"Existence and Uniqueness of Mild Solutions to the Stochastic Neural Field Equation with Discontinuous Firing Rate"

Abstract:

The neural field equation models the spatiotemporal evolution of neural activity in thin layers of cortical tissue from a macroscopic point of view. Under the influence of spatially extended additive noise (W^Q) it is given by

$$du(x,t) = \left[-u(x,t) + \int_{-\infty}^{\infty} w(x-y)\ F(u(y,t))\ dy\right] dt + \sigma\ dW^Q(x,t) \tag{1}$$

where u describes the activity of a population of neurons at position x and time t. The probability kernel w models the strength of nonlocal excitatory synaptic connections, whereas $F : \mathbb{R} \to \mathbb{R}$ constitutes a nonlinear firing rate function. In most of the applications F is taken to be a sigmoid function, under which assumption the existence and uniqueness of mild solutions (for suitable choices of the noise term) can be shown with standard arguments. In the case where the steepness of the sigmoid function tends to infinity i.e. when it converges to a Heaviside firing rate the question of existence and uniqueness of mild solutions has still been open for the deterministic as well as for the stochastic neural field equation.

We will prove existence via a fixed point iteration which yields a monotonically decreasing sequence of functions converging to a solution u of (1) in a pointwise sense. Furthermore, we see that this particular solution is maximal.

Concerning the issue of uniqueness different notions and approaches will be discussed. In the deterministic setting we will show that transverse solutions, i.e. solutions with non-vanishing gradient at θ, the point of discontinuity of the Heaviside function, are unique. However, nothing is known about how long a solution can be expected to remain transverse. In the stochastic setting we discuss the following alternative approaches: Having the maximal mild solution u as a supersolution, we construct a subsolution of (1) in a similar spirit. It is then essential to find conditions under which this sub- and supersolution coincide. Furthermore, we present a result of weak uniqueness on finite domain.