On the geometry of some rough paths

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Outline of course

Goal:
investigate geometry of individual (rough) paths, typically trajectories of stochastic processes, by means of techniques from the theory of dynamical systems

- **Lecture 1:** explain Ciesielski's isomorphism between path space and sequence space along series decompositions of paths
- **Lecture 2:** exemplify the use of series decompositions by looking at the Haar-Schauder expansion of Brownian motion
- **Lecture 3:** embed individual paths into dynamical systems and use stability theory to study the paths' geometry

Lecture 1: Ciesielski's isomorphism

Isomorphism between $\mathbb{C}^\alpha = C^\alpha([0,1],\mathbb{R})$, normed by

$$\|f\|_\alpha := \|f\|_\infty + \sup_{0\le s<t\le 1} \frac{|f_{s,t}|}{|t-s|^\alpha}, \quad f_{s,t} := f(t) - f(s),$$

and $\ell^\infty(\mathbb{R})$, based on *Haar functions* $(H_{pm}, p \geq 0, 1 \leq m \leq 2^p)$: $H_{00} \equiv 1$,

$$H_{pm}(t) := \begin{cases} \sqrt{2^p}, & t \in \left[\frac{m-1}{2^p}, \frac{2m-1}{2^{p+1}}\right), \\ -\sqrt{2^p}, & t \in \left[\frac{2m-1}{2^{p+1}}, \frac{m}{2^p}\right), \\ 0, & \text{otherwise.} \end{cases}$$

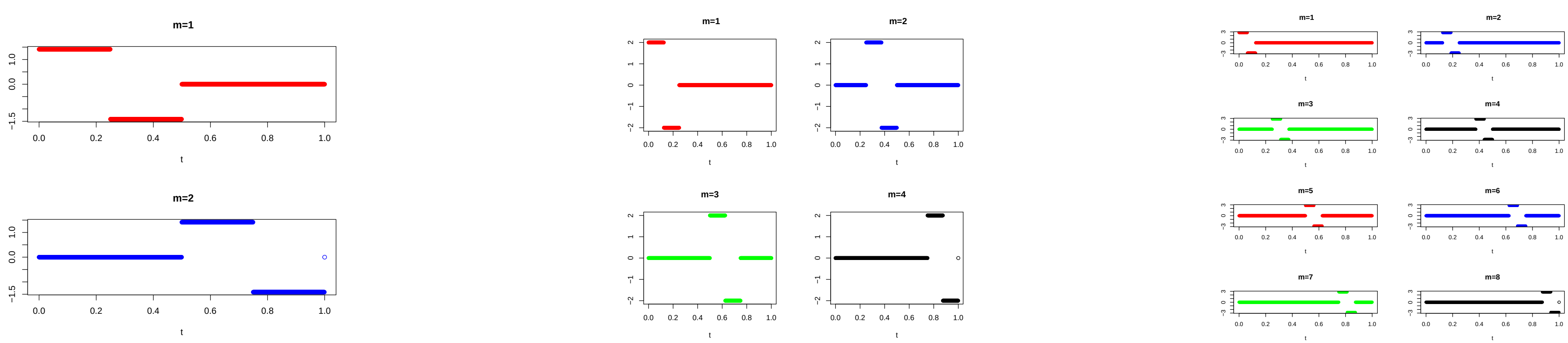

Figure 1: Haar functions: generations $p = 1, 2, 3$

Show: $(H_{pm})_{p\in\mathbb{N},1\leq m\leq 2^p}$ is a CONS in $L^2([0,1])$.
1. **Orthogonality: Case 1:** $p = q, n \neq m$. Then

$$\langle H_{pn}, H_{pm}\rangle = 0,$$

since the supports of the two functions are disjoint, and

$$\langle H_{pn}, H_{pn}\rangle = 2^p\lambda([\frac{m-1}{2^p}, \frac{m}{2^p}]) = 1.$$

Case 2: $p > q$: Then for $1 \leq n \leq 2^q$, $1 \leq m \leq 2^p$, either the supports $[\frac{n-1}{2^q}, \frac{n}{2^q}]$ and $[\frac{m-1}{2^p}, \frac{m}{2^p}]$ are disjoint. Or, $[\frac{m-1}{2^p}, \frac{m}{2^p}] \subset [\frac{n-1}{2^q}, \frac{n}{2^q}]$ and then $H_{qn}|_{[\frac{m-1}{2^p}, \frac{m}{2^p}]} = c$ is constant on the support of H_{pm}, and hence

$$\langle H_{pm}, H_{qn}\rangle = c\int_{\frac{m-1}{2^p}}^{\frac{m}{2^p}} H_{pm}(x)dx = 0.$$

2. **Completeness:** The linear hull of the set of indicator functions of intervals $[\frac{m-1}{2^p}, \frac{m}{2^p}]$ is dense in $L^2([0,1])$ (measure theory). Moreover $\mathbf{1}_{[\frac{m-1}{2^p}, \frac{m}{2^p}]}$ is in the linear hull of $(H_{pm})_{p\geq 0, 1\leq m\leq 2^p}$. To see this, use induction, starting by

$$\begin{aligned} 1_{[0,1]} &= H_{00}, \\ 1_{[0,\frac{1}{2}]} &= \frac{1}{2}(H_{00} + H_{01}), \\ 1_{[\frac{1}{2},1]} &= \frac{1}{2}(H_{00} - H_{01}). \end{aligned}$$

For convenience of notation: $H_{p0} \equiv 0$ for $p \geq 1$.

Primitives of Haar functions, *Schauder functions*: $G_{pm}(t) := \int_0^t H_{pm}(s)\mathrm{d}s$ for $t \in [0,1]$, $p \geq 0$, $0 \leq m \leq 2^p$.

We have $G_{00}(t) = t$ and for $p \geq 1$, $1 \leq m \leq 2^p$

$$G_{pm}(t) = \begin{cases} 2^{p/2}\left(t - \frac{m-1}{2^p}\right), & t \in \left[\frac{m-1}{2^p}, \frac{2m-1}{2^{p+1}}\right), \\ -2^{p/2}\left(t - \frac{m}{2^p}\right), & t \in \left[\frac{2m-1}{2^{p+1}}, \frac{m}{2^p}\right), \\ 0, & \text{otherwise.} \end{cases}$$

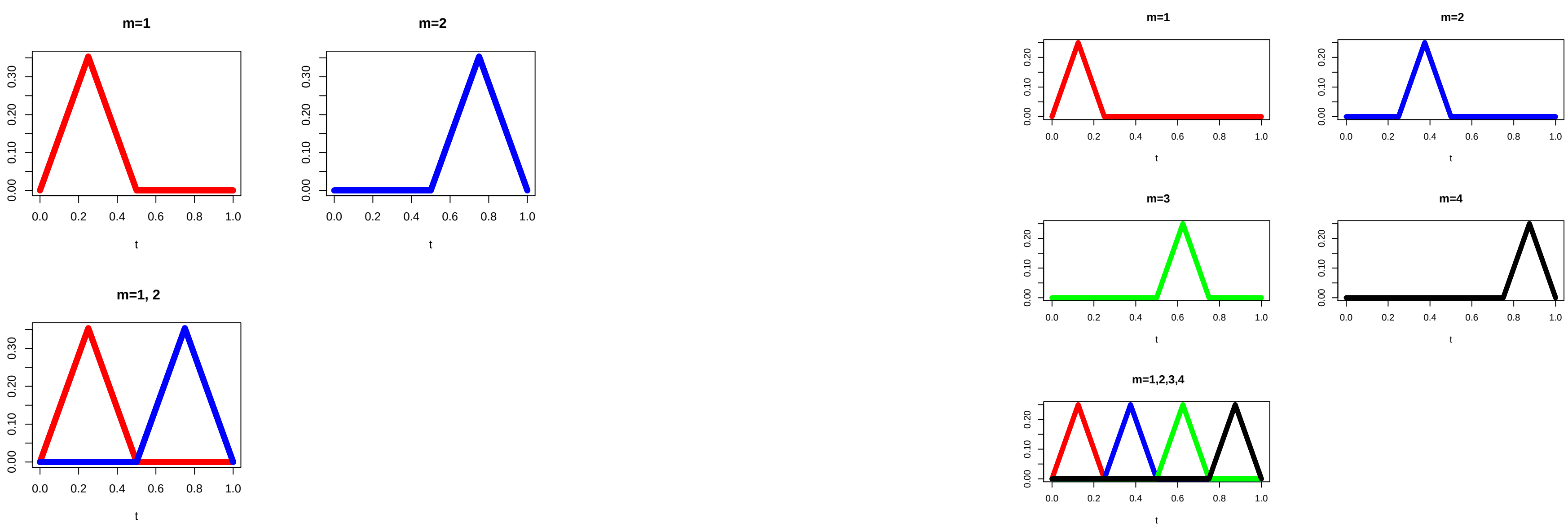

Figure 2: Schauder functions: generations $p = 1, 2$

Further for convenience: $G_{-10}(t) := 1$ for $t \in [0,1]$;

$$t^0_{pm} := \frac{m-1}{2^p}, \quad t^1_{pm} := \frac{2m-1}{2^{p+1}}, \quad t^2_{pm} := \frac{m}{2^p},$$

for $p \geq 1$ and $1 \leq m \leq 2^p$;

$t^0_{-10} := 0$, $t^1_{-10} := 0$, $t^2_{-10} := 1$, and $t^0_{00} := 0$, $t^1_{00} := 1$, $t^2_{00} := 1$, and $t^i_{p0} := 0$ for $p \geq 1$ and $i = 0, 1, 2$; definition for $i = 1$ simplifies statement of Lemma 1.

For $f \in C([0,1], \mathbb{R})$, $p \in \mathbb{N}$, and $1 \leq m \leq 2^p$, write

$$\begin{aligned} \langle H_{pm}, \mathrm{d}f \rangle &:= 2^{\frac{p}{2}} \left[\left(f\left(t^1_{pm}\right) - f\left(t^0_{pm}\right) \right) - \left(f\left(t^2_{pm}\right) - f\left(t^1_{pm}\right) \right) \right] \\ &= 2^{\frac{p}{2}} \left[2f\left(t^1_{pm}\right) - f\left(t^0_{pm}\right) - f\left(t^2_{pm}\right) \right] \end{aligned}$$

and $\langle H_{00}, \mathrm{d}f \rangle := f(1) - f(0)$, $\langle H_{-10}, \mathrm{d}f \rangle := f(0)$. We only defined G_{-10} and not H_{-10}.

Assume $f = f(0) + \int_0^\cdot \dot{f}(s)ds$, with $\dot{f} \in L^2([0,1])$. Then for $p \in \mathbb{N}, 1 \leq m \leq 2^p$

$$\begin{aligned}\langle H_{pm}, \dot{f} \rangle &= 2^{\frac{p}{2}}[\int_{t_{pm}^0}^{t_{pm}^1} \dot{f}(s)ds - \int_{t_{pm}^1}^{t_{pm}^2} \dot{f}(s)ds] \\ &= 2^{\frac{p}{2}}[(f(t_{pm}^1) - f(t_{pm}^0)) - (f(t_{pm}^2) - f(t_{pm}^1))] \\ &= \langle H_{pm}, \mathrm{d}f \rangle.\end{aligned}$$

Since (H_{pm}) is a CONS of $L^2([0,1],\mathbb{R}^d)$, we can further write

$$\dot{f} = \langle H_{00}, \dot{f}\rangle H_{00} + \sum_{p\geq 1, 1\leq m\leq 2^p} \langle H_{pm}, \dot{f}\rangle H_{pm}.$$

Integrating and interchanging limits gives

$$\begin{aligned} f &= f(0) + \int_0^\cdot \dot{f}(s)ds = \langle H_{-10}, df\rangle G_{-10} + \langle H_{00}, df\rangle \int_0^\cdot H_{00}(s)ds \\ &+ \sum_{p\geq 1, 1\leq m\leq 2^p} \langle H_{pm}, \dot{f}\rangle \int_0^\cdot H_{pm}(s)ds \\ &= \langle H_{-10}, df\rangle G_{-10} + \langle H_{00}, df\rangle G_{00} + \sum_{p\geq 1, 1\leq m\leq 2^p} \langle H_{pm}, df\rangle G_{pm}. \end{aligned} \quad (1)$$

The following Lemma shows that this can be generalized to (Hölder) continuous functions.

Lemma 1 ([8]). *1. For $f : [0,1] \to \mathbb{R}$ and $k \in \mathbb{N}$, the function*

$$f_k := \langle H_{-10}, \mathrm{d}f \rangle G_{-10} + \langle H_{00}, \mathrm{d}f \rangle G_{00} + \sum_{p=0}^{k} \sum_{m=1}^{2^p} \langle H_{pm}, \mathrm{d}f \rangle G_{pm}$$

$$= \sum_{p=-1}^{k} \sum_{m=0}^{2^p} \langle H_{pm}, \mathrm{d}f \rangle G_{pm}$$

is the linear interpolation of f between the points $t^1_{-10}, t^1_{00}, t^1_{pm}$, $0 \le p \le k, 1 \le m \le 2^p$.

2. If f is continuous, then $||f_k - f||_\infty \to 0$ as $k \to \infty$.

3. Let $\alpha \in (0,1)$. A continuous function $f : [0,1] \to \mathbb{R}$ is in $\mathbb{C}^\alpha$ if and only if

$$\sup_{p,m} 2^{p(\alpha-1/2)} |\langle H_{pm}, \mathrm{d}f \rangle| < \infty.$$

In this case

$$\sup_{p,m} 2^{p(\alpha-1/2)} |\langle H_{pm}, \mathrm{d}f\rangle| \simeq \|f\|_\alpha \text{ and} \tag{2}$$

$$\|f - f_{k-1}\|_\infty = \Big\| \sum_{p=k}^{\infty} \sum_{m=0}^{2^p} |\langle H_{pm}, \mathrm{d}f\rangle| G_{pm} \Big\|_\infty \lesssim \|f\|_\alpha 2^{-\alpha k}, \quad k \in \mathbb{N}.$$

Here $x \lesssim y$ means $x \leq Cy$ with a universal constant C. And $x \simeq y$ means $x \lesssim y$ as well as $y \lesssim x$.

Proof. 1. Let g_k be the linear interpolation of f between the points $t^1_{-10}, t^1_{00}, t^1_{pm}$, $0 \leq p \leq k, 1 \leq m \leq 2^p$. Then $g_k \in \mathbb{C}^\alpha$.
Show:
$g_k - f_k = 0$.

By (1) $f_n \to g_k$ as $n \to \infty$. But by definition of G_{pm} the contributions of dyadic generations bigger than k have to vanish at the points $t^1_{-10}, t^1_{00}, t^1_{pm}$, $0 \leq p \leq k, 1 \leq m \leq 2^p$.

2. follows from 1. and uniform continuity of f on $[0,1]$.

3. **Show:** For $f \in \mathcal{C}^\alpha$

$$\sup_{p,m} 2^{p(\alpha-1/2)} |\langle H_{pm}, \mathrm{d}f \rangle| < \infty.$$

Fix $p \geq 1, 1 \leq m \leq 2^p$. Then by definition of the Hölder norm

$$\begin{aligned}
|\langle H_{pm}, \mathrm{d}f \rangle| &\leq 2^{\frac{p}{2}} [(|f(t^1_{pm}) - f(t^0_{pm})| + |f(t^2_{pm}) - f(t^1_{pm})|] \\
&= 2^{\frac{p}{2}} 2^{-\alpha(p+1)} [\frac{|f(t^1_{pm}) - f(t^0_{pm})|}{|t^1_{pm} - t^0_{pm}|^\alpha} + \frac{|f(t^2_{pm}) - f(t^1_{pm})|}{|t^2_{pm} - t^1_{pm}|^\alpha}] \\
&\leq 2^{\frac{p}{2}+1-\alpha(p+1)} ||f||_\alpha.
\end{aligned}$$

Hence

$$\sup_{p,m} 2^{p(\alpha-1/2)} |\langle H_{pm}, \mathrm{d}f \rangle| \leq 2^{1-\alpha} \; ||f||_\alpha < \infty.$$

This proves one direction in claim 3., and one inequality in the equivalence of norms.

4. Show: For $k \in \mathbb{N}$ we have

$$\|f - f_{k-1}\|_\infty = \Big\| \sum_{p=k}^{\infty} \sum_{m=0}^{2^p} |\langle H_{pm}, \mathrm{d}f\rangle| G_{pm} \Big\|_\infty \lesssim \|f\|_\alpha 2^{-\alpha k}.$$

We fix $p \in \mathbb{N}$ and estimate the contribution of the pth dyadic generation.

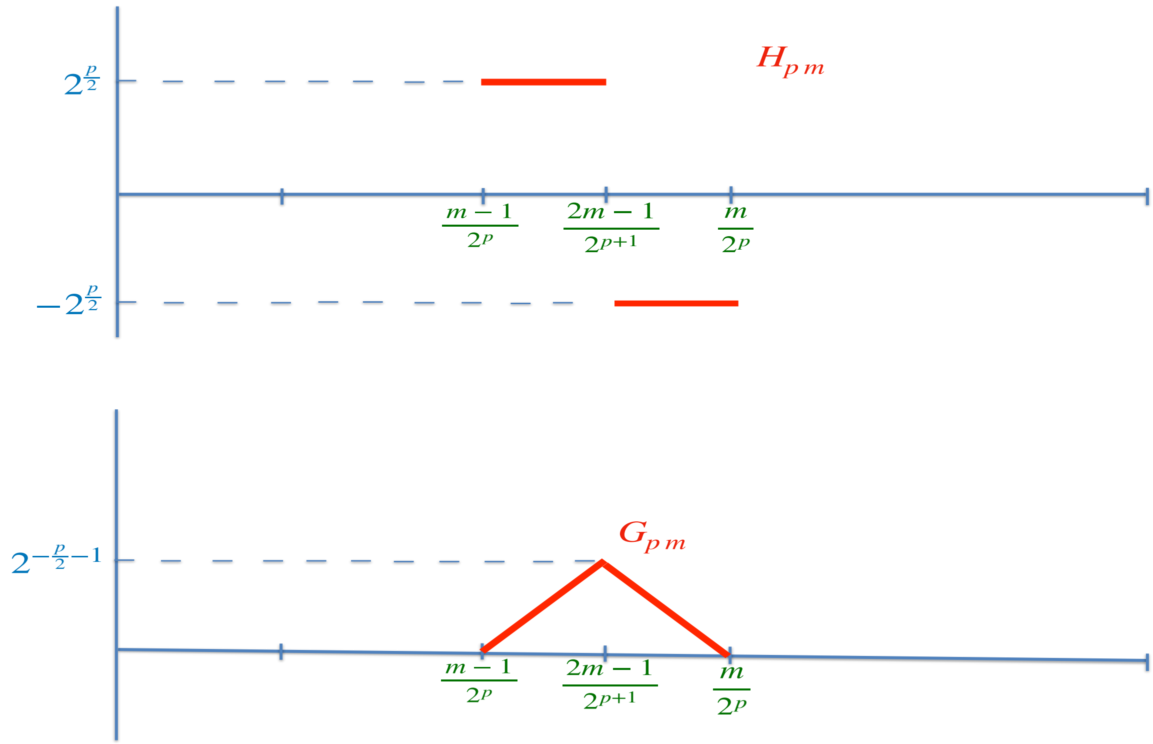

Figure 3: $||G_{pm}||_\infty$

Observe that by disjointness of the supports of $G_{pm}, 1 \leq m \leq 2^p$ and

$$||G_{pm}||_\infty = 2^{\frac{p}{2}} 2^{-(p+1)} = 2^{-\frac{p}{2}-1},$$

we have

$$|| \sum_{m=1}^{2^p} G_{pm} ||_\infty = 2^{-\frac{p}{2}-1}.$$

Hence by 3.

$$|| \sum_{m=0}^{2^p} \langle H_{pm}, \mathrm{d}f \rangle G_{pm} ||_\infty \lesssim ||f||_\alpha 2^{\frac{p}{2}+1-\alpha(p+1)} 2^{-\frac{p}{2}-1} = 2^{-\alpha(p+1)} ||f||_\alpha.$$

Therefore by 2. for $k \in \mathbb{N}$

$$\begin{aligned}
\|f - f_{k-1}\|_\infty &\leq \lim_{m\to\infty} \|f_m - f_{k-1}\|_\infty \\
&\leq \lim_{m\to\infty} \sum_{p=k}^{m-1} \|\sum_{m=0}^{2^p} \langle H_{pm}, \mathrm{d}f\rangle G_{pm}\|_\infty \\
&\lesssim \sum_{p=k}^{\infty} 2^{-\alpha p} \|f\|_\alpha \lesssim 2^{-\alpha k} \|f\|_\alpha.
\end{aligned}$$

5. **Show:** If f continuous and $K = \sup_{p,m} 2^{p(\alpha-1/2)} |\langle H_{pm}, \mathrm{d}f\rangle| < \infty$, then $f \in \mathcal{C}^\alpha$, and $\|f\|_\alpha \lesssim K$.

In fact, let $0 \leq s < t \leq 1$ be given. Assume that $q \in \mathbb{N}$ such that

$$2^{-(q+1)} \leq |t - s| \leq 2^{-q}.$$

Then by 2.

$$\begin{aligned}
\frac{|f(t)-f(s)|}{|t-s|^\alpha} &= \lim_{k\to\infty}\frac{|f_k(t)-f_k(s)|}{|t-s|^\alpha} \\
&\leq \sum_{p=0}^{\infty}\sup_{0\leq m\leq 2^p}|\langle H_{pm},\mathrm{d}f\rangle|\sum_{m=0}^{2^p}\frac{|G_{pm}(t)-G_{pm}(s)|}{|t-s|^\alpha} \\
&\lesssim K\sum_{p=0}^{\infty}2^{p(\frac{1}{2}-\alpha)}\sum_{m=0}^{2^p}\frac{|G_{pm}(t)-G_{pm}(s)|}{|t-s|^\alpha}.
\end{aligned}$$

To estimate the contributions of Schauder functions of generation p, distinguish cases:

Case 1: $p \leq q$

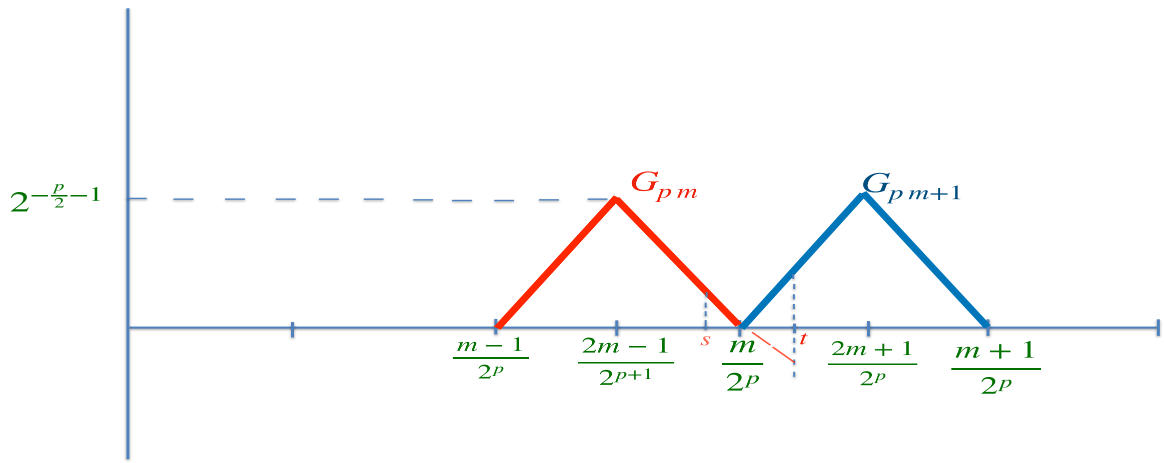

Figure 4: estimate of increments, small modes

Then either s and t belong to the same dyadic interval $[\frac{m_0-1}{2^p}, \frac{m_0}{2^p}]$ or to two adjacent ones. In both cases

$$\sum_{m=0}^{2^p} \frac{|G_{pm}(t) - G_{pm}(s)|}{|t-s|^\alpha} \leq 2^{\frac{p}{2}} \, |t-s|^{1-\alpha} \lesssim 2^{\frac{p}{2}-q(1-\alpha)}.$$

Case 2: $p > q$

In this case s and t each belong to a dyadic interval of generation p and hence

$$\sum_{m=0}^{2^p} \frac{|G_{pm}(t) - G_{pm}(s)|}{|t-s|^\alpha} \leq 2^{-\frac{p}{2}} |t-s|^{-\alpha} \lesssim 2^{-\frac{p}{2}+\alpha(q+1)}.$$

Summarizing, we obtain

$$\begin{aligned} \frac{|f(t)-f(s)|}{|t-s|^\alpha} &\lesssim K \, [\sum_{p=0}^{q} 2^{p(\frac{1}{2}-\alpha)} 2^{\frac{p}{2}-q(1-\alpha)} + \sum_{p=q+1}^{\infty} 2^{p(\frac{1}{2}-\alpha)} 2^{-\frac{p}{2}+\alpha(q+1)}] \\ &\lesssim K \, 2^{q(1-\alpha)-q(1-\alpha)} + 2^{-q\alpha+\alpha(q+1)} \simeq K. \end{aligned}$$

Consequently

$$\sup_{0 \leq s < t \leq 1} \frac{|f(t)-f(s)|}{|t-s|^\alpha} \lesssim K,$$

and $f \in \mathcal{C}^\alpha$. □

Theorem 2 ([8]). *Let* $0 < \alpha < 1$. *For* $p \geq 0, 1 \leq m \leq 2^p$ *let*

$$c_{pm}(\alpha) = 2^{p(\alpha - \frac{1}{2}) + \alpha - 1}, \quad c_{p0}(\alpha) = 1, \quad c_{-10}(\alpha) = 1.$$

Define

$$\begin{aligned} T_\alpha \quad &: \quad \mathcal{C}^\alpha \to l^\infty(\mathbb{R}) \\ f \quad &\mapsto \quad (c_{-10}(\alpha)\langle H_{-10}, \mathrm{d}f\rangle, c_{00}\langle H_{00}, \mathrm{d}f\rangle, (c_{pm}(\alpha)\,\langle H_{pm}, \mathrm{d}f\rangle)_{p\geq 1, 1\leq m\leq 2^p}). \end{aligned}$$

Then T_α is invertible and

$$\begin{aligned} T_\alpha^{-1} \quad &: \quad l^\infty(\mathbb{R}) \to \mathcal{C}^\alpha \\ (\eta_{-10}, \eta_{00}, (\eta_{pm})_{p\geq 1, 1\leq m\leq 2^p}) \quad &\mapsto \quad \eta_{-10} G_{-10} + \eta_{00} G_{00} + \sum_{p=1}^{\infty} \sum_{1\leq m\leq 2^p} \frac{1}{c_{pm}(\alpha)} \eta_{pm}\; G_{pm} \end{aligned}$$

T_α is an isomorphism, and for the operator norms we have the following inequalities

$$||T_\alpha|| = 1, \quad ||T_\alpha^{-1}|| \leq \frac{2}{(2^\alpha - 1)(2^{1-\alpha} - 1)}.$$

Proof. 1. **Show:** T_α well defined, $||T_\alpha|| = \sup_{||f||_\alpha \leq 1} \frac{||T_\alpha f||_\infty}{||f||_\alpha} \leq 1.$

By Lemma 1, for $p \geq 1, 1 \leq m \leq 2^p$ we have

$$|\langle H_{pm}, \mathrm{d}f\rangle| \leq 2^{-(p+1)\alpha + \frac{p}{2} + 1}||f||_\alpha = \frac{1}{c_{pm}(\alpha)}||f||_\alpha. \qquad (3)$$

This proves the claim.

2. **Show:** $||T_\alpha|| \geq 1$:

Note that for $p \geq 1, 1 \leq m \leq 2^p$ we have

$$||G_{pm}||_\alpha = 2^{\frac{p}{2}} 2^{(-p-1)(1-\alpha)} = 2^{p(\alpha - \frac{1}{2}) + \alpha - 1},$$

while

$$\langle H_{pm}, \mathrm{d}G_{pm}\rangle = \langle H_{pm}, H_{pm}\rangle = 1.$$

Hence

$$||G_{pm}||_\alpha = c_{pm}(\alpha)|\langle H_{pm}, \mathrm{d}G_{pm}\rangle| = ||T_\alpha(G_{pm})||_\infty.$$

3. Let S be the operator defined on $l^\infty(\mathbb{R})$ in the statement of the Theorem.

Show: S well defined, $S(T_\alpha(f)) = f, T_\alpha(S(\eta)) = \eta$ for $f \in \mathcal{C}^\alpha, \eta \in l^\infty(\mathbb{R})$.

The claims follow directly from Lemma 1. This implies that T_α is invertible and that S is its inverse.

4. **Show:** $||T_\alpha^{-1}|| \leq \frac{2}{(2^\alpha-1)(2^{1-\alpha}-1)}$.

Let $\eta = (\eta_{-10}, \eta_{00}, (\eta_{pm})_{p\geq 1, 1\leq m \leq 2^p}) \in l^\infty(\mathbb{R})$, choose $0 \leq s < t \leq 1$, and write $f = T_\alpha^{-1}(\eta)$. Then we have

$$|f(t) - f(s)| \leq ||\eta||_\infty [|t-s| + \sum_{p=1}^\infty \sum_{m=1}^{2^p} \frac{1}{c_{pm}(\alpha)} |G_{pm}(t) - G_{pm}(s)|]. \qquad (4)$$

Now choose $p_0 \geq 1$ such that

$$2^{-p_0-1} < |t-s| \leq 2^{-p_0}.$$

Case 1: $1 \leq p < p_0$: s and t can belong to at most two adjacent dyadic intervals of generation p. By inspection of the possible cases we get

$$\sum_{m=1}^{2^p} \frac{1}{c_{pm}(\alpha)} |G_{pm}(t) - G_{pm}(s)| \tag{5}$$
$$\leq \quad 2^{-p(\alpha-\frac{1}{2})-\alpha+1} \; 2^{\frac{p}{2}} \; |t-s|$$
$$\leq \quad 2^{p(1-\alpha)-\alpha+1-p_0(1-\alpha)} \; |t-s|^\alpha = (2^{1-\alpha})^{(1+p-p_0)} \; |t-s|^\alpha,$$

Case 2: $p \geq p_0$:

$$\sum_{m=1}^{2^p} \frac{1}{c_{pm}(\alpha)} |G_{pm}(t) - G_{pm}(s)| \tag{6}$$
$$\leq \quad 2^{-p(\alpha-\frac{1}{2})-\alpha+1} \; 2^{-\frac{p}{2}-1}$$
$$\leq \quad 2^{-p\alpha-\alpha+(p_0+1)\alpha} \; |t-s|^\alpha = (2^\alpha)^{(p_0-p)} \; |t-s|^\alpha.$$

Combining (4), (5) and (6), we obtain the estimate
$\frac{|f(t)-f(s)|}{|t-s|^\alpha} \leq \frac{2}{(2^\alpha-1)(2^{1-\alpha}-1)} ||\eta||_\infty$. □

Lecture 2: The Schauder representation of Brownian motion

Aim: description of (one-dimensional) Brownian motion in Haar-Schauder series.

If $X = (X_t)_{0 \leq t \leq 1}$ is Brownian motion, then

Show: $\langle H_{pm}, dX \rangle, p \geq 0, 1 \leq m \leq 2^p$ is an i.i.d. sequence of standard normal variables.

Proof. 1. First of all, note that since increments of Brownian motion are centered, we have

$$\mathbf{E}(\langle H_{pm}, dX \rangle) = 0, \quad p \geq 0, 1 \leq m \leq 2^p.$$

2. To calculate covariances, let $(p, m), (q, n)$ be given with $p, q \in \mathbb{N}, 1 \leq m \leq 2^p, 1 \leq n \leq 2^q$.

Case 1: $p = q, n = m$

In this case by independence of increments

$$\begin{aligned}\mathbf{E}(\langle H_{pm}, dX\rangle^2) &= 2^p[\mathbf{E}([(X(t^1_{pm}) - X(t^0_{pm}) - (X(t^2_{pm}) - X(t^1_{pm}))]^2)] \\ &= 2^p[\mathbf{E}((X(t^1_{pm}) - X(t^0_{pm}))^2) + \mathbf{E}((X(t^2_{pm}) - X(t^1_{pm})))^2) \\ &= 2^p[2^{-(p+1)} + 2^{-(p+1)}] = 1.\end{aligned}$$

Case 2: $p = q, m < n$

Here as a direct consequence of independence of increments

$$\mathbf{E}(\langle H_{pm}, dX\rangle\langle H_{pn}, dX\rangle) = 0.$$

Case 3: $p < q, [\frac{m-1}{2^p}, \frac{m}{2^p}] \cap [\frac{n-1}{2^q}, \frac{n}{2^q}] = \emptyset$

In this case, as in the preceding one by independence of increments

$$\mathbf{E}(\langle H_{pm}, dX\rangle\langle H_{qn}, dX\rangle) = 0.$$

Case 4: $p < q, [\frac{m-1}{2^p}, \frac{m}{2^p}] \supset [\frac{n-1}{2^q}, \frac{n}{2^q}]$

Here even w.l.o.g. $[\frac{m-1}{2^p}, \frac{2m-1}{2^{p+1}}] \supset [\frac{n-1}{2^q}, \frac{n}{2^q}]$. Hence

$$\begin{aligned}
&\mathbf{E}(\langle H_{pm}, dX\rangle\langle H_{qn}, dX\rangle) \\
&= 2^{\frac{p+q}{2}}[\mathbf{E}[(X(t^1_{pm}) - X(t^0_{pm}))(X(t^1_{qn}) - X(t^0_{qn}))] \\
&-\mathbf{E}[(X(t^1_{pm}) - X(t^0_{pm}))(X(t^2_{qn}) - X(t^1_{qn}))] \\
&-\mathbf{E}[(X(t^2_{pm}) - X(t^1_{pm}))(X(t^1_{qn}) - X(t^0_{qn}))] \\
&+\mathbf{E}[(X(t^2_{pm}) - X(t^1_{pm}))(X(t^2_{qn}) - X(t^1_{qn}))] \\
&= 2^{\frac{p+q}{2}+1}[\mathbf{E}[(X(t^1_{qn}) - X(t^0_{qn}))^2] \\
&\quad -\mathbf{E}[(X(t^2_{qn}) - X(t^1_{qn}))^2]] \quad \text{(independence of increments)} \\
&= 0 \quad \text{(equal length of intervals).}
\end{aligned}$$

This proves the claim. □

Given sequence of i.i.d. standard normal variables $(Z_{00}, (Z_{pm})_{p\geq 1, 1\leq m\leq 2^p})$ on $(\Omega, \mathcal{F}, \mathbb{P})$, define

$$B_t = Z_{00}G_{00}(t) + \sum_{p=1}^{\infty} \sum_{1\leq m\leq 2^p} Z_{pm}G_{pm}(t), \quad t \in [0,1]. \tag{7}$$

Aim: show that $B = (B_t)_{0\leq t\leq 1}$ is Brownian motion.

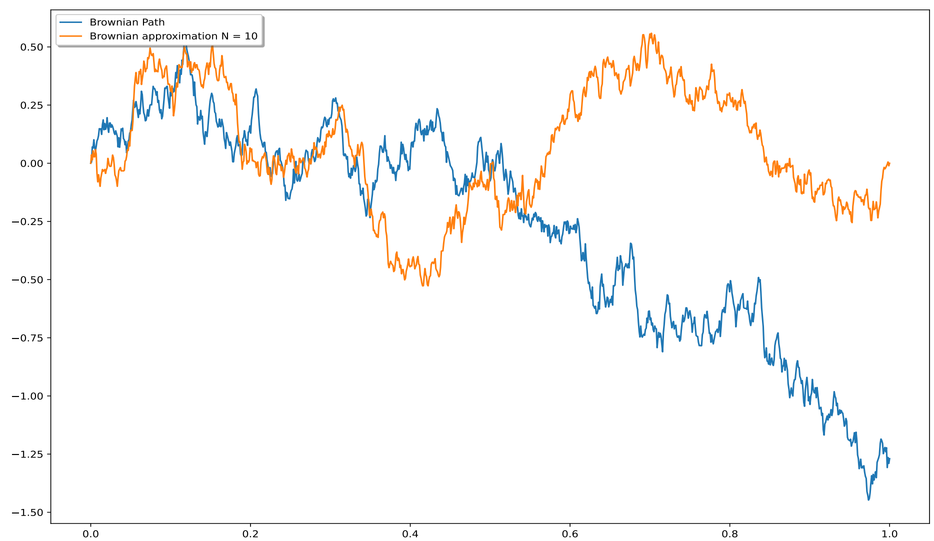

Figure 5: Brownian paths: Euler (blue), (7), $N = 10$ (orange)

Lemma 3. *There exists a real valued random variable C such that for $p \geq 1, 1 \leq m \leq 2^p$ we have*

$$|Z_{pm}| \leq C\sqrt{p \ln 2}. \tag{8}$$

Proof. 1. **Show:**

$$\mathbb{P}(|Z_{pm}| \geq \sqrt{2\beta \ln 2^p}) \leq \sqrt{\frac{2}{\pi}} 2^{-\beta p}.$$

For $x \geq 1, p \geq 1, 1 \leq m \leq 2^p$ we have

$$\mathbb{P}(|Z_{pm}| \geq x) = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-\frac{u^2}{2}} du \leq \sqrt{\frac{2}{\pi}} \int_x^\infty u e^{-\frac{u^2}{2}} du = \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}}.$$

Hence for $\beta > 1$

$$\mathbb{P}(|Z_{pm}| \geq \sqrt{2\beta \ln 2^p}) \leq \sqrt{\frac{2}{\pi}} e^{-\beta \ln 2^p} = \sqrt{\frac{2}{\pi}} 2^{-\beta p}.$$

2. **Show:** $|Z_{pm}| \leq \sqrt{4\beta p \ln 2}$ for a.a. $p \geq 1, 1 \leq m \leq 2^p$ with probability 1.

For $p \geq 0, 1 \leq m \leq 2^p$ let $A_{pm} = \{|Z_{pm}| \leq \sqrt{4\beta p \ln 2}\}$. Then by part 1.

$$\sum_{p=1}^{\infty} \sum_{1 \leq m \leq 2^p} \mathbb{P}(A^c_{pm}) \lesssim \sum_{p=0}^{\infty} 2^p 2^{-\beta p} < \infty,$$

since $\beta > 1$. So, the lemma of Borel-Cantelli yields

$$\mathbb{P}(\cap_{q \in \mathbb{N}} \cup_{p \geq q, 1 \leq m \leq 2^p} A^c_{pm}) = 0,$$

and so

$$\mathbb{P}(\cup_{q \in \mathbb{N}} \cap_{p \geq q, 1 \leq m \leq 2^p} A_{pm}) = 1.$$

This translates to: With probability 1 there exists $q \in \mathbb{N}$ such that for all $p \geq q$, all $1 \leq m \leq 2^p$ we have $|Z_{pm}| \leq \sqrt{4\beta \ln 2^p}$.

Hence

$$C = \sup_{p \geq 1, 1 \leq m \leq 2^p} \frac{|Z_{pm}|}{\sqrt{p \ln 2}}$$

is almost surely finite, and yields the desired inequality. □

According to the preceding Lemma, the convergence in (7) is absolute and therefore the process continuous. We now show that its law has the characteristics of the law of a Brownian motion.

Theorem 4. *The series in (7) converges absolutely in the uniform norm to a continuous process* B *which is a Brownian motion on* $[0, 1]$.

Proof. 1. **Show:** If $B_p(t) = Z_{00}G_{00}(t) + \sum_{k=1}^{p} \sum_{1 \leq m \leq 2^k} Z_{qm}G_{km}(t),\ p \in \mathbb{N}$, then we have

$$||B_p - B_q||_\infty \to 0 \quad \text{as} \quad p, q \to \infty.$$

Consequently, B is a.s. continuous.

Let $p, q \geq 1$ be such that $q \geq p$. Then with C of the preceding Lemma

$$\begin{aligned} ||B_p - B_q||_\infty &\leq \sum_{n=p}^{q} || \sum_{1 \leq m \leq 2^n} |Z_{nm}| G_{nm} ||_\infty \\ &\leq C \sum_{n=p}^{q} \sqrt{n \ln 2} \, || \sum_{1 \leq m \leq 2^n} G_{nm} ||_\infty \\ &\leq C \sum_{n=p}^{\infty} \sqrt{n} \, 2^{-\frac{n}{2}-1}, \end{aligned}$$

which converges to 0 as p tends to ∞.

2. **Show:** For $t \in [0, 1]$ $\mathbf{E}((B_q(t) - B_p(t))^2) \to 0$ as $p, q \to \infty$. In particular, $B(t)$ is square integrable for $t \in [0, 1]$.

In fact, for $t \in [0, 1]$, $p, q \geq 1$ such that $q \geq p$ by the law properties of $Z_{pm}, p \geq 1, 1 \leq m \leq 2^p$,

$$\begin{aligned}\mathbf{E}((B_q(t)-B_p(t))^2) &= \mathbb{E}([\sum_{n=p}^{q}\sum_{1\le m\le 2^n} Z_{nm}G_{nm}(t)]^2) \\ &= \sum_{n=p}^{q}\sum_{1\le m\le 2^n} G_{nm}(t)^2 \le \sum_{n=p}^{\infty} 2^{-n-2},\end{aligned}$$

which converges to 0 as $p \to \infty$.

3. **Show:** For $d \in \mathbb{N}$, $0 \le t_1 < \cdots < t_d \le 1$, and $\theta = (\theta_1, \cdots, \theta_d) \in \mathbb{R}^d$ the vector $(B(t_1), \cdots, B(t_d))$ is Gaussian with

$$\mathbf{E}(B(t_i)) = 0, \text{cov } (B(t_i), B(t_j)) = t_i \wedge t_j, \quad 1 \le i, j \le d.$$

We compute the Fourier transform $\varphi(\theta)$ of the vector $(B(t_1), \cdots, B(t_d))$ at θ.

By dominated convergence and the law properties of $Z_{pm}, p \geq 1, 1 \leq m \leq 2^p$, we have

$$\begin{aligned}
\varphi(\theta) &= \mathbb{E}(\exp(i \sum_{j=1}^{d} \theta_j B(t_j))) \\
&= \mathbb{E}(\exp(i \sum_{j=1}^{d} \theta_j \sum_{p=0}^{\infty} \sum_{0 \leq m \leq 2^p} Z_{pm} G_{pm}(t_j))) \\
&= \prod_{p=0}^{\infty} \prod_{0 \leq m \leq 2^p} \mathbb{E}(\exp(i Z_{pm} \sum_{j=1}^{d} \theta_j G_{pm}(t_j))) \\
&= \prod_{p=0}^{\infty} \prod_{0 \leq m \leq 2^p} \exp(-\frac{1}{2} (\sum_{j=1}^{d} \theta_j G_{pm}(t_j))^2) \\
&= \exp(-\frac{1}{2} \sum_{p=0}^{\infty} \sum_{0 \leq m \leq 2^p} (\sum_{j=1}^{d} \theta_j G_{pm}(t_j))^2) \\
&= \exp(-\frac{1}{2} \sum_{j,k=1}^{d} \theta_j \theta_k \sum_{p=0}^{\infty} \sum_{0 \leq m \leq 2^p} G_{pm}(t_j) G_{pm}(t_k)).
\end{aligned}$$

Now observe that Parseval's equation implies for $1 \leq j, k \leq d$

$$\begin{aligned} t_j \wedge t_k &= \langle 1_{[0,t_j]}, 1_{[0,t_k]} \rangle \\ &= \sum_{p=0}^{\infty} \sum_{0 \leq m \leq 2^p} \langle 1_{[0,t_j]}, H_{pm} \rangle \langle 1_{[0,t_k]}, H_{pm} \rangle \\ &= \sum_{p=0}^{\infty} \sum_{0 \leq m \leq 2^p} G_{pm}(t_j) G_{pm}(t_k). \end{aligned}$$

Therefore we finally obtain

$$\varphi(\theta) = \exp(-\frac{1}{2} \sum_{j,k=1}^{d} \theta_j \theta_k \; t_j \wedge t_k).$$

This implies the claimed properties.

□

Goal: Hölder continuity properties of B.

Theorem 5. *The Brownian motion $B = (B(t))_{0 \le t \le 1}$ is Hölder continuous of order $\alpha < 1/2$. Its trajectories are a.s. nowhere Hölder continuous of order $\alpha > 1/2$.*

Moreover we have (Lévy's modulus of continuity)

$$\mathbb{P}\Big(\sup_{0 \le s < t \le 1} \frac{|B(t) - B(s)|}{h(|t-s|)} < \infty \Big) = 1, \tag{9}$$

where $h(u) = \sqrt{u \log(1/u)}, u > 0$.

In particular, for $\alpha < \frac{1}{2}$, the trajectories of B are $\mathbb{P}$-a.s. contained in $\mathbb{C}^\alpha$.

Proof. 1. Let first $\alpha \in]0,1[$, $(c_{pm})_{p \ge 1, 1 \le m \le 2^p}$ be a sequence in $\mathbb{R}$ for which there exists $c \in \mathbb{R}$ such that for $p \ge 0, 1 \le m \le 2^p$ we have

$$|c_{pm}| \le c\sqrt{p}.$$

Let

$$f(t) = \sum_{p=1}^{\infty} \sum_{1 \le m \le 2^p} c_{pm} G_{pm}(t), \quad t \in [0,1].$$

The trajectories of B fulfill this inequality by Lemma 3.

Show: $\sup_{0\leq s<t\leq 1} \frac{|f(t)-f(s)|}{h(|t-s|)} < \infty.$

In fact, by continuity properties of G_{00}, we may assume $c_{00} = 0$. Then for $0 \leq s < t \leq 1$

$$|f(t) - f(s)| \leq \sum_{p=1}^{\infty} \sum_{m=1}^{2^p} |c_{pm}|\, |G_{pm}(t) - G_{pm}(s)|. \qquad (10)$$

Now choose $p_0 \geq 1$ such that

$$2^{-p_0-1} < |t - s| \leq 2^{-p_0}.$$

W.l.o.g. we can assume that $p_0 \geq 1$. Then for $1 \leq p < p_0$, s and t can belong to at most two adjacent dyadic intervals of generation p.

By inspection of the different cases we get

$$
\begin{aligned}
\sum_{m=1}^{2^p} |c_{pm}|\,|G_{pm}(t) - G_{pm}(s)| & \\
\leq \quad & c\sqrt{p}\, 2^{\frac{p}{2}}\, |t-s| \\
\leq \quad & c\sqrt{p}\, 2^{\frac{p-p_0}{2}}\, |t-s|^{\frac{1}{2}} \\
\leq \quad & \frac{c}{\sqrt{\ln 2}} \sqrt{\frac{p}{p_0}}\, 2^{\frac{p-p_0}{2}} \sqrt{|t-s| \ln \frac{1}{|t-s|}},
\end{aligned}
\tag{11}
$$

while for $p \geq p_0$

$$
\begin{aligned}
\sum_{m=1}^{2^p} |c_{pm}|\,|G_{pm}(t) - G_{pm}(s)| & \\
\leq \quad & c\,\sqrt{p}\, 2^{-\frac{p}{2}} \\
\leq \quad & \frac{c}{\sqrt{\ln 2}} \sqrt{\frac{p}{p_0}}\, 2^{\frac{p_0+1-p}{2}} \sqrt{|t-s| \ln \frac{1}{|t-s|}}.
\end{aligned}
\tag{12}
$$

Now

$$\sum_{0\le p\le p_0}\sqrt{\frac{p}{p_0}}2^{-\frac{p-p_0}{2}}\le\sum_{0\le p\le p_0}2^{-\frac{p-p_0}{2}}\lesssim 1.$$

And

$$\sum_{p>p_0}\sqrt{\frac{p}{p_0}}2^{\frac{p_0-p}{2}}\le\frac{2^{\frac{p_0}{2}}}{\sqrt{p_0}}\int_{p_0}^{\infty}\sqrt{x}2^{-\frac{x}{2}}dx.$$

Integration by parts for $y\ge 1$ gives

$$\begin{aligned}\int_y^{\infty}\sqrt{x}2^{-\frac{x}{2}}dx &= -\frac{2}{\ln 2}\sqrt{x}2^{-\frac{x}{2}}|_y^{\infty}+\frac{1}{\ln 2}\int_y^{\infty}\frac{1}{\sqrt{x}}2^{-\frac{x}{2}}dx\\ &\le \frac{2}{\ln 2}\sqrt{y}2^{-\frac{y}{2}}+\frac{1}{\ln 2}\int_y^{\infty}2^{-\frac{x}{2}}dx\\ &\le \frac{2}{\ln 2}\sqrt{y}2^{-\frac{y}{2}}+\frac{2}{(\ln 2)^2}2^{-\frac{y}{2}}.\end{aligned}$$

Now set $y = p_0$, to see

$$\sum_{p>p_0} \sqrt{\frac{p}{p_0}} 2^{\frac{p_0-p}{2}} \lesssim \frac{2^{\frac{p_0}{2}}}{\sqrt{p_0}} (\sqrt{p_0}+1)\ 2^{-\frac{p_0}{2}} \simeq 1.$$

Hence (10), (11) and (12) imply

$$\sup_{0\le s<t\le 1} \frac{|f(t)-f(s)|}{\sqrt{|t-s|\ \ln\frac{1}{|t-s|}}} \lesssim 1.$$

2. Part 1. implies all claims about Hölder continuity for $\alpha < \frac{1}{2}$.

$\square$

Lecture 3: Takagi functions via dynamical systems

Now study **Takagi function**

$$C(x) := \sum_{k=0}^{\infty} \gamma^k f(2^k x), \quad x \in [0,1], \gamma \in]\frac{1}{2}, 1[, \quad f = \text{dist}(\mathbf{Z}, \cdot).$$

Here is the Takagi base function $f(2^k \cdot)$ for $k = 0, 1, 2, 3$:

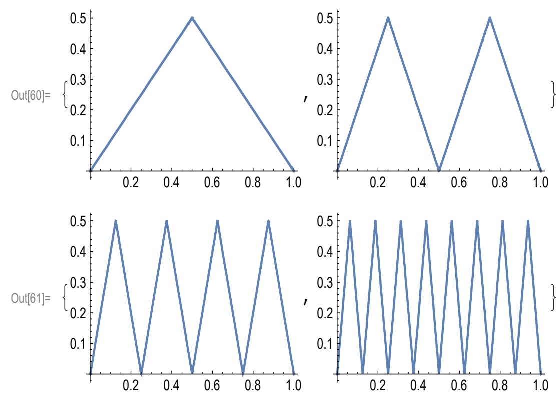

Figure 6: Takagi base function $x \mapsto f(2^k x), k = 0, 1, 2, 3$.

Hölder continuity

For $\gamma = 2^{-\frac{1}{2}}, k \geq 0$ we just obtain

$$\gamma^k f(2^k \cdot) = \sum_{1 \leq m \leq 2^k} G_{km}.$$

For $x, y \in [0,1], k \in \mathbb{N}$ s. th. $2^{-k} \leq |x-y| \leq 2^{-k+1}$, $\alpha \in]0,1[$ s. th. $\gamma = 2^{-\alpha}$:

$$\begin{aligned} |C(x) - C(y)| &\leq c\,[\sum_{l=0}^{k} \gamma^l 2^l |x-y| + \sum_{l=k+1}^{\infty} \gamma^l] \\ &\leq c[2^{k(1-\alpha)}|x-y| + 2^{-k\alpha}] \leq c|x-y|^\alpha. \end{aligned}$$

Hence C α-Hölder continuous.

Takagi function: geometry

Graph of T for $\gamma = 2^{-\frac{1}{2}}$ $(\alpha = \frac{1}{2})$:

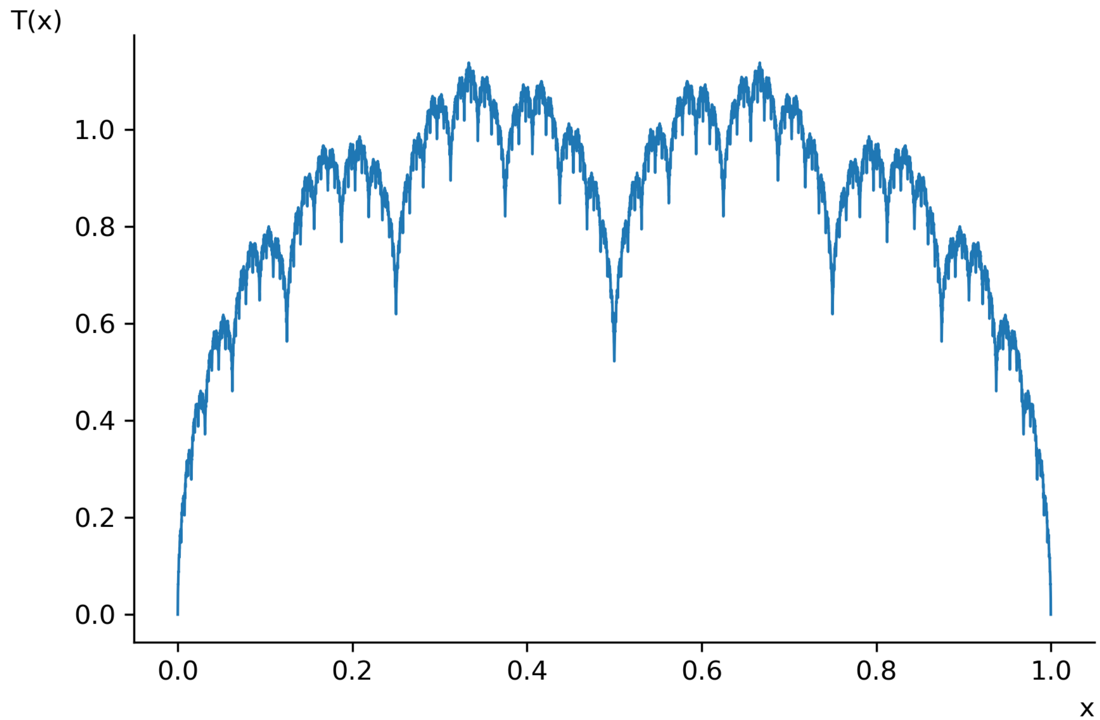

Figure 7: Graph of C: $\{(x, C(x)) : x \in [0,1]\} \subset \mathbb{R}^2$.

Goal: Investigate geometry of C, in particular local time.

A metric dynamical system

Lit: Baranski et al. [2], Keller [18], Shen [20].

Goal: Describe C **as attractor of a dynamical system** on $[0,1]^2$, alternatively $\Omega = \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.

For $\omega \in \Omega$, write $\omega = ((\omega_{-n})_{n\geq 0}, (\omega_n)_{n\geq 1})$, **F** product σ-field.

Canonical shift on Ω:

$$\theta : \Omega \to \omega, \omega \mapsto (\omega_{n+1})_{n\in\mathbb{Z}}, \quad \nu = \otimes_{n\in\mathbb{Z}}(\frac{1}{2}\delta_{\{0\}} + \frac{1}{2}\delta_{\{1\}})$$

the infinite product of Bernoulli measures.

$(\Omega, \mathbf{F}, \nu, \theta)$ metric dynamical system.

A metric dynamical system, baker's transformation

Now let

$$D = (D_1, D_2) : \Omega \to [0,1]^2, \quad \omega \mapsto \big(\sum_{n=0}^{\infty} \omega_{-n} 2^{-(n+1)}, \sum_{n=1}^{\infty} \omega_n 2^{-n}\big).$$

Then $\nu \circ D^{-1} = \lambda^2$ Lebesgue measure on $[0,1]^2$. D^{-1}: dyadic representation of components in $[0,1]^2$. Let

$$B = D \circ \theta \circ D^{-1} \qquad \text{baker's transformation.}$$

The ν-invariance of θ implies B-invariance of λ^2. For $(\xi, x) \in [0,1]^2$ denote

$$D^{-1}(\xi, x) = \big((\overline{\xi}_{-n})_{n \geq 0}, (\overline{x}_n)_{n \geq 1}\big).$$

For $(\xi, x) \in [0,1]^2$ and $k \geq 0$ resp. $k \geq 1$

$$B(\xi, x) = \Big(2\xi(\text{mod} 1), \frac{\overline{\xi}_0 + x}{2}\Big), \qquad B^{-1}(\xi, x) = \Big(\frac{\xi + \overline{x}_1}{2}, 2x(\text{mod} 1)\Big).$$

expansion contraction contraction expansion

Self affinity: C as attractor of a random dynamical system

Extend C from $[0,1]$ to $[0,1]^2$ by $C(\xi,x) = C(x), \quad \xi, x \in [0,1]$.

By definition of B we have $B_2^{-n}(\xi,x) = 2^n x$ (mod 1), and hence

$$C(x) = \sum_{n=0}^{\infty} \gamma^n f(2^n x) = \sum_{n=0}^{\infty} \gamma^n f(B_2^{-n}(\xi,x)).$$

Therefore, taking $k = n-1$,

$$\begin{aligned} C(B_2(\xi,x)) &= \sum_{n=0}^{\infty} \gamma^n f(B_2^{-n+1}(\xi,x)) \\ &= f(B_2(\xi,x)) + \gamma \sum_{k=0}^{\infty} \gamma^k f(B_2^{-k}(\xi,x)) \\ &= f(B_2(\xi,x)) + \gamma C(\xi,x). \end{aligned}$$

C as attractor of a random dynamical system

Define the map

$$\begin{aligned} F : [0,1]^2 \times \mathbb{R} &\to [0,1]^2 \times \mathbb{R}, \\ (\xi, x, y) &\mapsto \Big(B(\xi, x), f(B_2(\xi, x)) + \gamma y\Big), \end{aligned}$$

where $B = (B_1, B_2)$.

Then

$$\Big(B(\xi, x), C(B(\xi, x))\Big) = \Big(B(\xi, x), C(B_2(\xi, x))\Big) = F\Big(\xi, x, C(\xi, x)\Big).$$

Hence C is an attractor for F (on the skew product).

Here, with $\omega = (\xi, x)$: $A = (A(\omega) : \omega \in [0,1]^2) \subset \mathbf{R}$ compact **attractor** for F:

$$\begin{aligned} (i) \quad & F(\omega, A(\omega)) = (B(\omega), A(B(\omega)), \\ (ii) \quad & d(F_3^n D(B^{-n}(\omega)), A(\omega)) \to 0, \\ & \qquad D = (D(\omega) : \omega \in [0,1]^2) \subset \mathbf{R} \text{ compact}, \end{aligned}$$

Lyapunov exponents and invariant structures

Calculate Jacobian: for $\xi, x \in [0,1], y \in \mathbb{R}$

$$JF(\xi,x,y) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2}f'(B_2(\xi,x)) & \gamma \end{bmatrix}.$$

Hence Lyapunov exponents of F: $2, \frac{1}{2}, \gamma$. Invariant vector fields: for $\kappa = \frac{1}{2\gamma}$ $S(\xi,x) = -\sum_{n=1}^{\infty} \kappa^n f'(B_2^n(\xi,x))$:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad X(\xi,x) = \begin{pmatrix} 0 \\ 1 \\ S(\xi,x) \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Hence X spans invariant stable manifold: for $\xi, x \in [0,1], y \in \mathbb{R}$

$$JF(\xi,x,y)X(\xi,x) = \frac{1}{2}X\big(B(\xi,x)\big).$$

The Sinai-Bowen-Ruelle measure

Calculate action of S on B: for $\xi, x \in [0,1]$

$$\begin{aligned} S(B(\xi,x)) &= -\sum_{n=1}^{\infty} \kappa^n f'(B_2^{n+1}(\xi,x)) \\ &= -2\gamma \sum_{k=1}^{\infty} \kappa^k f'(B_2^k(\xi,x)) + f'(B_2(\xi,x)) \\ &= 2\gamma S(\xi,x) + f'(B_2(\xi,x)). \end{aligned}$$

So

$$JF(\xi,x,y)\,X(\xi,x) = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} f'(B_2(\xi,x)) + \gamma S(\xi,x) \end{pmatrix} = \frac{1}{2} X(B(\xi,x)).$$

Sinai-Bowen-Ruelle measure μ: marginals $\mu_x, x \in [0,1]$, given for $B \in \mathcal{B}([0,1])$ by

$$\mu_x(B) = \int_0^1 1_B(S(\xi,x))d\xi.$$

Smoothness of the SBR measure

Idea: use Fourier analysis to show absolute continuity of μ

Fourier transform of μ_x:

$$\phi_x(u) = \int_0^1 \exp\big(iuS(\xi, x)\big) d\xi, \quad u \in \mathbb{R}, x \in [0,1].$$

SBR measure absolutely continuous (with L^2 density) if

$$\int_0^1 \int_{\mathbb{R}} |\phi_x(u)|^2 du dx \quad = \quad \int_{\mathbb{R}} \int_{[0,1]^3} \exp\Big(iu\big(S(\xi, x) - S(\eta, x)\big)\Big) dx d\xi d\eta du < \infty.$$

Hence we are interested in properties of $S(\xi, x) - S(\eta, x)$, $(\xi, \eta, x) \in [0,1]^3$.

Self affinity and self similarity

For $\xi, x \in [0,1]$ (recall $\kappa = \frac{1}{2\gamma}$)

$$\begin{aligned} G_+(\xi, x) &= -\sum_{n\in\mathbb{Z}} \kappa^n \big[f'(B_2^n(\xi, x)) - f'(B_2^n(0, x))\big], \\ G_-(\xi, x) &= \sum_{n\in\mathbb{Z}} \gamma^n \big[f(B_2^{-n}(\xi, x)) - f(B_2^{-n}(\xi, 0))\big]. \end{aligned}$$

Then for $x, y, \xi, \eta \in [0,1]$

$$\begin{aligned} G_+(\xi, x) - G_+(\eta, x) &= -\sum_{n=1}^{\infty} \kappa^n [f'(B_2^n(\xi, x) - f'(B_2^n(\eta, x)] \\ &= S(\xi, x) - S(\eta, x), \end{aligned}$$

Self affinity and self similarity

$$\begin{aligned} G_-(\xi, y) - G_-(\xi, x) &= \sum_{k=0}^{\infty} \gamma^k [f(2^k y) - f(2^k x)] \\ &+ \sum_{n=1}^{\infty} \gamma^{-n} [f(B_2^n(\xi, y)) - f(B_2^n(\xi, x))] \\ &= C(y) - C(x) - \int_x^y S(\xi, z) dz. \end{aligned}$$

and *self similarity* holds:

$$\begin{aligned} G_+(B^{-1}(\xi, x)) &= \kappa G_+(\xi, x), \\ G_-(B(\xi, y)) - G_-(B(\xi, x)) &= \gamma [G_-(\xi, y) - G_-(\xi, x)]. \end{aligned}$$

Smoothness of the occupation measure

Idea: use Fourier analysis to show absolute continuity of occupation measure

$$\hat{C}(\xi, x) = C(x) - \int_0^x S(\xi, z)dz, \quad \nu_\xi(A) = \int_0^1 1_A(\hat{C}(\xi, x))dx,$$

$\xi, x \in [0,1], A \subset \mathbb{R}$ Borel.

Fourier transform of ν_ξ:

$$\psi_\xi(u) = \int_0^1 \exp\left(iu\hat{C}(\xi, x)\right)dx, \quad u \in \mathbb{R}, \xi \in [0,1].$$

Occupation measure absolutely continuous (with L^2 density) if

$$\int_0^1 \int_{\mathbb{R}} |\psi_\xi(u)|^2 du d\xi \quad = \quad \int_{\mathbb{R}} \int_{[0,1]^3} \exp\Big(iu\big(\hat{C}(\xi, y) - \hat{C}(\xi, x)\big)\Big) d\xi dx dy du < \infty.$$

Microscopic and macroscopic measures for S and C

Let

$$\begin{aligned}\rho(A) &= \lambda^3(\{(\xi,\eta,x)\in[0,1]^3 : G_+(\xi,x)-G_+(\eta,x)\in A\}),\\ &\hat{\rho}=\rho(\cdot|\frac{1}{2}<|\xi-\eta|),\\ \chi(A) &= \lambda^3(\{(\xi,x,y)\in[0,1]^3 : G_-(\xi,y)-G_-(\xi,x)\in A\}),\\ &\hat{\chi}=\chi(\cdot|\frac{1}{2}<|x-y|).\end{aligned}$$

Proposition 1 (microscopic-macroscopic transformation):
For A Borel set we have

$$\rho(A)=\sum_{n=0}^{\infty}2^{-n-1}\hat{\rho}(\kappa^{-n}A),\quad \chi(A)=\sum_{n=1}^{\infty}2^{-n-1}\hat{\chi}(\gamma^{-n}A).$$

Smoothness of the SBR measure

Have to show:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \exp(iux)\rho(dx)du < \infty.$$

By microscopic-macroscopic transformation

$$\begin{aligned}
\int_{\mathbb{R}} \int_{\mathbb{R}} \exp(iux)\rho(dx)du &= \sum_{n=0}^{\infty} 2^{-n-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \exp(iuy)\hat{\rho}(\kappa^{-n}dy)du \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{n=0}^{\infty} 2^{-n-1} \exp(iu\kappa^n y)\hat{\rho}(dy)du \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2} \sum_{n=0}^{\infty} \gamma^n \exp(iuy)\hat{\rho}(dy)du \\
&= \frac{1}{2(1-\gamma)} \int_{\mathbb{R}} \int_{\mathbb{R}} \exp(iuy)\hat{\rho}(dy)du.
\end{aligned}$$

To show finiteness, use (macroscopic) properties of $S(\xi,\cdot) - S(\eta,\cdot)$, i.e. properties on the set $\{\frac{1}{2} < |\xi - \eta|\}$.

Macroscopic properties of $S(\xi,\cdot)-S(\eta,\cdot)$

Fix $\xi,\eta\in[0,1]$, with dyadic sequences $\overline{\xi}_{-n},\overline{\eta}_{-n},n\geq 0$. For $n\in\mathbb{N}$ let

$$\tau_1=\inf\{\ell\geq 0:\bar{\xi}_{-\ell}\neq\bar{\eta}_{-\ell}\},\ \text{and}\ \tau_{n+1}=\inf\{\ell>\tau_n:\bar{\xi}_{-\ell}\neq\bar{\eta}_{-\ell}\}, \tag{13}$$

Note:

$$\xi=\sum_{k=0}^{\infty}(\frac{1}{2})^k\overline{\xi}_{-k},\qquad \xi-\eta=\sum_{\ell=1}^{\infty}(\frac{1}{2})^{\tau_\ell+1}(-1)^{(1-\bar{\xi}_{-\tau_\ell})}.$$

We can show

Proposition 2:
Let $\xi,\eta,x\in[0,1]$. Then

$$S(\xi,x)-S(\eta,x)=2\sum_{\ell=1}^{\infty}\kappa^{\tau_\ell+1}(-1)^{(1-\bar{\xi}_{-\tau_\ell})}. \tag{14}$$

Smoothness of the SBR measure

Now suppose $\frac{1}{2} < \xi - \eta$. Then $\overline{\xi}_0 = 1, \overline{\eta}_0 = 0$, and $\overline{\xi}_{-\tau_1} = \overline{\xi}_{-\tau_2} = 1$. Hence for $\kappa < \frac{1}{\sqrt{2}}$ on $\{1/2 < |\xi - \eta|\}$

$$\inf_{x \in [0,1]} |S(\xi, x) - S(\eta, x)| > 0,$$

and so there is $a > 0$ s. th. supp$(\hat{\rho}) \subset [-a, a]^c$.

Therefore

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \exp(iuy)\hat{\rho}(dy)du < \infty.$$

So we get

Theorem 6:
Let $\kappa \in [\frac{1}{2}, \frac{1}{\sqrt{2}}]$. For a. e. $x \in [0, 1]$, $\mu_x = \lambda \circ S(\cdot, x)^{-1}$ is absolutely continuous w. r. t. λ, with L^2 density. The SBR measure μ is absolutely continuous w. r. t. λ.

Macroscopic properties of $\hat{C}(\cdot, y) - \hat{C}(\cdot, x)$

Duality: SBR measure: $\lambda \circ G_+(\cdot, x)^{-1}$; occupation measure: $\lambda \circ G_-(\xi, \cdot)^{-1}$.

Recall: $G_-(\xi, y) - G_-(\xi, x) = C(y) - C(x) - \int_x^y S(\xi, z)dz = \hat{C}(\xi, y) - \hat{C}(\xi, x)$.

Fix $x, y \in [0, 1]$ with dyadic sequences $\overline{x}_n, \overline{y}_n, n \geq 1$. For $n \in \mathbb{N}$ let

$$\sigma_1 = \inf\{\ell \geq 1 : \bar{x}_\ell \neq \bar{y}_\ell\}, \text{ and } \sigma_{n+1} = \inf\{\ell > \sigma_n : \bar{x}_\ell \neq \bar{y}_\ell\}. \tag{15}$$

Then we have

Proposition 4:
Let $\xi, x, y \in [0, 1]$. Then

$$\hat{C}(\xi, y) - \hat{C}(\xi, x) = \sum_{\ell=1}^{\infty} \gamma^{\sigma_\ell} (-1)^{(1-\bar{y}_{\sigma_\ell})} S\big(B_1^{-\sigma_\ell}(\xi, y), 0\big)$$
$$+ 4\kappa \sum_{\ell=1}^{\infty} \gamma^{\sigma_\ell} \Phi(B_2^1(B_1^{-\sigma_\ell}(\xi, y), B_2^{-\sigma_\ell}(\xi, x)).$$

The existence of local time

For $\gamma < \frac{2}{3}$ this gives on $\{1/2 < |x-y|\}$

$$\inf_{\xi \in [0,1]} |\hat{C}(\xi, y) - \hat{C}(\xi, x)| = |C(y) - C(x) - (y-x)S(\xi, 0)| > 0$$

So analogous Fourier analytic argument yields

Theorem 7:
Let $\gamma \in [\frac{1}{2}, \frac{2}{3}]$. For a. e. $\xi \in [0,1]$

$$G_{-}(\xi, x) = C(x) - C(0) - \int_0^x S(\xi, z) dz$$

possesses a square integrable local time.

Conjecture: regularity of local time in ξ allows to remove *drift* $\int_0^{\cdot} S(\xi, z) dz$ and provides local time for C.

Outlook: Brownian paths as randomized Takagi functions

recall Takagi function:

$$C(x) := \sum_{k=0}^{\infty} \gamma^k f(2^k x), \quad x \in [0,1], \gamma \in]\frac{1}{2}, 1[, \quad f = \mathrm{dist}(\mathbf{Z}, \cdot).$$

Haar-Schauder expansion of Brownian motion: let $(Z_{km})_{k \geq 0, 1 \leq n \leq 2^k}$ be an array of i.i.d. standard normal variables; then the Haar-Schauder expansion of Brownian motion is given for $\gamma = 2^{-\frac{1}{2}}$ by

$$B(x) = \sum_{k=0}^{\infty} \gamma^k \sum_{1 \leq m \leq 2^k} Z_{km}\, f(2^k x)\, 1_{[\frac{m-1}{2^k}, \frac{m}{2^k}]}(x), \quad x \in [0,1].$$

Goal: investigate the geometry of individual Brownian paths by employing the analysis presented to randomized Takagi functions

The Hausdorff dimension of C

By microscopic-macroscopic conversion (of the measure χ) , Theorem 7 and recalling $\gamma = 2^{-\alpha}$ we get

Theorem 8:
For any $\gamma \in [\frac{1}{2}, \frac{2}{3}]$ the Hausdorff dimension of the graph of C is $2 - \alpha$.

Ongoing research:

- result on a.c. of the SBR measure for C holds for any $\kappa \in [\frac{1}{2}, 1[$
- by duality: a.c. of the occupation measure of $\hat{C}$ holds for any $\gamma \in [\frac{1}{2}, 1[$
- case C is replaced by a Weierstrass function: a.c. of SBR measure known for any $\kappa \in [\frac{1}{2}, 1[$ (Shen '17), by duality most likely transfers to occupation measures

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