DYNKIN GAME WITH ASYMMETRIC INFORMATION

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Abstract. We consider a Dynkin game where the seller possesses additional information compared to the buyer. The additional information is described by a random variable taking finitely many values. We show that the game possesses a value and we provide a necessary and sufficient condition for the existence of a Nash equilibrium. Results are illustrated with an explicit example.
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1. **Introduction**

A Dynkin game is a zero-sum stochastic stopping game first introduced by Dynkin [8]. The game is set up between two players A and B. Each player can stop the game at any time for an observable payoff. If a player chooses to stop the game, B receives some premium from A. In such a game, B attempts to maximize the amount he receives, while A tries to minimize the payout. As B receives a payoff from A, he has to pay a price to enter the game which can also be considered as the value of the game. A primary problem in such a Dynkin game is to identify suitable conditions on the payoff which ensure the existence of a fair value for the game. A related question is the existence of saddle points i.e. stopping times which are optimal for each player irrespective of the actions of the other. These questions have been well investigated in the literature, see [7], [9], [12], [17], [21], [22], [23], [24], [25].

Dynkin games found applications in finance in the valuation of game contingent claims. A game contingent claim (GCC) is a generalization of an American contingent claim which enables the seller to terminate the contract before maturity, but at the expense of a penalty. Kifer [16] showed that in a setting of a complete market model, a game contingent claim has a unique price given by the value of a Dynkin game. Thus, pricing a game contingent claim in this context reduces to identifying the value of a Dynkin game. In incomplete markets, the concept of neutral derivative price process [15] of a game contingent claim is given by the dynamic value of a Dynkin game.

In stochastic games, the decisions of the players are based on the information available to them. In a perfect information game, all players have

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full knowledge. In this paper, we consider a Dynkin game in which one party has additional information compared to the other one. While the latter makes his decisions based on the public information flow $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, the informed player possesses the additional information modeled by some random variable G known to him from the very beginning. Thus the information flow $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ of the informed player is given by the initial enlargement of $\mathbb{F}$ with G i.e. $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(G),\ t \in [0,T]$. We study the effect of this information asymmetry on the existence of the value and saddle points of the Dynkin game. A special case of information asymmetry is considered in [18]. There the authors assumed that G is independent of $\mathbb{F}$ and worked in a Markovian framework. They show the existence of a value and provide sufficient conditions for the existence of a saddle point. We emphasize that we do not assume independence, and the game is studied in a general framework. Moreover, we address in this paper the related question of the cost of additional information, see [2, 3] for optimal investment problems and [10] for optimal stopping problems.

We restrict ourselves in this paper to the simple case in which G takes finitely many values. We show that the Dynkin game with asymmetric information has a value. The key tool is a suitable representation of $\mathbb{G}-$stopping times in terms of $\mathbb{F}-$stopping times. This representation allows us to express the conditional value of the game as a sum of the values of Dynkin games in the reference filtration $\mathbb{F}$ parametrized by the values of G. We show that a saddle point exists if and only if each parametrized Dynkin game has a saddle point.

The paper is organized as follows. In Section 2, we introduce some notations and recall some results related to the classical Dynkin game problem. Our formulation of a Dynkin game with asymmetric information is given in Section 3. We also state our main result Theorem 3.9 which states the existence of the Dynkin game which is introduced. In Section 4 we establish a one-to-one correspondence between saddle points for the Dynkin game with asymmetric information and saddle points of parametrized Dynkin games in the reference filtration. Finally, we present an example to illustrate our results.

2. **Preliminaries**

2.1. **Setup and notations.** Let $T > 0$ represent a finite time. We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is the reference filtration satisfying the usual conditions of right-continuity and completeness. Moreover we assume that $\mathcal{F}_0$ is trivial, i.e. $\mathbb{P}(F)$ equals either 0 or 1 for all $F \in \mathcal{F}_0$. Equations resp. inequalities involving random variables are usually understood in the almost sure sense. Let $t \in \mathbb{R}^+$ and $\mathbb{H}$ a filtration in $\mathcal{F}$. We denote by $\mathcal{T}_{t,T}(\mathbb{H})$ the set of $\mathbb{H}-$stopping times with values in $[t,T]$. We write $\mathcal{T}(\mathbb{H})$ for $\mathcal{T}_{0,T}(\mathbb{H})$. We denote by $\mathcal{S}(\mathbb{H})$

the set of real-valued $\mathbb{H}$-adapted processes Y with càdlàg paths such that $\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t|\right] < +\infty$.

2.2. **Dynkin game problem.** We will recall some definitions and results pertaining to Dynkin games. Let $U, L \in \mathcal{S}(\mathbb{F})$. We assume the following :

Assumption 2.1. $0 \leq L \leq U < \infty$ and $L_T = U_T$.

A Dynkin game of two players A (seller) and B (buyer) associated to the pair of processes (U, L) is a game with objective

$$J_0(\gamma,\tau) = \mathbb{E}\left[U_\gamma 1_{\{\gamma<\tau\}} + L_\tau 1_{\{\tau\leq\gamma\}}\right],\ (\gamma,\tau) \in \mathcal{T}(\mathbb{F}) \times \mathcal{T}(\mathbb{F}). \tag{2.1}$$

In (2.1), γ is a stopping time chosen by A and τ a stopping time chosen by B at time 0. At time $\gamma \wedge \tau$ when the game stops, B receives from A the amount $U_\gamma 1_{\{\gamma<\tau\}} + L_\tau 1_{\{\tau\leq\gamma\}}$. The goal of A is to choose γ so as minimize (2.1) for all possible choices of τ. This leads to the upper value of the game

$$\overline{V}_0 := \inf_{\gamma\in\mathcal{T}(\mathbb{F})} \sup_{\tau\in\mathcal{T}(\mathbb{F})} J_0(\gamma,\tau). \tag{2.2}$$

The goal of B on the other hand is to choose τ so as to maximize (2.1) for all possible choices of γ. This corresponds to the following lower value of the game

$$\underline{V}_0 := \sup_{\tau\in\mathcal{T}(\mathbb{F})} \inf_{\gamma\in\mathcal{T}(\mathbb{F})} J_0(\gamma,\tau). \tag{2.3}$$

Definition 2.2. The Dynkin game with objective (2.1) has a value if $\underline{V}_0 = \overline{V}_0$. We denote by V_0 this common value.

The game can also be started at some time $t \in [0, T]$. In this case one chooses the stopping times in the set $\mathcal{T}_{t,T}(\mathbb{F})$. The corresponding lower and upper values $\underline{V}_t$,$\overline{V}_t$ are given by

$$\overline{V}_t = \operatorname*{ess\,inf}_{\gamma\in\mathcal{T}_{t,T}(\mathbb{F})} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}(\mathbb{F})} \mathbb{E}\left[R(\gamma,\tau)\middle|\mathcal{F}_t\right], \tag{2.4a}$$

$$\underline{V}_t = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}(\mathbb{F})} \operatorname*{ess\,inf}_{\gamma\in\mathcal{T}_{t,T}(\mathbb{F})} \mathbb{E}\left[R(\gamma,\tau)\middle|\mathcal{F}_t\right], \tag{2.4b}$$

where

$$R(\gamma,\tau) = U_\gamma 1_{\{\gamma<\tau\}} + L_\tau 1_{\{\tau\leq\gamma\}},\ \gamma,\tau \in \mathcal{T}_{t,T}(\mathbb{F}). \tag{2.5}$$

We refer to (2.5) as the payoff function of the Dynkin game.

Definition 2.3. The Dynkin game associated to the processes L, U has a value at time $t \in [0, T]$ if $\overline{V}_t = \underline{V}_t$. The common value is denoted by V_t.

Definition 2.4. Let $t \in [0, T]$. A pair $(\bar{\gamma}, \bar{\tau}) \in \mathcal{T}_{t,T}(\mathbb{F}) \times \mathcal{T}_{t,T}(\mathbb{F})$ is a saddle point at time t for a Dynkin game with objective function (2.1) if and only if for every $(\gamma, \tau) \in \mathcal{T}_{t,T}(\mathbb{F}) \times \mathcal{T}_{t,T}(\mathbb{F})$ we have

$$\mathbb{E}\left[R(\bar{\gamma},\tau)\middle|\mathcal{F}_t\right] \leq \mathbb{E}\left[R(\bar{\gamma},\bar{\tau})\middle|\mathcal{F}_t\right] \leq \mathbb{E}\left[R(\gamma,\bar{\tau})\middle|\mathcal{F}_t\right]. \tag{2.6}$$

If a saddle point exists, then $\overline{V}_t = \underline{V}_t$. However, the converse is not true in general (see Example 3.1 in [9]). A classical assumption in the literature ([1, 4, 5]) that guarantees the existence of a value is the so-called Mokoboski condition, i.e. there exists two supermartingales ϕ and ψ such that

$$L \leq \psi - \phi \leq U. \tag{2.7}$$

The condition (2.7) is quite difficult to check. However it is very useful for the connection between Dynkin games and doubly reflected backward stochastic differential equations (DRBSDEs), see [7]. In [19], the authors establish an existence result for the value of Dynkin games without Mokoboski's condition, and they also provide sufficient conditions for the existence of a saddle point. The results are derived using deep-rooted facts from the general theory of stochastic processes. First we provide some additional definitions before stating the results.

Definition 2.5. An $\mathbb{F}$-optional process ϕ is said to be *left-upper semicontinuous in expectation along stopping times (LUSCE)* if for each $\mathbb{F}-$stopping time γ and each sequence of $\mathbb{F}-$stopping times $(\gamma^n)_{n\in\mathbb{N}}$ such that $\gamma^n \uparrow \gamma$ we have

$$\mathbb{E}\left[\phi_\gamma\right] \geq \limsup_{n\to\infty} \mathbb{E}\left[\phi_{\gamma_n}\right].$$

The following result from Lepeltier et al. [19] requiring the integrability assumptions concerning Y of subsection 2.1 guarantees the existence of a value under Assumption 2.1.

Theorem 2.6. *Let $L, U \in \mathcal{S}(\mathbb{F})$ satisfy Assumption 2.1. Then there exists a unique right-continuous $\mathbb{F}$-adapted process V such that*

$$V_t = \underline{V}_t = \overline{V}_t,\ t \in [0,T]. \tag{2.8}$$

Moreover, if L and $-U$ are LUSCE along stopping times then the Dynkin game with criteria (2.1) has a saddle point.

Our goal in this paper is to investigate the existence of the value of a Dynkin game if the players have different information flows.

3. Dynkin game with asymmetric information

3.1. **Model.** The probability space is the same as in Subsection 2.1. In this section, we consider a Dynkin game with asymmetric information. This is a Dynkin game similar to the classical one with the exception that one player has more information than the other. The ordinary player chooses his strategy based on the public information flow modeled by the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. The informed player possesses from the beginning extra information about the outcome of some discrete finite valued random variable G. Thus his information flow $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ is given by the initial enlargement of $\mathbb{F}$ with G i.e. $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(G),\ t \in [0,T]$.

In the sequel, we assume for simplicity that G takes only two different values. More precisely:

Assumption 3.1. G is a discrete random variable taking the two values a and b such that $0 < \mathbb{P}(G = a) < 1$ and $\mathbb{P}(\{G = a\} \cup \{G = b\}) = 1$.

Remark 3.2. Let η be the law of G. As G takes finitely many values, Jacod's density hypothesis [13, 14] is satisfied i.e. for $t \in [0, T]$, the regular conditional distribution of G given $\mathcal{F}_t$ is absolutely continuous w.r.t η for $\mathbb{P}$-almost all $\omega \in \Omega$ (see for instance [20]). Under Assumption 3.1 $\mathbb{F}$-semimartingales are also $\mathbb{G}$-semimartingales and the filtration $\mathbb{G}$ is right-continuous (see [11]).

Remark 3.3. We have $\mathcal{G}_0 = \sigma(G)$. This is due to the fact that $\mathbb{F}$ is right-continuous and $\mathcal{F}_0$ is trivial.

We introduce the processes $\alpha(a)$ and $\alpha(b)$ where

$$\alpha_t(u) = \mathbb{E}\left[1_{\{G=u\}}\,|\mathcal{F}_t\right], t \in [0, T], u \in \{a, b\}. \tag{3.1}$$

Note that $\alpha_0(u) = \mathbb{P}(G = u) > 0, u \in \{a, b\}$. We consider two payoff processes $L, U \in \mathcal{S}(\mathbb{F})$. The criteria of our Dynkin game with asymmetric information is given by

$$J_0^G(\gamma, \tau) := \mathbb{E}\left[U_\gamma 1_{\{\gamma<\tau\}} + L_\tau 1_{\{\tau\leq\gamma\}}\middle|\mathcal{G}_0\right],\ (\gamma, \tau) \in \mathcal{T}(\mathbb{G}) \times \mathcal{T}(\mathbb{F}). \tag{3.2}$$

The lower resp. upper values $\underline{V}_0^G$ and $\overline{V}_0^G$ are given by

$$\underline{V}_0^G := \operatorname*{ess\,sup}_{\tau\in\mathcal{T}(\mathbb{F})} \operatorname*{ess\,inf}_{\gamma\in\mathcal{T}(\mathbb{G})} J_0^G(\gamma, \tau), \tag{3.3a}$$

$$\overline{V}_0^G := \operatorname*{ess\,inf}_{\gamma\in\mathcal{T}(\mathbb{G})} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}(\mathbb{F})} J_0^G(\gamma, \tau). \tag{3.3b}$$

It is clear that $\underline{V}_0^G \leq \overline{V}_0^G \leq \operatorname*{ess\,inf}_{\gamma\in\mathcal{T}(\mathbb{F})} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}(\mathbb{F})} J_0^G(\gamma, \tau)$.

We say that the Dynkin game with criteria (3.2) has a value if $\underline{V}_0^G = \overline{V}_0^G$. In the sequel, we denote by $\underline{V}^G$ and $\overline{V}^G$ the lower and upper value respectively. Note that $J_0^G(\gamma, \tau) = \mathbb{E}\left[R(\gamma, \tau)|\,\mathcal{G}_0\right]$ where R is the payoff function defined in (2.5) which is extended naturally to $\mathcal{T}(\mathbb{G}) \times \mathcal{T}(\mathbb{G})$ as follows:

$$R(\gamma, \tau) = U_\gamma 1_{\{\gamma<\tau\}} + L_\tau 1_{\{\tau\leq\gamma\}},\ \gamma, \tau \in \mathcal{T}(\mathbb{G}).$$

A pair $(\bar\gamma, \bar\tau) \in \mathcal{T}(\mathbb{G}) \times \mathcal{T}(\mathbb{F})$ is a saddle point if and only if for all $(\gamma, \tau) \in \mathcal{T}(\mathbb{G}) \times \mathcal{T}(\mathbb{F})$ we have,

$$\mathbb{E}\left[R(\bar\gamma, \tau)|\,\mathcal{G}_0\right] \leq \mathbb{E}\left[R(\bar\gamma, \bar\tau)|\,\mathcal{G}_0\right] \leq \mathbb{E}\left[R(\gamma, \bar\tau)|\,\mathcal{G}_0\right]. \tag{3.4}$$

The conditioning on $\mathcal{G}_0$ in (3.3a) and (3.3b) entails that this extra information is reachable and ready at the beginning of trading (which of course could be obtained for a certain price). But the less informed player does not know it, or does not care about this additional information as it never affects his own strategy, or maybe he is not allowed to use it. This is the case for example if A and B are traders, B has previously worked for the same firm as A but now worked for a firm in competition. The condition $\tau \in \mathcal{T}(\mathbb{F})$ in the definition of $\overline{V}^G$ and $\underline{V}^G$ implies the ordinary player's ignorance or passivity to the additional information.

The condition $\tau \in \mathcal{T}(\mathbb{F})$ in the definition of $\overline{V}^G$ and $\underline{V}^G$ plays the role of a constraint imposed on the ordinary player. Due to this restriction, it is not obvious if $\overline{V}^G = \underline{V}^G$ for an arbitrary random variable G.

In the sequel, we will show that there is a value for the Dynkin game with criteria (3.2). We will also provide a sufficient condition for the existence of a saddle point. We will deal with the first issue in this section, while the second one is treated in the following section.

3.2. **Existence of a value.** In this section, we derive a formula representing the value function of a Dynkin game with asymmetric information. Our main idea is to look for a suitable factorization of $\mathbb{G}-$stopping times in terms of $\mathbb{F}-$stopping times, and then reduce the problem to a Dynkin game without information asymmetry. The following result gives a complete characterization of $\mathbb{G}-$stopping times. It is a consequence of [10, Proposition 3.3].

Proposition 3.4. *Under Assumption 3.1, γ is a $\mathbb{G}-$stopping time if and only if it is of the form*

$$\gamma = \gamma^a 1_{\{G=a\}} + \gamma^b 1_{\{G=b\}}$$

where γ^a and γ^b are $\mathbb{F}-$stopping times.

The random variables $\overline{V}^G$ and $\underline{V}^G$ are $\mathcal{G}_0-$measurable. Due to the structure of $\mathcal{G}_0$, there exists real numbers $\overline{V}(a), \overline{V}(b), \underline{V}(a)$ and $\underline{V}(b)$ such that

$$\begin{aligned} \overline{V}^G = &\ \overline{V}(a) 1_{\{G=a\}} + \overline{V}(b) 1_{\{G=b\}}, \\ \underline{V}^G = &\ \underline{V}(a) 1_{\{G=a\}} + \underline{V}(b) 1_{\{G=b\}}. \end{aligned}$$

To show that $\underline{V}^G = \overline{V}^G$, it suffices to show that $\overline{V}(a) = \underline{V}(a)$ and $\overline{V}(b) = \underline{V}(b)$. Intuitively, we expect the latter equations to hold. This can be seen as follows: Given $(\gamma, \tau) \in \mathcal{T}(\mathbb{G}) \times \mathcal{T}(\mathbb{F})$, $\mathbb{E}\left[R(\gamma,\tau) \middle| \mathcal{G}_0\right]$ is a linear combination of the functions $1_{\{G=a\}}$ and $1_{\{G=b\}}$ due to Remark 3.3 and Proposition 3.4. Moreover, the coefficients of the linear combination depend only on $\mathbb{F}-$stopping times. Since $\{G=a\}$ and $\{G=b\}$ are disjoint, $\overline{V}(u), \underline{V}(u), u \in \{a,b\}$, should be the values of the Dynkin game with payoff given by the coefficients of $\mathbb{E}\left[R(\gamma,\tau) \middle| \mathcal{G}_0\right]$ on $\{G=u\}, u \in \{a,b\}$. This intuition will be made precise in Theorem 3.9.

We consider the function $h : \mathcal{T}(\mathbb{F}) \times \mathcal{T}(\mathbb{F}) \times \{a,b\} \to \mathbb{R}$ where for $(\gamma,\tau,u) \in \mathcal{T}(\mathbb{F}) \times \mathcal{T}(\mathbb{F}) \times \{a,b\}$ we have

$$h(\gamma,\tau,u) = \mathbb{E}\left[U_\gamma \alpha_\gamma(u) 1_{\{\gamma<\tau\}} + L_\tau \alpha_\tau(u) 1_{\{\tau\le\gamma\}}\right]. \tag{3.5}$$

The following simple lemma is a consequence of the density hypothesis (see [6] p. 5.).

Lemma 3.5. *We suppose that Assumption 3.1 holds. Let* $\gamma = \gamma^a 1_{\{G=a\}} + \gamma^b 1_{\{G=b\}}, \sigma = \sigma^a 1_{\{G=a\}} + \sigma^b 1_{\{G=b\}} \in \mathcal{T}(\mathbb{G})$. *We have*

$$\mathbb{E}\left[R(\gamma,\sigma)\middle|\mathcal{G}_0\right] = \frac{h(\gamma^a,\sigma^a,a)}{\alpha_0(a)}1_{\{G=a\}} + \frac{h(\gamma^b,\sigma^b,b)}{\alpha_0(b)}1_{\{G=b\}}. \tag{3.6}$$

Proof. Since $\mathcal{G}_0 = \sigma(G), \mathbb{E}\left[R(\gamma,\sigma) \mid \mathcal{G}_0\right] = l(a)1_{\{G=a\}} + l(b)1_{\{G=b\}}$ with $l(a)$ and $l(b)$ real numbers. As $\{G=a\}$ and $\{G=b\}$ are disjoint, we have

$$\begin{aligned}\{\sigma \leq \gamma\} = &\quad \{\sigma^a \leq \gamma^a\} \cap \{G=a\} \bigcup \{\sigma^b \leq \gamma^b\} \cap \{G=b\},\\ \{\sigma < \gamma\} = &\quad \{\sigma^a < \gamma^a\} \cap \{G=a\} \bigcup \{\sigma^b < \gamma^b\} \cap \{G=b\}.\end{aligned}$$

We deduce that

$$R(\gamma,\sigma) = R(\gamma^a,\sigma^a)1_{\{G=a\}} + R(\gamma^b,\sigma^b)1_{\{G=b\}}.$$

Hence $l(u) = \frac{\mathbb{E}\left[R(\gamma^u,\sigma^u)1_{\{G=u\}}\right]}{\mathbb{P}(G=u)} = \frac{h(\gamma^u,\sigma^u,u)}{\alpha_0(u)}, u \in \{a,b\}$. □

Using Lemma 3.5, we can prove the following simple equation.

Proposition 3.6. *We suppose that Assumptions 2.1 and 3.1 hold. Let* $\gamma = \gamma^a 1_{\{G=a\}} + \gamma^b 1_{\{G=b\}} \in \mathcal{T}(\mathbb{G})$. *Then we have*

$$\mathcal{K} := \operatorname*{ess\,sup}_{\tau\in\mathcal{T}(\mathbb{F})} \mathbb{E}\left[R(\gamma,\tau)|\mathcal{G}_0\right] = \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}(\mathbb{G})} \mathbb{E}\left[R(\gamma,\sigma)|\mathcal{G}_0\right]. \tag{3.7}$$

Proof. Since $\mathcal{T}(\mathbb{F}) \subseteq \mathcal{T}(\mathbb{G})$, we have

$$\mathcal{K} = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}(\mathbb{F})} \mathbb{E}\left[R(\gamma,\tau)|\mathcal{G}_0\right] \leq \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}(\mathbb{G})} \mathbb{E}\left[R(\gamma,\sigma)|\mathcal{G}_0\right]. \tag{3.8}$$

By Lemma 3.5, we have

$$\mathcal{K} = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}(\mathbb{F})} \left(\frac{h(\gamma^a,\tau,a)}{\alpha_0(a)}1_{\{G=a\}} + \frac{h(\gamma^b,\tau,b)}{\alpha_0(b)}1_{\{G=b\}}\right). \tag{3.9}$$

For $u \in \{a,b\}$, the family $\left\{\frac{h(\gamma^u,\tau,u)}{\alpha_0(u)}1_{\{G=u\}}, \tau \in \mathcal{T}(\mathbb{F})\right\}$ is stable under the operation of taking the supremum. Since the sets $\{G=a\}$ and $\{G=b\}$ are disjoint, we have

$$\mathcal{K} = \left(\sup_{\tau\in\mathcal{T}(\mathbb{F})} \frac{h(\gamma^a,\tau,a)}{\alpha_0(a)}\right)1_{\{G=a\}} + \left(\sup_{\tau\in\mathcal{T}(\mathbb{F})} \frac{h(\gamma^b,\tau,b)}{\alpha_0(b)}\right)1_{\{G=b\}}. \tag{3.10}$$

Noting that $\mathcal{T}(\mathbb{G})$ can be identified with $\mathcal{T}(\mathbb{F}) \times \mathcal{T}(\mathbb{F})$, we deduce from Lemma 3.5 and the properties of ess sup that

$$\begin{aligned} \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}(\mathbb{G})} \mathbb{E}\left[R(\gamma,\sigma)|\mathcal{G}_0\right] \leq & \left(\sup_{\tau\in\mathcal{T}(\mathbb{F})} \frac{h(\gamma^a,\tau,a)}{\alpha_0(a)}\right) 1_{\{G=a\}} \\ + & \left(\sup_{\tau\in\mathcal{T}(\mathbb{F})} \frac{h(\gamma^b,\tau,b)}{\alpha_0(b)}\right) 1_{\{G=b\}}. \end{aligned} \tag{3.11}$$

Thus (3.7) holds. □

Remark 3.7. Let $\gamma = \gamma^a 1_{\{G=a\}} + \gamma^b 1_{\{G=b\}} \in \mathcal{T}(\mathbb{G})$. We have from Lemma 5 in [19] that

$$\mathcal{K} = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}(\mathbb{G})} \mathbb{E}\left[\left(U_\gamma 1_{\{\gamma\leq\tau\}} + L_\tau 1_{\{\tau<\gamma\}}\right)|\mathcal{G}_0\right]. \tag{3.12}$$

Thus $\mathcal{K}$ is the value of a classical optimal stopping problem with the random terminal time γ. Equation (3.7) shows that given $\mathcal{G}_0$, the value of the optimal stopping problem (3.12) is unchanged if one uses only $\mathbb{F}-$stopping times. Due to (3.10), the value $\mathcal{K}$ can be achieved by an $\mathbb{F}-$stopping time $\bar{\tau}$ if and only if

$$\sup_{\tau\in\mathcal{T}(\mathbb{F})} h(\gamma^u,\tau,u) = h(\gamma^u,\bar{\tau},u),\ u\in\{a,b\}. \tag{3.13}$$

This is the case if G is independent of $\mathbb{F}$ since then $\alpha(u)$ is constant for each $u \in \{a,b\}$.

From the above proposition, we deduce the following corollary.

Corollary 3.8. *We suppose that Assumption 2.1 and 3.1 hold. Then*

$$\begin{aligned} V^{*G} &:= \operatorname*{ess\,inf}_{\gamma\in\mathcal{T}(\mathbb{G})} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}(\mathbb{G})} \mathbb{E}\left[R(\gamma,\tau)\ |\mathcal{G}_0\right] \\ &= \operatorname*{ess\,inf}_{\gamma\in\mathcal{T}(\mathbb{G})} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}(\mathbb{F})} \mathbb{E}\left[R(\gamma,\tau)\ |\mathcal{G}_0\right] = \overline{V}^G. \end{aligned}$$

Proof. For every $\gamma \in \mathcal{T}(\mathbb{G})$, we know from Proposition 3.6 that

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{T}(\mathbb{F})} \mathbb{E}\left[R(\gamma,\tau)\ |\mathcal{G}_0\right] = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}(\mathbb{G})} \mathbb{E}\left[R(\gamma,\tau)\ |\mathcal{G}_0\right].$$

The assertion then follows by taking ess inf on both sides of the above equation. □

Before stating our main result, we introduce some auxiliary functions. For $u \in \{a,b\}$ let

$$\begin{aligned} V(u) = & \inf_{\gamma\in\mathcal{T}(\mathbb{F})} \sup_{\tau\in\mathcal{T}(\mathbb{F})} \mathbb{E}\left[U_\gamma \alpha_\gamma(u) 1_{\{\gamma<\tau\}} + L_\tau \alpha_\tau(u) 1_{\{\tau\leq\gamma\}}\right] \\ = & \sup_{\tau\in\mathcal{T}(\mathbb{F})} \inf_{\gamma\in\mathcal{T}(\mathbb{F})} \mathbb{E}\left[U_\gamma \alpha_\gamma(u) 1_{\{\gamma<\tau\}} + L_\tau \alpha_\tau(u) 1_{\{\tau\leq\gamma\}}\right]. \end{aligned} \tag{3.14}$$

The possibility to interchange inf and sup in (3.14) is due to Theorem 2.6. $V(u), u \in \{a,b\}$, is the value of a classical Dynkin game in the reference filtration $\mathbb{F}$. It is the value of the game obtained by weighing the payoff at each time $t \in [0,T]$ with the conditional probability of G given $\mathcal{F}_t$. Using the function h defined in (3.5), we can rewrite $V(u), u \in \{a,b\}$, as follows:

$$V(u) = \inf_{\gamma \in \mathcal{T}(\mathbb{F})} \sup_{\tau \in \mathcal{T}(\mathbb{F})} h(\gamma,\tau,u) = \sup_{\tau \in \mathcal{T}(\mathbb{F})} \inf_{\gamma \in \mathcal{T}(\mathbb{F})} h(\gamma,\tau,u). \tag{3.15}$$

The following theorem is the main result of this work.

Theorem 3.9. *We suppose that Assumptions 2.1 and 3.1 hold. The Dynkin game with objective (3.2) has the value* $V^G := \overline{V}^G = \underline{V}^G = V^{*G}$. *Moreover, we have the following decomposition formula:*

$$V^G = \frac{V(a)}{\alpha_0(a)} 1_{\{G=a\}} + \frac{V(b)}{\alpha_0(b)} 1_{\{G=b\}}. \tag{3.16}$$

Proof. From Corollary 3.8, we have $\overline{V}^G = V^{*G}$. We have that $\overline{V}^G \geq \underline{V}^G$. To show the reverse inequality, we will show that

$$\underline{V}^G \geq \frac{V(a)}{\alpha_0(a)} 1_{\{G=a\}} + \frac{V(b)}{\alpha_0(b)} 1_{\{G=b\}} \geq \overline{V}^G.$$

We shall first show that $\underline{V}^G \geq \frac{V(a)}{\alpha_0(a)} 1_{\{G=a\}} + \frac{V(b)}{\alpha_0(b)} 1_{\{G=b\}}$. For this purpose, let $\tau \in \mathcal{T}(\mathbb{F})$ and $\gamma = \gamma^a 1_{\{G=a\}} + \gamma^b 1_{\{G=b\}} \in \mathcal{T}(\mathbb{G})$. We recall that $R(\gamma,\tau) = R(\gamma^a,\tau)1_{\{G=a\}} + R(\gamma^b,\tau)1_{\{G=b\}}$. Moreover, we have

$$\begin{aligned}
\mathbb{E}\left[R(\gamma,\tau) \mid \mathcal{G}_0\right] &= \frac{h(\gamma^a,\tau,a)}{\alpha_0(a)} 1_{\{G=a\}} + \frac{h(\gamma^b,\tau,b)}{\alpha_0(b)} 1_{\{G=b\}} \\
&\geq \inf_{\delta \in \mathcal{T}(\mathbb{F})} \frac{h(\delta,\tau,a)}{\alpha_0(a)} 1_{\{G=a\}} + \inf_{\delta \in \mathcal{T}(\mathbb{F})} \frac{h(\delta,\tau,b)}{\alpha_0(b)} 1_{\{G=b\}} \\
&= \frac{J_\tau(a)}{\alpha_0(a)} 1_{\{G=a\}} + \frac{J_\tau(b)}{\alpha_0(b)} 1_{\{G=b\}},
\end{aligned}$$

where $J_\tau(u) = \inf_{\delta \in \mathcal{T}(\mathbb{F})} h(\delta,\tau,u), u \in \{a,b\}$. With the above estimate, we deduce that

$$\operatorname*{ess\,inf}_{\gamma \in \mathcal{T}(\mathbb{G})} \mathbb{E}\left[R(\gamma,\tau) \mid \mathcal{G}_0\right] \geq \frac{J_\tau(a)}{\alpha_0(a)} 1_{\{G=a\}} + \frac{J_\tau(b)}{\alpha_0(b)} 1_{\{G=b\}}.$$

Hence,

$$\begin{aligned}
\underline{V}^G &\geq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}(\mathbb{F})} \left(\frac{J_\tau(a)}{\alpha_0(a)} 1_{\{G=a\}} + \frac{J_\tau(b)}{\alpha_0(b)} 1_{\{G=b\}} \right) \\
&= \frac{1}{\alpha_0(a)} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}(\mathbb{F})} J_\tau(a) 1_{\{G=a\}} + \frac{1}{\alpha_0(b)} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}(\mathbb{F})} J_\tau(b) 1_{\{G=b\}} \\
&= \frac{1}{\alpha_0(a)} 1_{\{G=a\}} \sup_{\tau \in \mathcal{T}(\mathbb{F})} J_\tau(a) + \frac{1}{\alpha_0(b)} 1_{\{G=b\}} \sup_{\tau \in \mathcal{T}(\mathbb{F})} J_\tau(b) \\
&= \frac{V(a)}{\alpha_0(a)} 1_{\{G=a\}} + \frac{V(b)}{\alpha_0(b)} 1_{\{G=b\}}.
\end{aligned}$$

Let us next prove that $\overline{V}^G \le \frac{V(a)}{\alpha_0(a)} 1_{\{G=a\}} + \frac{V(b)}{\alpha_0(b)} 1_{\{G=b\}}$. In fact, let $\gamma = \gamma^a 1_{\{G=a\}} + \gamma^b 1_{\{G=b\}} \in \mathcal{T}(\mathbb{G})$. By Proposition 3.6 and (3.10), we have

$$\begin{aligned} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}(\mathbb{F})} \mathbb{E}\left[R(\gamma,\tau) \middle| \mathcal{G}_0 \right] \le & \left(\sup_{\tau \in \mathcal{T}(\mathbb{F})} \frac{h(\gamma^a, \tau, a)}{\alpha_0(a)} \right) 1_{\{G=a\}} \\ & + \left(\sup_{\tau \in \mathcal{T}(\mathbb{F})} \frac{h(\gamma^b, \tau, b)}{\alpha_0(b)} \right) 1_{\{G=b\}}. \end{aligned} \tag{3.17}$$

Using once more the fact that $\{G = a\}$ and $\{G = b\}$ are disjoint, the correspondence between $\mathcal{T}(\mathbb{G})$ and $\mathcal{T}(\mathbb{F}) \times \mathcal{T}(\mathbb{F})$, the definition of $\overline{V}^G$ and $V(u), u \in \{a, b\}$, we obtain

$$\overline{V}^G \le \frac{V(a)}{\alpha_0(a)} 1_{\{G=a\}} + \frac{V(b)}{\alpha_0(b)} 1_{\{G=b\}}. \tag{3.18}$$

We deduce that $\underline{V}^G = \overline{V}^G$. □

Corollary 3.8 together with Theorem 3.9 show that in case that one party has initial additional information modeled by a discrete random variable, the Dynkin game with the criteria (3.2) admits a value. Moreover, this value coincides with the value of the Dynkin game in which both players exploit full information. The difference between this value and the value of the same Dynkin game without additional information can be considered as the price of this extra information paid by the informed party to obtain it. For game contingent claims, the seller has to select a stopping time $\gamma \in \mathcal{T}_{0,T}(\mathbb{F})$ at which he exercises his claim in such a way that the expected payoff is minimized. If he has privileged information, he has access to a larger set of exercise times leading to a lower expected payoff. In the sequel, we show that this price is positive on average. To investigate this, we denote the cost of the extra information with CEI, and define more formally

Definition 3.10.

$$CEI := V_0 - V^G = \inf_{\gamma \in \mathcal{T}(\mathbb{F})} \sup_{\tau \in \mathcal{T}(\mathbb{F})} \mathbb{E}[R(\gamma,\tau)] - \operatorname*{ess\,inf}_{\gamma \in \mathcal{T}(\mathbb{G})} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}(\mathbb{F})} \mathbb{E}\left[R(\gamma,\tau) \ | \mathcal{G}_0 \right],$$

Proposition 3.11. *We suppose that Assumptions 2.1 and 3.1 hold. Furthermore, we assume $(\bar\gamma, \bar\tau) \in \mathcal{T}(\mathbb{G}) \times \mathcal{T}(\mathbb{F})$ is a saddle point for the Dynkin game with asymmetric information. Then we have $\mathbb{E}[CEI] \ge 0$.*

Proof. We have

$$\begin{aligned} CEI &= \left(\inf_{\gamma \in \mathcal{T}(\mathbb{F})} \sup_{\tau \in \mathcal{T}(\mathbb{F})} \mathbb{E}[R(\gamma,\tau)] - \inf_{\gamma \in \mathcal{T}(\mathbb{G})} \sup_{\tau \in \mathcal{T}(\mathbb{F})} \mathbb{E}[R(\gamma,\tau)] \right) \\ &+ \left(\inf_{\gamma \in \mathcal{T}(\mathbb{G})} \sup_{\tau \in \mathcal{T}(\mathbb{F})} \mathbb{E}[R(\gamma,\tau)] - \operatorname*{ess\,inf}_{\gamma \in \mathcal{T}(\mathbb{G})} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}(\mathbb{F})} \mathbb{E}\left[R(\gamma,\tau) \ | \mathcal{G}_0 \right] \right). \end{aligned}$$

The first expression is a non-negative random variable. Using the fact $(\bar{\gamma},\bar{\tau}) \in \mathcal{T}(\mathbb{G})\times\mathcal{T}(\mathbb{F})$ is a saddle point for the Dynkin game with asymmetric information, we prove that the expectation of the second expression is also positive and thus $\mathbb{E}[CEI]$ is a positive quantity. We have

$$V^G = \operatorname*{ess\,inf}_{\gamma\in\mathcal{T}(\mathbb{G})} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}(\mathbb{F})} \mathbb{E}\left[R(\gamma,\tau)\ |\mathcal{G}_0\right] = \mathbb{E}\left[R(\bar{\gamma},\bar{\tau})\ |\mathcal{G}_0\right] \le \mathbb{E}\left[R(\gamma,\bar{\tau})\ |\mathcal{G}_0\right].$$

By the tower property of the conditional expectation we have

$$\mathbb{E}(V^G) \le \mathbb{E}\left[R(\gamma,\bar{\tau})\right] \le \sup_{\tau\in\mathcal{T}(\mathbb{F})} \mathbb{E}[R(\gamma,\tau)],$$

Therefore we obtain that $\mathbb{E}(V^G) \le \inf_{\gamma\in\mathcal{T}(\mathbb{G})}\sup_{\tau\in\mathcal{T}(\mathbb{F})}\mathbb{E}[R(\gamma,\tau)]$ and then $\mathbb{E}(V^G) \le \mathbb{E}(V_0)$. □

4. **Existence of saddle points**

We now turn to the existence of saddle points. The following proposition gives a necessary and sufficient condition to characterize a saddle point.

Proposition 4.1. *Let $(\bar{\gamma},\bar{\tau}) \in \mathcal{T}(\mathbb{G})\times\mathcal{T}(\mathbb{F})$ with $\bar{\gamma} = \bar{\gamma}^a 1_{\{G=a\}} + \bar{\gamma}^b 1_{\{G=b\}}$. Then $(\bar{\gamma},\bar{\tau})$ is a saddle point for the Dynkin game with objective (3.2) if and only if for $u\in\{a,b\}$, $(\bar{\gamma}^u,\bar{\tau})$ is a saddle point for (3.14).*

Proof. Let $(\bar{\gamma},\bar{\tau}) \in \mathcal{T}(\mathbb{G})\times\mathcal{T}(\mathbb{F})$. Then $(\bar{\gamma},\bar{\tau})$ is a saddle point if and only if for any $(\gamma,\tau)\in\mathcal{T}(\mathbb{G})\times\mathcal{T}(\mathbb{F})$, we have

$$\mathbb{E}\left[R(\bar{\gamma},\tau)\middle|\mathcal{G}_0\right] \le \mathbb{E}\left[R(\bar{\gamma},\bar{\tau})\middle|\mathcal{G}_0\right] \le \mathbb{E}\left[R(\gamma,\bar{\tau})\middle|\mathcal{G}_0\right]. \tag{4.1}$$

Using Lemma 3.5, we see that (4.1) is equivalent to the following inequality for $u\in\{a,b\}$

$$h(\bar{\gamma}^u,\tau,u) \le h(\bar{\gamma}^u,\bar{\tau},u) \le h(\gamma^u,\bar{\tau},u),\ \forall\,(\gamma^u,\tau)\in\mathcal{T}(\mathbb{F})\times\mathcal{T}(\mathbb{F}). \tag{4.2}$$

By definition of $V(u), u\in\{a,b\}$, the above inequality is equivalent to : $(\bar{\gamma}^u,\bar{\tau})$ is a saddle point for (3.14). The proof is complete. □

Proposition 4.1 yields the following corollary which provides a sufficient condition for the existence of a saddle point.

Corollary 4.2. *Assume that G is independent of $\mathbb{F}$, L and $-U$ are LUSCE. Then a saddle point $(\bar{\gamma},\bar{\tau})$ for the game with objective (3.2) exists.*

Proof. As G is independent of $\mathbb{F}$, $(t,u)\mapsto\alpha_t(u)$ is constant in t for each $u\in\{a,b\}$. From the representation (3.14), we deduce that

$$\frac{V(a)}{\alpha_0(a)} = \frac{V(b)}{\alpha_0(b)} = \inf_{\gamma\in\mathcal{T}(\mathbb{F})}\sup_{\tau\in\mathcal{T}(\mathbb{F})} \mathbb{E}\left[U_\gamma 1_{\{\gamma<\tau\}} + L_\tau 1_{\{\tau\le\gamma\}}\right]. \tag{4.3}$$

Since L and $-U$ are LUSCE, we infer from Theorem 2.6 that the Dynkin game with value (4.3) admits a saddle point $(\bar{\gamma},\bar{\tau})\in\mathcal{T}(\mathbb{F})\times\mathcal{T}(\mathbb{F})$. As $\alpha_t(u)$ is constant in t for each $u\in\{a,b\}$, we deduce that $(\bar{\gamma},\bar{\tau})$ is a saddle point for (3.14). By Proposition 4.1, we also have a saddle point for the game with objective (3.2).

□

5. **A simple case**

5.1. **Explicit example.** A well-known example of a game contingent claim is a callable put which is a put option whose seller (issuer) can terminate it before its excersie time for a certain penalty $\delta = (\delta_t)_{0\le t\le T}$ where δ is a positive process. In [**?**], the callable put is characterised as a composite exotic option, and the value function is studied. Let $S = (S_t)_{t\in[0,T]}$ denote the price process of the underlying asset and K the strike price. Then a callable (game) put option has the payoff function $R(\gamma,\tau)$ in (2.5) where $L = (S-K)^+$ and $U = (S-K)^+ + \delta$. In other words, assuming that the seller (issuer) chooses to terminate the contract at a stopping time $\gamma \in \mathcal{T}(\mathbb{F})$ and the buyer (holder) exercises his option at $\tau \in \mathcal{T}(\mathbb{F})$, at time $\gamma\wedge\tau$, the seller pays to the buyer an amount $R(\gamma,\tau)$.

Now suppose that the asset price process S is given by the SDE $dS_t = \mu S_t dt + \sigma S_t dB_t$, $t \in [0,T]$, where μ is the drift and $\sigma > 0$ the volatility. Then S is a geometric Brownian motion given by $S_t = S_0 e^{\sigma B_t + (\mu - \frac{1}{2}\sigma^2)t}, t \in [0,T]$. Moreover, assume that $\delta \in S(\mathbb{F})$ is such that $\delta_t > 0$ for $t \in [0,T)$ and $\delta_T = 0$. Then, since $e^{\sigma B}$ is a continuous function, and $\mathbb{E}(e^{\sigma B_t}) = e^{\frac{1}{2}t^2\sigma^2} < \infty,\ t \in [0,T]$, we have $L, U \in S(\mathbb{F})$ and the processes satisfy Assumption 2.1.

As a special case, consider the situation in which G contains the information on whether the endpoint of a one-dimensional $\mathbb{F}-$Brownian motion lies in some given interval, i.e., $G := 1_{\{B_T\in[a,b]\}}$ for some $a < b$. If the interval $[a,b]$ is given such that $0 < \mathbb{P}(B_T \in [a,b]) < 1$, then from Theorem 3.9, the game call option with asymmetric information which is modeled by G has value $V^G = \overline{V}^G = \underline{V}^G$ and we have the following decomposition formula for V^G

$$V^G = \frac{V(0)}{\alpha_0(0)} 1_{\{G=0\}} + \frac{V(1)}{\alpha_0(1)} 1_{\{G=1\}}, \tag{5.1}$$

where

$$\alpha_0(0) = \mathbb{P}(B_T \notin [a,b]), \quad \alpha_0(1) = \mathbb{P}(B_T \in [a,b]),$$

and for $u \in \{0,1\}$, $V(u)$ is given by the following equations from the definition of α_t,

$$\begin{aligned} V(0) &= \inf_{\gamma\in\mathcal{T}(\mathbb{F})} \sup_{\tau\in\mathcal{T}(\mathbb{F})} \mathbb{E}\left[R(\gamma,\tau)1_{\{B_T\notin[a,b]\}}\right] \\ &= \sup_{\tau\in\mathcal{T}(\mathbb{F})} \inf_{\gamma\in\mathcal{T}(\mathbb{F})} \mathbb{E}\left[R(\gamma,\tau)1_{\{B_T\notin[a,b]\}}\right] \end{aligned}$$

and

$$V(1) = \inf_{\gamma \in \mathcal{T}(\mathbb{F})} \sup_{\tau \in \mathcal{T}(\mathbb{F})} \mathbb{E}\left[R(\gamma,\tau)1_{\{B_T \in [a,b]\}}\right]$$
$$= \sup_{\tau \in \mathcal{T}(\mathbb{F})} \inf_{\gamma \in \mathcal{T}(\mathbb{F})} \mathbb{E}\left[R(\gamma,\tau)1_{\{B_T \in [a,b]\}}\right].$$

5.2. **Numerical illustration.** To illustrate the results numerically, we consider 400 time steps and fix the parameters by $\mu = 0.14, \sigma = 0.4, K = 100, T = 0.5, \delta = 5$, as is considered in [?] in which a pathwise GCC pricing algorithm is presented, and the callable put is studied as one example. These values will serve to describe a benchmark against which the values of the Dynkin game without extra information are compared. With this choice, the values are given in Figure 1.

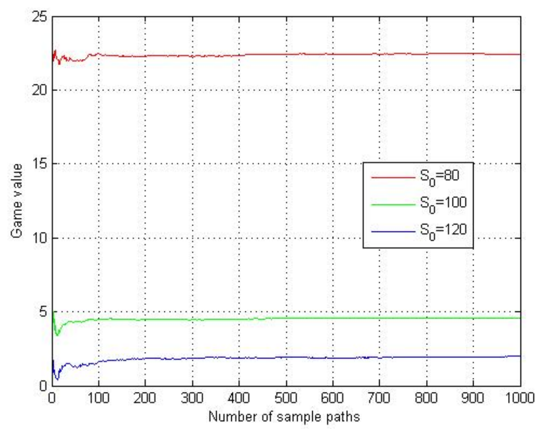


Figure 1. Value of the Dynkin game without extra information for $S_0 = 80, 100, 120$.

We also investigate the sensitivity of this value with respect to the penalty that the seller has to pay for early exercise. The resulting graph (Figure 2) shows that the value of the game is an increasing function of the penalty. Moreover, we observe that whenever the penalty exceeds a threshold value (over $\delta = 40$ in this simulation), the game degenerates to a simple American option since the seller will never exercise. Moreover, we simulate the value functions of this classic Dynkin game for different initial prices $S_0 = 80, 100, 120$ and 3000 sample paths. The graphs are presented in Figure 3.

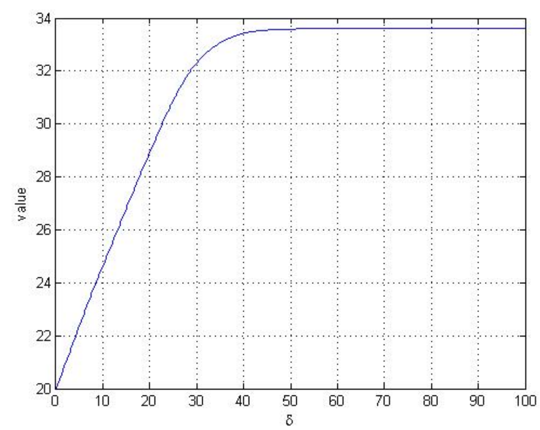


FIGURE 2. Value of the Dynkin game without extra information for $S_0 = 80$ when the penalty of the early excercise increases

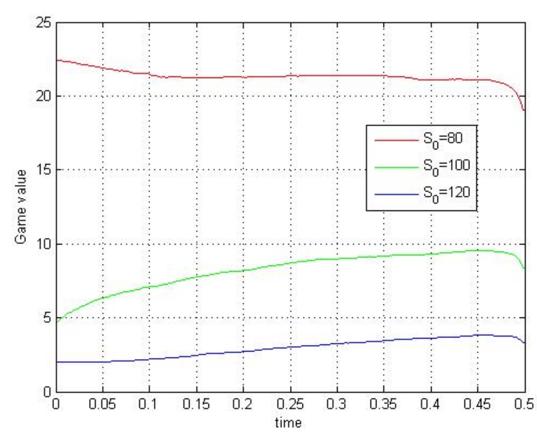
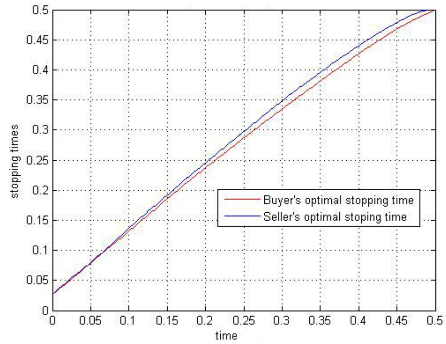
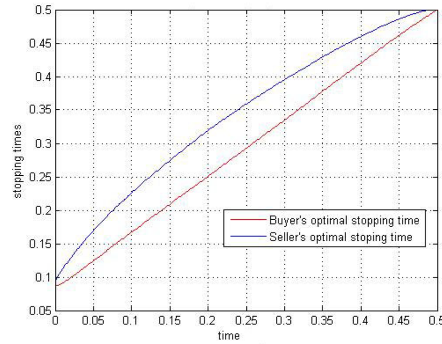


FIGURE 3. Value function of the Dynkin game without extra information.

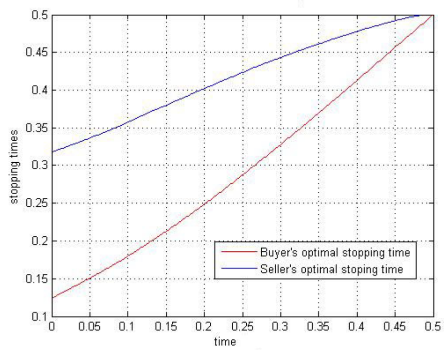
In the sequel, we ran 3000 simulations to estimate an average of the optimal stopping times for this classical Dynkin game. To achieve this, we chose different initial prices. The resulting graphs (Figure 4) show that if the callable put option is out of money, the seller does not try to exercise it before the buyer.



(A) $S_0 = 80$



(B) $S_0 = 100$



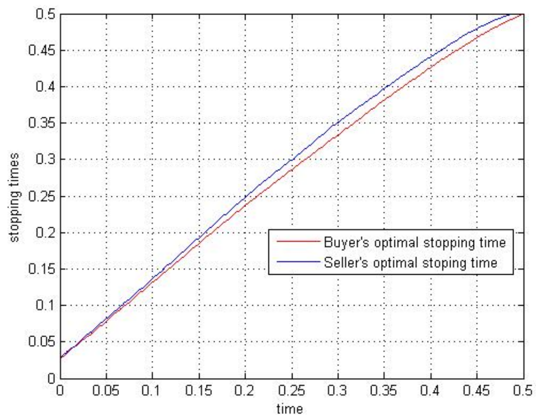
(C) $S_0 = 120$

FIGURE 4. Optimal stopping times for different initial prices

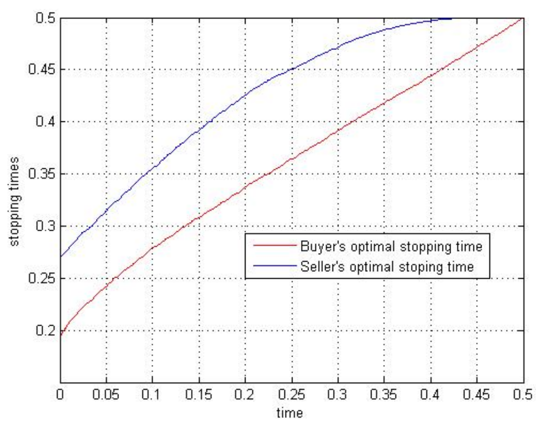
Furthermore, we study the effect of increasing the penalty of early exercise on the optimal stopping times. The results are presented in Figure 5. For this, we fix $S_0 = 80$ and simulate the optimal stopping times for $\delta = 0.001, 5, 20$. Again, the number of sample paths is 3000. The result indicates that if the penalty increases then the optimal stopping time for the seller is significantly smaller than the optimal stopping time for the buyer.

Finally, we obtain numerical results for simulation of the callable put option in the case that the seller knows whether B_T is in $[0, 1]$ or not. The value of the game with asymmetric information is calculated by (5.1). Figure 6 is a comparison between the value of the game with and without this extra information for initial prices $S_0 = 80, 100, 120$. The graph shows that the value of the game with extra information for the seller is lower than the value without this information. This is in line with Proposition 3.11 where we observe that the inequality $V_0 = \mathbb{E}[V_0] \geq \mathbb{E}[V^G]$ holds in this case since according to Proposition 4.1, the Dynkin game with asymmetric information has a saddle point $(\gamma, \tau) \in \mathcal{T}(\mathbb{G}) \times \mathcal{T}(\mathbb{F})$.

(A) $\delta = 0.001$



(B) $\delta = 5$



(C) $\delta = 20$

FIGURE 5. Effect of the penalty of early cancelation on optimal stopping times

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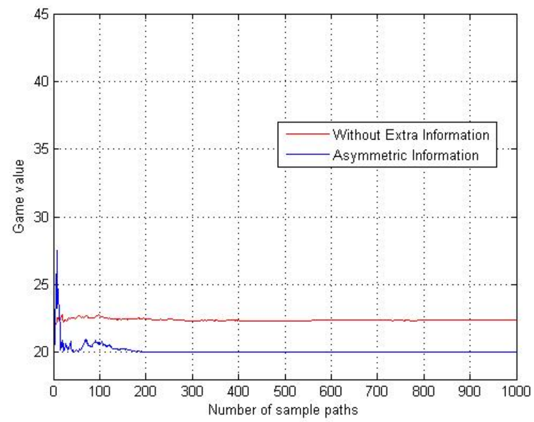
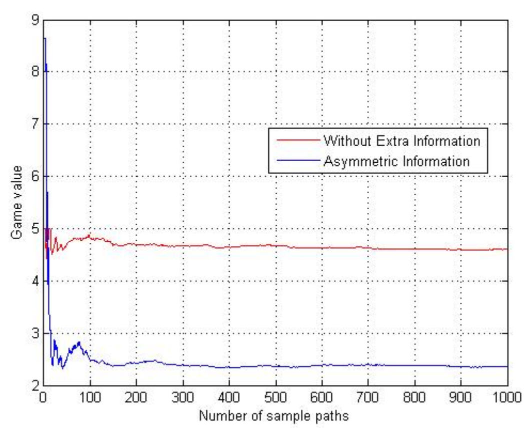
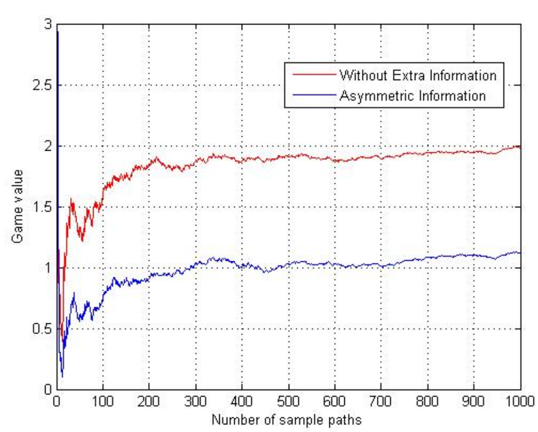


(A) $S_0 = 80$



(B) $S_0 = 100$



(C) $S_0 = 120$

FIGURE 6. Value of the Dynkin game with and without extra information for the seller.

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