STOCHASTIC ANALYSIS
BANACH CENTER PUBLICATIONS, VOLUME 105
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2015

COMPARISON PRINCIPLE APPROACH TO UTILITY MAXIMIZATION

PETER IMKELLER and VICTOR NZENGANG

Institut für Mathematik, Humboldt-Universität zu Berlin
Unter den Linden 6, D-10099 Berlin, Germany
E-mail: imkeller@math.hu-berlin.de, nzengang@math.hu-berlin.de

Abstract. We consider the problem of optimal investment for maximal expected utility in an incomplete market with trading strategies subject to closed constraints. Under the assumption that the underlying utility function has constant sign, we employ the comparison principle for BSDEs to construct a family of supermartingales leading to a necessary and sufficient condition for optimality. As a consequence, the value function is characterized as the initial value of a BSDE with Lipschitz growth.

Introduction. The problem of optimal investment or utility maximization with or without constraints or liabilities has been studied intensively since the early days of stochastic finance. Various mathematical tool boxes have contributed to a number of methods. The most classical and prominent one is based on duality theory. Its intrinsic link to convex analysis also indicates natural limitations, in particular when facing constraints not given by convex sets. A tool that has gained some popularity during the last two decades features backward stochastic differential equations (BSDE). As opposed to convex duality theory, it does not need convexity assumptions and is therefore able to deal with scenarios in which constraints are formulated on closed, but not necessarily convex sets. This *primal stochastic* approach has been shown to yield intrinsically stochastic, but not explicit descriptions of optimal investment strategies in terms of the control components of solution pairs of BSDEs tailor made for the underlying preference or utility function (see El Karoui et al. [REK00], Hu et al. [HIM05]). For classical utility functions without exogenous liability, simple BSDEs have been designed that lead to solutions of the pricing

2010 *Mathematics Subject Classification*: Primary 60H07, 91G10; secondary 60G44, 60H07, 60H15, 93E03, 93E20.

Key words and phrases: financial market; incomplete market; maximal utility; portfolio optimization; exponential utility; power utility; supermartingale; stochastic differential equation; backwards stochastic differential equation; comparison principle.

The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc105-0-10

and hedging part of the maximization problem. For rather general utility functions, and in presence of exogenous liabilities, various approaches have shown that in general the forward dynamics of the asset prices on the underlying financial market and the backward (control) dynamics expressed by the BSDE are fully coupled. As a consequence, the primal stochastic approach leads to a system of coupled forward-backward stochastic equations (FBSDE) (see Horst et al. [HHI$^+$]), or, if formulated in terms of the dynamics of the associated value function, to a system of backward stochastic partial differential equations (BSPDE) (see Mania et al. [MT03]). For systems of FBSDEs, finding solutions is essentially more involved, and explicitly solvable systems easily accessible to numerical algorithms are even rarer.

Typically, generators of BSDEs arising in the primal stochastic approach of optimal investment problems are quadratic in the control variable, and therefore lack the important global Lipschitz continuity property. In this paper, we present a new version of the primal stochastic (BSDE) approach for class of optimal investment problems characterized by utility functions that possess a definite sign, and to which liabilities are coupled in a multiplicative form. In our approach we decompose the generator into a family of linear generators depending on individual admissible investment strategies. We then associate to each one of them an individual BSDE with linear generator for which the solution is explicitly available. In order to return to the original optimization problem, we finally heavily rely on the comparison principle for BSDEs: maximization in the generator corresponds to maximization in the solution process describing the optimal portfolio and price. Particularly attractive features of the global solution thus obtained are the following: it can be represented by a BSDE with at least Lipschitz continuous (if not linear) generator, and it contains a necessary and sufficient condition for optimality of the optimal investment process. It is therefore easily accessible for numerical approximation, given some pre-knowledge on the optimal investment strategy that can be recursively obtained along an algorithm.

The paper is organized as follows. In Section 1 we explain the basic concepts and fix notation. In Section 2 our financial market model is set up. We explain the utility maximization problem with multiplicative liabilities, and constraints defined by a random progressive family of closed sets (closed *multifunction*) that are not necessarily convex. In the main Section 3 we design our approach based on a combination of a family of BSDEs with linear generators and the comparison principle. To make it work, the price we have to pay is Assumption 3.1, which includes conditions on relative risk aversion and asymptotic elasticity, on the sign of the utility function, and boundedness of the constraint sets. In the main result (Theorem 3.7) we show that under this assumption the optimal investment problem is solved by a single BSDE with linear generator associated to the optimal investment process. In the final Section 4 we discuss our main result in the cases of exponential and power utilities.

1. Notation and preliminaries. The purpose of this section is to fix notation and review basic concepts that will be used throughout this paper. Let $T \in (0, +\infty)$ be the finite time horizon and $m, d \in \mathbb{N}$ with $m \leq d$. Throughout this paper, we work with a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. We suppose that $\mathbb{F}$ is the filtration generated by a d-dimensional Brownian motion $W = (W^1, \dots, W^d)$, completed with the $\mathbb{P}$-null sets. We denote by $\mathcal{P}$ the σ-field of predictable subsets of $\Omega \times [0, T]$. λ denotes the Lebesgue measure on $[0, T]$ and $\mathbb{P} \otimes \lambda$ the product measure on $\Omega \times [0, T]$. $\mathbb{E}[\,\cdot\,]$ is the symbol used for the expectation with respect to $\mathbb{P}$. For $x, y \in \mathbb{R}^{m\times 1}$ (resp. $\mathbb{R}^{d\times 1}$), we denote by $|\cdot|$ the Euclidean norm, and by $\langle\cdot,\cdot\rangle$ the inner product on $\mathbb{R}^{m\times 1}$ (resp. $\mathbb{R}^{d\times 1}$). Throughout this work, an $\mathbb{R}^{m\times 1}$ vector $a = (a_1, \dots, a_m)^{\mathrm{tr}}$ will be identified with its $\mathbb{R}^{d\times 1}$ extension $a = (a_1, \dots, a_m, 0, \dots, 0)^{\mathrm{tr}}$. We adopt the same identification with $\mathbb{R}^{m\times 1}$-valued processes. For an $\mathbb{R}^{d\times 1}$ (resp. $\mathbb{R}^{m\times 1}$)-valued predictable process Z, we denote by $\int_0^\cdot Z_t\,dW_t$ the stochastic integral of Z with respect to W. For two $\mathbb{R}^{d\times 1}$- (resp. $\mathbb{R}^{m\times 1}$)-valued semimartingales X, Y, we write $\langle X, Y\rangle$ for the co-variation process of X and Y. For a semimartingale M, we denote by $\mathcal{E}(M)$ its Doleans–Dade exponential. The vector space $\mathcal{H}^m$ is defined by

$$\mathcal{H}^m := \{A \subseteq \mathbb{R}^m : A \text{ is nonempty and compact}\}.$$

Let $A \subseteq \mathbb{R}^m$, $b \in \mathbb{R}^m$. We denote by $\mathrm{dist}_A(b)$ the distance between b and A, defined as

$$\mathrm{dist}_A(b) = \inf_{a\in A} |a - b|.$$

We endow $\mathcal{H}^m$ with the Hausdorff metric d_H defined by

$$d_H(A, B) := \max\big\{\sup_{a\in A} \mathrm{dist}_B(a), \sup_{b\in B} \mathrm{dist}_A(b)\big\}.$$

$(\mathcal{H}^m, d_H)$ is a complete metric space, and we write $\mathcal{B}(\mathcal{H}^m)$ for its Borel σ-algebra. In the sequel, the following spaces will play an important role:

- $\mathbb{H}^{2,d}_T$ (or $\mathbb{H}^{2,m}_T$), the space of $\mathbb{R}^{d\times 1}$- (resp. $\mathbb{R}^{m\times 1}$)-valued predictable processes such that $\mathbb{E}\big[\int_0^T |Z_t|^2\,dt\big] < +\infty$;
- L^2_T, the space of $\mathcal{F}_T$-measurable real valued random variables X that are square integrable, i.e $\mathbb{E}[|X|^2] < +\infty$.

2. The model. Our financial market consists of one bond S^0 with zero interest rate and m stocks with price processes $(S^1, S^2, \dots, S^m)$. The dynamics of the price processes is described by

$$dS^i_t := S^i_t(\theta^i_t\,dt + dW^i_t), \quad t \in [0, T],\ i \in \{1, \dots, m\}, \tag{1}$$

where θ is $\mathbb{R}^m$-valued predictable. To guarantee that our model is free of arbitrage, we make the assumption

Assumption 2.1. *The process θ is uniformly bounded, i.e. there exists a constant $C > 0$ such that $|\theta_t| \le C$, $\forall t \in [0, T]$, $\mathbb{P}$-a.s.*

By Theorem 2.3 in [Kaz94], $\mathcal{E}(\int_0^\cdot \theta_t\,dW_t)$ is a uniformly integrable martingale and thus defines a probability measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$. Under this measure, the price process S is a martingale. The set of equivalent martingale measures is therefore nonempty, and so our market model is free of arbitrage according to [DS94]. Note that if $m < d$, our market model is incomplete.

We consider an investor endowed with the initial capital $x > 0$, and who buys and sells the risky assets S according to trading strategies $\pi \in \mathbb{H}_T^{2,m}$. The wealth process $X^{\pi,x}$ associated to π possesses the dynamics

$$X_t^{\pi,x} := x + \int_0^t \Sigma(X_s^{\pi,x}, \pi_s) \frac{dS_s}{S_s}, \quad t \in [0,T], \tag{2}$$

with $\Sigma(X^{\pi,x}, \pi) = X^{\pi,x}\pi$ or π. The trading strategy π is interpreted as the amount of money invested into the stocks if $\Sigma(X^{\pi,x}, \pi) = \pi$. If $\Sigma(X^{\pi,x}, \pi) = X^{\pi,x}\pi$, π is interpreted as the proportion of wealth invested into the stocks. $X^{\pi,x}$ is well defined due to the integrability property imposed on π. Trading strategies are subject to constraints due to regulations or management policies. We model the constraints by a measurable multifunction

$$\mathcal{C} : \Omega \times [0,T] \to \mathcal{H}^m. \tag{3}$$

$\mathcal{C}$ is a $\mathcal{P} - \mathcal{B}(\mathcal{H}^m)$ measurable mapping. A process h is said to be *dynamically constrained* to $\mathcal{C}$ and we write $h \in \mathcal{C}$, if $h_t \in \mathcal{C}(t,\omega)$ for $\mathbb{P} \otimes \lambda$-a.a $(\omega, t) \in \Omega \times [0,T]$. So our admissible trading strategies are elements of $\mathbb{H}_T^{2,m}$ that are dynamically constrained to $\mathcal{C}$. Their collection is denoted by $\mathcal{A}$, formally

$$\mathcal{A} = \{\pi \in \mathbb{H}_T^{2,m} \text{ such that } \pi \in \mathcal{C}\}.$$

One natural scenario in which dynamic constraint sets appear is for example by the re-parametrization of strategies according to [HIM05, Mor09].

Remark 2.2. Our model for stock price has constant volatility. This is by no means restrictive. Consider a stock $\tilde{S}$ with the dynamics

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = \mu_t \, dt + \sigma_t \, dW_t, \; t \in [0,T],$$

where μ (resp. σ) is an $\mathbb{R}^m$- (resp. $\mathbb{R}^{m \times d}$)-valued predictable process. Assume that the coefficients of μ and σ are uniformly bounded and σ is of full rank. Let $\theta = \sigma^{\mathrm{tr}}.(\sigma.\sigma^{\mathrm{tr}})^{-1}\mu$. Then clearly we have $\frac{d\tilde{S}}{\tilde{S}} = \sigma \frac{dS}{S}$. Hence parametrizing investment strategies through $\rho = \pi^{\mathrm{tr}}.\sigma$ puts us into the setup of constant volatility.

The objective of the investor is to maximize his expected utility from terminal wealth eventually depending on a liability F, i.e. an $\mathcal{F}_T$ measurable random variable. In other words, he wants to solve the optimization problem

$$V(x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\big[\mathcal{U}(X_T^{\pi,x}, F)\big], \tag{4}$$

with $\mathcal{U} : \mathrm{dom}(\mathcal{U}) \to \mathbb{R}$, and $\mathrm{dom}(\mathcal{U}) \subset \mathbb{R} \times \mathbb{R}$. A trading strategy $\nu \in \mathcal{A}$ is said to be optimal if it attains the maximum in (4). In the sequel, we refer to V as the value function. If the liability is of additive type, $\mathcal{U}$ has the form $\mathcal{U}(x,y) = U(x-y)$ or $\mathcal{U}(x,y) = U(x+y)$, for $(x,y) \in \mathrm{dom}(\mathcal{U})$, where U is a deterministic utility function, i.e. a strictly increasing and concave function. The problem (4) with additive liability has been well investigated in the literature using convex duality arguments, see [CSW01] for bounded F, and [HK04] for the unbounded case. In this context, solving (4) leads to a pricing rule for F, see [REK00, HK04]. The liability is of multiplicative type if $\mathcal{U}(x,y) = U(xy)$, for $(x,y) \in \mathrm{dom}(\mathcal{U})$, again with a deterministic utility function U. Here F can be seen as a liability that scales the wealth of our investor by a random portion at maturity. In this setting, (4) has been investigated in [Zar01, IRZ11] for the power utility. In this paper, we are interested in the following form of $\mathcal{U} : \mathcal{U}(x, y) = U(x)\tilde{y}$, where U is a utility function and $\tilde{y}$ a deterministic function of y. The problem (4) therefore has the equivalent formulation

$$V(x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}[U(X_T^{\pi,x})H] \tag{5}$$

where $U : \mathrm{dom}(U) \subseteq \mathbb{R} \to \mathbb{R}$ is a deterministic utility function, and H an $\mathcal{F}_T$-measurable random variable. For $H = 1$, (5) is the classical portfolio optimization problem. Clearly, utility maximization with additive liability and exponential utility function or multiplicative liability with power utility function are particular cases of (5). The special form (5) first appears in [Zar01] in the context of stochastic factors models. There H is a deterministic function of the terminal value of the stochastic factor process. The author employs the classical PDE approach to solve (5) and shows how the solution leads to a suitable bound on the prices of non-traded assets. There are other reasons for studying (5). For instance, in case the investor wants to maximize his utility taking into account only some particular scenarios, this corresponds to $H = 1_A$, where $A \in \mathcal{F}_T$ describes the set of scenarios. If $H > 0$, (5) can be seen as a classical portfolio optimization under the equivalent probability with density $\frac{H}{\mathbb{E}[H]}$.

Remark 2.3. For $\mathbb{P} \otimes \lambda$-a.a. $(\omega, t) \in \Omega \times [0, T]$, $\mathcal{C}(\omega, t)$ is compact, but not necessarily convex. Hence $\mathcal{A}$ may not be convex. Therefore, an optimal trading strategy to (5) may not exists. If it exists, it need not be unique.

Throughout this paper, we work under the assumption

Assumption 2.4. *$V(x) < +\infty$.*

Among the classical tools to deal with stochastic optimal control problems there is the dynamic programming principle (DPP). It involves considering the following family of processes $\{V(\cdot, X^{\pi,x}), \pi \in \mathcal{A}\}$, with

$$V(t, X_t^{\pi,x}) = \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}(t, X_t^{\pi,x})} \mathbb{E}[U(X_T^{\alpha,x})H|\mathcal{F}_t], \quad \pi \in \mathcal{A},\ t \in [0, T], \tag{6}$$

where

$$\mathcal{A}(t, X_t^{\pi,x}) = \big\{\alpha \in \mathcal{A} \text{ such that } X_s^{\pi,x} = X_s^{\alpha,x}, s \in [0, t]\big\}. \tag{7}$$

The following proposition gives a necessary and sufficient condition for optimality.

Proposition 2.5. *Assume* 2.4. *For every $\pi \in \mathcal{A}$, $V(\cdot, X^{\pi,x})$ is a supermartingale. A trading strategy ν is optimal if and only if $V(\cdot, X^{\nu,x})$ is a martingale.*

Proof. See [EK81, Theorem 1.15, Theorem 1.17]. ■

In a Markovian setting, it is well known that the dynamic value function $V(\cdot, x)$ satisfies a Hamilton–Jacobi–Bellman (HJB) equation. In a more general framework, Mania et al. [MT03] derive a backward stochastic partial differential equation (BSPDE) for $V(\cdot, x)$ under some strong regularities assumptions. Except for the classical utility functions, there exists in general no closed form expression for $V(\cdot, x)$. In these particular cases, the closed form expressions are obtained using solutions of BSDEs with quadratic growth, see [REK00, HIM05, Mor09]. In the following section, for a particular class of utility functions we employ the tool of comparison principle for BSDEs to construct a family of processes similar to (6) providing a necessary and sufficient condition for optimality.

3. The comparison principle approach. The comparison principle has been shown to play an efficient role for stochastic control problems, see [EKPQ97, Section 3]. In this section, we use this tool for the solution of (5). To achieve this, we will transform (5) into a stochastic control problem expressed by a family of BSDEs with linear generators. We begin by formulating some assumptions on the function U, the trading strategies and the claim H that will turn out to be helpful for our approach.

ASSUMPTION 3.1. *There exists a constant $K > 0$ such that*

(A1) *$\mathcal{C}$ is uniformly bounded by K,*
(A2) *U has a constant sign and is twice continuously differentiable,*
(A3) *for all $\pi \in \mathcal{A}$, the processes $\frac{U''(X^{\pi,x})}{U'(X^{\pi,x})}\Sigma(X^{\pi,x},\pi)$ and $\frac{U'(X^{\pi,x})}{U(X^{\pi,x})}\Sigma(X^{\pi,x},\pi)$ are uniformly bounded by K,*
(A4) *H is a strictly positive square integrable random variable,*
(A5) $\mathbb{E}\big[\sup_{0\le t\le T} |U(X_t^{\pi,x})|^2\big] < +\infty$ *for every $\pi \in \mathcal{A}$.*

(A1) states that trading strategies are uniformly bounded. The assumption (A2) on the sign of U is restrictive as it excludes the logarithmic utility function. If $\Sigma(X^{\pi,x},\pi) = X^{\pi,x}\pi$, then assumption (A3) states that risk aversion is bounded. If $\Sigma(X^{\pi,x},\pi) = \pi$, (A3) amounts to saying that the absolute risk aversion is bounded. (A5) will ensure that the processes we construct later have nice integrability properties. Under (A1), (A5) is satisfied for the exponential and power utility functions.

From now on, we will suppose that $U > 0$. The arguments are similar when $U < 0$ and the results in this case are obtained from the ones given by changing $\operatorname{ess\,sup}$ to $\operatorname{ess\,inf}$.

3.1. Motivation and approach. In this subsection, we reformulate (5) as a stochastic control problem featuring a family of BSDEs with linear generators. Once this is done, we apply the comparison principle for BSDEs to determine the value function of (5) expressed by the solution of a BSDE with Lipschitz growth. Along with the family of linear BSDEs comes a suitable representation of (6), see (9). Let us briefly explain our approach. Fix $\pi \in \mathcal{A}$. Since the random variable $U(X_T^{\pi,x})H$ is strictly positive and integrable, the martingale $M^{\pi,x}$ defined by $M_t^{\pi,x} = [U(X_T^{\pi,x})H|\mathcal{F}_t]$, $t \in [0,T]$, is strictly positive. By the strict positivity of U, there exists a unique adapted and continuous process $Y^{\pi,x}$ such that $U(X^{\pi,x})Y^{\pi,x} = M^{\pi,x}$. The martingale property of the family of processes $M^{\pi,x}$, $\pi \in \mathcal{A}$ gives $V(x) = U(x)\Lambda(x)$ with

$$\Lambda(x) = \sup_{\pi\in\mathcal{A}} Y_0^{\pi,x}. \tag{8}$$

The family defined by (6) has a representation in terms of $Y^{\pi,x}, \pi \in \mathcal{A}$. To see this, for $\pi \in \mathcal{A}$, $t \in [0,T]$, $\alpha \in \mathcal{A}(t, X_t^{\pi,x})$, write $Y^{\alpha}(t, X_t^{\pi,x})$ for $Y_t^{\alpha,x}$ to make its dependence on $X_t^{\pi,x}$ explicit. From the martingale property of $M^{\alpha,x}$, $\alpha \in \mathcal{A}(t, X_t^{\pi,x})$, we deduce that

$$V(t, X_t^{\pi,x}) = U(X_t^{\pi,x})Y(t, X_t^{\pi,x}),\ \mathbb{P}\text{-a.s.} \tag{9}$$

with

$$Y(t, X_t^{\pi,x}) = \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}(t, X_t^{\pi,x})} Y^{\alpha}(t, X_t^{\pi,x}), \ \mathbb{P}\text{-a.s.} \tag{10}$$

Clearly $Y(\cdot, x)$ is the value process of the dynamic version of (8). (9) states a one-to-one correspondence between the value function process $V(\cdot, x)$ and $Y(\cdot, x)$. Namely, we have $V(t, x) = U(x)Y(t, x)$. Unfortunately, computing $Y(\cdot, x)$ is as challenging as calculating $V(\cdot, x)$. For every $\pi \in \mathcal{A}$ and $t \in [0, T]$, we have $\mathcal{A}(t, X_t^{\pi,x}) \subseteq \mathcal{A}$. We therefore get the following upper bound for $Y(t, X_t^{\pi,x})$, $t \in [0, T]$:

$$Y(t, X_t^{\pi,x}) \leq \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}} Y_t^{\alpha,x}, \ \mathbb{P}\text{-a.s.} \tag{11}$$

We focus on constructing a process Y that coincides almost surely with the right hand side of (11). Since equality is given in (11) at time 0, Y will clearly serve to identify a necessary and sufficient condition for optimality. The martingale representation theorem and the dynamics of $X^{\pi,x}$ will be used to describe the dynamical properties of $Y^{\pi,x}$. It turns out to be the value process of a BSDE with a standard pair of parameters (F^{π}, H), with F^{π} defined by (12). Hence $\Lambda(x)$ in (8) can be computed by solving a family of BSDEs to which we can apply the comparison principle.

For $\pi \in \mathcal{A}$, the generator F^{π} has the following structure

$$\begin{aligned} F^{\pi}(\omega, t, y, z) = {} & \frac{U'(X_t^{\pi,x}(\omega))}{U(X_t^{\pi,x}(\omega))} \langle \Sigma(X_t^{\pi,x}(\omega), \pi_t(\omega)), \theta_t(\omega) y + z \rangle \\ & + \frac{1}{2} \frac{U''(X_t^{\pi,x}(\omega))}{U(X_t^{\pi,x}(\omega))} |\Sigma(X_t^{\pi,x}(\omega), \pi_t(\omega))|^2 y, \end{aligned} \tag{12}$$

$(\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d$.

F^{π} is linear in (y, z). Furthermore, the coefficients are uniformly bounded by Assumptions 2.1 and 3.1. Hence the generator is uniformly Lipschitz continuous and the Lipschitz constant is independent of π. We denote it by K. Moreover, $F^{\pi}(\cdot, \cdot, 0, 0) = 0$. The pair (F^{π}, H) is therefore a standard pair of parameters. The following proposition guarantees existence and gives a characterization of the solution pair $(Y^{\pi,x}, Z^{\pi,x})$ of the associated BSDE for every $\pi \in \mathcal{A}$. The proof essentially relies on Itô's formula and the verification of some integrability properties.

Proposition 3.2. *Assume that* 2.1 *and* 3.1 *hold. Then, for every $\pi \in \mathcal{A}$, the solution pair $(Y^{\pi,x}, Z^{\pi,x})$ of the BSDE*

$$-dY_t^{\pi,x} = F^{\pi}(\cdot, t, Y_t^{\pi,x}, Z_t^{\pi,x})\, dt - Z_t^{\pi,x}\, dW_t, \quad Y_T^{\pi,x} = H, \tag{13}$$

is such that $R^{\pi,x} = U(X^{\pi,x})Y^{\pi,x}$ is a martingale. $Y^{\pi,x}$ is strictly positive.

Proof. Let $\pi \in \mathcal{A}$ and F^{π} as defined by (12). The pair (F^{π}, H) is a standard pair of parameters. So by Theorem 2.1 in [EKPQ97], (13) admits a unique solution $(Y^{\pi,x}, Z^{\pi,x}) \in \mathbb{H}_T^2 \times \mathbb{H}_T^{2,d}$. From Itô's formula and (2), we deduce the following dynamics for $U(X^{\pi,x})$:

$$\begin{aligned} d[U(X_t^{\pi,x})] = {} & \Big[U'(X_t^{\pi,x}) \langle \Sigma(X_t^{\pi,x}, \pi_t) \theta_t \rangle + \frac{1}{2} U''(X_t^{\pi,x}) |\Sigma(X_t^{\pi,x}, \pi_t)|^2 \Big] dt \\ & + U'(X_t^{\pi,x}) \Sigma(X_t^{\pi,x}, \pi_t)\, dW_t, \quad \text{for } t \in [0, T]. \end{aligned}$$

Applying Itô's formula to the product $R^{\pi,x} = U(X^{\pi,x})Y^{\pi,x}$, we obtain for $t \in [0,T]$

$$\begin{aligned} dR_t^{\pi,x} &= Y_t^{\pi,x}\, dU(X_t^{\pi,x}) + U(X_t^{\pi,x})\, dY_t^{\pi,x} + d\langle Y^{\pi,x}, U(X^{\pi,x})\rangle_t \\ &= \big(U'(X_t^{\pi,x})Y_t^{\pi,x}\Sigma(X_t^{\pi,x},\pi_t) + U(X_t^{\pi,x})Z_t^{\pi,x})\big)\, dW_t. \end{aligned}$$

This shows that $R^{\pi,x}$ is a local martingale. To show that it is a true martingale, it suffices to show that $\mathbb{E}\big[\langle R^{\pi,x}\rangle_T^{1/2}\big] < +\infty$. Then the Burkholder–Davis–Gundy (BDG) inequality allows us to conclude. In fact, we have

$$\begin{aligned} &\mathbb{E}\big[\langle R^{\pi,x}\rangle_T^{1/2}\big] = \mathbb{E}\Big[\int_0^T \big|U'(X_t^{\pi,x})\Sigma(X_t^{\pi,x},\pi_t)Y_t^{\pi,x} + U(X_t^{\pi,x})Z_t^{\pi,x}\big|^2\, dt\Big]^{1/2} \\ &\le \mathbb{E}\bigg[\sup_{0\le t\le T}|U(X_t^{\pi,x})|^2 \int_0^T \Big|Y_t^{\pi,x}\frac{U'(X_t^{\pi,x})}{U(X_t^{\pi,x})}\,\Sigma(X_t^{\pi,x},\pi_t) + Z_t^{\pi,x}\Big|^2\, dt\bigg]^{1/2} \\ &\le \frac{1}{2}\,\mathbb{E}\big[\sup_{0\le t\le T}|U(X_t^{\pi,x})|^2\big] + \frac{1}{2}\,\mathbb{E}\bigg[\int_0^T \Big|\frac{U'(X_t^{\pi,x})}{U(X_t^{\pi,x})}\,\Sigma(X_t^{\pi,x},\pi_t)Y_t^{\pi,x} + Z_t^{\pi,x}\Big|^2\, dt\bigg] < +\infty, \end{aligned}$$

by Assumption 3.1 and the square integrability property of solutions of BSDEs.

The martingale $U(X^{\pi,x})Y^{\pi,x}$ is strictly positive since its terminal value $U(X_T^{\pi,x})H$ is. Moreover, $U(X^{\pi,x})$ is strictly positive, so that this property transfers to $Y^{\pi,x}$. ∎

Proposition 3.2 has the following corollary that represents $V(x)$ as the value function of a stochastic optimal control problem in BSDE language.

COROLLARY 3.3. *Under Assumptions* 2.1 *and* 3.1*, the following equation holds*

$$V(x) = \sup_{\pi\in\mathcal{A}} \mathbb{E}[U(X_T^{\pi,x})H] = U(x)\sup_{\pi\in\mathcal{A}} Y_0^{\pi,x}, \tag{14}$$

where $(Y^{\pi,x}, Z^{\pi,x})$*,* $\pi\in\mathcal{A}$*, are given by Proposition* 3.2*. Furthermore, a trading strategy* $\bar{\pi}\in\mathcal{A}$ *is optimal if and only if* $V(x) = U(x)Y_0^{\bar{\pi},x}$*.*

Proof. From Proposition 3.2 $U(X^{\pi,x})Y^{\pi,x}$ is a martingale for every $\pi\in\mathcal{A}$. Hence, we have $\mathbb{E}[U(X_T^{\pi,x})H] = \mathbb{E}[U(x)Y_0^{\pi,x}] = U(x)Y_0^{\pi,x}$. Therefore

$$V(x) = \sup_{\pi\in\mathcal{A}} \mathbb{E}[U(X_T^{\pi,x})H] = \sup_{\pi\in\mathcal{A}} \mathbb{E}[U(x)Y_0^{\pi,x}] = \sup_{\pi\in\mathcal{A}} U(x)Y_0^{\pi,x}.$$

We now prove the second assertion of the proposition.

Let first $\bar{\pi}\in\mathcal{A}$ be an optimal trading strategy for (2). By definition of optimality, $V(x) = \mathbb{E}[U(X_T^{\bar{\pi},x})H]$. The random variable $U(X_T^{\bar{\pi},x})H$ is the terminal value of the martingale $U(X^{\bar{\pi},x})Y^{\bar{\pi},x}$. Hence

$$V(x) = \mathbb{E}[U(X^{\bar{\pi},x})H] = U(x)Y_0^{\bar{\pi},x}.$$

To show the converse, let $\bar{\pi}\in\mathcal{A}$ such that $V(x) = U(x)Y_0^{\bar{\pi},x}$. By Proposition 3.2, $U(X^{\bar{\pi},x})Y^{\bar{\pi},x}$ is a martingale with mean $U(x)Y_0^{\bar{\pi},x}$. Therefore $V(x) = \mathbb{E}[U(X_T^{\bar{\pi},x})H]$. Hence $\bar{\pi}$ is optimal. ∎

Corollary 3.3 demonstrates that to determine $V(x)$, it is sufficient and necessary to determine $\Lambda(x)$ defined in (8). The relation (14) shows that (5) and (8) have the same optimal trading strategy. Corollary 3.3 tells us that a trading strategy $\bar{\pi}$ is optimal if $Y^{\bar{\pi}}$ dominates the family $Y^{\pi,x}$, $\pi\in\mathcal{A}$, at time $t=0$. The following proposition gives a condition under which this domination pertains to the entire trajectory.

Proposition 3.4 (Sufficient condition for optimality). *Suppose that Assumptions* 2.1 *and* 3.1 *hold. Let* $\bar\pi \in \mathcal{A}$*. The following are equivalent*:

(C1) $U(X^{\pi,x})Y^{\bar\pi,x}$ *is a supermartingale for every* $\pi \in \mathcal{A}$,
(C2) $F^{\bar\pi}(\cdot,\cdot,Y^{\bar\pi,x},Z^{\bar\pi,x}) = \operatorname{ess\,sup}_{\pi\in\mathcal{A}} F^{\pi}(\cdot,\cdot,Y^{\bar\pi,x},Z^{\bar\pi,x})$, $\mathbb{P}\otimes\lambda$*-a.e.*

If (C1) *or* (C2) *holds then* $\bar\pi$ *is optimal and we have for* $t \in [0,T]$

$$Y_t^{\bar\pi,x} = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} Y_t^{\pi,x}, \;\; \mathbb{P}\text{-}a.s. \tag{15}$$

Proof. First assume that (C1) holds. Let $\bar\pi \in \mathcal{A}$ for which (C1) is satisfied. Let $\pi \in \mathcal{A}$. Then the process $U(X^{\pi,x})Y^{\bar\pi,x}$ is a continuous supermartingale since $U(X^{\pi,x})$ and $Y^{\bar\pi,x}$ have continuous paths. It has the unique decomposition $U(X^{\pi,x})Y^{\bar\pi,x} = U(x)Y_0^{\bar\pi,x} + N^{\bar\pi,\pi}+A^{\bar\pi,\pi}$ where $N^{\bar\pi,\pi}$ is a continuous local martingale and $A^{\bar\pi,\pi}$ a decreasing predictable process starting at 0. Applying Itô's formula to $U(X^{\pi,x})Y^{\bar\pi,x}$, we obtain for $t\in[0,T]$

$$\begin{aligned} d\big[U(X_t^{\pi,x})Y_t^{\bar\pi,x}\big] &= Y_t^{\bar\pi,x}\, dU(X_t^{\pi,x}) + U(X_t^{\pi,x})\, dY_t^{\bar\pi,x} + d\langle U(X^{\pi,x}), Y^{\bar\pi,x}\rangle_t, \\ &= U(X_t^{\pi,x})\big[F^{\pi}(\cdot,t,Y_t^{\bar\pi,x},Z_t^{\bar\pi,x}) - F^{\bar\pi}(\cdot,t,Y_t^{\bar\pi,x},Z_t^{\bar\pi,x})\big]\, dt \\ &\quad + \big[Y_t^{\bar\pi,x}U'(X_t^{\pi,x})\Sigma(X_t^{\pi,x},\pi_t) + U(X_t^{\pi,x})Z_t^{\bar\pi,x}\big]\, dW_t. \end{aligned} \tag{16}$$

To identify $N^{\bar\pi,\pi}$ with the stochastic integral part of (16) and $A^{\bar\pi,\pi}$ with the corresponding finite variation part, we only need to show that the stochastic integral part in (16) is well defined. Once this is proven, the identification follows from the uniqueness of Doob's decomposition, see [Pro04, Theorem 13, Chapter 3]. The processes $U(X^{\pi,x})$ and $Y^{\bar\pi,x}$ have continuous paths $\mathbb{P}$-a.s. They are pathwise bounded since $[0,T]$ is compact. By definition of BSDE solutions, $\int_0^T |Z_t^{\bar\pi,x}|^2\, dt < +\infty$, $\mathbb{P}$-a.s. Using Assumption 3.1 and the integrability property of $U(X^{\pi,x})$, $Y^{\bar\pi,x}$ and $Z^{\bar\pi,x}$, we have for $C^{\bar\pi,\pi} = Y^{\bar\pi,x}U'(X^{\pi,x})\Sigma(X^{\pi,x},\pi) + U(X^{\pi,x})Z^{\bar\pi,x}$

$$\begin{aligned} \int_0^T |C_t^{\bar\pi,\pi}|^2\, dt &= \int_0^T |U(X_t^{\pi,x})|^2 \left| Y_t^{\bar\pi,x} \frac{U'(X_t^{\pi,x})}{U(X_t^{\pi,x})}\, \Sigma(X_t^{\pi,x},\pi_t) + Z_t^{\bar\pi,x}\right|^2 dt \\ &\le \sup_{0\le t\le T} |U(X_t^{\pi,x})|^2 \int_0^T \left| Y_t^{\bar\pi,x}\, \frac{U'(X_t^{\pi,x})}{U(X_t^{\pi,x})}\, \Sigma(X_t^{\pi,x},\pi_t) + Z_t^{\bar\pi,x}\right|^2 dt < +\infty, \;\; \mathbb{P}\text{-a.s.} \end{aligned}$$

The process $N^{\bar\pi,\pi} := \int_0^{\cdot} C_t^{\bar\pi,\pi}\, dW_t$ is therefore a well defined local martingale. It follows from the uniqueness of decomposition of supermartingales that the finite variation part of (16) is a modification of $A^{\bar\pi,\pi}$. The process $A^{\bar\pi,\pi}$ is decreasing, hence we have $\mathbb{P}\otimes\lambda$-a.e. the inequality

$$U(X^{\pi,x})\big[F^{\pi}(\cdot,\cdot,Y^{\bar\pi,x},Z^{\bar\pi,x}) - F^{\bar\pi}(\cdot,\cdot,Y^{\bar\pi,x},Z^{\bar\pi,x})U(X^{\pi,x})\big] \le 0.$$

Dividing by $U(X^{\pi,x}) > 0$, we obtain the inequality

$$F^{\bar\pi}(\cdot,\cdot,Y^{\bar\pi,x},Z^{\bar\pi,x}) \ge F^{\pi}(\cdot,\cdot,Y^{\bar\pi,x},Z^{\bar\pi,x}), \;\; \mathbb{P}\otimes\lambda\text{-a.e.} \tag{17}$$

(17) is valid for every $\pi\in\mathcal{A}$. Therefore we have

$$F^{\bar\pi}(\cdot,\cdot,Y^{\bar\pi,x},Z^{\bar\pi,x}) \ge \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} F^{\pi}(\cdot,\cdot,Y^{\bar\pi,x},Z^{\bar\pi,x}), \;\; \mathbb{P}\otimes\lambda\text{-a.e.}$$

Since $\bar\pi\in\mathcal{A}$, we deduce (C2).

Now assume that $\bar\pi \in \mathcal{A}$ satisfies (C2). Let $\pi \in \mathcal{A}$. From (16),

$$U(X^{\pi,x})Y^{\bar\pi,x} = U(x)Y_0^{\bar\pi,x} + N^{\bar\pi,\pi} + A^{\bar\pi,\pi},$$

where

$$A^{\bar\pi,\pi} = \int_0^{\cdot} U(X_t^{\pi,x})\big[F^{\pi}(\cdot,t,Y_t^{\bar\pi,x},Z_t^{\bar\pi,x}) - F^{\bar\pi}(\cdot,t,Y_t^{\bar\pi,x},Z_t^{\bar\pi,x})\big]\,dt,$$

$$N^{\bar\pi,\pi} = \int_0^{\cdot} \big[Y_t^{\bar\pi,x}U'(X_t^{\pi,x})\Sigma(X_t^{\pi,x},\pi_t) + U(X_t^{\pi,x})Z_t^{\bar\pi,x}\big]\,dW_t.$$

By (C2), $U(X^{\pi,x})\big[F^{\pi}(\cdot,Y^{\bar\pi,x},Z^{\bar\pi,x}) - F^{\bar\pi}(\cdot,Y^{\bar\pi,x},Z^{\bar\pi,x})\big]$ is non-positive. Using Cauchy–Schwarz's inequality and Assumptions (A3), (A5), we obtain the integrability property

$$\mathbb{E}\Big[\Big|\int_0^T U(X_t^{\pi,x})\big[F^{\pi}(\cdot,t,Y^{\bar\pi_t,x},Z^{\bar\pi_t,x}) - F^{\bar\pi}(\cdot,t,Y_t^{\bar\pi,x},Z_t^{\bar\pi,x})\big]\,dt\Big|\Big] < +\infty.$$

We deduce that $A^{\bar\pi,\pi}$ is integrable, predictable and decreasing. Therefore to show that $U(X^{\pi,x})Y^{\bar\pi,x}$ is a supermartingale, it suffices to show that the local martingale $N^{\bar\pi,\pi}$ is a martingale. Using Assumption 3.1 and the integrability properties of the processes $U(X^{\pi,x}), Y^{\bar\pi,x}$ and $Z^{\bar\pi,x}$, we get

$$\begin{aligned}
\mathbb{E}\big[\langle N^{\pi}\rangle_T^{1/2}\big] &= \mathbb{E}\Big[\Big(\int_0^T |Y_t^{\bar\pi,x}U'(X_t^{\pi,x})\Sigma(X_t^{\pi,x},\pi_t) + U(X_t^{\pi,x})Z_t^{\bar\pi,x}|^2\,dt\Big)^{1/2}\Big]\\
&\le \mathbb{E}\Big[\Big(\sup_{0\le t\le T}|U(X_t^{\pi,x})|\int_0^T\Big|Y_t^{\bar\pi,x}\,\frac{U'(X_t^{\pi,x})}{U(X_t^{\pi,x})}\,\Sigma(X_t^{\pi,x},\pi_t) + Z_t^{\bar\pi,x}\Big|^2\,dt\Big)^{1/2}\Big]\\
&\le \frac{1}{2}\,\mathbb{E}\big[\sup_{0\le t\le T}|U(X_t^{\pi,x})|^2\big] + \frac{1}{2}\,\mathbb{E}\Big[\int_0^T\Big|Y_t^{\bar\pi,x}\,\frac{U'(X_t^{\pi,x})}{U(X_t^{\pi,x})}\,\Sigma(X_t^{\pi,x},\pi_t) + Z_t^{\bar\pi,x}\Big|^2\,dt\Big] < +\infty.
\end{aligned}$$

By the inequality of Burkholder–Davis–Gundy, $\mathbb{E}\big[\sup_{0\le t\le T}|N_t^{\pi}|\big] \le C\mathbb{E}[\langle N^{\pi}\rangle_T^{1/2}] < +\infty$. Hence $N^{\bar\pi,\pi}$ is uniformly integrable, thus $U(X^{\pi,x})Y^{\bar\pi,x}$ is a supermartingale as the sum of a martingale and a predictable, integrable, and decreasing process. We deduce that (C2) implies (C1).

Let $\bar\pi \in \mathcal{A}$ such that (C2) holds. The relation (15) follows from the comparison principle. Clearly, (15) implies that $\bar\pi$ is optimal. ∎

In general, (15) does not imply (C2), see [BCH+00]. It will be shown in Theorem 3.7 that under Assumption 3.1 they are equivalent. This will be a consequence of the optimality property of $\bar\pi$ and the comparison principle. Proposition 3.4 will then serve as a necessary condition for optimality.

3.2. Main result. Consider $F : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d$ defined by

$$F(\cdot,t,y,z) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} F^{\pi}(\cdot,t,y,z), \tag{18}$$

with the ess sup in (18) taken with respect to the product measure $\mathbb{P}\otimes\lambda$.

F will be used as the generator of the BSDE that characterizes the value function Λ. The following lemma shows that F has the properties of a generator.

LEMMA 3.5. *F defined by* (18) *is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}^d) - \mathcal{B}(\mathbb{R})$ measurable. Moreover, the pair (F,H) is a standard parameter.*

Proof. For every $(\omega, t) \in \Omega \times [0, T]$, we have $F(\omega, t, 0, 0) = 0$. For every $\pi \in \mathcal{A}$, F^π is linear in (y, z) and the coefficients are uniformly bounded by the constant K. We deduce that F is uniformly Lipschitz continuous in (y, z) with Lipschitz constant K. By definition of $\operatorname{ess\,sup} F(\cdot, \cdot, y, z)$ is $\mathcal{P} - \mathcal{B}(\mathbb{R})$-measurable for every $(y, z) \in \mathbb{R} \times \mathbb{R}^d$. We deduce from Theorem 2 in [Gow72] that F is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) - \mathcal{B}(\mathbb{R})$ measurable.

The pair (F, H) is standard as H is square integrable. ∎

Now consider the BSDE

$$\begin{aligned} -dY_t &= F(\cdot, t, Y_t, Z_t)\, dt - Z_t\, dW_t, \quad t \in [0, T], \\ Y_T &= H. \end{aligned} \tag{19}$$

Since (F, H) is standard, a solution pair (Y, Z) exists and is unique. Let $L_+(\mathbb{F})$ be the set of nonnegative càdlàg $\mathbb{F}$-adapted processes. We denote by $L^{\mathrm{s}}_+(\mathbb{F})$ the set of processes $\widetilde{Y} \in L_+(\mathbb{F})$ with terminal value H and such that $U(X^{\pi,x})\widetilde{Y}$ is a supermartingale for every $\pi \in \mathcal{A}$. The following proposition collects some properties of the process Y.

PROPOSITION 3.6. *Assume* 2.1 *and* 3.1. *Let* (Y, Z) *be the solution of the BSDE with parameter* (F, H). *Then* $Y \in L^s_+(\mathbb{F})$ *and it is the minimal process in this class. Moreover,* Y *has the following representation*

$$Y = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}} Y^{\pi,x}, \quad \mathbb{P} \otimes \lambda\text{-}a.e. \tag{20}$$

Proof. Since (Y, Z) is the solution of a BSDE, Y is continuous and adapted. Moreover, $H > 0$ and $F(\cdot, \cdot, 0, 0) = 0$, hence $Y \geq 0$. We deduce that $Y \in L_+(\mathbb{F})$. Let $\pi \in \mathcal{A}$. Proceeding as in the proof of Proposition 3.4, we see that $U(X^{\pi,x})Y$ has the decomposition $U(X^{\pi,x})Y = U(x)Y_0 + N^\pi + A^\pi$ with

$$\begin{aligned} A^\pi &= \int_0^\cdot U(X_t^{\pi,x}) \big[F^\pi(\cdot, t, Y_t, Z_t) - F(\cdot, t, Y_t, Z_t)\big]\, dt, \\ N^\pi &= \int_0^\cdot \big[Y_t U'(X_t^{\pi,x}) \Sigma(X_t^{\pi,x}, \pi_t) + U(X_t^{\pi,x}) Z_t\big]\, dW_t. \end{aligned}$$

Arguments of Proposition 3.4 show that N^π is a continuous martingale and A^π an integrable, predictable process. Moreover A^π is decreasing by the choice of F. Hence $U(X^{\pi,x})Y$ is a supermartingale. Since π is arbitrary, we deduce that $Y \in L^{\mathrm{s}}_+(\mathbb{F})$. The equality (20) is a consequence of the comparison principle. To see that Y is minimal in this set, let $\widetilde{Y} \in L^{\mathrm{s}}_+(\mathbb{F})$. For $\pi \in \mathcal{A}$, $U(X^{\pi,x})\widetilde{Y}$ is a supermartingale with terminal value $U(X_T^{\pi,x})H$. Since $U(X^{\pi,x})Y^{\pi,x}$ is a martingale with the same terminal value, we deduce for all $t \in [0, T]$ that $\widetilde{Y}_t \geq Y^{\pi,x}$, $\mathbb{P}$-a.s. From the representation (20), we infer that for all $t \in [0, T]$ we have $\widetilde{Y}_t \geq Y_t$, $\mathbb{P}$-a.s. Since Y and $\widetilde{Y}$ are càdlàg, we have $\widetilde{Y} \geq Y$. Hence Y is the smallest process in $L^{\mathrm{s}}_+(\mathbb{F})$. ∎

We can now state the main theorem of this paper which gives a necessary and sufficient condition for optimality via the process Y.

THEOREM 3.7. *We suppose that Assumption* 2.1 *and* 3.1 *are valid. Let* (Y, Z) *be the solution of the BSDE with parameter* (F, H). *Then we have* $V(x) = U(x)Y_0$. *Moreover,* $\nu \in \mathcal{A}$ *is optimal if and only if* $F(\cdot, \cdot, Y, Z) = F^\nu(\cdot, \cdot, Y, Z)$, $\mathbb{P} \otimes \lambda$-*a.e. or* $Y = Y^\nu$.

Proof. By (20), $Y_0 = \sup_{\pi\in\mathcal{A}} Y_0^{\pi,x}$. The equality $V(x) = U(x)Y_0$ follows from (14). Let $\nu \in \mathcal{A}$ be an optimal trading strategy. Then $Y_0^\nu = Y_0$. By definition of F, we have

$$F(\cdot,\cdot,Y,Z) \geq F^\nu(\cdot,\cdot,Y,Z), \quad \mathbb{P}\otimes\lambda\text{-a.e.} \tag{21}$$

Since the comparison is strict and we have $Y_0^\nu = Y_0$, we deduce that (21) is an equality. Hence $Y^\nu = Y$.

Suppose conversely that $Y^\nu = Y$ or (21) is an equation. Then $V(x) = U(x)Y_0 = U(x)Y_0^\nu$. We deduce that ν is optimal. ∎

Theorem 3.7 shows that to solve our optimal control problem, we only need to solve the BSDE (19). The generator F of (19) is of Lipschitz growth. Hence it can be approximated efficiently using numerical schemes for BSDEs, see [MPSMT02]. Our necessary and sufficient condition for optimality is therefore easier to check than the condition provided by the stochastic maximum principle. This one involves adjoint equations of second order. Alternatively, approaching the problem from the perspective of the dynamic programming principle requires to solve a highly nonlinear PDE or a backward stochastic PDE in terms of Mania et al. [MT03]. We do not tackle the problem of existence of an optimal trading strategy. In the examples that follow, we will identify the optimal trading strategy as measurable selection of certain multifunctions.

4. Applications to particular cases. In this section, we apply Theorem 3.7 to solve (5) in the case of exponential and power utility. $\mathbb{R}^+$ denotes the positive real line. Our market model is identical to that in Section 2.

4.1. The case of exponential utility with additive liability. Let x be the initial endowment of our investor. To a trading strategy $\pi \in \mathbb{H}_T^{2,m}$, we associate the wealth process $X^{\pi,x}$ described by the SDE

$$X_t^{\pi,x} = x + \int_0^t \langle \pi_s, \theta_s\, ds + dW_s\rangle, \quad t\in[0,T].$$

In our notation of Section 2, here we consider $\Sigma(X^{\pi,x},\pi) = \pi$. The utility function U is given by $U(y) = -\exp(-\alpha y)$, $y\in\mathbb{R}$, with $\alpha > 0$. As $\mathrm{dom}(U) = \mathbb{R}$, the set of admissible trading strategies is given by

$$\mathcal{A} = \big\{\pi \in \mathbb{H}_T^{2,m} \text{ such that } \pi_t(\omega) \in \mathcal{C}(\omega,t), \text{ for } \mathbb{P}\otimes\lambda\text{-a.e. } (\omega,t)\in\Omega\times[0,T]\big\}.$$

Let $\widetilde{H}$ be a real valued $\mathcal{F}_T$-measurable random variable. We deal with the optimization problem

$$V(x) = \sup_{\pi\in\mathcal{A}} \mathbb{E}[U(X_T^{\pi,x} - \widetilde{H})] = \sup_{\pi\in\mathcal{A}} \mathbb{E}[U(X_T^{\pi,x})H], \tag{22}$$

with $H = \exp(\alpha\widetilde{H})$. We suppose that H is square integrable. So (A4) is satisfied. By the exponential structure of U and the boundedness of the trading strategies, (A2), (A3) and (A5) are satisfied. Hence altogether Assumption 3.1 holds, provided H is square integrable. Let $\pi\in\mathcal{A}$. The generator F^π defined in (12) has the form

$$F^\pi(\omega,t,y,z) = \frac{\alpha^2}{2}\, y|\pi_t(\omega)|^2 - \langle \alpha\pi_t(\omega), \theta_t(\omega)y + z\rangle, \ (\omega,t,y,z)\in\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d.$$

Note that $U < 0$. So we replace $\operatorname{ess\,sup}$ by $\operatorname{ess\,inf}$ in the definition of F given by (18). We have

$$F(\cdot,\cdot,y,z) = \operatorname*{ess\,inf}_{\pi\in\mathcal{A}} F^{\pi}(\cdot,\cdot,y,z) = \operatorname*{ess\,inf}_{\pi\in\mathcal{A}} \Big[\frac{\alpha^2}{2}\, y|\pi|^2 - \langle \alpha\pi, \theta y + z\rangle\Big],\ (y,z)\in\mathbb{R}\times\mathbb{R}^d.$$

The following theorem gives a representation of the optimal trading strategy as a projection onto the predictable multifunction $\mathcal{C}$.

THEOREM 4.1. *The value function V of problem* (22) *is given by*

$$V(x) = U(x)Y_0, \quad x > 0,$$

where (Y,Z) is the solution of the BSDE

$$\begin{aligned} -dY_t &= \operatorname*{ess\,inf}_{\pi\in\mathcal{A}} \Big[\frac{\alpha^2}{2}\, Y_t|\pi_t|^2 - \alpha\langle \pi_t, \theta_t Y_t + Z_t\rangle\Big]\, dt - Z_t\, dW_t, \\ Y_T &= H. \end{aligned} \tag{23}$$

Every measurable selection $\bar{\pi}$ of the multifunction $P : \Omega\times[0,T]\to\mathbb{R}^m$ with values in the closed subsets of $\mathbb{R}^m$, defined by

$$P(\omega,t) = \arg\min_{a\in\mathcal{C}(\omega,t)} \Big|\frac{1}{\alpha}\Big(\theta_t(\omega) + \frac{Z_t}{Y_t}(\omega)\Big) - a\Big|^2, \quad (\omega,t)\in\Omega\times[0,T], \tag{24}$$

is an optimal trading strategy.

Proof. By Lemma 3.5, the pair (F,H) is standard. Hence the BSDE (23) admits a unique solution (Y,Z). By Theorem 3.7, $V = U\cdot Y_0$. The process Y is strictly positive since $H > 0$. First we show that there is at least one measurable selection. Consider the map $\Pi : \Omega\times[0,T]\times\mathbb{R}^m\to\mathbb{R}$ defined by

$$\Pi(\omega,t,a) = \Big|a - \frac{1}{\alpha}\Big(\theta_t(\omega) + \frac{Z_t}{Y_t}(\omega)\Big)\Big|^2.$$

Clearly, the map Π is measurable in (ω,t) and continuous in a. Since $\mathcal{C}$ is a measurable multifunction with values in the closed sets of $\mathbb{R}^m$, by the theorem of the measurable selection [Roc76, Theorem 2.K], P is a measurable multifunction with values in the closed sets. By the measurable selection theorem [Roc76, Corollary 1.C], it admits at least one measurable selection ν. It is then straightforward to see that the $\operatorname{ess\,inf}$ in (23) is attained at ν. Hence $F(\cdot,Y,Z) = F^{\nu}(\cdot,Y,Z)$. We deduce from Theorem 3.7 that ν is an optimal trading strategy. ■

REMARK 4.2.

1. In the case under investigation, the family of supermartingales constructed is exactly the one defined by El Karoui in the context of convex constraints. The construction has already been achieved by Hu et al. [HIM05], Rouge et al. [REK00] in the Brownian setting and by Morlais [Mor09] in the general continuous martingale setting. In these papers, the family of supermartingales is constructed by means of quadratic BSDEs. In the Brownian setting, the BSDE has the structure

$$-dP_t = f(t,Q_t)\, dt - Q_t\, dW_t,\ P_T = \frac{1}{\alpha}\log H \tag{25}$$

with

$$f(\omega,t,q) = -\frac{\alpha}{2}\operatorname{dist}^2\Big(q+\frac{\theta_t(\omega)}{\alpha}, \mathcal{C}(\omega,t)\Big) + q\theta_t(\omega) + \frac{|\theta_t(\omega)|^2}{2\alpha}\,,$$

for all $(\omega, t, q) \in \Omega \times [0,T] \times \mathbb{R}^d$. The solution (P,Q) to (25) is related to the solution (Y,Z) of (23) via the logarithmic transform

$$P = \frac{1}{\alpha}\log Y,\ Q = \frac{Z}{Y}\,. \tag{26}$$

In [REK00, HIM05, Mor09], the claim H is assumed to be bounded while in our approach we only require an exponential moment of order 2α.

2. A BSDE similar to (23) has been obtained by Lim et al [LQ11] in a model of an incomplete market with defaults. It is derived using arguments related to the dynamic programming principle.

4.2. The case of the power utility with multiplicative liability. In this subsection, we consider the positive utility function $U(y) = y^\gamma$, $y \geq 0$, with $\gamma \in (0,1)$. The market model is the same as in the previous subsection, just with a slightly different meaning for trading strategies: they are measured in the proportion of wealth invested into the stocks. This corresponds to choosing $\Sigma(X^{\pi,x},\pi) = X^{\pi,x}\cdot\pi$ in Section 2. Hence the wealth process $X^{\pi,x}$ associated to π is described by the SDE

$$X_t^{\pi,x} = x + \int_0^t X_s^{\pi,x}\langle \pi_s, \theta_s\,ds + dW_s\rangle, \qquad x>0,\ t\in[0,T].$$

Clearly, $X^{\pi,x} > 0$ for $\pi \in \mathbb{H}^{2,m}$. The constraints on the trading strategies are described by the multifunction $\mathcal{C}$ defined in (3). Hence the set of admissible trading strategies is given by

$$\mathcal{A} = \big\{\pi\in\mathbb{H}_T^{2,m} \text{ such that } \pi_t(\omega)\in\mathcal{C}(\omega,t) \text{ for } \mathbb{P}\otimes\lambda\text{-a.e. } (\omega,t)\in\Omega\times[0,T]\big\}.$$

Let $\widetilde{H}$ be a nonnegative $\mathcal{F}_T$-measurable random variable. We study the optimization problem

$$V(x) = \sup_{\pi\in\mathcal{A}}\mathbb{E}[U(X_T^{\pi,x}\widetilde{H})] = \sup_{\pi\in\mathcal{A}}\mathbb{E}[U(X_T^{\pi,x})H], \quad x>0, \tag{27}$$

with $H = \widetilde{H}^\gamma$. We suppose that H is square integrable. Due to the boundedness assumptions on $\mathcal{C}$, Assumption 3.1 holds. The generators F^π, $\pi\in\mathcal{A}$, defined in (12), have the form

$$F^\pi(\cdot,t,y,z) = \frac{\gamma(\gamma-1)}{2}\,|\pi_t(\omega)|^2 y + \gamma\langle\pi_t(\omega),\theta_t(\omega)y+z\rangle,$$

for all $(\omega,t,y,z)\in\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d$. Hence F defined in (18) takes the simple form

$$F(\cdot,\cdot,y,z) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}}\Big[\frac{\gamma(\gamma-1)}{2}\,|\pi|^2 y + \gamma\langle\pi,\theta y+z\rangle\Big], \quad (y,z)\in\mathbb{R}\times\mathbb{R}^d.$$

The following theorem states the existence of an optimal trading strategy and characterizes it as a measurable selection.

THEOREM 4.3. *Assume* 2.1 *and* 3.1. *The value function* V *of problem* (27) *is given by*

$$V = U \cdot Y_0,$$

where (Y, Z) *is the solution of the BSDE*

$$-dY_t = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \Big[\frac{(\gamma-1)\gamma}{2} |\pi|_t^2 Y_t + \gamma\langle \pi_t, \theta_t Y_t + Z_t\rangle \Big]\, dt - Z_t\, dW_t, \qquad (28)$$
$$Y_T = H.$$

Every measurable selection $\hat{\pi}$ *of the multifunction* $P : \Omega \times [0,T] \to \mathbb{R}^m$ *with values in the closed sets of* $\mathbb{R}^m$*, defined by*

$$P(\omega,t) = \arg \min_{a\in\mathcal{C}(\omega,t)} \left| \frac{\theta_t(\omega) + \frac{Z_t}{Y_t}(\omega)}{1-\gamma} - a \right|^2, \quad (\omega,t) \in \Omega \times [0,T],$$

is an optimal trading strategy.

Proof. The proof is analogous to the proof of Theorem 4.1. ■

REMARK 4.4.

1. In [Zar01], Zariphopoulou addresses (27) in the Markovian setting using PDE methods. It was investigated by BSDE methods in [HIM05, Mor09] with $\tilde{H} = 1$ and in [IRZ11, Section 2.2] with $0 < \tilde{H} < 1$. In both cases, the BSDE obtained is of quadratic growth and its solution (P, Q) is related to the solution (Y, Z) of (28) via a similar logarithmic transform as in (26).

2. BSDE (28) has already been obtained in [LQ11] in the context of incomplete markets with defaults via the dynamic programming principle.

3. The sign constraint on the utility function prevents us from studying simple log utility. To get rid of this restriction, one may construct a family of martingales $R^{\pi,x}$ of the form $R^{\pi,x} = U(X^{\pi,x}Y^{\pi,x})$ for utility functions defined on the positive real line. Unfortunately, in this case the BSDEs attached to the processes $Y^{\pi,x}$ are of quadratic growth. We therefore aim at a suitable comparison principle for BSDE with quadratic growth to guarantee the validity of versions of Proposition 3.4 and Proposition 3.6. This is subject of work in progress.

Acknowledgements. Victor is very grateful to the Berlin Mathematical School for granting financial support.

References

[BCH+00] P. Briand, F. Coquet, Y. Hu, J. Mémin, S. Peng, *A converse comparison theorem for BSDEs and related properties of g-expectation*, Electron. Comm. Probab. 5 (2000), 101–117.

[CSW01] J. Cvitanić, W. Schachermayer, H. Wang, *Utility maximization in incomplete markets with random endowment*, Finance Stoch. 5 (2001), 259–272.

[DS94] F. Delbaen, W. Schachermayer, *A general version of the fundamental theorem of asset pricing*, Math. Ann. 300 (1994), 463–520.

[EK81] N. El Karoui, *Les aspects probabilistes du contrôle stochastique*, In: École d'Été de Probabilités de Saint-Flour IX–1979, Lecture Notes in Math. 876, Springer, Berlin 1981, 73–238.

[EKPQ97] N. El Karoui, S. Peng, M.-C. Quenez, *Backward stochastic differential equations in finance*, Math. Finance 7 (1997), 1–71.

[Gow72] K. Gowrisankaran, *Measurability of functions in product spaces*, Proc. Amer. Math. Soc. 31 (1972), 485–488.

[HHI+] U. Horst, Y. Hu, P. Imkeller, A. Réveillac, J. Zhang, *Forward-backward systems for expected utility maximization*, Stochastic Process. Appl. 124 (2014), 1813–1848.

[HIM05] Y. Hu, P. Imkeller, M. Müller, *Utility maximization in incomplete markets*, Ann. Appl. Probab. 15 (2005), 1691–1712.

[HK04] J. Hugonnier, D. Kramkov, *Optimal investment with random endowments in incomplete markets*, Ann. Appl. Probab. 14 (2004), 845–864.

[IRZ11] P. Imkeller, A. Réveillac, J. Zhang, *Solvability and numerical simulation of BSDEs related to BSPDEs with applications to utility maximization*, Int. J. Theor. Appl. Finance 14 (2011), 635–667.

[Kaz94] N. Kazamaki, *Continuous Exponential Martingales and BMO*, Lecture Notes in Math. 1579, Springer, Berlin 1994.

[LQ11] T. Lim, M.-C. Quenez, *Exponential utility maximization in an incomplete market with defaults*, Electron. J. Probab. 16 (2011), 1434–1464.

[MPSMT02] J. Ma, P. Protter, J. San Martin, S. Torres, *Numerical method for backward stochastic differential equations*, Ann. Appl. Probab. 12 (2002), 302–316.

[MT03] M. Mania, R. Tevzadze, *Backward stochastic PDE and imperfect hedging*, Int. J. Theor. Appl. Finance 6 (2003), 663–692.

[Mor09] M.-A. Morlais, *Quadratic BSDE driven by a continuous martingale and applications to the utility maximization problem*, Finance Stoch. 13 (2009), 121–150.

[Pro04] P. Protter, *Stochastic Integration and Differential Equations: Version 2.1*, Stoch. Model. Appl. Probab. 21, Springer, Berlin 2005.

[Roc76] R. T. Rockafellar, *Integral Functionals, Normal Integrands and Measurable Selections*, Lectures Notes in Math. 543, Springer, Berlin 1976.

[REK00] R. Rouge, N. El Karoui, *Pricing via utility maximization and entropy*, Math. Finance 10 (2000), 259–276.

[Zar01] T. Zariphopoulou, *A solution approach to valuation with unhedgeable risks*, Finance Stoch. 5 (2001), 61–82.