\mathfrak{L}_g$ of g-pointed smooth curves $[C, x_1, \ldots, x_g] \in \mathcal{M}_{g,g}$ with the property $h^0\big(C, \mathcal{O}_C(x_1 + \cdots + x_g)\big) \geq 2$. For the class of the closure $\overline{\mathfrak{L}}_g$ inside $\overline{\mathcal{M}}_{g,g}$, we refer either to [16, Theorem 5.4], or, for an alternative proof, to [6, Theorem 4.6]:

$$[\overline{\mathfrak{L}}_g] = -\lambda + \psi - \sum_{j,s\geq 0} \binom{|s-j|+1}{2} \delta_{j:s} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,g})^{\mathfrak{S}_g}. \tag{5}$$

The coefficient of $\delta_{0:2}$ in this formula is equal to -3, whereas that of δ_{irr} is equal to zero.

We then let $\mathcal{E}_{g+1}$ be (up to rescaling by the factor $\frac{1}{g+1}$), the locus of pointed curves $[C, x_1, \ldots, x_{g+1}] \in \mathcal{M}_{g,g+1}$ such that there exists an index $1 \leq \ell \leq g+1$, with

$$h^0\left(C, \mathcal{O}_C\left(\sum_{j=1}^{g+1} x_j - x_\ell\right)\right) \geq 2.$$

Denoting for $\ell = 1, \ldots, g+1$ by $\pi_\ell : \overline{\mathcal{M}}_{g,g+1} \to \overline{\mathcal{M}}_g$ the map forgetting the marked point labeled by x_ℓ, we have the following equality of $\mathbb{Q}$-divisors on $\overline{\mathcal{M}}_{g,g+1}$

$$\overline{\mathcal{E}}_{g+1} := \frac{1}{g+1} \sum_{\ell=1}^{g+1} \pi_\ell^*(\overline{\mathfrak{L}}_g).$$

In particular, using the pull-back formulas [16, Theorem 2.3], we find that

$$[\overline{\mathcal{E}}_{g+1}] = -\lambda + \frac{g}{g+1}\psi - \sum_{j,s\geq 0} \frac{(g+1-s)c_{j:s} + s c_{j:s-1}}{g+1} \delta_{j:s} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,g+1})^{\mathfrak{S}_{g+1}}, \tag{6}$$

where $c_{j:s} := \binom{|s-j|+1}{2}$ is the coefficient of $-\delta_{j:s}$ in the expression (5) of the class $[\bar{\mathfrak{L}}_g]$. Furthermore, we use the convention $\delta_{0:1} := -\psi$, therefore the coefficient of $-\delta_{0:2}$ in the expansion of $[\overline{\mathcal{E}}_{g+1}]$ is equal to $\frac{3g-1}{g+1}$.

Essential in the proof of Theorem 1.1 in the odd genus case $g = 2i-1$, is the following effective divisor $\overline{\mathcal{U}}_g$ on $\overline{\mathcal{M}}_{g,g+1}$ having $\mathfrak{S}_i \times \mathfrak{S}_i$-symmetry. We consider the locus of pointed curve $[C, x_1, \ldots, x_i, y_1, \ldots, y_i] \in \mathcal{M}_{g,g+1}$ possessing a pencil $A \in W^1_g(C)$ containing all the points $\{x_j\}_{j=1}^i$ and $\{y_j\}_{j=1}^i$ respectively in two distinct fibres, that is,

$$H^0\left(C, A\left(-\sum_{j=1}^{i} x_j\right)\right) \neq 0 \quad \text{and} \quad H^0\left(C, A\left(-\sum_{j=1}^{i} y_j\right)\right) \neq 0.$$

This divisor has already been considered in [7, Section 4], where the following alternative geometric characterization of its points is shown:

$$[C, x_1, \ldots, x_i, y_1, \ldots, y_i] \in \mathcal{U}_g \Longleftrightarrow h^0\left(C, \bigwedge^{i-1} Q_C\left(\sum_{j=1}^{i} x_j - \sum_{j=1}^{i} y_j\right)\right) \neq 0.$$

Fibrewise, $\mathcal{U}_g$ is the pull-back of the Raynaud theta divisor $\Theta_{\bigwedge^{i-1} Q_C} \subset \mathrm{Pic}^0(C)$ under the difference map $\phi_C : C_i \times C_i \to \mathrm{Pic}^0(C)$ considered in Sect. 1.

Before computing the class $[\overline{\mathcal{U}}_g]$, we record the following enumerative fact.

Lemma 2.3 *Let C be a general curve of genus $2i-1$ and fix general points $x_2, \ldots, x_i, y_1, \ldots, y_i \in C$. The number of pencils $A \in W^1_{2i-1}(C)$ such that*

$$H^0\left(C, A\left(-\sum_{j=2}^{i} x_j\right)\right) \neq 0, \quad \textit{and} \quad H^0\left(C, A(-\sum_{j=1}^{i} y_j)\right) \neq 0,$$

is equal to $\frac{1}{2}\binom{2i}{i} - 1$.

Proof First we observe that the divisors $x_2 + \cdots + x_i$ and $y_1 + \cdots + y_i$ cannot appear in the same fibre of A, for else, $A = \mathcal{O}_C\left(\sum_{j=2}^{i} x_j + \sum_{j=1}^{i} y_j\right)$, and then the $2i-1$ points $x_2, \ldots, x_i, y_2, \ldots, y_i$ move in a pencil; this contradicts the generality assumption on the marked points. We let the points $x_2, \ldots, x_i$ and $y_1, \ldots, y_i$ respectively, come together. One is led to compute, for a general 2-pointed curve $[C, x, y] \in \mathcal{M}_{2i-1,2}$, the number of pencils $A \in W^1_{2i-1}(C)$, such that $H^0(C, A(-(i-1)x)) \neq 0$ and $H^0(C, A(-iy)) \neq 0$. By a standard argument, see [4], or [6, Theorem 4.6], or [16, Theorem 3.2], we degenerate $[C, x, y]$ to a flag curve consisting of a smooth rational spine R and g elliptic tails meeting R at general points $p_1, \ldots, p_{2i-1} \in R$, such that the marked points x and y specialize to general points $x, y \in R$. The number in question equals the number of linear series $\mathfrak{g}^1_{2i+1}$ on $\mathbf{P}^1$ with cusps at $p_1, \ldots, p_{2i-1}$ and ramification sequence $(0, i-1)$ at x and $(0, i-2)$ at y respectively. This equals the intersection product $\sigma_{(0,1)}^{2i-1} \cdot \sigma_{(0,i-1)} \cdot \sigma_{(0,i-2)}$ of Schubert cycles in the cohomology ring $H^*(\mathbf{G}(1, 2i-1), \mathbb{Q})$ of the Grassmannian of lines in $\mathbf{P}^{i-1}$. Applying the Pieri formula we write

$$\sigma_{(0,i-1)} \cdot \sigma_{(0,i-2)} = \sum_{a=0}^{i-2} \sigma_{a,2i-3-a} \in H^*(\mathbf{G}(1, 2i-1), \mathbb{Q}),$$

hence after using [12, Example 14.7.11], the sought-after number is equal to

$$\sum_{a=0}^{i-2} \sigma_{(0,1)}^{2i-1} \cdot \sigma_{(a,2i-3-a)} = \sum_{a=0}^{i-2} \binom{2i}{a+1} \frac{i-a-1}{i} = \frac{1}{2}\binom{2i}{i} - 1.$$

□

Theorem 2.4 *For $i \geq 2$, the following formula holds in $Pic(\overline{\mathcal{M}}_{2i-1,2i})^{\mathfrak{S}_i \times \mathfrak{S}_i}$:*

$$[\overline{\mathcal{U}}_{2i-1}] = \binom{2i-3}{i-2}\left(\frac{6i-4}{2i-3}\lambda + (\psi_x + \psi_y) - \frac{i-1}{2i-3}\delta_{\rm irr} - 4(\delta_{0:xx} + \delta_{0:yy}) - \cdots\right) - \delta_{0:xy}.$$

We point out that the remaining coefficients of $\delta_{j:s}$ that do not appear explicitly are all non-positive; the coefficient of $\delta_{0:xy}$ is equal to -1.

Proof The coefficients of λ, ψ_x, ψ_y and $\delta_{\rm irr}$ in the expression $[\overline{\mathcal{U}}_g]$ are determined in the course of proving Theorem 4.1 in [7]. By the $\mathbb{Z}_2$-symmetry in the vectors $\sum_{j=1}^{i} x_j$ and $\sum_{j=1}^{i} y_j$ in the construction of $\overline{\mathcal{U}}_g$, the coefficients of $\delta_{0:xx}$ and $\delta_{0:yy}$ are equal. We first consider the forgetful map $\pi_{x_1} : \overline{\mathcal{M}}_{g,g+1} \to \overline{\mathcal{M}}_{g,g}$ dropping the point x_1 and we claim we have the following equality of effective divisors on $\overline{\mathcal{M}}_{g,g}$

$$(\pi_{x_1})_* \left([\overline{\mathcal{U}}_g] \cdot \delta_{0:x_1 y_1}\right) = \mathfrak{L}_g + \overline{\mathcal{Z}}_g, \tag{7}$$

where $\mathfrak{L}_g$ is Logan's divisor on $\overline{\mathcal{M}}_{g,g}$ already considered before, and $\mathcal{Z}_g$ is the locus consisting of smooth pointed curves $[C, x_2, \ldots, x_i, y_1, \ldots, y_i] \in \mathcal{M}_{g,g}$ for which there exists a pencil

$A \in W^1_{g-1}(C)$ containing the points $x_2, \ldots, x_i$ (respectively $y_2, \ldots, y_i$) in two distinct fibres. Granting (7), it follows that the $\delta_{0:xy}$-coefficient in $[\overline{\mathcal{U}}_g]$ is equal to -1. Indeed, on one hand, one has the identity $(\pi_{x_1})_*(\delta^2_{0:x_1y_1}) = -\psi_{y_1}$ and no other push-forward class of type $(\pi_{x_1})_*(\xi \cdot \delta_{0:x_1y_1})$, where $\xi \in \text{Pic}(\overline{\mathcal{M}}_{g,g+1})$ is a tautological class, contains ψ_{y_1}. Therefore, the $\delta_{0:x_1y_1}$-coefficient of $[\overline{\mathcal{U}}_g]$ is equal to the $(-\psi_{y_1})$-coefficient of $[\mathfrak{L}_g]+[\overline{\mathcal{Z}}_g]$. The definition of $\overline{\mathcal{Z}}_g$ makes no reference to the marked point y_1, hence $\overline{\mathcal{Z}}_g$ is the pull-back of an effective divisor from $\overline{\mathcal{M}}_{g,g-1}$ under the map $\pi_{y_1} : \overline{\mathcal{M}}_{g,g} \to \overline{\mathcal{M}}_{g,g-1}$ dropping y_1. In particular the ψ_{y_1}-coefficient of $[\overline{\mathcal{Z}}_g]$ is equal to zero. The conclusion now follows from relation (5).

We now prove the equality of divisors (7). We choose a general point in the intersection $\overline{\mathcal{U}}_g \cap \Delta_{0:x_1y_1}$. It corresponds to a stable curve $C \cup_p \mathbf{P}^1$, where C is a smooth curve of genus g, and to distinct marked points $x_1, y_1 \in \mathbf{P}^1 - \{p\}$ and $x_2, \ldots, x_i, y_2, \ldots, y_i \in C - \{p\}$ respectively. From the definition of $\overline{\mathcal{U}}_g$ it follows that there exists a limit linear series $l := (l_C, l_{\mathbf{P}^1})$ of type $\mathfrak{g}^1_g$ on $C \cup_p \mathbf{P}^1$, together with pairs of sections $(\sigma_C, \sigma_{\mathbf{P}^1})$ and $(\tau_C, \tau_{\mathbf{P}^1})$ of l, satisfying the following conditions:

$$\text{div}(\sigma_C) \geq x_2 + \cdots + x_i, \qquad \text{div}(\tau_C) \geq y_2 + \cdots + y_i,$$
$$\text{div}(\sigma_{\mathbf{P}^1}) \geq x_1, \quad \text{div}(\tau_{\mathbf{P}^1}) \geq y_1, \quad \text{and}$$
$$\text{ord}_p(\sigma_C) + \text{ord}_p(\sigma_{\mathbf{P}^1}) \geq g, \quad \text{ord}_p(\tau_C) + \text{ord}_p(\tau_{\mathbf{P}^1}) \geq g.$$

Note that both sections $\sigma_{\mathbf{P}^1}$ and $\tau_{\mathbf{P}^1}$ vanish somewhere else apart from p, hence σ_C and τ_C both vanish at p. Assuming $\sigma_C \neq \tau_C$, it follows that the linear system $l_C \in G^1_g(C)$ has a base point at p. Subtracting it, we obtain a pencil $l'_C \in G^1_{g-1}(C)$ having the divisors $x_2 + \cdots + x_i$ and $y_2 + \cdots + y_i$ appear in two of its fibres. This shows that

$$[C, x_2, \ldots, x_i, p = y_1, y_2, \ldots, y_i] \in \mathcal{Z}_g.$$

If, on the other hand, σ_C and τ_C are equal (up to scalar multiplication), then the underlying line bundle of l_C is $\mathcal{O}_C(p + \sum_{j=2}^i (x_j + y_j))$. In particular, we conclude that $[C, x_2, \ldots, x_i, p = y_1, \ldots, y_i] \in \mathfrak{L}_g$. This proves the set-theoretic inclusion

$$(\pi_{x_1})_* \left([\overline{\mathcal{U}}_g] \cdot \Delta_{0:x_1y_1}\right) \subset \mathfrak{L}_g \cup \overline{\mathcal{Z}}_g.$$

The reverse inclusion, as well as the fact that these points appear with multiplicity one in the equality (7), are standard exercise in admissible coverings, see also [7, Proposition 4.6].

To determine the $\delta_{0:xx}$-coefficient in $[\overline{\mathcal{U}}_g]$, we use a fibral curve of π_{x_1}. We fix a general curve C of genus g, general points $x_2, \ldots, x_i, y_1, \ldots, y_i \in C$, and denote by $F_{x_1} \subset \overline{\mathcal{M}}_{g,g+1}$ the fibre $\pi^{-1}_{x_1}([C, x_2, \ldots, x_i, y_1, \ldots, y_i]) \subset \overline{\mathcal{M}}_{g,g+1}$. Then one has

$$F_{x_1} \cdot \psi_{x_1} = 2g-2+2i-1 = 6i-5, \; F_{x_1} \cdot \psi_{x_2} = \cdots = F_{x_1} \cdot \psi_{x_i} = F_{x_1} \cdot \psi_{y_1} = \cdots = F_{x_1} \cdot \psi_{y_i} = 1.$$

Furthermore, $F_{x_1} \cdot \delta_{0:xx} = i-1$, $F_{x_1} \cdot \delta_{0:xy} = 1$ and $F_{x_1} \cdot \delta_{0:yy} = 0$. The intersection of F_{x_1} with all other tautological classes in $\text{Pic}(\overline{\mathcal{M}}_{g,g+1})$ is obviously equal to zero. On the other hand, $F_{x_1} \cdot \overline{\mathcal{U}}_g$ is equal to i times the number of pencils $A \in W^1_{2i-1}(C)$ containing the divisors $x_2 + \cdots + x_i$ and $y_1 + \cdots + y_i$ respectively, in two different fibres. Applying Lemma 2.3 this number is equal to $\frac{1}{2}\binom{2i}{i} - 1$. In this we obtain a relation determining the δ_{xx}-coefficient of $[\overline{\mathcal{U}}_g]$. □

The last ingredient in the proof of Theorem 1.1 is the following result:

Proposition 2.5 *The divisor class* $\psi := \psi_x + \psi_y \in Pic(\overline{\mathcal{M}}_{g,2i})$ *descends to a big and nef divisor class on* $\overline{\mathfrak{Diff}}_g$.

Proof The class ψ is $\mathfrak{S}_{2i}$-invariant, hence there exists a class $N_{g,2i} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,2i}/\mathfrak{S}_{2i})$, which pulls back to ψ. It is proved in [10, Proposition 1.2] that this class $N_{g,2i}$ is big and nef. Consider the sequence of finite maps $\overline{\mathcal{M}}_{g,2i} \xrightarrow{\pi} \overline{\mathfrak{Diff}}_g \xrightarrow{\nu} \overline{\mathcal{M}}_{g,2i}/\mathfrak{S}_{2i}$. Then $\nu^*(N_{g,2i}) \in \mathrm{Pic}(\overline{\mathfrak{Diff}}_g)$ is still big and nef and has the property that $\pi^*(\nu^*(N_{g,2i})) = \psi$, which finishes the proof. □

We can finally determine the Kodaira dimension of $\overline{\mathfrak{Diff}}_g$ for $g > 13$. Theorem 1.1 will follow from the following two, more precise statements given in terms of the slope

$$s(\overline{\mathcal{M}}_g) := \inf_{D\in \mathrm{Eff}(\overline{\mathcal{M}}_g)} s(D)$$

of the moduli space of curves. Recall that the Brill–Noether divisors $\overline{\mathcal{M}}^r_{g,d}$ of curves C with $W^r_d(C) \neq \emptyset$ for $\rho(g,r,d) = -1$, yield the upper bound $s(\overline{\mathcal{M}}_g) \leq 6 + \frac{12}{g+1}$, for any g such that $g+1$ is composite, see [4]. For any even genus $g = 2k-2$, the *Petri divisors* provide the slightly weaker upper bound $s(\overline{\mathcal{M}}_g) \leq \frac{6k^2+k-6}{k(k-1)}$. Stronger lower bounds are given for an infinite sequence of genera g in [6] using *Koszul divisors*, though these will not be needed in this paper.

Theorem 2.6 *Set $g := 2i$. The universal difference variety $\overline{\mathfrak{Diff}}_g$ is of general type, whenever $s(\overline{\mathcal{M}}_g) \leq \frac{41}{6}$.*

Proof Recall that $\overline{\mathcal{D}}_2$ is the effective divisor on $\overline{\mathcal{M}}_{g,i}$ whose class is computed by (4). Applying Proposition 2.1, we write the following formula

$$\pi_x^*([\overline{\mathcal{D}}_2]) + \pi_y^*([\overline{\mathcal{D}}_2]) = -2\lambda + 3(\psi_x+\psi_y) - 6\delta_{0:xy} -10(\delta_{0:xx}+\delta_{0:yy}) - \cdots \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,2i})^{\mathfrak{S}_i\times\mathfrak{S}_i};$$

the coefficient of δ_{irr} in this formula is equal to zero and all the other boundary classes appear with non-positive coefficients.

The $\mathfrak{S}_g$-invariant class $\overline{\mathfrak{L}}_g$ on $\overline{\mathcal{M}}_{g,g}$ constructed in [16] and has the following class $[\overline{\mathfrak{L}}_g] = -\lambda+\psi_x+\psi_y-3(\delta_{0:xy}+\delta_{0:xx}+\delta_{0:yy})-\cdots$. We form the following effective $\mathbb{Q}$-combination in $\mathrm{Eff}(\overline{\mathcal{M}}_{g,2i})^{\mathfrak{S}_i\times\mathfrak{S}_i}$

$$F := \frac{1}{4}\left(\pi_x^*([\overline{\mathcal{D}}_2]) + \pi_y^*([\overline{\mathcal{D}}_2])\right) + \frac{1}{6}[\overline{\mathfrak{L}}_g] = -\frac{2}{3}\lambda + \frac{11}{12}(\psi_x+\psi_y) - 2\delta_{0:xy} - 3(\delta_{0:xx}+\delta_{0:yy}) - \cdots,$$

where the coefficient of δ_{irr} in this expression is equal to zero and all the other boundary classes have non-positive coefficients. We now fix an effective divisor $D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$ of slope $s(D) = s$ as small as possible, for instance a Brill–Noether or Petri divisor; after possibly rescaling by a positive rational number, we write

$$[D] = s\lambda - \delta_{\mathrm{irr}} - \sum_{j=1}^{\lfloor \frac{g}{2}\rfloor} b_j\delta_j \in \mathrm{Pic}(\overline{\mathcal{M}}_g).$$

Using the explicit formulas for $[D]$ from [4], we have that $b_j \geq 1$ for $j \geq 1$. Then the class

$$(2s-1)\phi^*([D]) + F = \left(2s - \frac{2}{3}\right)\lambda + \frac{11}{12}(\psi_x+\psi_y) - 2\delta_{0:xy} - 3(\delta_{0:xx}+\delta_{0:yy}) - \cdots$$

is effective. By comparing this class against that of $\pi^*(K_{\overline{\mathfrak{Diff}}_g})$, from (2), it follows that whenever $2s - \frac{2}{3} \le 13$, the universal difference variety $\overline{\mathfrak{Diff}}_g$ has general type. This last condition is satisfied when $s(\overline{\mathcal{M}}_g) \le \frac{41}{6}$. □

Theorem 2.6 coupled with the above mentioned upper bounds on $s(\overline{\mathcal{M}}_g)$ implies Theorem 1.1 for even $g \ge 14$. In genus $g = 12$, we only have partial results, via the existence of the global Abel–Jacobi generically finite map $\overline{\mathfrak{Diff}}_g \dashrightarrow \overline{\mathfrak{Pic}}^g_g$. In particular, we have the inequality $\mathrm{kod}(\overline{\mathfrak{Diff}}_{12}) \ge \mathrm{kod}(\overline{\mathfrak{Pic}}^{12}_{12})$ and it is shown in [9] that $\mathrm{kod}(\overline{\mathfrak{Pic}}^{12}_{12}) = 33$.

Remark 2.7 In view of Tan's [18] lower bound $s(\overline{\mathcal{M}}_{12}) \ge \frac{41}{6}$ for the slope of $\overline{\mathcal{M}}_{12}$, the geometry of $\overline{\mathfrak{Diff}}_{12}$ appears quite intriguing. Note also that $g = 12$ is the smallest genus when $s(\overline{\mathcal{M}}_g)$ is not known. It is an interesting open question to construct an effective divisor on $\overline{\mathcal{M}}_{12}$ having slope $\frac{41}{6}$. The known effective divisor on $\overline{\mathcal{M}}_{12}$ of smallest slope is the one from [11], giving the bound $s(\overline{\mathcal{M}}_{12}) \le \frac{4415}{642}$.

Theorem 2.8 *Set* $g := 2i + 1$. *The universal difference variety* $\overline{\mathfrak{Diff}}_g$ *is a variety of general type for* $g \ge 13$.

Proof This time we consider the divisor $[\overline{\mathcal{U}}_g]$ and the $\mathfrak{S}_{g+1}$-invariant class $[\overline{\mathcal{E}}_{g+1}]$ on $\overline{\mathcal{M}}_{g,g+1}$ whose class is computed in (6). We set the positive constants

$$\alpha := \frac{i}{3i-2} \cdot \frac{8\binom{2i-3}{i-2} - 3}{4\binom{2i-3}{i-2} - 1} \quad \text{and} \quad \beta := \frac{\binom{2i-3}{i-2}}{4\binom{2i-3}{i-2} - 1},$$

then choose an effective divisor D on $\overline{\mathcal{M}}_g$, where $[D] = s\lambda - \delta_{\mathrm{irr}} - \cdots \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$. We form the effective $\mathbb{Q}$-linear combination

$$F := \left(2 - \beta\frac{i-1}{2i-3}\right) \cdot \phi^*([D]) + \alpha \cdot [\overline{\mathcal{E}}_{g+1}] + \beta \cdot [\overline{\mathcal{U}}_g] \in \mathrm{Eff}(\overline{\mathcal{M}}_{g,g+1})^{\mathfrak{S}_i \times \mathfrak{S}_i},$$

whose δ_{irr}, $\delta_{0:xx}$, $\delta_{0:yy}$ and $\delta_{0:xy}$-coefficients respectively are all equal to those of the class

$$\pi^*(K_{\overline{\mathfrak{Diff}}_g}) = 13\lambda + \psi_x + \psi_y - 2\delta_{0:xy} - 3(\delta_{0:xx} + \delta_{0:yy}) - \cdots,$$

whereas the coefficient of $\psi_x + \psi_y$ is smaller than 1. It follows that the class $\pi^*(K_{\overline{\mathfrak{Diff}}_g}) - F$ is big if and only if the following inequality holds

$$s(\overline{\mathcal{M}}_g) \le \frac{(2i-3)(\alpha + 13) - 2\beta(3i-2)}{2(2i-3) - \beta(i-1)}.$$

This inequality is satisfied for $i \ge 7$, which shows that $\overline{\mathfrak{Diff}}_g$ is of general type for odd $g \ge 13$. For $i = 6$, we obtain the bound $s \le 6.907...$, but since $s(\overline{\mathcal{M}}_{11}) \ge 7$, see for instance [8], there is no effective divisor on $\overline{\mathcal{M}}_{11}$ satisfying this condition. □

We make no prediction concerning the Kodaira dimension of $\overline{\mathfrak{Diff}}_{11}$. We complete the proof of Theorem 1.1 by dealing with the case $g = 10$.

Theorem 2.9 *The Kodaira dimension of* $\overline{\mathfrak{Diff}}_{10}$ *is equal to zero.*

Proof We consider the divisor $\overline{\mathcal{K}}_{10}$ on $\overline{\mathcal{M}}_{10}$ consisting of curves lying on a $K3$ surface. It follows from [8] that $s(\overline{\mathcal{K}}_{10}) = 7$; furthermore, the Kodaira–Iitaka dimension of the linear system $|\overline{\mathcal{K}}_{10}|$ is equal to zero. It is shown in [9] Theorem 0.1 that the Kodaira dimension of the universal symmetric product $\overline{\mathcal{M}}_{10,10}/\mathfrak{S}_{10}$ is equal to 0. Since there is a finite map

$\nu : \overline{\mathfrak{Diff}}_{10} \to \overline{\mathcal{M}}_{10,10}/\mathfrak{S}_{10}$, one obtains the inequality $\mathrm{kod}(\overline{\mathfrak{Diff}}_{10}) \geq 0$. We now establish the opposite inequality and show that $\mathrm{kod}(\overline{\mathcal{M}}_{10,10}, \pi^*(K_{\overline{\mathfrak{Diff}}_{10}})) = 0$.

Applying (2) and (5), we write the following equality in $\mathrm{Pic}(\overline{\mathcal{M}}_{10,10})^{\mathfrak{S}_5\times\mathfrak{S}_5}$

$$\pi^*\left(K_{\overline{\mathfrak{Diff}}_{10}}\right) = 2[\phi^*(\overline{\mathcal{K}}_{10})] + [\overline{\mathfrak{L}}_{10}] + \delta_{0:xy} + \sum_{j,s\geq 0} d_{j:s}\delta_{j:s}, \tag{8}$$

where the coefficients $d_{j:s}$ are all non-negative and $d_{0:2} = 0$. For each pair of distinct indices $1 \leq i < j \leq 5$, a covering family Γ_{ij} for the divisor $\overline{\mathfrak{L}}_{10}$ on $\overline{\mathcal{M}}_{10,10}$ was constructed in [9, Proposition 1.4]; starting with a general curve $[C, x_1, \ldots, x_5, y_1, \ldots, y_5] \in \mathfrak{L}_{10}$, the 1-nodal curve of genus 11 obtained from C by identifying x_i and x_j lies on a $K3$ surface S. Moving this curve in a pencil of 1-nodal curves on S, desingularizing the entire family and finally making a base change of order 2 to distinguish between the sections corresponding to the nodes, one obtains a sweeping curve $\Gamma_{ij} \subset \overline{\mathfrak{L}}_{10}$, having the following numerical features (recall that $g = 10$):

$$\Gamma_{ij}\cdot\lambda = 2(g+1),\ \ \Gamma_{ij}\cdot\delta_{\mathrm{irr}} = 2(6g+17),\ \ \Gamma_{ij}\cdot\psi_z = 2 \text{ for } z \in \{x_i, x_j\}^c,$$
$$\Gamma_{ij}\cdot\psi_{x_i} = \Gamma_{ij}\cdot\psi_{x_j} = 5,\ \Gamma_{ij}\cdot\delta_{0:\{x_i,x_j\}} = 2,\ \ \Gamma_{ij}\cdot\delta_{\ell:T} = 0 \text{ for } 0 \leq \ell \leq 10, \text{ and } T \subset \{x_i, x_j\}^c.$$

We find that $\Gamma_{ij}\cdot\delta_{0:xx} = 4$ and $\Gamma\cdot\delta_{0:xy} = \Gamma_{ij}\cdot\delta_{0:yy} = 0$. We further calculate $\Gamma\cdot\phi^*(\overline{\mathcal{K}}_{10}) = 0$ and $\Gamma_{ij}\cdot\overline{\mathfrak{L}}_{10} = -2 < 0$.

Since Γ_{ij} fills-up the divisor $\overline{\mathfrak{L}}_{10}$, this coupled with relation (8) implies that for all integers $n \geq 0$, one has an equality of linear series on $\overline{\mathcal{M}}_{10,10}$

$$\left|n\pi^*\left(K_{\overline{\mathfrak{Diff}}_{10}}\right)\right| = \left|n\pi^*\left(K_{\overline{\mathfrak{Diff}}_{10}}\right) - n\overline{\mathfrak{L}}_{10}\right|.$$

To show that the remaining linear system has Kodaira dimension 0 one follows the last lines of the proof of Theorem 0.1 in [9], using the extremality of the divisor $\overline{\mathcal{K}}_{10}$ on $\overline{\mathcal{M}}_{10}$. □

3 An effective divisor on $\overline{\mathcal{M}}_{g,g-3}$

The aim of this section is to prove Theorem 1.2. We begin by solving the following enumerative question which comes up repeatedly in the process of computing $[\overline{\mathcal{D}}_g]$.

Theorem 3.1 *Let $[C, p] \in \mathcal{M}_{g,1}$ be a general pointed curve of genus g and a fixed integer $0 \leq \gamma \leq g-3$. Then there exist a finite number of pairs $(L, x) \in W^2_g(C) \times C$ such that*

$$H^0\left(C, L(-\gamma\, x - (g-3-\gamma)\, p)\right) \geq 1.$$

Their number is computed by the formula

$$N(g,\gamma) := \frac{g(g-1)(g-5)}{3}\gamma(\gamma g - 3\gamma - 1).$$

Proof We introduce auxiliary maps $\chi : C \times C_3 \to C_{\gamma+3}$ and $\iota : C_{\gamma+3} \to C_g$ given by,

$$\chi(x, D) := \gamma\cdot x + D, \quad \text{and} \quad \iota(E) := E + (g-3-\gamma)\cdot p.$$

The number we evaluate is $N(g,\gamma) := \chi^*\iota^*([C^2_g])$, where $C^2_g := \{D \in C_g : \dim|D| \geq 3\}$. The cohomology class of this variety of special divisors is computed in [2, p. 326]:

$$\left[C^2_g\right] = \frac{\theta^4}{12} - \frac{x\theta^3}{3} + \frac{x^2\theta^2}{6} \in H^8(C_{g-3}, \mathbb{Q}).$$

Noting that $\iota^*(\theta) = \theta$ and $\iota^*(x) = x$, one needs to estimate the pull-backs of the tautological monomials $x^\alpha \theta^{4-\alpha}$. For this purpose, we use [2, p. 358]:

$$\chi^*(x^\alpha \theta^{4-\alpha}) = \frac{g!}{(g-4+\alpha)!} \Big[(1+\gamma t_1 + t_2)^\alpha \cdot \big(1+\gamma^2 t_1 + t_2\big)^{4-\alpha}\Big]_{t_1 t_2^3},$$

where the last symbol indicates the coefficient of the monomial $t_1 t_2^3$ in the polynomial appearing on the right side of the formula. The rest follows after a routine evaluation. □

The second enumerative ingredient in the proof of Theorem 1.2 is the following result, which can be proved by degeneration using Schubert calculus. We skip details and refer instead to [4,6], or to the proof of Lemma 2.3 in this paper:

Proposition 3.2 *For a general curve $[C] \in \mathcal{M}_{g-1}$, there exist a finite number of pairs $(L, x) \in W^2_g(C) \times C$ satisfying the conditions*

$$h^0(C, L(-2x)) \geq 2, \quad \text{and} \quad h^0(C, L(-(g-2)x)) \geq 1.$$

Each pair corresponds to a complete linear series L. The number of such pairs is equal to

$$n(g-1) := (g-1)(g-2)(g-3)(g-4)^2.$$

Proof of Theorem 1.2 We expand $[\overline{\mathcal{D}}_g] \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,g-3})$, and begin the calculation by determining the coefficients of λ, δ_{irr} and $\psi := \sum_{i=1}^{g-3} \psi_i$ respectively. It is useful to observe that if $\pi_n : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$ is the map forgetting the marked point labeled by n for some $n \geq 1$ and D is any divisor class on $\overline{\mathcal{M}}_{g,n}$, then for distinct labels $i, j \neq n$, the λ, δ_{irr} and ψ_j coefficients of the divisors D on $\overline{\mathcal{M}}_{g,n}$ and $(\pi_n)_*(D \cdot \delta_{0:in})$ on $\overline{\mathcal{M}}_{g,n-1}$ respectively, coincide. The divisor $(\pi_n)_*(D \cdot \delta_{0:in})$ can be thought of as the locus of points $[C, x_1, \ldots, x_n] \in D$ where the points x_i and x_n are allowed to come together. By iteration, the divisor $\overline{\mathcal{D}}_g^{g-3}$ on $\overline{\mathcal{M}}_{g,1}$ obtained by letting all points $x_1, \ldots, x_{g-3}$ coalesce, has the same λ and δ_{irr} coefficients as $\overline{\mathcal{D}}_g$. But obviously

$$\overline{\mathcal{D}}_g^{g-3} = \Big\{[C, x] \in \mathcal{M}_{g,1} : \exists L \in W^2_g(C) \text{ such that } h^0(C, L(-(g-3)x)) \geq 1\Big\},$$

and note that this is a "pointed Brill–Noether divisor" in the sense of Eisenbud–Harris. The cone of Brill–Noether divisors on $\overline{\mathcal{M}}_{g,1}$ is 2-dimensional, see [5, Theorem 4.1], and there exist constants $\mu, \nu \in \mathbb{Q}$, such that $[\overline{\mathcal{D}}_g^{g-3}] = \mu \cdot \mathfrak{BN} + \nu \cdot [\overline{\mathcal{W}}]$, where

$$\mathfrak{BN} := (g+3)\lambda - \frac{g+1}{6}\delta_{\mathrm{irr}} - \sum_{j=1}^{g-1} j(g-j)\delta_{j:1} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,1})$$

is the pull-back from $\overline{\mathcal{M}}_g$ of the Brill–Noether divisor class and $\overline{\mathcal{W}}$ is the divisor of Weierstrass points in $\overline{\mathcal{M}}_{g,1}$ with class given by the formula

$$[\overline{\mathcal{W}}] = -\lambda + \psi - \sum_{j=1}^{g-1} \binom{g-j+1}{2} \delta_{j:1} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,1}).$$

The coefficients μ and ν are computed by intersecting both sides of the previous identity with explicit curves inside $\overline{\mathcal{M}}_{g,1}$. First we fix a genus g curve C and let the marked point vary along C. If $C_x := \phi^{-1}([C]) \subset \overline{\mathcal{M}}_{g,1}$ denotes the induced curve in moduli, then the only generator of $\mathrm{Pic}(\overline{\mathcal{M}}_{g,1})$ which has non-zero intersection number with C_x is ψ, and $C_x \cdot \psi = 2g-2$. On the other hand one has $C_x \cdot \overline{\mathcal{D}}_g^{g-3} = N(g, g-3)$, that is,

$$\nu = \frac{N(g, g-3)}{g(g-1)(g+1)}.$$

To compute μ, we construct a curve inside $\Delta_{1:1}$ as follows: Fix a 2-pointed elliptic curve $[E, x, y] \in \mathcal{M}_{1,2}$ such that the class $x - y \in \mathrm{Pic}^0(E)$ is not torsion, and a general curve $[C] \in \mathcal{M}_{g-1}$. We define the family $\bar{C}_1 := \{[C \cup_y E, x]\}_{y \in C}$, obtained by varying the point of attachment along C, while keeping the marked point fixed on E. The only generator of $\mathrm{Pic}(\overline{\mathcal{M}}_{g,1})$ meeting $\bar{C}_1$ non-trivially is $\delta_{1:1} = \delta_{g-1:\emptyset}$, in which case $\bar{C}_1 \cdot \delta_{1:1} = -2g+4$. On the other hand, $\bar{C}_1 \cdot \overline{\mathcal{D}}_g^{g-3}$ is equal to the number of limit linear series $\mathfrak{g}^2_g$ on curves of type $C \cup_y E$, having vanishing sequence at least $(0, 1, g-3)$ at $x \in E$. This can happen only if this linear series is refined and its C-aspect has vanishing sequence at the point of attachment $y \in C$ equal to either (i) $(1, 2, g-3)$, or (ii) $(0, 2, g-2)$. In both cases, the E-aspect being uniquely determined, we obtain that $\bar{C}_1 \cdot \overline{\mathcal{D}}_g^{g-3} = N(g-1, g-4) + n(g-1)$. This leads to $\mu = 3(g-3)(g-4)/(g+1)$.

Next, let $\overline{\mathcal{D}}_g^{g-4}$ be the divisor on $\overline{\mathcal{M}}_{g,2}$ obtained from $\overline{\mathcal{D}}_g$ by letting all marked points except one, come together. Precisely, $\overline{\mathcal{D}}_g^{g-4}$ is the closure of the locus of pointed curves $[C, x, y] \in \mathcal{M}_{g,2}$ such that there exists $L \in W^2_g(C)$ with $h^0\big(C, L(-x-(g-4)y)\big) \geq 1$. We express $[\overline{\mathcal{D}}_g^{g-4}] = c_x \psi_x + c_y \psi_y - e\delta_{0:xy} - \cdots \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,2})$, and observe that c_x equals the ψ-coefficient of $\overline{\mathcal{D}}_g$, whereas the coefficient $e = \nu\binom{g+1}{2}$ has already been calculated. We fix a general curve $[C] \in \overline{\mathcal{M}}_g$ and define test curves $C_x := \{[C, x, y] : x \in C\} \subset \overline{\mathcal{M}}_{g,2}$ and $C_y := \{[C, x, y] : y \in C\} \subset \overline{\mathcal{M}}_{g,2}$, by fixing one general marked point on C and letting the other vary freely. By intersecting $\overline{\mathcal{D}}_g^{g-4}$ with these curves we obtain the formulas:

$$(2g-1)c_x + c_y - e = C_x \cdot \overline{\mathcal{D}}_g^{g-4} = N(g, 1)$$
$$\text{and} \quad c_x + (2g-1)c_y - e = C_y \cdot \overline{\mathcal{D}}_g^{g-4} = N(g, g-4).$$

Solving this system, determines c_x. Finally, the $\delta_{0:2}$-coefficient of $\overline{\mathcal{D}}_g$ is computed by intersecting $\overline{\mathcal{D}}_g$ with the test curve $\phi_{g-3}^{-1}([C, x_1, \ldots, x_{g-4}]) \subset \overline{\mathcal{M}}_{g,g-3}$, obtained by fixing $g-4$ marked points on a general curve, and letting the remaining point vary. □

As an application, we bound the effective cone of the symmetric product of degree $g-3$ on a general curve $[C] \in \mathcal{M}_g$. Let $u : C_{g-3} \dashrightarrow \overline{\mathcal{M}}_{g,g-3}/\mathfrak{S}_{g-3}$ the (rational) fibre map and $\widetilde{\mathcal{D}}_g$ the effective divisor on $\overline{\mathcal{M}}_{g,g-3}/\mathfrak{S}_{g,g-3}$ to which $\overline{\mathcal{D}}_g$ descends. Then $\mathcal{D}_g[C] := u^*(\widetilde{\mathcal{D}}_g)$ is an effective divisor on C_{g-3}:

Theorem 3.3 *The cohomology class of the codimension one locus inside* C_{g-3}

$$\mathcal{D}_g[C] := \{D \in C_{g-3} : \exists L \in W^2_g(C) \text{ with } h^0\big(C, L(-D)\big) \geq 1\} \quad \text{equals}$$
$$[\mathcal{D}_g(C)] = \frac{(g-5)(g-3)(g-1)}{3}\Big(\theta - \frac{g}{g-3}x\Big).$$

It is natural to wonder whether the class $\theta - \frac{g}{g-3}x$ is extremal in $\mathrm{Eff}(C_{g-3})$. If so, $\mathcal{D}_g[C]$ together with the diagonal class $\delta_C \equiv -\theta + (2g-4)x$ would generate the effective cone inside the 2-dimensional space $N^1(C_{g-3})_{\mathbb{Q}}$. We refer to [14] Theorem 3, for a proof that δ_C spans an extremal ray, which shows that in order to compute $\mathrm{Eff}(C_{g-3})$, one only has to determine the slope of $\mathrm{Eff}(C_{g-3})$ in the fourth quadrant of the (θ, x)-plane. A similar description of the effective cone of C_{g-2} was given in [17]. We have a partial result in this direction, showing that all effective divisors of slope higher than $\frac{g}{g-3}$ (if any), must contain a geometric codimension one subvariety of $\mathcal{D}_g[C]$.

Proposition 3.4 *Any irreducible effective divisor on C_{g-3} with class proportional to $\theta-\alpha x \in H^2(C_{g-3}, \mathbb{Q})$, where $\alpha > \frac{g}{g-3}$, contains the codimension two locus inside C_{g-3}*

$$Z_{g-3}[C] := \{D \in C_{g-3} : \exists A \in W^1_{g-2}(C) \text{ with } H^0(C, A(-D)) \neq 0\}.$$

Proof For a pencil $A \in W^1_{g-2}(C)$, we denote by $V^1_{g-3}(A)$ the 1-cycle of effective divisors $D \in C_{g-3}$ that appear in a fibre of the pencil. The class $[V^1_{g-3}(A)]$ is computed in [2, page 342]. By direct calculation, we find that the inequality $[V^1_{g-3}(A)] \cdot (\theta - \alpha x) < 0$ holds, whereas $[V^1_{g-3}(A)] \cdot \mathcal{D}_g[C] = 0$. □

References

1. Arbarello, E., Cornalba, M.: The Picard groups of the moduli spaces of curves. Topology **26**, 153–171 (1987)
2. Arbarello, E., Cornalba, M., Griffiths, P., Harris, J.: Geometry of algebraic curves. In: Grundlehren der mathematischen Wissenschaften, vol. 267. Springer, New York (1985)
3. Bini, G., Fontanari, C., Viviani, F.: On the birational geometry of the universal Picard variety. Int. Math. Res. Not. 4, 740–780 (2012)
4. Eisenbud D., Harris, J.: The Kodaira dimension of the moduli space of curves of genus 23. Invent. Math. **90**, 359–387 (1987)
5. Eisenbud, D., Harris, J.: Irreducibility of some families of linear series with Brill–Noether number −1. Ann. Sci. École Normale Supérieure **22**, 33–53 (1989)
6. Farkas, G.: Koszul divisors on moduli spaces of curves. Am. J. Math. **131**, 819–869 (2009)
7. Farkas, G., Mustaţă, M., Popa, M.: Divisors on $\mathcal{M}_{g,g+1}$ and the minimal resolution conjecture for points on canonical curves. Ann. Sci. École Normale Supérieure **36**, 553–581 (2003)
8. Farkas, G., Popa, M.: Effective divisors on $\overline{\mathcal{M}}_g$, curves on $K3$ surfaces and the slope conjecture. J. Algebr. Geom. **14**, 151–174 (2005)
9. Farkas, G., Verra, A.: The classification of the universal Jacobian over the moduli space of curves. Comm. Math. Helvetici (2013). arXiv:1005.5354
10. Farkas, G., Verra, A.: The universal theta divisor over the moduli space of curves. J. Math. Pures et Appliquées (2013). arXiv:1009.0184
11. Farkas, G., Verra, A.: The geometry of the moduli space of odd spin curves. arXiv:1004.0278
12. Fulton, W.: Intersection theory. In: Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 2, 2nd edn. Springer, Berlin (1998)
13. Harris, J., Mumford, D.: On the Kodaira dimension of $\overline{\mathcal{M}}_g$. Invent. Math. **67**, 23–88 (1982)
14. Kouvidakis, A.: Divisors on symmetric products of curves. Trans. Am. Math. Soc. **337**, 117–128 (1993)
15. Laszlo, Y.: Un théorème de Riemann pour les diviseurs thêta sur les espaces de modules de fibrés stables sur une courbe. Duke Math. J. **64**, 333–347 (1991)
16. Logan, A.: The Kodaira dimension of moduli spaces of curves with marked points. Am. J. Math. **125**, 105–138 (2003)
17. Mustopa, Y.: Kernel bundles, syzygies and the effective cone of C_{g-2}. Int. Math. Res. Not. **6**, 1417–1437 (2011)
18. Tan, S.-L.: On the slopes of the moduli spaces of curves. Int. J. Math. **9**, 119–127 (1998)
19. Voisin, C.: Green's canonical syzygy conjecture for generic curves of odd genus. Compos. Math. **141**, 1163–1190 (2005)