Fourier analysis and (stochastic) integration
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Abstract

We show how Haar and Schauder functions and more generally Fourier analysis may be used to understand basic problems in stochastic analysis at depth. We start by a Haar-Schauder development of the Brownian motion, revealing its regularity properties. We prove that this development can be used for a simple and efficient proof of Schilder's theorem. Following [GIP15], we then show that it can be used on a pathwise level to explain Young's integral. Combined with the concept of controlled paths, it can be extended to provide the Stratonovich type integral of rough path analysis.

1 Motivation of the Fourier analytic approach of integration

This course deals with integration, in particular the concepts used in stochastic analysis, where continuous, but *rough* functions have to be integrated against each other, e.g. in the notion of *Itô's integral.* Let us briefly review basic concepts of integration, and give a motivation of our approach using Fourier analytic tools. We will, as in the entire course, concentrate on functions on the unit interval $[0, 1]$.

Integration w.r.t. increasing functions

Let $f, g : [0, 1] \to \mathbb{R}$ be functions, g increasing, right continuous, bounded.

Goal: define

$$\int_0^1 f(x)dg(x).$$

This is done as follows in classical integration theory. Let

$$\mu_g(]x,y]) = g(y) - g(x), \quad 0 \le x \le y \le 1.$$

Then μ_g can be extended to a *finite measure* on the Borel sets $\mathcal{B}([0,1])$ of the unit interval. Let $\mathcal{B}$ be the Borel sets of the real line. To obtain the integral, $f : [0,1] \to \mathbb{R}$ has to be $\mathcal{B}([0,1]) - \mathcal{B}$ *-measurable*, i.e.

$$f^{-1}(B) \in \mathcal{B}([0,1]),$$

for any $B \in \mathcal{B}$. In this case proceed as follows:

Case 1: $0 \le f$

Step 1: f step function

$0 \le f$ is a *step function*, if

$$f = \sum_{i=1}^n a_i \mathbf{1}_{B_i}$$

where $B_i \in \mathcal{B}([0,1])$, $((B_i)_{1\le i\le n}$ pairwise disjoint, $0 \le a_i \in \mathbb{R}, 1 \le i \le n$,and $n \in \mathbb{N}$. Then let

$$\int f \; d\mu_g := \sum_{i=1}^n a_i \mu_g(B_i).$$

This is well defined (does not depend on the representation of f).
Step 2: $0 \le f$

In this case

$$\int f \; d\mu_g := \sup\left\{ \int h \; d\mu_g : \; 0 \le h \le f, \; h \text{ step function}\right\} \; (\le \infty).$$

The abstract sup of the definition can be approximated by a simple sequence of step function integrals.

Lemma 1.1. *Let* $0 \le f$ $\mathcal{B}([0,1]) - \mathcal{B}$*-measurable. Then there exists a sequence* $(f_n)_{n\in\mathbb{N}}$ *of simple functions with* $0 \le f_n \uparrow f$*. For each such sequence*

$$\int f_n \; d\mu_g \uparrow \int f \; d\mu_g.$$

Proof. 1. *construction of sequence*:
For $n \in \mathbb{N}$ and $1 \le k \le n2^n$ let

$$B_{kn} = \{\frac{k-1}{2^n} \le f < \frac{k}{2^n}\},$$

and

$$f_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \cdot 1_{B_{kn}}.$$

Then f_n step function, and $0 \le f_n \uparrow f$, since

$$\begin{aligned} f_n 1_{\{\frac{k-1}{2^n} \le f < \frac{k}{2^n}\}} &= \frac{k-1}{2^n} 1_{\{\frac{k-1}{2^n} \le f < \frac{k}{2^n}\}} \\ &\le \frac{k-1}{2^n} 1_{\{\frac{k-1}{2^n} \le f < \frac{2k-1}{2^{n+1}}\}} + \frac{2k-1}{2^{n+1}} 1_{\{\frac{2k-1}{2^{n+1}} \le f < \frac{k}{2^n}\}} \\ &= f_{n+1} 1_{\{\frac{k-1}{2^n} \le f < \frac{k}{2^n}\}}. \end{aligned}$$

2. **Show:** $0 \le h \le f$, h step function; then $\int h \, d\mu_g < \infty \Rightarrow \int h \, d\mu_g \le \sup_{n\in\mathbb{N}} \int f_n \, d\mu_g$.

Let $h = \sum_{i=1}^n a_i \mathbf{1}_{B_i}$, $B = \bigcup_{i=1}^n B_i$. Let $\varepsilon > 0$, for $n \in \mathbb{N}$ let

$$E_n = \{x \in [0,1] : \; f_n(x) < h(x) - \varepsilon\}.$$

Then $B \supset E_n \downarrow \emptyset$, therefore

$$\begin{aligned} \int f_n \, d\mu_g &\ge \int f_n \mathbf{1}_{E_n^c \cap B} \, d\mu_g \\ &\ge \int (h-\varepsilon) \mathbf{1}_{E_n^c \cap B} \, d\mu_g \quad \text{(monotonicity)} \\ &= \sum_{i=1}^n a_i \mu(B_i \cap E_n^c) - \varepsilon \mu_g(E_n^c \cap B) \\ &\overset{n\to\infty}{\longrightarrow} \int h \, d\mu_g - \varepsilon \mu(B) \quad (\sigma\text{-continuity of } \mu_g). \end{aligned}$$

Hence $\sup \int f_n \, d\mu_g \ge \int h \, d\mu_g$ (ε arbitrary).

3. From 2. we get

$$\sup_{n\in\mathbb{N}} \int f_n \, d\mu_g \ge \int f \, d\mu_g.$$

On the other hand

$$\sup_{n\in\mathbb{N}} \int f_n \, d\mu_g \le \int f \, d\mu_g$$

since $f_n \le f$, $n \in \mathbb{N}$ by monotonicity. □

Case 2: f measurable

Decompose $f = f^+ - f^-$ with $0 \le f^+, f^-$; suppose $\int f^+ d\mu_g < \infty$ or $\int f^- d\mu_g < \infty$. Then f is called *μ_g-integrable* and

$$\int_0^1 f(x) dg(x) := \int f d\mu_g = \int f^+ d\mu_g - \int f^- d\mu_g.$$

Integration with respect to functions of bounded variation

Let now $f, g : [0,1] \to \mathbb{R}$ be such that g is right continuous and *of bounded variation*, i.e.

$$\sup\{\sum_{1\leq i\leq n} |g(x_i) - g(x_{i-1})| : 0 \leq x_0 \leq x_1 \cdots \leq x_n \leq 1, \quad n \in \mathbb{N}\} < \infty.$$

Then

$$g = g^+ - g^-,$$

with g^+, g^- right continuous increasing functions. Then we may define

$$\int_0^1 f(x)dg(x) := \int f d\mu_{g^+} - \int f d\mu_{g^-},$$

if the latter exist and are finite.

The Young integral

Let again $f, g : [0,1] \to \mathbb{R}$ be functions. If g is rougher than of bounded variation, to define $\int_0^1 f(x)dg(x)$ we have to require some smoothness also from the integrand f. Let us consider, as in the entire course, *Hölder continuous functions*. For $\alpha \in]0,1[$ let $\mathcal{C}^\alpha = C^\alpha([0,1], \mathbb{R}^d)$ be the space of *(α)-Hölder continuous functions* on $[0,1]$ with finite Hölder norm

$$\|f\|_\alpha := \|f\|_\infty + \sup_{0\leq s<t\leq 1} \frac{|f_{s,t}|}{|t-s|^\alpha},$$

where we denote

$$f_{s,t} := f(t) - f(s), \quad 0 \leq s \leq t \leq 1.$$

Assume $f \in \mathcal{C}^\alpha, g \in \mathcal{C}^\beta$ with $\alpha + \beta > 1$. Then

$$\int_0^1 f(x)dg(x)$$

can be defined as *Young integral*. This will be explained below.

The rough path integral

While the Young integral, as will be seen, can be defined canonically, the situation changes drastically if $f \in \mathcal{C}^\alpha, g \in \mathcal{C}^\beta$ with $\alpha + \beta \leq 1$. This is typically the case if f and g are trajectories of the *Brownian motion* in scenarios of stochastic analysis, as will be explained below. Then we can still define the *rough path integral*

$$\int_0^1 f(x)dg(x).$$

But this time, different versions are possible, e.g. the *Itô integral* or the *Stratonovich integral*. We shall define below a version of the Stratonovich integral, via the following *Fourier analytic approach*.

Idea of the Fourier analytic approach

Let again $f, g : [0,1] \to \mathbb{R}$ be (Hölder) continuous functions.
Consider $L^2([0,1]) = L^2([0,1], dt)$ with scalar product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\, dx, \quad f, g \in L^2([0,1]).$$

Assume that there exist $\dot{f}, \dot{g} \in L^2([0,1])$ such that

$$f = f(0) + \int_0^{\cdot} \dot{f}(s)ds, \quad g = g(0) + \int_0^{\cdot} \dot{g}(s)ds.$$

Assume that a CONS $(\Phi_n)_{n\geq 0}$ of $L^2([0,1])$ is given (definition below), so that

$$\dot{f} = \sum_{n\geq 0} \langle \Phi_n, \dot{f} \rangle \Phi_n, \quad \dot{g} = \sum_{n\geq 0} \langle \Phi_n, \dot{g} \rangle \Phi_n.$$

Then with the primitives $(\Psi_n)_{n\geq 0}$ we have

$$f = f(0) + \sum_{n\geq 0} \langle \Phi_n, \dot{f} \rangle \Psi_n, \quad g = g(0) + \sum_{n\geq 0} \langle \Phi_n, \dot{g} \rangle \Psi_n.$$

If $\Psi_n, n \geq 0$, is sufficiently smooth, we could formally define

$$\int_0^1 f(x)dg(x) = \sum_{k,n\geq 0} \langle \Phi_k, \dot{f} \rangle \langle \Phi_n, \dot{g} \rangle \int_0^1 \Psi_k(x) d\Psi_n(x).$$

The advantage: $\int_0^1 \Psi_k(x)d\Psi_n(x)$ may be easy to get, for instance (as below) if Φ_n, Ψ_n are of bounded variation.

The challenge: *control the convergence of the double series.*

The principal goal of this course is to develop a theory of integration based on this idea, with $(\Phi_n, \Psi_n)_{n\geq 0}$ given by the *Haar-Schauder system* that explains both the *Young integral* and the *Stratonovich version of the rough path integral.*

2 Preliminaries on Fourier expansions

We start by defining complete orthonormal systems.

Definition 2.1. $(\Phi_n)_{n\geq 0} \subset L^2([0,1])$
complete orthonormal system (CONS), if

(i) $\langle \Phi_n, \Phi_m \rangle = \delta_{nm}$, $n, m \geq 0$,

(ii) $\operatorname{span}\{\Phi_n : \ n \geq 0\}$ dense in $L^2([0,1])$.

In this case for $f \in L^2([0,1])$:

$$f = \sum_{n=0}^{\infty} \langle f, \Phi_n \rangle \Phi_n,$$

i.e.

$$\sum_{k=0}^{n} \langle f, \Phi_k \rangle \Phi_k \overset{n\to\infty}{\longrightarrow} f \quad \text{in } L^2([0,1]).$$

To see this, assume that

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_k \Phi_k = f,$$

with some $a_k \in \mathbb{R}, k \in \mathbb{N}$. Then for $0 \leq l$

$$\langle f, \Phi_l \rangle = \lim_{n\to\infty} \langle \sum_{k=0}^{n} a_k \Phi_k, \Phi_l \rangle = \lim_{n\to\infty} \sum_{k=0}^{n} a_k \langle \Phi_k, \Phi_l \rangle = a_l.$$

This proves the convergence claim.
Hence with $f_n = \sum_{k=0}^{n} \langle f, \Phi_k \rangle \Phi_k$ we have

$$\|f_n - f\|^2 \overset{n\to\infty}{\longrightarrow} 0.$$

In other words, for $\mathbb{N} \ni n \leq m \in \mathbb{N}$ with $n, m \to \infty$

$$\begin{aligned} \Big\| \sum_{k=n+1}^{m} \langle f, \Phi_k \rangle \Phi_k \Big\|^2 \quad &= \quad \sum_{k,l=n+1}^{m} \langle f, \Phi_k \rangle \langle f, \Phi_k \rangle \langle \Phi_k, \Phi_l \rangle \\ &= \quad \sum_{k=n+1}^{m} \langle f, \Phi_k \rangle^2 \\ &\overset{n\to\infty}{\longrightarrow} \quad 0. \end{aligned}$$

This implies that

$$\sum_{k=n+1}^{\infty} \langle f, \Phi_k \rangle^2 \to 0$$

as $n \to \infty$. In particular

$$\|f_n\|_2^2 = \sum_{k=0}^{n} \langle f, \Phi_k \rangle^2 \to \|f\|_2^2 = \sum_{k=0}^{\infty} \langle f, \Phi_k \rangle^2.$$

We obtain **Parseval's equation**: For $f, g \in L^2([0,1])$ we have

$$\begin{aligned} \langle f, g \rangle &= \frac{1}{4}[\|f+g\|^2 - \|f-g\|^2] \\ &= \frac{1}{4}\left[\sum_{k=0}^{\infty} \langle f+g, \Phi_k \rangle^2 - \sum_{k=0}^{\infty} \langle f-g, \Phi_k \rangle^2\right] \\ &= \sum_{k=0}^{\infty} \langle f, \Phi_k \rangle \langle g, \Phi_k \rangle. \end{aligned}$$

3 Ciesielski's isomorphism

Much of our approach consists in a quantitative translation of smoothness (roughness) properties of *functions on* $[0,1]$ into the language of *sequence spaces*. This translation is comprised in *Ciesielski's isomorphism* between $\mathcal{C}^\alpha$, the space of α*-Hölder continuous functions* on $[0,1]$ and the space of *bounded sequences* in $\mathbb{R}^d$

$$\ell^\infty(\mathbb{R}^d) = \{x = (x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{R}^d,\ n \in \mathbb{N},\ ||x||_\infty := \sup_{n\in\mathbb{N}} ||x_n|| < \infty\},$$

with which we start our approach.

For this, we consider a particular CONS in $L^2([0,1])$, the *Haar functions* $(H_{pm}, p \geq 0, 1 \leq m \leq 2^p)$. They form a so-called *wavelet*: Wavelets are systems of functions which are created by translations and scalings from one *mother* function.
Mother function:

$$H_{01}(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq t \leq 1, \\ 0, & \text{else}. \end{cases}$$

(define on $\mathbb{R}$, then restrict to $[0,1]$). Therewith

$$\begin{aligned}
H_{00}(t) &= \mathbf{1}_{[0,1]}, \\
H_{11}(t) &= \sqrt{2}H_{01}(2t), \\
H_{12}(t) &= \sqrt{2}H_{01}(2t-1), \\
H_{21}(t) &= 2^{\frac{2}{2}}H_{01}(4t), \\
H_{22}(t) &= 2^{\frac{2}{2}}H_{01}(4t-1), \\
H_{23}(t) &= 2^{\frac{2}{2}}H_{01}(4t-2), \\
H_{24}(t) &= 2^{\frac{2}{2}}H_{01}(4t-3).
\end{aligned}$$

In general for $p \in \mathbb{N}$, $1 \leq m \leq 2^p$:

$$H_{pm}(t) := \begin{cases} 2^{\frac{p}{2}}, & t \in \left[\frac{m-1}{2^p}, \frac{2m-1}{2^{p+1}}\right), \\ -2^{\frac{p}{2}}, & t \in \left[\frac{2m-1}{2^{p+1}}, \frac{m}{2^p}\right), \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $(H_{pm})_{p\in\mathbb{N},1\leq m\leq 2^p}$ is an ONS in $L^2([0,1])$. To see this, observe that

$$\langle H_{pn}, H_{pm}\rangle = 0,$$

if $m \neq n$, since the supports of the two functions are disjoint, and

$$\langle H_{pn}, H_{pn}\rangle = 2^p\lambda([\frac{m-1}{2^p}, \frac{m}{2^p}]) = 1.$$

And if $p, q \in \mathbb{N}$ are different, then for $1 \leq n \leq 2^q$, $1 \leq m \leq 2^p$, either the supports $[\frac{n-1}{2^q}, \frac{n}{2^q}]$ and $[\frac{m-1}{2^p}, \frac{m}{2^p}]$ are disjoint. Or, if $p > q$, $[\frac{m-1}{2^p}, \frac{m}{2^p}] \subset [\frac{n-1}{2^q}, \frac{n}{2^q}]$ and then $H_{qn}|_{[\frac{m-1}{2^p}, \frac{m}{2^p}]} = c$ is constant on the support of H_{pm}, and hence

$$\langle H_{pm}, H_{qn}\rangle = c\int_{\frac{m-1}{2^p}}^{\frac{m}{2^p}} H_{pm}(x)dx = 0.$$

The system is also complete: the linear hull of the set of indicator functions of dyadic intervals of the form $[\frac{m-1}{2^p}, \frac{m}{2^p}]$ is dense in $L^2([0,1])$ (measure theory, definition of integral through integrals of step functions). Moreover $\mathbf{1}_{[\frac{m-1}{2^p}, \frac{m}{2^p}]}$ is in the linear hull of $(H_{pm})_{p\geq 0,1\leq m\leq 2^p}$. To see this, we use induction, starting by

$$\begin{aligned}
1_{[0,1]} &= H_{00}, \\
1_{[0,\frac{1}{2}]} &= \frac{1}{2}(H_{00} + H_{01}), \\
1_{[\frac{1}{2},1]} &= \frac{1}{2}(H_{00} - H_{01}).
\end{aligned}$$

For convenience of notation, we also define $H_{p0} \equiv 0$ for $p \geq 1$.
We need representations of the primitives of the Haar functions, the *Schauder functions*. They are given by

$$G_{pm}(t) := \int_0^t H_{pm}(s)\mathrm{d}s, \quad t \in [0,1], p \geq 0, 0 \leq m \leq 2^p.$$

More explicitly, $G_{00}(t) = t$ and for $p \geq 1$, $1 \leq m \leq 2^p$

$$G_{pm}(t) = \begin{cases} 2^{p/2}\left(t - \frac{m-1}{2^p}\right), & t \in \left[\frac{m-1}{2^p}, \frac{2m-1}{2^{p+1}}\right), \\ -2^{p/2}\left(t - \frac{m}{2^p}\right), & t \in \left[\frac{2m-1}{2^{p+1}}, \frac{m}{2^p}\right), \\ 0, & \text{otherwise.} \end{cases}$$

Since every G_{pm} satisfies $G_{pm}(0) = 0$, we are only able to expand functions f with $f(0) = 0$ in terms of this family (G_{pm}). Therefore, we define

$$G_{-10}(t) := 1, \quad t \in [0,1].$$

To abbreviate notation, we denote the endpoints resp. the midpoints of the supports of H_{pm} by t^i_{pm}, $i = 0, 1, 2$, as

$$t^0_{pm} := \frac{m-1}{2^p}, \quad t^1_{pm} := \frac{2m-1}{2^{p+1}}, \quad t^2_{pm} := \frac{m}{2^p}, \quad p \geq 1, 1 \leq m \leq 2^p.$$

For convenience of notation:

$$\begin{aligned} t^0_{-10} &:= 0, \quad t^1_{-10} := 0, \quad t^2_{-10} := 1, \\ t^0_{00} &:= 0, \quad t^1_{00} := 1, \quad t^2_{00} := 1, \\ t^i_{p0} &:= 0, \quad p \geq 1,\, i = 0,1,2. \end{aligned}$$

Definition of t^i_{-10} and t^i_{00} for $i \neq 1$ arbitrary; but definition for $i = 1$ simplifies statement of Lemma 3.1 below.

Let $f \in C([0,1], \mathbb{R}^d)$. Then define

$$\begin{aligned} \langle H_{pm}, \mathrm{d}f \rangle &:= 2^{\frac{p}{2}}\left[\left(f\left(t^1_{pm}\right) - f\left(t^0_{pm}\right)\right) - \left(f\left(t^2_{pm}\right) - f\left(t^1_{pm}\right)\right)\right] \\ &= 2^{\frac{p}{2}}\left[2f\left(t^1_{pm}\right) - f\left(t^0_{pm}\right) - f\left(t^2_{pm}\right)\right], \quad p \in \mathbb{N}, 1 \leq m \leq 2^p, \\ \langle H_{00}, \mathrm{d}f \rangle &:= f(1) - f(0), \\ \langle H_{-10}, \mathrm{d}f \rangle &:= f(0). \end{aligned}$$

Note that we only defined G_{-10} and not H_{-10}.

Assume $f = f(0) + \int_0^\cdot \dot{f}(s)ds$, with $\dot{f} \in L^2([0,1])$. Then for $p \in \mathbb{N}, 1 \le m \le 2^p$

$$\begin{aligned}\langle H_{pm}, \dot{f} \rangle &= 2^{\frac{p}{2}} [\int_{t_{pm}^0}^{t_{pm}^1} \dot{f}(s)ds - \int_{t_{pm}^1}^{t_{pm}^2} \dot{f}(s)ds] \\ &= 2^{\frac{p}{2}} [(f(t_{pm}^1) - f(t_{pm}^0)) - (f(t_{pm}^2) - f(t_{pm}^1))] \\ &= \langle H_{pm}, \mathrm{d}f \rangle.\end{aligned}$$

Since (H_{pm}) is a CONS of $L^2([0,1], \mathbb{R}^d)$, we can further write

$$\dot{f} = \langle H_{00}, df \rangle H_{00} + \sum_{p \ge 1, 1 \le m \le 2^p} \langle H_{pm}, \dot{f} \rangle H_{pm}.$$

Integrating and interchanging limits gives

$$\begin{aligned} f &= f(0) + \int_0^\cdot \dot{f}(s)ds = \langle H_{-10}, df \rangle G_{-10} + \langle H_{00}, df \rangle \int_0^\cdot H_{00}(s)ds && (1) \\ &+ \sum_{p \ge 1, 1 \le m \le 2^p} \langle H_{pm}, \dot{f} \rangle \int_0^\cdot H_{pm}(s)ds && (2) \\ &= \langle H_{-10}, df \rangle G_{-10} + \langle H_{00}, df \rangle G_{00} + \sum_{p \ge 1, 1 \le m \le 2^p} \langle H_{pm}, df \rangle G_{pm}. && (3)\end{aligned}$$

The following Lemma shows that this can be generalized to (Hölder) continuous functions.

Lemma 3.1 ([Cie60]). *1. For $f \colon [0,1] \to \mathbb{R}^d$ and $k \in \mathbb{N}$, the function*

$$\begin{aligned} f_k &:= \langle H_{-10}, \mathrm{d}f \rangle G_{-10} + \langle H_{00}, \mathrm{d}f \rangle G_{00} + \sum_{p=0}^{k} \sum_{m=1}^{2^p} \langle H_{pm}, \mathrm{d}f \rangle G_{pm} \\ &= \sum_{p=-1}^{k} \sum_{m=0}^{2^p} \langle H_{pm}, \mathrm{d}f \rangle G_{pm} \end{aligned}$$

is the linear interpolation of f between the points
$t_{-10}^1, t_{00}^1, t_{pm}^1$, $0 \le p \le k, 1 \le m \le 2^p$.

2. If f is continuous, then $||f_k - f||_\infty \to 0$ as $k \to \infty$.

3. Let $\alpha \in (0,1)$. A continuous function $f : [0,1] \to \mathbb{R}^d$ is in $\mathcal{C}^\alpha$ if and only if

$$\sup_{p,m} 2^{p(\alpha-1/2)} |\langle H_{pm}, \mathrm{d}f\rangle| < \infty.$$

In this case

$$\sup_{p,m} 2^{p(\alpha-1/2)} |\langle H_{pm}, \mathrm{d}f\rangle| \simeq \|f\|_\alpha \text{ and} \tag{4}$$

$$\|f - f_{k-1}\|_\infty = \Big\| \sum_{p=k}^{\infty} \sum_{m=0}^{2^p} |\langle H_{pm}, \mathrm{d}f\rangle| G_{pm} \Big\|_\infty \lesssim \|f\|_\alpha 2^{-\alpha k}, \quad k \in \mathbb{N}.$$

Here $x \lesssim y$ means $x \leq Cy$ with a universal constant C. And $x \simeq y$ means $x \lesssim y$ as well as $y \lesssim x$.

Proof. 1. Let g_k be the linear interpolation of f between the points $t^1_{-10}, t^1_{00}, t^1_{pm}$, $0 \leq p \leq k, 1 \leq m \leq 2^p$. Then $g_k \in \mathcal{C}^\alpha$.

Show:
$g_k - f_k = 0$.

By (1) $f_n \to g_k$ as $n \to \infty$. But by definition of G_{pm} the contributions of dyadic generations bigger than k have to vanish at the points $t^1_{-10}, t^1_{00}, t^1_{pm}$, $0 \leq p \leq k, 1 \leq m \leq 2^p$.

2. follows from 1. and uniform continuity of f on $[0,1]$.

3. **Show:** For $f \in \mathcal{C}^\alpha$

$$\sup_{p,m} 2^{p(\alpha-1/2)} |\langle H_{pm}, \mathrm{d}f\rangle| < \infty.$$

Fix $p \geq 1, 1 \leq m \leq 2^p$. Then by definition of the Hölder norm

$$\begin{aligned}
|\langle H_{pm}, \mathrm{d}f\rangle| &\leq 2^{\frac{p}{2}}[(|f(t^1_{pm}) - f(t^0_{pm})| + |f(t^2_{pm}) - f(t^1_{pm})|] \\
&= 2^{\frac{p}{2}} 2^{-\alpha(p+1)} [\frac{|f(t^1_{pm}) - f(t^0_{pm})|}{|t^1_{pm} - t^0_{pm}|^\alpha} + \frac{|f(t^2_{pm}) - f(t^1_{pm})|}{|t^2_{pm} - t^1_{pm}|^\alpha}] \\
&\leq 2^{\frac{p}{2}+1-\alpha(p+1)} ||f||_\alpha.
\end{aligned}$$

Hence

$$\sup_{p,m} 2^{p(\alpha-1/2)}|\langle H_{pm}, \mathrm{d}f\rangle| \leq 2^{1-\alpha} \ ||f||_\alpha < \infty.$$

This proves one direction in claim 3., and one inequality in the equivalence of norms.

4. **Show:** For $k \in \mathbb{N}$ we have

$$\|f - f_{k-1}\|_\infty = \Big\| \sum_{p=k}^{\infty} \sum_{m=0}^{2^p} |\langle H_{pm}, \mathrm{d}f\rangle| G_{pm} \Big\|_\infty \lesssim \|f\|_\alpha 2^{-\alpha k}.$$

We fix $p \in \mathbb{N}$ and estimate the contribution of the pth dyadic generation. Observe that by disjointness of the supports of $G_{pm}, 1 \leq m \leq 2^p$ and

$$||G_{pm}||_\infty = 2^{\frac{p}{2}} 2^{-(p+1)} = 2^{-\frac{p}{2}-1},$$

we have

$$|| \sum_{m=1}^{2^p} G_{pm}||_\infty \leq 2^{-\frac{p}{2}-1}.$$

Hence by 3.

$$|| \sum_{m=0}^{2^p} \langle H_{pm}, \mathrm{d}f\rangle G_{pm}||_\infty \lesssim ||f||_\alpha 2^{\frac{p}{2}+1-\alpha(p+1)} 2^{-\frac{p}{2}-1} = 2^{-\alpha(p+1)}||f||_\alpha.$$

Therefore by 2. for $k \in \mathbb{N}$

$$\begin{aligned} ||f - f_{k-1}||_\infty &\leq \lim_{m\to\infty} ||f_m - f_{k-1}||_\infty \\ &\leq \lim_{m\to\infty} \sum_{p=k}^{m-1} || \sum_{m=0}^{2^p} \langle H_{pm}, \mathrm{d}f\rangle G_{pm}||_\infty \\ &\lesssim \sum_{p=k}^{\infty} 2^{-\alpha p} \, ||f||_\alpha \lesssim 2^{-\alpha k} \, ||f||_\alpha. \end{aligned}$$

5. **Show:** If f continuous and $K = \sup_{p,m} 2^{p(\alpha-1/2)}|\langle H_{pm}, \mathrm{d}f\rangle| < \infty$, then $f \in \mathcal{C}^\alpha$, and $||f||_\alpha \lesssim K$.

In fact, let $0 \leq s < t \leq 1$ be given. Assume that $q \in \mathbb{N}$ such that

$$2^{-(q+1)} \leq |t - s| \leq 2^{-q}.$$

Then by 2.

$$\begin{aligned} \frac{|f(t) - f(s)|}{|t - s|^\alpha} &= \lim_{k\to\infty} \frac{|f_k(t) - f_k(s)|}{|t - s|^\alpha} \\ &\leq \sum_{p=0}^{\infty} \sup_{0\leq m\leq 2^p} |\langle H_{pm}, \mathrm{d}f\rangle| \sum_{m=0}^{2^p} \frac{|G_{pm}(t) - G_{pm}(s)|}{|t - s|^\alpha} \\ &\lesssim K \sum_{p=0}^{\infty} 2^{p(\frac{1}{2}-\alpha)} \sum_{m=0}^{2^p} \frac{|G_{pm}(t) - G_{pm}(s)|}{|t - s|^\alpha}. \end{aligned}$$

To estimate the contributions of Schauder functions of generation p, distinguish cases:

Case 1: $p \leq q$

Then either s and t belong to the same dyadic interval $[\frac{m_0-1}{2^p}, \frac{m_0}{2^p}]$ or to two adjacent ones. In both cases

$$\sum_{m=0}^{2^p} \frac{|G_{pm}(t) - G_{pm}(s)|}{|t-s|^\alpha} \leq 2^{\frac{p}{2}} \ |t-s|^{1-\alpha} \lesssim 2^{\frac{p}{2}-q(1-\alpha)}.$$

Case 2: $p > q$

In this case s and t each belong to a dyadic interval of generation p and hence

$$\sum_{m=0}^{2^p} \frac{|G_{pm}(t) - G_{pm}(s)|}{|t-s|^\alpha} \leq 2^{-\frac{p}{2}} \ |t-s|^{-\alpha} \lesssim 2^{-\frac{p}{2}+\alpha(q+1)}.$$

Summarizing, we obtain

$$\begin{aligned} \frac{|f(t)-f(s)|}{|t-s|^\alpha} &\lesssim K \ [\sum_{p=0}^{q} 2^{p(\frac{1}{2}-\alpha)} 2^{\frac{p}{2}-q(1-\alpha)} + \sum_{p=q+1}^{\infty} 2^{p(\frac{1}{2}-\alpha)} 2^{-\frac{p}{2}+\alpha(q+1)}] \\ &\lesssim K \ 2^{q(1-\alpha)-q(1-\alpha)} + 2^{-q\alpha+\alpha(q+1)} \simeq K. \end{aligned}$$

Consequently

$$\sup_{0 \leq s < t \leq 1} \frac{|f(t)-f(s)|}{|t-s|^\alpha} \lesssim K,$$

and $f \in \mathcal{C}^\alpha$.

□

Theorem 3.2 ([Cie60]). *Let* $0 < \alpha < 1$. *For* $p \geq 0, 1 \leq m \leq 2^p$ *let*

$$c_{pm}(\alpha) = 2^{p(\alpha-\frac{1}{2})+\alpha-1}, \quad c_{p0}(\alpha) = 1, \quad c_{-10}(\alpha) = 1.$$

Define

$$\begin{aligned} T_\alpha \quad &: \quad \mathcal{C}^\alpha \to l^\infty(\mathbb{R}^d) \\ f \quad &\mapsto \quad (c_{-10}(\alpha)\langle H_{-10}, \mathrm{d}f\rangle, c_{00}\langle H_{00}, \mathrm{d}f\rangle, (c_{pm}(\alpha)\ \langle H_{pm}, \mathrm{d}f\rangle)_{p\geq 1, 1\leq m\leq 2^p}). \end{aligned}$$

Then T_α *is invertible and*

$$\begin{aligned} T_\alpha^{-1} \quad &: \quad l^\infty(\mathbb{R}^d) \to \mathcal{C}^\alpha \\ (\eta_{-10}, \eta_{00}, (\eta_{pm})_{p\geq 1, 1\leq m\leq 2^p}) \quad &\mapsto \quad \eta_{-10} G_{-10} + \eta_{00} G_{00} + \sum_{p=1}^{\infty} \sum_{1\leq m\leq 2^p} \frac{1}{c_{pm}(\alpha)} \eta_{pm} \ G_{pm}. \end{aligned}$$

T_α is an isomorphism, and for the operator norms we have the following inequalities

$$||T_\alpha|| = 1, \quad ||T_\alpha^{-1}|| \leq \frac{2}{(2^\alpha - 1)(2^{1-\alpha} - 1)}.$$

Proof. 1. **Show:** T_α well defined, $||T_\alpha|| = \sup_{||f||_\alpha \leq 1} \frac{||T_\alpha||_\infty}{||f||_\alpha} \leq 1$.

By Lemma 3.1, for $p \geq 1, 1 \leq m \leq 2^p$ we have

$$|\langle H_{pm}, \mathrm{d}f \rangle| \leq 2^{-(p+1)\alpha + \frac{p}{2} + 1} ||f||_\alpha = \frac{1}{c_{pm}(\alpha)} ||f||_\alpha. \tag{5}$$

This proves the claim.

2. **Show:** $||T_\alpha|| \geq 1$:

Note that for $p \geq 1, 1 \leq m \leq 2^p$ we have

$$||G_{pm}||_\alpha = 2^{\frac{p}{2}} 2^{(-p-1)(1-\alpha)} = 2^{p(\alpha - \frac{1}{2}) + \alpha - 1},$$

while

$$\langle H_{pm}, \mathrm{d}G_{pm} \rangle = \langle H_{pm}, H_{pm} \rangle = 1.$$

Hence

$$||G_{pm}||_\alpha = c_{pm}(\alpha) |\langle H_{pm}, \mathrm{d}G_{pm} \rangle| = ||T_\alpha(G_{pm})||_\infty.$$

3. Let S be the operator defined on $l^\infty(\mathbb{R}^d)$ in the statement of the Theorem.

Show: S well defined, $S(T_\alpha(f)) = f, T_\alpha(S(\eta)) = \eta$ for $f \in \mathcal{C}^\alpha, \eta \in l^\infty(\mathbb{R}^d)$.

The claims follow directly from Lemma 3.1. This implies that T_α is invertible and that S is its inverse.

4. **Show:** $||T_\alpha^{-1}|| \leq \frac{2}{(2^\alpha - 1)(2^{1-\alpha} - 1)}$.

Let $\eta = (\eta_{-10}, \eta_{00}, (\eta_{pm})_{p \geq 1, 1 \leq m \leq 2^p}) \in l^\infty(\mathbb{R}^d)$, choose $0 \leq s < t \leq 1$, and write $f = T_\alpha^{-1}(\eta)$.
Then we have

$$|f(t) - f(s)| \leq ||\eta||_\infty [|t - s| + \sum_{p=1}^{\infty} \sum_{m=1}^{2^p} \frac{1}{c_{pm}(\alpha)} |G_{pm}(t) - G_{pm}(s)|]. \tag{6}$$

Now choose $p_0 \geq 1$ such that

$$2^{-p_0 - 1} < |t - s| \leq 2^{-p_0}.$$

Then for $1 \leq p < p_0$, s and t can belong to at most two adjacent dyadic intervals of generation p. By inspection of the possible cases we get

$$\begin{aligned} & \sum_{m=1}^{2^p} \frac{1}{c_{pm}(\alpha)} |G_{pm}(t) - G_{pm}(s)| \qquad (7) \\ \leq \; & 2^{-p(\alpha - \frac{1}{2}) - \alpha + 1} \; 2^{\frac{p}{2}} \; |t - s| \\ \leq \; & 2^{p(1-\alpha) - \alpha + 1 - p_0(1-\alpha)} \; |t - s|^\alpha = (2^{1-\alpha})^{(1 + p - p_0)} \; |t - s|^\alpha, \end{aligned}$$

while for $p \geq p_0$

$$\begin{aligned}
&\sum_{m=1}^{2^p} \frac{1}{c_{pm}(\alpha)} \, |G_{pm}(t) - G_{pm}(s)| \qquad (8)\\
&\leq \; 2^{-p(\alpha-\frac{1}{2})-\alpha+1} \; 2^{-\frac{p}{2}-1}\\
&\leq \; 2^{-p\alpha-\alpha+(p_0+1)\alpha} \; |t-s|^\alpha = (2^\alpha)^{(p_0-p)} \; |t-s|^\alpha.
\end{aligned}$$

Combining (6), (7) and (8), we obtain the estimate

$$\frac{|f(t)-f(s)|}{|t-s|^\alpha} \leq \frac{2}{(2^\alpha-1)(2^{1-\alpha}-1)} \, ||\eta||_\infty,$$

and therefore

$$||T_\alpha^{-1}|| \leq \frac{2}{(2^\alpha-1)(2^{1-\alpha}-1)}.$$

□

We can extend the isomorphism of Theorem 3.2 to subspaces of Hölder continuous functions which will arise later in the study of the LDP for Brownian motion. For $0 < \alpha \leq 1$ let $\mathcal{C}_0^\alpha$ be the subspace of $\mathcal{C}([0,1])$ composed of all functions f for which $f(0) = 0$ and

$$\lim_{\delta\to 0} \sup_{0\leq s<t\leq 1, |t-s|\leq\delta} \frac{|f(t)-f(s)|}{|t-s|^\alpha} = 0.$$

The isomorphism of Theorem 3.2 will then be restricted to the subspace $\mathcal{C}_0$ of all sequences $\eta = (\eta_{pm})_{p\geq 1, 1\leq m\leq 2^p}$ in $\mathcal{C}$ which converge to 0 as $p \to \infty$ as a target space. The following Theorem holds, with a slightly, not essentially different proof.

Theorem 3.3. *Let $0 < \alpha < 1$. Let $c_{-10}, c_{00}, c_{pm}(\alpha), p \geq 1, 1 \leq m \leq 2^p$, be defined as in Theorem 3.2. Define*

$$\begin{aligned}
T_{\alpha,0} \quad &: \quad \mathcal{C}_0^\alpha \to \mathcal{C}_0,\\
f \quad &\mapsto \quad (c_{00}\langle H_{00}, \mathrm{d}f\rangle, (c_{pm}(\alpha) \, \langle H_{pm}, df\rangle)_{p\geq 1, 1\leq m\leq 2^p}).
\end{aligned}$$

Then $T_{\alpha,0}$ is invertible and

$$\begin{aligned}
T_{\alpha,0}^{-1} \quad &: \quad \mathcal{C}_0 \to \mathcal{C}_0^\alpha,\\
((\eta_{00}, (\eta_{pm})_{p\geq 1, 1\leq m\leq 2^p}) \quad &\mapsto \quad \eta_{00}G_{00} + \sum_{p=1}^\infty \sum_{1\leq m\leq 2^p} \frac{1}{c_{pm}(\alpha)} \eta_{pm} \; G_{pm}.
\end{aligned}$$

$T_{\alpha,0}$ is an isomorphism, and for the operator norms we have the following inequalities

$$||T_{\alpha,0}|| = 1, \quad ||T_{\alpha,0}^{-1}|| \leq \frac{2}{(2^\alpha-1)(2^{1-\alpha}-1)}.$$

4 The Schauder representation of Brownian motion

We shall now present an approach of the study of one-dimensional Brownian motion which is close to Wiener's representation of Brownian motion by Fourier series with trigonometric functions as a basis. So in this section we set $d = 1$. Our basis will be given by the Haar-Schauder system of the preceding section. In fact, the trajectories of Brownian motion will be described just as in the preceding section continuous functions were isomorphically described by sequences.
Let us first recall the definition of Brownian motion. We again restrict our attention to processes on the interval $[0, 1]$.

Definition 4.1. A stochastic process $(B(t))_{0\leq t\leq 1}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called *Brownian motion* on $[0, 1]$, if

(i) $B(0) = 0$,

(ii) for $0 \leq t_1 < t_2 < \cdots < t_n \leq 1$ the random variables $B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1})$ are independent (*independent increments*),

(iii) for $0 \leq s < t \leq 1$ the increment $B(t) - B(s)$ has the law $N(0, t - s)$ (*Gaussian with mean zero and variance* $t - s$),

(iv) $t \mapsto B(t)(\omega)$ is continuous for $\omega \in \Omega$.

We first characterize Brownian motion it as a *Gaussian process.* We need some basic facts about Gaussian processes.

4.1 Covariances and characteristic functions

Let $X = \begin{bmatrix} X_1 \\ \vdots \\ X_d \end{bmatrix}$ be a random vector (i.e. $X_1, \ldots, X_d$ are (real valued) random variables) with values in $\mathbb{R}^d$. Define

$$\mu = \mathbb{E}(X) = \begin{bmatrix} \mathbb{E}(X_1) \\ \vdots \\ \mathbb{E}(X_d) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_d \end{bmatrix}$$

mean (in case X_i is integrable, $1 \leq i \leq d$),

$$C = (c_{ij})_{1\leq i,j\leq d}, \quad c_{ij} = \mathbb{E}([X_i - \mu_i][X_j - \mu_j]), \quad 1 \leq i, j \leq d,$$

covariance matrix (in case X_i is square integrable, $1 \leq i \leq d$).

Definition 4.2. A random vector X is called *Gaussian* with *mean* μ, *covariance* C (C positive definite, invertible), if the density of X is given by

$$f_X(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{\sqrt{\det C}} \exp\Big(-\frac{1}{2}(x-\mu)^* C^{-1}(x-\mu)\Big), \quad x \in \mathbb{R}^d$$

($*$: transposition).

Obviously for any Gaussian vector with mean μ, covariance C, $1 \leq i \leq d$, $1 \leq j \leq d$:

$$\begin{aligned}
\mathbb{E}(X_i) &= \int_{\mathbb{R}^d} x_i f_X(x)\ dx \\
&= \int_{\mathbb{R}^d} (x_i - \mu_i) f_X(x)\ dx + \mu_i \\
& \boxed{y = x - \mu} \\
&= \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{\sqrt{\det C}} y_i \exp\Big(-\frac{1}{2} y^* C^{-1} y\Big)\ dy + \mu_i \\
& \boxed{y \to -y} \\
&= -\int_{\mathbb{R}^d} \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{\sqrt{\det C}} y_i \exp\Big(-\frac{1}{2} y^* C^{-1} y\Big)\ dy + \mu_i \\
&= \mu_i, \\
\mathbb{E}([X_i - \mu_i][X_j - \mu_j]) &= \int_{\mathbb{R}^d} (x_i - \mu_i)(x_j - \mu_j) f_X(x)\ dx \\
& \boxed{y = C^{-\frac{1}{2}}(x - \mu)} \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} (C^{\frac{1}{2}} y)_i (C^{\frac{1}{2}} y)_j \exp\Big(-\frac{1}{2} y^* y\Big)\ dy \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \sum_{k=1}^d C^{\frac{1}{2}}_{ik} y_k \sum_{l=1}^d C^{\frac{1}{2}}_{jl} y_l \exp\Big(-\frac{1}{2} y^* y\Big)\ dy \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{k=1}^d C^{\frac{1}{2}}_{ik} C^{\frac{1}{2}}_{kj} \int_{\mathbb{R}^d} y_k^2 \exp\Big(-\frac{1}{2} y^* y\Big)\ dy \\
&= \sum_{k=1}^d C^{\frac{1}{2}}_{ik} C^{\frac{1}{2}}_{kj} \\
&= c_{ij}.
\end{aligned}$$

Hence the meaning of the notions and the aim of the definition are consistent. For Gaussian vectors *independence* has a very simple characterization:

Lemma 4.3. *Let* $X = \begin{bmatrix} X_1 \\ \vdots \\ X_d \end{bmatrix}$ *be Gaussian. Then* $X_1, \ldots, X_d$ *independent if and only if (iff)* C *is a diagonal matrix (*X uncorrelated*).*

Proof. "⇒" $i \neq j$:

$$\begin{aligned} c_{ij} &= \mathbb{E}([X_i - \mu_i][X_j - \mu_j]) \\ &= \mathbb{E}([X_i - \mu_i]) \cdot \mathbb{E}([X_j - \mu_j]) \quad (X_i, X_j \text{ independent}) \\ &= 0. \end{aligned}$$

"⇐" C diagonal with $C = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{bmatrix}$. Then for $x \in \mathbb{R}^d$

$$\begin{aligned} f_X(x) &= \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{(\prod_{i=1}^d \lambda_i)^{\frac{1}{2}}} \exp\Big(-\frac{1}{2}(x-\mu)^* C^{-1}(x-\mu)\Big) \\ &= \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{(\prod_{i=1}^d \lambda_i)^{\frac{1}{2}}} \exp\Big(-\frac{1}{2}\sum_{j=1}^d (x_j-\mu_j)^2 \frac{1}{\lambda_j}\Big) \\ &= \prod_{j=1}^d \underbrace{\frac{1}{\sqrt{2\pi\lambda_j}} \exp\Big(-\frac{1}{2}(x_j-\mu_j)^2 \frac{1}{\lambda_j}\Big)}_{\text{density of a } N(\mu_j, \lambda_j)\text{-distributed random variable}} . \end{aligned}$$

$\Rightarrow X_1, \ldots, X_d$ independent.

□

To characterize Gaussian laws we use Fourier transforms or characteristic functions.

Lemma 4.4. *Let X be Gaussian with mean μ, covariance C. Then for $\theta \in \mathbb{R}^d$ we have*

$$\varphi_X(\theta) = \mathbb{E}(\exp(i\theta^* X)) = \exp\Big(i\theta^*\mu - \frac{1}{2}\theta^* C\theta\Big).$$

Proof. 1) Let first Z be a standard normal variable. Then for $u \in \mathbb{R}$ we have

$$\varphi_Z(u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(iux) \exp(-\frac{x^2}{2})dx.$$

Consequently

$$\begin{aligned} \varphi_Z'(u) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} ix \ \exp(iux) \exp(-\frac{x^2}{2})dx \\ &= \frac{i}{\sqrt{2\pi}}[-\exp(iux) \exp(-\frac{x^2}{2})|_{-\infty}^{\infty} + \int_{\mathbb{R}} iu \ \exp(iux) \exp(-\frac{x^2}{2})dx] \\ &= -u\varphi_Z(u), \end{aligned}$$

$\varphi_Z(0) = 1$. This differential equation has the unique solution $\varphi_Z(u) = \exp(-\frac{u^2}{2})$.

2) For $\theta \in \mathbb{R}^d$ we have

$$\begin{aligned}
\varphi_X(\theta) &= \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{\sqrt{\det C}} \exp\Big(i\theta^* x - \frac{1}{2}(x-\mu)^* C^{-1}(x-\mu)\Big)\, dx \\
& \boxed{y = x - \mu} \\
&= \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{(\det C)^{\frac{1}{2}}} \exp(i\theta^*\mu) \exp\Big(i\theta^* y - \frac{1}{2} y^* C^{-1} y\Big)\, dy \\
& \boxed{z = C^{-\frac{1}{2}} y} \\
&= \exp(i\theta^*\mu) \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \exp\Big(i\theta^* C^{\frac{1}{2}} z - \frac{1}{2} z^* z\Big)\, dz \\
&= \exp(i\theta^*\mu) \prod_{j=1}^d \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\Big(i \underbrace{\sum_{k=1}^d \theta_k C^{\frac{1}{2}}_{kj}}_{=:u_j} z_j - \frac{1}{2} z_j^2\Big)\, dz_j \\
&= \exp(i\theta^*\mu) \prod_{j=1}^d \exp\Big(-\frac{1}{2} u_j^2\Big) \\
& \boxed{\textstyle\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp(iux) \exp(-\frac{x^2}{2})\, dx = \exp(-\frac{1}{2}u^2),\ u \in \mathbb{R} \text{ (part 1))}} \\
&= \exp(i\theta^*\mu) \exp\Big(-\frac{1}{2} \sum_{j=1}^d \Big(\sum_{j=1}^d \theta_j C^{\frac{1}{2}}_{kj}\Big)^2\Big) \\
&= \exp(i\theta^*\mu) \exp\Big(-\frac{1}{2} \theta^* C \theta\Big).
\end{aligned}$$

□

In what follows, we shall use the fact known from elementary probability that a law of a random vector is uniquely determined by its characteristic function.

Propostion 4.5. *X is Gaussian $\Leftrightarrow$ For all $\theta \in \mathbb{R}^d$ the random variable $\theta^* X$ is Gaussian.*

Proof. "$\Rightarrow$" μ mean, C covariance, for $\theta \in \mathbb{R}^d$ let $Z_\theta = \theta^* X$. Then for $t \in \mathbb{R}$

$$\begin{aligned}
\varphi_{Z_\theta}(t) &= \mathbb{E}(\exp(it\theta^* X)) \\
&= \mathbb{E}(\exp(i(t\theta)^* X)) \\
&= \exp\Big(it\theta^*\mu - \frac{1}{2} t^2 \theta^* C \theta\Big).
\end{aligned}$$

Therefore Z_θ is Gaussian with mean $\theta^*\mu$, variance $\theta^* C \theta$.

"$\Leftarrow$" Let Z_θ be Gaussian for $\theta \in \mathbb{R}^d$. Choose $\theta_j = (0, \ldots, 0, 1, 0 \ldots, 0)$ (jth position); then $Z_{\theta_j} = X_j$ is Gaussian, i.e. X_j is square integrable;

hence the mean vector μ of X and the covariance matrix C of X exist. We further have

$$\begin{aligned}
\mathbb{E}(Z_\theta) &= \sum_{j=1}^{d} \mathbb{E}(X_j\theta_j) \\
&= \sum_{j=1}^{d} \theta_j\mu_j \\
&= \theta^*\mu, \\
\operatorname{var}(Z_\theta) &= \mathbb{E}((\theta^*X - \theta^*\mu)^2) \\
&= \mathbb{E}((\theta^*(X-\mu))^2) \\
&= \sum_{i,j=1}^{d} \mathbb{E}(\theta_i(X_i-\mu_i)(X_j-\mu_j)\theta_j)) \\
&= \sum_{i,j=1}^{d} \theta_i\theta_j c_{ij} \\
&= \theta^*C\theta.
\end{aligned}$$

Consequently we have

$$\begin{aligned}
\varphi_{Z_\theta}(t) &= \exp\Big(it\theta^*\mu - \frac{1}{2}t^2\theta^*C\theta\Big) \\
\varphi_X(\theta) &= \varphi_{Z_\theta}(1) = \exp\Big(i\theta^*\mu - \frac{1}{2}\theta^*C\theta\Big).
\end{aligned}$$

Apply Lemma 4.4: X Gaussian with mean μ, covariance C (uniqueness of characteristic functions).

□

Definition 4.6. A stochastic process $(X(t))_{0\le t\le 1}$ is called *Gaussian*, if for $0 \le t_1 < \cdots < t_d \le 1$ we have: $(X(t_1), \ldots, X(t_d))$ is Gaussian.

Remark 4.7. 1. By definition, Gaussian processes are uniquely determined by

$$\begin{aligned}
\mu(t) &= \mathbb{E}(X(t)), \quad 0 \le t \le 1, \\
C(s,t) &= \operatorname{cov}(X(s), X(t)) = \mathbb{E}([X(s)-\mu(s)][X(t)-\mu(t)]), \quad 0 \le s,t \le 1.
\end{aligned}$$

2. By definition, a Brownian motion is Gaussian. Reason: for $0 \le t_1 < \cdots < t_d$ the vector $(X(t_1), X(t_2)-X(t_1), \ldots, X(t_d)-X(t_{d-1}))$ is Gaussian by definition. Hence for $\theta \in \mathbb{R}^d$

$$\sum_{i=1}^{d} \theta_i(X(t_i) - X(t_{i-1})) \qquad (t_0 := 0)$$

is Gaussian. Therefore

$$\begin{aligned}\sum_{i=1}^{d}\theta_i X(t_i) \quad - \quad \sum_{i=1}^{d}\theta_i X(t_{i-1}) &= \sum_{i=1}^{d}\theta_i X(t_i) - \sum_{i=1}^{d}\theta_{i+1} X(t_i)\\ &= \sum_{i=1}^{d}(\theta_i - \theta_{i+1})X(t_i)\end{aligned}$$

with $\theta_{d+1} = 0$ is Gaussian. And consequently for $\eta \in \mathbb{R}^d$

$$\sum_{i=1}^{d}\eta_i X(t_i)$$

is Gaussian, so by proposition 4.5 $(X(t_1), \ldots, X(t_d))$ is Gaussian. We have:

$$\begin{aligned}\mu(t) &= \mathbb{E}(X(t)) = 0,\\ C(s,t) &= \mathbb{E}(X(t)\cdot X(s)) \quad (0 \le s \le t \le 1)\\ &= \mathbb{E}((X(t) - X(s) + X(s))\cdot X(s))\\ &= \mathbb{E}((X(t)-X(s))\cdot X(s)) + \mathbb{E}(X(s)^2)\\ &= 0 + s \quad \text{(definition of Brownian motion).}\\ &= s \wedge t.\end{aligned}$$

Propostion 4.8. *Let $(X(t))_{0\le t\le 1}$ be a Gaussian process with $\mu(t) = \mathbb{E}(X(t)) = 0$, $C(s,t) = s \wedge t$, $0 \le s,t \le 1$. Then this process has independent increments. If X is continuous and $X(0) = 0$, then X is a Brownian motion on $[0,1]$.*

Proof. Let $0 \le t_1 < \cdots < t_n \le 1$. Then $(X(t_1), \ldots, X(t_d))^*$ is Gaussian. Moreover $(X(t_1), X(t_2) - X(t_1), \ldots, X(t_d) - X(t_{d-1}))$ is Gaussian (proposition 4.5). Further for $1 \le i,j \le n$, $i < j$, $t_0 = 0$ we have

$$\begin{aligned}& \mathbb{E}((X(t_i) - X(t_{i-1}))(X(t_j) - X(t_{j-1})))\\ =\ & \mathbb{E}(X(t_i)X(t_j)) - \mathbb{E}(X(t_i)X(t_{j-1})) - \mathbb{E}(X(t_{i-1})X(t_j)) + \mathbb{E}(X(t_{i-1})X(t_{j-1}))\\ =\ & t_i - t_i - t_{i-1} + t_{i-1}\\ =\ & 0.\end{aligned}$$

Hence $(X(t_1), X(t_2) - X(t_1), \ldots, X(t_d) - X(t_{d-1}))$ is independent (Lemma 4.3). Let $0 \le s < t \le 1$. Then $X(t) - X(s)$ is Gaussian and we have

$$\begin{aligned}\mathbb{E}(X(t) - X(s)) &= \mu(t) - \mu(s) = 0\\ \operatorname{var}(X(t) - X(s)) &= \mathbb{E}((X(t) - X(s))^2)\\ &= \mathbb{E}(X(t)^2) - 2\mathbb{E}(X(t)X(s)) + \mathbb{E}(X_(s)^2)\\ &= t - 2s + s\\ &= t - s.\end{aligned}$$

Therefore by definition X is a Brownian motion on $[0,1]$.

□

4.2 The Schauder representation

Given a Brownian motion X indexed by the unit interval, with the same notation as in the preceding section, we write it sample by sample as a series with coefficients $\langle H_{00}, \mathrm{d}X\rangle, \langle H_{pm}, \mathrm{d}X\rangle, p \geq 1, 1 \leq m \leq 2^p$.

Show: $\langle H_{pm}, dX\rangle, p \geq 0, 1 \leq m \leq 2^p$ is an i.i.d. sequence of standard normal variables.

Proof. 1. First of all, note that since increments of Brownian motion are centered, we have

$$\mathbb{E}(\langle H_{pm}, dX\rangle) = 0, \quad p \geq 0, 1 \leq m \leq 2^p.$$

2. To calculate covariances, let $(p,m),(q,n)$ be given with $p,q \in \mathbb{N}, 1 \leq m \leq 2^p, 1 \leq n \leq 2^q$.

Case 1: $p = q, n = m$

In this case by independence of increments

$$\begin{aligned}
\mathbb{E}(\langle H_{pm}, dX\rangle^2) &= 2^p[\mathbb{E}([(X(t^1_{pm}) - X(t^0_{pm}) - (X(t^2_{pm}) - X(t^1_{pm}))]^2)] \\
&= 2^p[\mathbb{E}((X(t^1_{pm}) - X(t^0_{pm}))^2) + \mathbb{E}((X(t^2_{pm}) - X(t^1_{pm})))^2) \\
&= 2^p[2^{-(p+1)} + 2^{-(p+1)}] = 1.
\end{aligned}$$

Case 2: $p = q, m < n$

Here as a direct consequence of independence of increments

$$\mathbb{E}(\langle H_{pm}, dX\rangle\langle H_{pn}, dX\rangle) = 0.$$

Case 3: $p < q, [\frac{m-1}{2^p}, \frac{m}{2^p}] \cap [\frac{n-1}{2^q}, \frac{n}{2^q}] = \emptyset$

In this case, as in the preceding one by independence of increments

$$\mathbb{E}(\langle H_{pm}, dX\rangle\langle H_{qn}, dX\rangle) = 0.$$

Case 4: $p < q, [\frac{m-1}{2^p}, \frac{m}{2^p}] \supset [\frac{n-1}{2^q}, \frac{n}{2^q}]$

Here even w.l.o.g. $[\frac{m-1}{2^p}, \frac{2m-1}{2^{p+1}}] \supset [\frac{n-1}{2^q}, \frac{n}{2^q}]$. Hence

$$\begin{aligned}
&\mathbb{E}(\langle H_{pm}, dX\rangle\langle H_{qn}, dX\rangle) \\
&\quad = 2^{\frac{p+q}{2}}[\mathbb{E}((X(t^1_{pm}) - X(t^0_{pm}))(X(t^1_{qn}) - X(t^0_{qn}))) \\
&\quad -\mathbb{E}((X(t^1_{pm}) - X(t^0_{pm}))(X(t^2_{qn}) - X(t^1_{qn}))) \\
&\quad -\mathbb{E}((X(t^2_{pm}) - X(t^1_{pm}))(X(t^1_{qn}) - X(t^0_{qn}))) \\
&\quad +\mathbb{E}((X(t^2_{pm}) - X(t^1_{pm}))(X(t^2_{qn}) - X(t^1_{qn}))) \\
&\quad = \mathbb{E}((X(t^1_{qn}) - X(t^0_{qn}))^2) \\
&\qquad -\mathbb{E}((X(t^2_{qn}) - X(t^1_{qn}))^2) \quad \text{(independence of increments)} \\
&= 0 \quad \text{(equal length of intervals)}.
\end{aligned}$$

This proves the claim. □

This, in turn, allows us to construct Brownian motion indexed by the unit interval by taking any sequence of i.i.d. standard normal variables $(Z_{00}, (Z_{pm})_{p\geq 1, 1\leq m\leq 2^p})$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and defining the stochastic process

$$B(t) = Z_{00}G_{00}(t) + \sum_{p=1}^{\infty} \sum_{1\leq m\leq 2^p} Z_{pm}G_{pm}(t), \quad t \in [0, 1]. \tag{9}$$

To get information about the quality of convergence of this Fourier series, we need to control the size of the random sequence $(Z_{pm})_{p\geq 1, 1\leq m\leq 2^p}$ in the following Lemma.

Lemma 4.9. *There exists a real valued random variable C such that for $p \geq 1, 1 \leq m \leq 2^p$ we have*

$$|Z_{pm}| \leq C\sqrt{p \ln 2}. \tag{10}$$

Proof. 1. **Show:**

$$\mathbb{P}(|Z_{pm}| \geq \sqrt{2\beta \ln 2^p}) \leq \sqrt{\frac{2}{\pi}} 2^{-\beta p}.$$

For $x \geq 1, p \geq 1, 1 \leq m \leq 2^p$ we have

$$\mathbb{P}(|Z_{pm}| \geq x) = \sqrt{\frac{2}{\pi}} \int_x^{\infty} e^{-\frac{u^2}{2}} du \leq \sqrt{\frac{2}{\pi}} \int_x^{\infty} u e^{-\frac{u^2}{2}} du = \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}}.$$

Hence for $\beta > 1$

$$\mathbb{P}(|Z_{pm}| \geq \sqrt{2\beta \ln 2^p}) \leq \sqrt{\frac{2}{\pi}}\, e^{-\beta \ln 2^p} = \sqrt{\frac{2}{\pi}} 2^{-\beta p}.$$

2. **Show:** $|Z_{pm}| \leq \sqrt{4\beta p \ln 2}$ for almost all $p \geq 1, 1 \leq m \leq 2^p$ with probability 1.

For $p \geq 0, 1 \leq m \leq 2^p$ let

$$A_{pm} = \{|Z_{pm}| \leq \sqrt{4\beta p \ln 2}\}.$$

Then by part 1.

$$\sum_{p=1}^{\infty} \sum_{1 \leq m \leq 2^p} \mathbb{P}(A_{pm}^c) \lesssim \sum_{p=0}^{\infty} 2^p 2^{-\beta p} < \infty,$$

since $\beta > 1$. So, the lemma of Borel-Cantelli yields

$$\mathbb{P}(\cap_{q \in \mathbb{N}} \cup_{p \geq q, 1 \leq m \leq 2^p} A_{pm}^c) = 0,$$

and so

$$\mathbb{P}(\cup_{q \in \mathbb{N}} \cap_{p \geq q, 1 \leq m \leq 2^p} A_{pm}) = 1.$$

This translates to: With probability 1 there exists $q \in \mathbb{N}$ such that for all $p \geq q$, all $1 \leq m \leq 2^p$ we have $|Z_{pm}| \leq \sqrt{4\beta \ln 2^p}$.

Hence

$$C = \sup_{p \geq 1, 1 \leq m \leq 2^p} \frac{|Z_{pm}|}{\sqrt{p \ln 2}}$$

is almost surely finite, and yields the desired inequality. □

The preceding Lemma enables us to state that the convergence in (9) is absolute and therefore the process continuous. Its law has the characteristics of the law of a Brownian motion, as the following Theorem shows.

Theorem 4.10. *The series in (9) converges absolutely in the uniform norm to a continuous process B which is a Brownian motion on $[0, 1]$.*

Proof. 1. **Show:** If $B_p(t) = Z_{00}G_{00}(t) + \sum_{k=1}^{p} \sum_{1 \leq m \leq 2^k} Z_{qm}G_{km}(t),\ p \in \mathbb{N}$, then we have

$$||B_p - B_q||_\infty \to 0 \quad \text{as} \quad p, q \to \infty.$$

Consequently, B is a.s. continuous.

Let $p, q \geq 1$ be such that $q \geq p$. Then with C of the preceding Lemma

$$\begin{aligned}
||B_p - B_q||_\infty &\leq \sum_{n=p}^{q} || \sum_{1 \leq m \leq 2^n} |Z_{nm}| G_{nm}||_\infty \\
&\leq C \sum_{n=p}^{q} \sqrt{n \ln 2}\ || \sum_{1 \leq m \leq 2^n} G_{nm}||_\infty \\
&\leq C \sum_{n=p}^{\infty} \sqrt{n}\ 2^{-\frac{n}{2}-1},
\end{aligned}$$

which converges to 0 as p tends to ∞.

2. **Show:** For $t \in [0,1]$ $\mathbb{E}((B_q(t) - B_p(t))^2) \to 0$ as $p, q \to \infty$. In particular, $B(t)$ is square integrable for $t \in [0,1]$.

In fact, for $t \in [0,1]$, $p, q \geq 1$ such that $q \geq p$ by the law properties of $Z_{pm}, p \geq 1, 1 \leq m \leq 2^p$,

$$\begin{aligned}
\mathbb{E}((B_q(t) - B_p(t))^2) &= \mathbb{E}([\sum_{n=p}^{q} \sum_{1 \leq m \leq 2^n} Z_{nm} G_{nm}(t)]^2) \\
&= \sum_{n=p}^{q} \sum_{1 \leq m \leq 2^n} G_{nm}(t)^2 \leq \sum_{n=p}^{\infty} 2^{-n-2},
\end{aligned}$$

which converges to 0 as $p \to \infty$.

3. **Show:** For $d \in \mathbb{N}$, $0 \leq t_1 < \cdots < t_d \leq 1$, and $\theta = (\theta_1, \cdots, \theta_d) \in \mathbb{R}^d$ the vector $(B(t_1), \cdots, B(t_d))$ is Gaussian with

$$\mathbb{E}(B(t_i)) = 0, \text{cov } (B(t_i), B(t_j)) = t_i \wedge t_j, \quad 1 \leq i, j \leq d.$$

We compute the Fourier transform $\varphi(\theta)$ of the vector $(B(t_1), \cdots, B(t_d))$ at θ. By dominated convergence and the law properties of $Z_{pm}, p \geq 1, 1 \leq m \leq 2^p$, we have

$$\begin{aligned}
\varphi(\theta) &= \mathbb{E}(\exp(i \sum_{j=1}^{d} \theta_j B(t_j))) \\
&= \mathbb{E}(\exp(i \sum_{j=1}^{d} \theta_j \sum_{p=0}^{\infty} \sum_{0 \leq m \leq 2^p} Z_{pm} G_{pm}(t_j))) \\
&= \prod_{p=0}^{\infty} \prod_{0 \leq m \leq 2^p} \mathbb{E}(\exp(i Z_{pm} \sum_{j=1}^{d} \theta_j G_{pm}(t_j))) \\
&= \prod_{p=0}^{\infty} \prod_{0 \leq m \leq 2^p} \exp(-\frac{1}{2} (\sum_{j=1}^{d} \theta_j G_{pm}(t_j))^2) \\
&= \exp(-\frac{1}{2} \sum_{p=0}^{\infty} \sum_{0 \leq m \leq 2^p} (\sum_{j=1}^{d} \theta_j G_{pm}(t_j))^2) \\
&= \exp(-\frac{1}{2} \sum_{j,k=1}^{d} \theta_j \theta_k \sum_{p=0}^{\infty} \sum_{0 \leq m \leq 2^p} G_{pm}(t_j) G_{pm}(t_k)).
\end{aligned}$$

Now observe that Parseval's equation implies for $1 \le j, k \le d$

$$\begin{aligned} t_j \wedge t_k &= \langle 1_{[0,t_j]}, 1_{[0,t_k]} \rangle \\ &= \sum_{p=0}^{\infty} \sum_{0 \le m \le 2^p} \langle 1_{[0,t_j]}, H_{pm} \rangle \langle 1_{[0,t_k]}, H_{pm} \rangle \\ &= \sum_{p=0}^{\infty} \sum_{0 \le m \le 2^p} G_{pm}(t_j) G_{pm}(t_k). \end{aligned}$$

Therefore we finally obtain

$$\varphi(\theta) = \exp(-\frac{1}{2} \sum_{j,k=1}^{d} \theta_j \theta_k \ t_j \wedge t_k).$$

This implies the claimed properties, by Proposition 4.5.

4. It remains to apply Proposition 4.8. □

We now use the Schauder representation of Brownian motion to show its Hölder continuity properties.

Theorem 4.11. *The Brownian motion $B = (B(t))_{0 \le t \le 1}$ is Hölder continuous of order $\alpha < 1/2$. Its trajectories are a.s. nowhere Hölder continuous of order $\alpha > 1/2$.*

Moreover we have (Lévy's modulus of continuity)

$$\mathbb{P}\Big(\sup_{0 \le s < t \le 1} \frac{|B(t) - B(s)|}{h(|t-s|)} < \infty \Big) = 1, \tag{11}$$

where $h(u) = \sqrt{u \log(1/u)}, u > 0$.

In particular, for $\alpha < \frac{1}{2}$, the trajectories of B are $\mathbb{P}$-a.s. contained in $\mathcal{C}_0^\alpha$.

Proof. 1. Let first $\alpha \in]0,1[$, $(c_{pm})_{p \ge 1, 1 \le m \le 2^p}$ sequence in $\mathbb{R}$ for which there exists $c \in \mathbb{R}$ such that for $p \ge 0, 1 \le m \le 2^p$ we have

$$|c_{pm}| \le c\sqrt{p}.$$

Let

$$f(t) = \sum_{p=1}^{\infty} \sum_{1 \le m \le 2^p} c_{pm} \, G_{pm}(t), \quad t \in [0,1].$$

The trajectories of B fulfill this inequality by Lemma 4.9.

Show: $\sup_{0 \le s < t \le 1} \frac{|f(t) - f(s)|}{h(|t-s|)} < \infty$.

In fact, by continuity properties of G_{00}, we may assume $c_{00} = 0$. Then for $0 \le s < t \le 1$

$$|f(t) - f(s)| \le \sum_{p=1}^{\infty} \sum_{m=1}^{2^p} |c_{pm}|\,|G_{pm}(t) - G_{pm}(s)|. \tag{12}$$

Now choose $p_0 \ge 1$ such that

$$2^{-p_0-1} < |t-s| \le 2^{-p_0}.$$

W.l.o.g. we can assunme that $p_0 \ge 1$. Then for $1 \le p < p_0$, s and t can belong to at most two adjacent dyadic intervals of generation p. By inspection of the different cases we get

$$\begin{aligned}
&\sum_{m=1}^{2^p} |c_{pm}|\,|G_{pm}(t) - G_{pm}(s)| \qquad (13)\\
&\le c\sqrt{p}\, 2^{\frac{p}{2}}\, |t-s| \\
&\le c\sqrt{p}\, 2^{\frac{p-p_0}{2}}\, |t-s|^{\frac{1}{2}} \\
&\le \frac{c}{\sqrt{\ln 2}} \sqrt{\frac{p}{p_0}}\, 2^{\frac{p-p_0}{2}} \sqrt{|t-s|\, \ln \frac{1}{|t-s|}},
\end{aligned}$$

while for $p \ge p_0$

$$\begin{aligned}
&\sum_{m=1}^{2^p} |c_{pm}|\,|G_{pm}(t) - G_{pm}(s)| \qquad (14)\\
&\le c\,\sqrt{p}\, 2^{-\frac{p}{2}} \\
&\le \frac{c}{\sqrt{\ln 2}} \sqrt{\frac{p}{p_0}}\, 2^{\frac{p_0+1-p}{2}} \sqrt{|t-s|\, \ln \frac{1}{|t-s|}}.
\end{aligned}$$

Now

$$\sum_{0 \le p \le p_0} \sqrt{\frac{p}{p_0}} 2^{-\frac{p-p_0}{2}} \le \sum_{0 \le p \le p_0} 2^{-\frac{p-p_0}{2}} \lesssim 1.$$

And

$$\sum_{p > p_0} \sqrt{\frac{p}{p_0}} 2^{\frac{p_0-p}{2}} \le \frac{2^{\frac{p_0}{2}}}{\sqrt{p_0}} \int_{p_0}^{\infty} \sqrt{x} 2^{-\frac{x}{2}}\, dx.$$

Integration by parts for $y \ge 1$ gives

$$\begin{aligned}
\int_y^{\infty} \sqrt{x} 2^{-\frac{x}{2}}\, dx &= -\frac{2}{\ln 2}\sqrt{x} 2^{-\frac{x}{2}}\Big|_y^{\infty} + \frac{1}{\ln 2} \int_y^{\infty} \frac{1}{\sqrt{x}} 2^{-\frac{x}{2}}\, dx \\
&\le \frac{2}{\ln 2}\sqrt{y} 2^{-\frac{y}{2}} + \frac{1}{\ln 2} \int_y^{\infty} 2^{-\frac{x}{2}}\, dx \\
&\le \frac{2}{\ln 2}\sqrt{y} 2^{-\frac{y}{2}} + \frac{2}{(\ln 2)^2} 2^{-\frac{y}{2}}.
\end{aligned}$$

Now set $y = p_0$, to see

$$\sum_{p>p_0} \sqrt{\frac{p}{p_0}} 2^{\frac{p_0-p}{2}} \lesssim \frac{2^{\frac{p_0}{2}}}{\sqrt{p_0}} (\sqrt{p_0}+1) \ 2^{-\frac{p_0}{2}} \simeq 1.$$

Hence (12), (13) and (14) imply

$$\sup_{0\le s<t\le 1} \frac{|f(t)-f(s)|}{\sqrt{|t-s| \ \ln \frac{1}{|t-s|}}} \lesssim 1.$$

2. Part 1. implies all claims about Hölder continuity for $\alpha < \frac{1}{2}$.

3. Let us next fix $\alpha > \frac{1}{2}$. For $c > 0, \epsilon > 0$ let

$$\begin{aligned}\Gamma(\alpha, c, \epsilon) = \{\omega \in \Omega : \ \exists \ s \in [0,1] \ \forall \ t \in [0,1], \ |s-t| \le \epsilon : \\ |B(t)(\omega) - B(s)(\omega)| \le c|s-t|^\alpha\}.\end{aligned}$$

Show: $\mathbb{P}(\Gamma(\alpha, c, \epsilon)) = 0$ for all $c, \epsilon > 0$. Consequently, B is a. s. nowhere Hölder continuous of order α.

3.1. To this end, for all $m, n \in \mathbb{N}, m \le n$, and $0 \le k < n$ let

$$X_{m,k} = \max\{|B(\frac{j}{n}) - B(\frac{j+1}{n})| : k \le j < m+k\}.$$

Show: For $m, n \in \mathbb{N}$ such that $\frac{m}{n} \le \epsilon$ we have

$$\Gamma(\alpha, c, \epsilon) \subset \{\min_{0\le k\le n-m} X_{m,k} \le 2c(\frac{m}{n})^\alpha\}. \tag{15}$$

Let $\omega \in \Gamma(\alpha, c, \epsilon)$. Let $s \in [0,1]$ such that for all $t \in [0,1]$ with $|s-t| \le \epsilon$ we have $|B(t)(\omega) - B(s)(\omega)| \le c|s-t|^\alpha$.

Choose $0 \le k \le n-m$ such that $\frac{k}{n} \le s < \frac{k+m}{n}$. Then for $k \le j < k+m$

$$\begin{aligned}|B(\frac{j}{n})(\omega) - B(\frac{j+1}{n})(\omega)| &\le |B(\frac{j}{n})(\omega) - B(s)(\omega)| + |B(s)(\omega) - B(\frac{j+1}{n})(\omega)| \\ &\le c|\frac{j}{n} - s|^\alpha + c|s - \frac{j+1}{n}|^\alpha \le 2c(\frac{m}{n})^\alpha.\end{aligned}$$

Hence

$$X_{n,k}(\omega) \le 2c(\frac{m}{n})^\alpha.$$

This proves the claim.

3.2. **Show:** $\mathbb{P}(\Gamma(\alpha, c, \epsilon)) = 0$ for all $c, \epsilon > 0$.

Using independence and stationarity of the laws of the increments of B, and its scaling properties, we get for $m, n \in \mathbb{N}$ such that $\frac{m}{n} \leq \epsilon$

$$\begin{aligned}
\mathbb{P}(\min_{0\leq k\leq n-m} X_{m,k} \leq 2c(\frac{m}{n})^\alpha) &\leq n\mathbb{P}(X_{m,1} \leq 2c(\frac{m}{n})^\alpha) \\
&\leq n\mathbb{P}(|B(\frac{1}{n})| \leq 2c(\frac{m}{n})^\alpha)^m \\
&= n\mathbb{P}(|B(1)| \leq 2c\sqrt{n}(\frac{m}{n})^\alpha)^m \\
&\leq n[\sqrt{\frac{2}{\pi}}2c\sqrt{n}(\frac{m}{n})^\alpha]^m = n^{1+(\frac{1}{2}-\alpha)m}[\sqrt{\frac{2}{\pi}}2cm^\alpha]^m.
\end{aligned}$$

Now choose m so that $1 + (\frac{1}{2} - \alpha)m < 0$. Then let $n \to \infty$ to obtain that

$$\mathbb{P}(\Gamma(\alpha, c, \epsilon)) \leq \lim_{n\to\infty} \mathbb{P}(\min_{0\leq k\leq n-m} X_{m,k} \leq 2c(\frac{m}{n})^\alpha) = 0,$$

as desired. □

5 Basic concepts of large deviations theory

We aim at giving, as a striking example of the power of the Fourier analytic approach of rough functions or stochastic processes, an approach of Schilder's theorem by means of Fourier analysis. Schilder's theorem describes the large deviations properties of standard Brownian motion. In preparation, we have to briefly discuss basic concepts of the focal techniques of large deviations in this section, following [DZ98].

5.1 Concept and basic properties

To state the large deviation principle, and investigate its basic properties, let $(\mu_\epsilon)_{\epsilon>0}$ be a family of probability measures on a topological (Hausdorff) space $(\mathbf{X}, \mathcal{B})$ ($\mathcal{B}$ is the Borel σ-algebra). Think of μ_ϵ as law of $\sqrt{\epsilon}B$ for $\epsilon > 0$.

Typical examples for the topological space $\mathbf{X}$: $\mathcal{C}([0,1])$ or $\mathcal{C}_0^\alpha([0,1])$, in which the functions vanish at 0.

The LDP concerns the limiting behavior of exponential rates of $(\mu_\epsilon)_{\epsilon>0}$ as $\epsilon \to 0$ in terms of a *rate function.*

Definition 5.1. A *rate function* is a lower semicontinuous function $I : \mathbf{X} \to [\mathbf{0}, \infty]$, i.e. for all $\alpha \in [0, \infty[$, the *level sets*

$$\Psi_I(\alpha) = \{x \in \mathbf{X} : \mathbf{I}(\mathbf{x}) \leq \alpha\}$$

are closed. I is called *good rate function*, if all level sets are compact.

Remark 5.2. If the topology of $\mathbf{X}$ has a countable basis, lower semicontinuity of I is equivalent to the property

$$\liminf_{n\to\infty} I(x_n) \geq I(x)$$

for all sequences $(x_n)_{n\in\mathbb{N}} \subset \mathbf{X}$ converging to $x \in \mathbf{X}$.

Proof. 1. Let $\{y : I(y) \leq \alpha\}$ be closed for $\alpha \in [0, \infty[$, $(x_n)_{n\in\mathbb{N}}$ a sequence in $\mathbf{X}$ converging to x. Then for $\epsilon > 0$

$$x \in \{y : I(y) > I(x) - \epsilon\} \quad \text{is open,}$$

hence there exists $m \in \mathbb{N}$ such that the set contains $x_n, n \geq m$. Consequently

$$\liminf_{n\to\infty} I(x_n) \geq I(x) - \epsilon.$$

Let $\epsilon \to 0$.

2. Assume that for any sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbf{X}$ converging to $x \in \mathbf{X}$ we have

$$\liminf_{n\to\infty} I(x_n) \geq I(x).$$

Assume that for some $\alpha \in [0, \infty[$ the set

$$\{y : I(y) > \alpha\} \quad \text{is not open.}$$

Let $x \in \{y : I(y) > \alpha\}$ such that for a countable basis of neighborhoods $(U_n)_{n\in\mathbb{N}}$ we have $U_n \cap \{y : I(y) \leq \alpha\} \neq \emptyset$. Pick $x_n \in U_n \cap \{y : I(y) \leq \alpha\}$. Then $x_n \to x$, and for any $n \in \mathbb{N}$ we have $I(x_n) \leq \alpha$, and hence $\liminf_{n\to\infty} I(x_n) \leq \alpha < I(x)$, which contradicts the hypothesis. □

Definition 5.3. Let I be a rate function. A family of probability measures $(\mu_\epsilon)_{\epsilon>0}$ on $(\mathbf{X}, \mathcal{B})$ *satisfies the large deviation principle (LDP) with rate function* I if for all $\Gamma \in \mathcal{B}$ we have

$$-\inf_{x\in\Gamma^o} I(x) \leq \liminf_{\epsilon\to 0} \epsilon \log \mu_\epsilon(\Gamma) \leq \limsup_{\epsilon\to 0} \epsilon \log \mu_\epsilon(\Gamma) \leq -\inf_{x\in\overline{\Gamma}} I(x).$$

Here Γ^o resp. $\overline{\Gamma}$ denote the open kernel resp. the closed hull of Γ.

The following equivalent characterization is evident, but often more practical to prove.

Remark 5.4. $(\mu_\epsilon)_{\epsilon>0}$ satisfies a LDP with rate function I iff the following conditions are satisfied.
(a) For every $\alpha < \infty$ and every $\Gamma \in \mathcal{B}$ such that $\inf_{x\in\overline{\Gamma}} I(y) \geq \alpha$ we have

$$\limsup_{\epsilon\to 0} \epsilon \log \mu_\epsilon(\Gamma) \leq -\alpha.$$

(b) For $x \in \mathbf{X}$ with $I(x) < \infty$ and any $\Gamma \in \mathcal{B}$ with $x \in \Gamma^o$ we have

$$\liminf_{\epsilon\to 0} \epsilon \log \mu_\epsilon(\Gamma) \geq -I(x).$$

Proof. 1. Assume the LDP fulfilled, and let $\alpha < \infty, \Gamma \in \mathcal{B}$ with $\inf_{x\in\bar{\Gamma}} I(x) \geq \alpha$ be given. Then

$$\limsup_{\epsilon\to 0} \epsilon \ln \mu_\epsilon(\Gamma) \leq -\inf_{x\in\bar{\Gamma}} I(x) \leq -\alpha.$$

Let $x \in \mathbf{X}, \mathbf{I}(\mathbf{x}) < \infty$, and $\Gamma \in \mathcal{B}$ with $x \in \Gamma^o$. Then

$$\liminf_{\epsilon\to 0} \epsilon \ln \mu_\epsilon(\Gamma) \geq -\inf_{y\in\Gamma^o} I(y) \geq -I(x).$$

2. Assume that (a) and (b) hold, and let $\Gamma \in \mathcal{B}$. Then

$$\limsup_{\epsilon\to 0} \epsilon \ln \mu_\epsilon(\Gamma) \leq \inf\{-\alpha : \inf_{x\in\bar{\Gamma}} I(x) \geq \alpha\} = -\inf_{x\in\bar{\Gamma}} I(x).$$

The second inequality follows from taking $\sup_{x\in\Gamma^o}(-I(x))$. □

Remark 5.5. Let $(\mu_\epsilon)_{\epsilon>0}$ be a family of probability measures, I a rate function. Then the LDP is equivalent to the following statements:
(a) for $F \subset \mathbf{X}$ closed we have

$$\limsup_{\epsilon\to 0} \epsilon \log \mu_\epsilon(F) \leq -\inf_{x\in F} I(x),$$

(b) for $G \subset \mathbf{X}$ open we have

$$\liminf_{\epsilon\to 0} \epsilon \log \mu_\epsilon(G) \geq -\inf_{x\in G} I(x).$$

Proof. 1. The LDP evidently implies (a) and (b).
2. Assume that (a) and (b) are satisfied, and let $\Gamma \in \mathcal{B}$. By (a) we have

$$\limsup_{\epsilon\to 0} \epsilon \log \mu_\epsilon(\Gamma) \leq \limsup_{\epsilon\to 0} \epsilon \log \mu_\epsilon(\overline{\Gamma}) \leq -\inf_{x\in\overline{\Gamma}} I(x).$$

By (b) we have

$$\liminf_{\epsilon\to 0} \epsilon \log \mu_\epsilon(\Gamma) \geq \liminf_{\epsilon\to 0} \epsilon \log \mu_\epsilon(\Gamma^o) \geq -\inf_{x\in\Gamma^o} I(x).$$

Combining the two inequalities gives the defining property. □

To illustrate the notions, let us give an example. The large deviation rate for a one-dimensional Gaussian unit variable can be directly calculated. Consider Z with standard normal law, and let μ_ϵ be the law of $\sqrt{\epsilon}Z$. Then the following statement holds.

Theorem 5.6. *Let*

$$I(x) = \frac{x^2}{2}, \quad x \in \mathbb{R}.$$

Then for any open set $G \subset \mathbb{R}$ and any closed set $F \subset \mathbb{R}$ we have

$$-\inf_{x\in G} I(x) \leq \liminf_{\epsilon\to 0} \epsilon \log \mu_\epsilon(G),$$

$$-\inf_{x\in F} I(x) \geq \limsup_{\epsilon\to 0} \epsilon \log \mu_\epsilon(F).$$

I is a good rate function.

Proof. 1. Let $F \subset \mathbb{R}$ be closed.

Let $a = \inf\{|x| : x \in F\}$. The case $a = 0$ is trivial. We therefore assume $a > 0$. By symmetry we may assume that there exists $b \geq a$ such that $F \subset]-\infty, -b] \cup [a, \infty[$.

Hence for $\epsilon > 0$

$$\mu_\epsilon(F) \leq \mu_\epsilon([a,\infty[) + \mu_\epsilon(]-\infty,-b]) \leq \frac{2}{\sqrt{2\pi}} \int_{\frac{a}{\sqrt{\epsilon}}}^\infty \exp(-\frac{x^2}{2}) dx.$$

For $u > 1$ we have the auxiliary inequality

$$\int_u^\infty \exp(-\frac{x^2}{2}) dx \leq \int_u^\infty x \exp(-\frac{x^2}{2}) dx = \exp(-\frac{1}{2}u^2).$$

Hence for $\epsilon < a^2$

$$\epsilon \ln \mu_\epsilon(F) \leq \epsilon[\ln(\frac{2}{\sqrt{2\pi}}) - \frac{a^2}{2\epsilon}],$$

and therefore

$$\limsup_{\epsilon\to 0} \epsilon \log \mu_\epsilon(F) \leq -\frac{a^2}{2} = -\inf_{x\in F} I(x).$$

2. Let $G \subset \mathbb{R}$ be open.

We need a different auxiliary inequality. Integration by parts gives for $u > 1$

$$\int_u^\infty \exp(-\frac{x^2}{2}) dx = \frac{1}{u}\exp(-\frac{1}{2}u^2) - \int_u^\infty \frac{1}{x^2} \exp(-\frac{x^2}{2}) dx,$$

hence

$$\frac{1}{u}\exp(-\frac{1}{2}u^2) \leq (1 + \frac{1}{u^2}) \int_u^\infty \exp(-\frac{x^2}{2}) dx$$

and

$$\frac{u}{1+u^2}\exp(-\frac{1}{2}u^2) \leq \int_u^\infty \exp(-\frac{x^2}{2})dx.$$

Now let $G \subset \mathbb{R}$ be open, $y \in G$. By symmetry, we may assume that $y > 0$. Let, moreover, $a, b > 0$ such that $y \in]a, b[\subset G$. Then, for ϵ small enough we have

$$\begin{aligned}\mu_\epsilon(G) &\geq \mu_\epsilon(]a,\infty[) - \mu_\epsilon([b,\infty[) = \frac{1}{\sqrt{2\pi}}[\int_{\frac{a}{\sqrt{\epsilon}}}^\infty \exp(-\frac{x^2}{2})dx - \int_{\frac{b}{\sqrt{\epsilon}}}^\infty \exp(-\frac{x^2}{2})dx] \\ &\geq \frac{1}{\sqrt{2\pi}}[\frac{\frac{a}{\sqrt{\epsilon}}}{1+\frac{a^2}{\epsilon}}\exp(-\frac{a^2}{2\epsilon}) - \exp(-\frac{b^2}{2\epsilon})] \\ &\geq \frac{1}{\sqrt{2\pi}}\frac{\frac{a}{2\sqrt{\epsilon}}}{1+\frac{a^2}{\epsilon}}\exp(-\frac{a^2}{2\epsilon}).\end{aligned}$$

Therefore

$$\liminf_{\epsilon\to 0} \epsilon \ln \mu_\epsilon(G) \geq -\frac{a^2}{2} \geq -\frac{y^2}{2} = -I(y).$$

This implies the lower bound.

3. For $\alpha \geq 0$ we have

$$\Psi_\alpha(I) = [-\sqrt{2\alpha}, \sqrt{2\alpha}].$$

Hence I is a good rate function.

□

In practise, often a weaker form of LDP is given.

Definition 5.7. We say that $(\mu_\epsilon)_{\epsilon>0}$ satisfies a *weak LDP* with rate function I, if in (a) of the preceding remark *closed* is replaced by *compact.*

To strengthen a weak LDP to an LDP, some *tightness* condition is needed.

Definition 5.8. A family $(\mu_\epsilon)_{\epsilon>0}$ is said to be *exponentially tight* if for every $\alpha < \infty$ there exists a compact set $K_\alpha \subset \mathbf{X}$ such that

$$\limsup_{\epsilon\to 0} \epsilon \log \mu_\epsilon(K_\alpha^c) < -\alpha.$$

We show how exponential tightness can be used to deduce a LD principle.

Lemma 5.9. *Let $(\mu_\epsilon)_{\epsilon>0}$ be an exponentially tight family of probability measures. Then we have:*
*(a) The condition (*upper bound*)*

$$\limsup_{\epsilon\to 0} \epsilon \log \mu_\epsilon(K) \leq -\inf_{x\in K} I(x), \quad K \subset \mathbf{X} \text{ compact}$$

implies the lower bound for closed sets $F \subset \mathbf{X}$.
*(b) The condition (*lower bound*)*

$$\limsup_{\epsilon \to 0} \epsilon \log \mu_\epsilon(G) \geq - \inf_{x \in G} I(x), \quad G \subset \mathbf{X} \text{ open}$$

implies that I *is a good rate function.*

Proof. 1. Let $F \subset \mathbf{X}$ be closed, and $\alpha < \infty$ such that $\inf_{x \in F} I(x) \geq \alpha$. Choose K_α according to the definition of exponential tightness. Then for any $\epsilon > 0$

$$\mu_\epsilon(F) \leq \mu_\epsilon(F \cap K_\alpha) + \mu_\epsilon(K_\alpha^c).$$

Now, for $a, b : \mathbb{R}_+ \to \mathbb{R}_+$, due to $\ln(a(\epsilon) + b(\epsilon)) \leq \ln(2a(\epsilon)) \vee \ln(2b(\epsilon)) = \ln(a(\epsilon)) \vee \ln(b(\epsilon)) + \ln 2$ we have $\lim_{\epsilon \to 0} \epsilon \ln(a(\epsilon) + b(\epsilon)) \leq \lim_{\epsilon \to 0} \epsilon \ln(a(\epsilon)) \vee \ln(b(\epsilon))$. Similar statements hold for $\liminf_{\epsilon \to 0}$, $\limsup_{\epsilon \to 0}$.

Therefore by hypothesis applied to $F \cap K_\alpha$

$$\begin{aligned}
\limsup_{\epsilon \to 0} \epsilon \log \mu_\epsilon(F) &\leq \limsup_{\epsilon \to 0} \epsilon \log[\mu_\epsilon(F \cap K_\alpha) + \mu_\epsilon(K_\alpha^c)] \\
&\leq \limsup_{\epsilon \to 0} \epsilon \log[\mu_\epsilon(F \cap K_\alpha)] \vee \limsup_{\epsilon \to 0} \epsilon \log[\mu_\epsilon(K_\alpha^c)] \\
&= [\limsup_{\epsilon \to 0} \epsilon \log[\mu_\epsilon(F \cap K_\alpha)] \vee (-\alpha) \\
&\leq - \inf_{x \in F \cap K_\alpha} I(x) \vee (-\alpha) \leq - \inf_{x \in F} I(x) \vee (-\alpha) = - \inf_{x \in F} I(x).
\end{aligned}$$

2. For $\alpha < \infty$ let K_α be chosen according to the definition of exponential tightness. We have to show that $\Psi_I(\alpha)$ is compact. Apply the lower bound to the open set K_α^c. Then we have

$$- \inf_{x \in K_\alpha^c} I(x) \leq \liminf_{\epsilon \to 0} \epsilon \log \mu_\epsilon(K_\alpha^c) < -\alpha,$$

i.e.

$$\inf_{x \in K_\alpha^c} I(x) > \alpha,$$

which means that $I(x) \leq \alpha$ implies $x \in K_\alpha$. Hence $\Psi_I(\alpha) \subset K_\alpha$ is compact.

□

5.2 Construction of LDP from exponential rates of elementary sets

Large deviations principles state exponential rates for all open and closed sets of a topological space. Suppose that originally the rates are only known for some simple sets for instance belonging to a basis of the topology. We shall now give a sufficient criterion under which from those rates one can obtain an LDP. In fact, we start with discussing a weak LDP.

Theorem 5.10. *Let $\mathcal{G}_0$ be a collection of open sets in $(\mathbf{X}, \mathcal{B})$ such that for each open set G and each $y \in G$ there is $G_0 \in \mathcal{G}_0$ such that $y \in G_0 \subset G$.*

Let I be a rate function, $(\mu_\epsilon)_{\epsilon>0}$ a family of probability measures. Assume that for $G \in \mathcal{G}_0$ we have

$$-\inf_{x \in G} I(x) = \lim_{\epsilon \to 0} \epsilon \ln \mu_\epsilon(G).$$

Then $(\mu_\epsilon)_{\epsilon>0}$ satisfies a weak LDP with rate function I.

Proof. 1. Let us first prove the **lower bound.**

In fact, let G be open. Choose $x \in G$, and a basis set G_0 such that $x \in G_0 \subset G$. Then evidently

$$\liminf_{\epsilon \to 0} \epsilon \ln \mu_\epsilon(G) \geq \liminf_{\epsilon \to 0} \epsilon \ln \mu_\epsilon(G_0) = -\inf_{y \in G_0} I(y) \geq -I(x).$$

Now the lower bound follows readily by taking the sup of $-I(x), x \in G$, on the right hand side, the left hand side not depending on x.

2. For the **upper bound**, fix a compact $K \subset \mathbf{X}$. For $\delta > 0$ denote

$$I^\delta(x) = (I(x) - \delta) \wedge \frac{1}{\delta}, \quad x \in \mathbf{X}.$$

For $x \in K$, use lower semicontinuity of I, more precisely that $\{y \in \mathbf{X} : \mathbf{I(y)} > \mathbf{I}^\delta(\mathbf{x})\}$ containing x is open, to choose a set $G_x \in \mathcal{G}_0$ containing x such that

$$-I^\delta(x) \geq -\inf_{y \in G_x} I(y) \geq \limsup_{\epsilon \to 0} \epsilon \ln \mu_\epsilon(G_x).$$

Use compactness of K to extract from the open cover $K \subset \cup_{x \in K} G_x$ a finite subcover $K \subset \cup_{i=1}^n G_{x_i}$. Then with an argument as in the proof of Lemma 5.9 we obtain

$$\begin{aligned} \limsup_{\epsilon \to 0} \epsilon \ln \mu_\epsilon(K) &\leq \max_{1 \leq i \leq n} \limsup_{\epsilon \to 0} \epsilon \ln \mu_\epsilon(G_{x_i}) \\ &\leq -\min_{1 \leq i \leq n} I^\delta(x_i) \leq -\inf_{x \in K} I^\delta(x). \end{aligned}$$

Now let $\delta \to 0$, to complete the proof. □

Corollary 5.11. *Let $\mathcal{G}_0$ be a collection of open sets in $(\mathbf{X}, \mathcal{B})$ such that for each open G and each $y \in G$ there is $G_0 \in \mathcal{G}_0$ such that $y \in G_0 \subset G$.*

Let I be a rate function, $(\mu_\epsilon)_{\epsilon>0}$ an exponentially tight family of probability measures.

Assume that for every $G \in \mathcal{G}_0$ we have

$$-\inf_{x \in G} I(x) = \lim_{\epsilon \to 0} \epsilon \ln \mu_\epsilon(G).$$

Then $(\mu_\epsilon)_{\epsilon>0}$ satisfies an LDP with good rate function I.

Proof. Apply Lemma 5.9 to Theorem 5.10. □

5.3 Transformations of LDP

Assume an LDP for a family of probability measures $(\mu_\epsilon)_{\epsilon>0}$ on a topological space $(\mathbf{X}, \mathcal{B})$ is given, and $f : \mathbf{X} \to \mathbf{Y}$ is a continuous map to a topological (Hausdorff) space $(\mathbf{Y}, \mathcal{C})$. We will show that the family $(\nu_\epsilon = \mu_\epsilon \circ f^{-1} : \epsilon > 0)$ also satisfies an LDP.

Theorem 5.12 (contraction principle)**.** *Let* $(\mathbf{X}, \mathcal{B}), (\mathbf{Y}, \mathcal{C})$ *be topological spaces,* $f : \mathbf{X} \to \mathbf{Y}$ *continuous. Let* $I : \mathbf{X} \to [\mathbf{0}, \infty]$ *be a good rate function.*

(a) For $y \in \mathbf{Y}$ *let*

$$I'(y) = \inf\{I(x) : x \in \mathbf{X}, \mathbf{y} = \mathbf{f}(\mathbf{x})\}.$$

Then I' *is a good rate function on* $\mathbf{Y}$*.*

(b) Suppose $(\mu_\epsilon)_{\epsilon>0}$ *satisfies an LDP with rate function* I*, and* $\nu_\epsilon = \mu_\epsilon \circ f^{-1}, \epsilon > 0$*. Then* $(\nu_\epsilon)_{\epsilon>0}$ *satisfies an LDP with rate function* I'*.*

Proof. (a) **We have to show:** For $\alpha < \infty$

$$\Psi_{I'}(\alpha) = \{y \in Y : I'(y) \leq \alpha\} \text{ is compact.}$$

Show: $\Psi_{I'}(\alpha) = f(\Psi_I(\alpha))$.

" $\subset$ **"**

Let $y \in \Psi_{I'}(\alpha)$, $(x_n)_{n\in\mathbb{N}}$ a sequence in $f^{-1}(y)$ such that $I(x_n) \to I'(y)$. Then by compactness of the level sets of I we can assume that $x_n \to x \in f^{-1}(y)$. By lower semicontinuity of I we have $I(x) \leq \liminf_{n\to\infty} I(x_n) = I'(y) \leq \alpha$. Hence $y \in f(\Psi_I(\alpha))$.

" $\supset$ **"**:

This is trivial.

Since f is continuous, with $\Psi_I(\alpha)$ also $f(\Psi_I(\alpha)) = \Psi_{I'}(\alpha)$ is compact.

(b) Let $H \subset \mathbf{Y}$ be open. Then $f^{-1}(H) \subset \mathbf{X}$ is open, and we have

$$\begin{aligned}\liminf_{\epsilon\to 0} \epsilon \log \nu_\epsilon(H) &= \liminf_{\epsilon\to 0} \epsilon \log \mu_\epsilon(f^{-1}(H)) \\ &\geq -\inf_{x\in f^{-1}(H)} I(x) = -\inf_{y\in H} \inf_{x\in f^{-1}(y)} I(x) = -\inf_{y\in H} I'(y).\end{aligned}$$

An analogous statement holds for closed sets. □

6 Large deviations for Brownian motion

In this section, we shall exhibit the strength of the Fourier analysis induced calculus on sequence spaces to derive an LDP for Brownian motion, usually comprised in *Schilder's theorem.* In fact we will establish how the sequence space perspective allows to reduce the calculation of the rate functions to the one for simple one-dimensional Gaussian variables given above.

This remarkable approach was presented in Baldi and Roynette [BR92].

Again we have $d = 1$. Let B be a one-dimensional Brownian motion indexed by $[0, 1]$, described by

$$B = Z_{00}G_{00}(t) + \sum_{p=1}^{\infty} \sum_{1 \leq m \leq 2^p} Z_{pm}\, G_{pm},$$

with a sequence $(Z_{00}, (Z_{pm})_{p \geq 1, 1 \leq m \leq 2^p})$ of i.i.d standard normal variables, and the Schauder functions $(G_{pm})_{p \geq 0, 0 \leq m \leq 2^p}$, as described in section 4.

Recall the Haar functions $(H_{pm})_{p \geq 0, 0 \leq m \leq 2^p}$ and the sequences

$$(c_{-10}, c_{00}, (c_{pm}(\alpha))_{p \geq 1, 1 \leq m \leq 2^p})$$

appearing in Ciesielski's isomorphism in Theorem 3.2 for $0 < \alpha < 1$, given by

$$c_{pm}(\alpha) = 2^{p(\alpha - \frac{1}{2}) + \alpha - 1}, \quad c_{00}(\alpha) = 1, \quad c_{-10}(\alpha) = 1, \tag{16}$$

if $p \geq 1, 1 \leq m \leq 2^p$.

We investigate the asymptotic behavior of the family of probability measures $(\mu_\epsilon)_{\epsilon > 0}$, where μ_ϵ is the law of $\sqrt{\epsilon} B$, $\epsilon > 0$. We remark that according to Theorem 4.11 for any $\epsilon > 0, 0 < \alpha < \frac{1}{2}$ we have

$$\mu_\epsilon(\mathcal{C}_0^\alpha) = 1. \tag{17}$$

The large deviation rates for Brownian motion will crucially depend on the following function space, the *Cameron-Martin space* of absolutely continuous functions.

Definition 6.1. Let

$$\begin{aligned} \mathcal{H}_1 &= \Big\{ f | f : [0,1] \to \mathbb{R},\ f(0) = 0,\ f \text{ abs. cont., density } \dot{f} \in L^2([0,1]) \Big\} \\ &= \Big\{ f | f = \int_0^t \dot{f}(s) ds,\ \dot{f} \in L^2([0,1]) \Big\}. \end{aligned} \tag{18}$$

By means of (18) we can define the rate function for Brownian motion.

Definition 6.2. For $f \in \mathcal{C}([0,1])$ with $f(0)=0$ let

$$I(f) = \begin{cases} \frac{1}{2}\int_0^1 (\dot{f})^2(u)du, & \text{if } f \in \mathcal{H}_1, \\ \infty, & \text{otherwise.} \end{cases} \tag{19}$$

In the following Theorem the rate function for an LDP for Brownian motion is calculated for basic sets of a topology that is finer than the supremum norm topology usually employed on Wiener space. Using standard arguments, it can be enhanced to an LDP w.r.t. the supremum norm topology in *Schilder's Theorem*.

We consider the following basic sets of the Hölder topology. Recall the Ciesielski isomorphisms $T_{\alpha,0}$ from section 3. Let $\alpha \in]0,1[$, and define

$$\mathcal{G}_0 = \{G(\psi,\delta)|\ G(\psi,\delta) = T_{\alpha,0}^{-1}(B_\delta^\infty(T_{\alpha,0}(\psi))), \quad \delta > 0, \psi \in \mathcal{C}_0^\alpha\},$$

where for $\xi \in \ell^\infty = \ell^\infty(\mathbb{R})$ we denote $B_\delta^\infty(\xi)$ the ball of radius δ in the topology of l^∞. Note that $\mathcal{G}_0$ satisfies the hypotheses of Corollary 5.11 on $\mathcal{C}_0^\alpha$.

Theorem 6.3. *Let $0 < \alpha < 1/2$, $\delta > 0$ and $\psi \in (\mathcal{C}_0^\alpha, \|\cdot\|_\alpha)$. Then with the rate function I defined by (19)*

$$\lim_{\epsilon \to 0} \epsilon \ln \mu_\epsilon(G(\psi,\delta)) = -\inf_{f \in G(\psi,\delta)} I(f), \tag{20}$$

$$\lim_{\epsilon \to 0} \epsilon \ln \mu_\epsilon(\overline{G(\psi,\delta)}) = -\inf_{f \in \overline{G(\psi,\delta)}} I(f). \tag{21}$$

Proof. 0. We prove (20). Arguments for (21) are almost identical.

Note that

$$\mu_\epsilon(G(\psi,\delta)) = \mathbb{P}((\sqrt{\epsilon}B)^{-1}(G(\psi,\delta))) = \mathbb{P}((\sqrt{\epsilon}B)^{-1}(T_{\alpha,0}^{-1}(B_\delta^\infty(\xi)))),$$

with

$$\xi = T_{\alpha,0}(\psi) = (\xi_{00}, (\xi_{pm})_{1\le p, 1 \le m \le 2^p}).$$

Also note that by the Schauder representation of B (cf. Theorem 3.2) for $\varepsilon > 0$

$$T_{\alpha,0}(\sqrt{\varepsilon}B) = (\sqrt{\varepsilon}Z_{00}, (\sqrt{\varepsilon}c_{pm}(\alpha)Z_{pm})_{1\le p,1\le m\le 2^p}.$$

1. Denoting for notational simplicity $Z_{p0} = 0, \xi_{p0} = 0, c_{p0} = 1, p \ge 1$.

Show:

$$(\sqrt{\varepsilon}B)^{-1}[T_{\alpha,0}^{-1}(B_\delta^\infty(\xi))] = \bigcap_{p \ge 0, 0 \le m \le 2^p} \Big\{\sqrt{\varepsilon}c_{pm}(\alpha)Z_{pm} \in]\xi_{pm} - \delta, \xi_{pm} + \delta[\Big\}.$$

In fact,

$$\begin{aligned}\sqrt{\varepsilon}B \in T_{\alpha,0}^{-1}(B_\delta^\infty(\xi)) &\iff T_{\alpha,0}(\sqrt{\varepsilon}B) \in B_\delta^\infty(\xi)\\ &\iff \sup_{p\geq 0, 0\leq m\leq 2^p} |\sqrt{\varepsilon}c_{pm}(\alpha)Z_{pm} - \xi_{pm}| < \delta.\end{aligned}$$

This implies the desired equation.

2. Since $(Z_{pm})_{p\geq 0,0\leq m\leq 2^p}$ is a family of independent random variables, we obtain for $\epsilon > 0$ from 1.

$$\begin{aligned}\mu_\epsilon(G(\psi,\delta)) &= \prod_{p\geq 0,0\leq m\leq 2^p} \mathbb{P}\Big(\sqrt{\varepsilon}c_{pm}(\alpha)Z_{pm} \in]\xi_{pm}-\delta, \xi_{pm}+\delta[\Big)\\ &= \prod_{p\geq 0,0\leq m\leq 2^p} P_{pm}(\varepsilon).\end{aligned}$$

We split the index set of the probabilities $(P_{pm}(\varepsilon))_{p\geq 0,0\leq m\leq 2^p}$ into four different parts to be treated separately:

$$\begin{aligned}&\Lambda_1 = \Big\{(p,m) : p\geq 0, 0\leq m\leq 2^p,\ 0\notin [\xi_{pm}-\delta,\xi_{pm}+\delta]\Big\},\\ &\Lambda_2 = \Big\{(p,m) : p\geq 0, 0\leq m\leq 2^p,\ \xi_{pm} = \pm\delta\Big\},\\ &\Lambda_3 = \Big\{(p,m) : p\geq 0, 0\leq m\leq 2^p, [\xi_{pm}-\delta,\xi_{pm}+\delta] \supset \Big[-\tfrac{\delta}{2},\tfrac{\delta}{2}\Big]\Big\},\\ &\Lambda_4 = (\Lambda_3)^c \setminus (\Lambda_1\cup\Lambda_2).\end{aligned}$$

Let us recall that $(\xi_{pm})_{p\geq 0,0\leq m\leq 2^p} \in \mathcal{C}_0$, so Λ_3 contains almost all $(p,m), p\geq 0, 0\leq m\leq 2^p$, and hence $\Lambda_1\cup\Lambda_2\cup\Lambda_4 = (\Lambda_3)^c$ is finite.

3. **Show:**

$$\lim_{\varepsilon\to 0} \prod_{(p,m)\in\Lambda_3} P_{pm}(\varepsilon) = 1. \tag{22}$$

Since $(Z_{00}, Z_{pm})_{p\geq 1,1\leq m\leq 2^p}$ are standard normal variables, we have

$$\begin{aligned}\prod_{(p,m)\in\Lambda_3} P_{pm}(\varepsilon) &\geq \prod_{(p,m)\in\Lambda_3,p\geq 1} \mathbb{P}\Big(Z_{pm} \in \Big]-\frac{\delta}{2c_{pm}(\alpha)\sqrt{\varepsilon}}, \frac{\delta}{2c_{pm}(\alpha)\sqrt{\varepsilon}}\Big[\Big)\\ &= \prod_{(p,m)\in\Lambda_3,p\geq 1} \Big(1-\sqrt{\frac{2}{\pi}}\int_{\delta/(2c_{pm}(\alpha)\sqrt{\varepsilon})}^\infty e^{-u^2/2}du\Big)\\ &\geq \prod_{(p,m)\in\Lambda_3,p\geq 1} \Big(1-\sqrt{\frac{2}{\pi}}\exp(-\frac{\delta^2}{8c_{pm}(\alpha)^2\epsilon})\Big).\end{aligned}$$

The last inequality is due to (16) and our choice of α, $c_{pm}(\alpha)\leq 1$, $\lim_{p\to\infty} c_{pm}(\alpha) = 0$. Therefore, for $\epsilon > 0$ such that $\epsilon < \delta^2$ and all $p\geq 1, 1\leq$

$m \leq 2^p$ we may estimate (see proof of Theorem 5.6)

$$\int_{\delta/(2c_{pm}(\alpha)\sqrt{\varepsilon})}^{\infty} \exp(-\frac{x^2}{2})dx \leq \exp(-\frac{\delta^2}{8c_{pm}(\alpha)^2\epsilon}).$$

Now by the elementary inequality

$$\ln(\frac{1}{1-x}) = \ln(1+\frac{x}{1-x}) \leq \frac{x}{1-x}, \quad x \in]0,1[,$$

we have for ε small enough

$$\begin{aligned}
\ln\Big(\frac{1}{\prod_{(p,m)\in\Lambda_3} P_{pm}(\varepsilon)}\Big) &\leq \sum_{(p,m)\in\Lambda_3, p\geq 1} \ln \frac{1}{1-\sqrt{\frac{2}{\pi}}\exp(-\frac{\delta^2}{8c_{pm}(\alpha)^2\epsilon})} \\
&\lesssim \sum_{(p,m)\in\Lambda_3, p\geq 1} \exp(-\frac{\delta^2}{8c_{pm}(\alpha)^2\varepsilon}).
\end{aligned}$$

By (16) the right hand side converges to 0 as $\varepsilon \to 0$. We deduce, using the continuity of ln

$$\lim_{\varepsilon\to 0} \prod_{(p,m)\in\Lambda_3} P_{pm}(\varepsilon) = 1. \tag{23}$$

4. **Show:**

$$\lim_{\varepsilon\to 0} \prod_{(p,m)\in\Lambda_4} P_{pm}(\varepsilon) = 1. \tag{24}$$

Indeed, by definition $[\xi_{pm} - \delta, \xi_{pm} + \delta]$ contains a small neighborhood of the origin for any $(p,m) \in \Lambda_4$. Hence for any $(p,m) \in \Lambda_4$

$$\lim_{\epsilon\to 0} P_{pm}(\epsilon) = 1.$$

Moreover, $|\Lambda_4| < \infty$. This implies the claimed convergence.

5. **Show:**

$$\lim_{\varepsilon\to 0} \prod_{(p,m)\in\Lambda_2} P_{pm}(\varepsilon) = 2^{-|\Lambda_2|}. \tag{25}$$

By definition of Λ_2, we have

$$\lim_{\epsilon\to 0} P_{pm}(\epsilon) = \frac{1}{2}.$$

Since $|\Lambda_2| < \infty$, this implies the desired convergence.

6. We define

$$\bar{\xi}_{pm} = \begin{cases} \xi_{pm} - \delta, & \text{if } \xi_{pm} - \delta > 0, \\ -(\xi_{pm} + \delta), & \text{if } \xi_{pm} + \delta < 0. \end{cases}$$

Show:

$$\lim_{\epsilon\to 0}\epsilon\ln\prod_{(p,m)\in\Lambda_1}P_{pm}(\varepsilon) = -\sum_{(p,m)\in\Lambda_1}\frac{\bar{\xi}_{pm}^2}{2c_{pm}(\alpha)^2}. \tag{26}$$

Since for $(p,m)\in\Lambda_1$ Z_{pm} has a standard normal law, Theorem 5.6 implies

$$\lim_{\epsilon\to 0}\epsilon\ln P_{pm}(\varepsilon) = -\frac{\bar{\xi}_{pm}^2}{2c_{pm}(\alpha)^2}.$$

Since $|\Lambda_1|<\infty$, we therefore have the claimed equation.

7. **Show:** For $f\in\mathcal{C}_0^\alpha\cap\mathcal{H}_1$ with Schauder representation

$$f = \eta_{00}G_{00} + \sum_{p\geq 1, 1\leq m\leq 2^p}\frac{\eta_{pm}}{c_{pm}(\alpha)}G_{pm}, \quad (\eta_{00},(\eta_{pm})_{p\geq 0,1\leq m\leq 2^p})\in l^\infty,$$

we have

$$\frac{1}{2}\int_0^1\dot{f}(s)^2\,ds = \sum_{p\geq 0,1\leq m\leq 2^p}\frac{\eta_{pm}^2}{2c_{pm}(\alpha)^2}.$$

In fact, the derivative satisfies

$$\dot{f} = \eta_{00}H_{00} + \sum_{p\geq 1,1\leq m\leq 2^p}\frac{\eta_{pm}}{c_{pm}(\alpha)}H_{pm}.$$

Since
$(H_{00},(H_{pm})_{p\geq 1,1\leq m\leq 2^p})$, is an CONS in $L^2([0,1])$, we obtain

$$\frac{1}{2}\int_0^1\dot{f}(s)^2\,ds = \sum_{p\geq 0,1\leq m\leq 2^p}\frac{\eta_{pm}^2}{2c_{pm}(\alpha)^2}.$$

8. **Show:**

$$\liminf_{\epsilon\to 0}\mu_\epsilon(G(\eta,\delta)) = -\inf_{f\in G(\eta,\delta)}I(f).$$

Using (23), (24), (25) and (26), we deduce

$$\liminf_{\epsilon\to 0}\mu_\epsilon(G(\eta,\delta)) = -\sum_{(p,m)\in\Lambda_1}\frac{\bar{\xi}_{pm}^2}{2c_{pm}(\alpha)^2}.$$

So by part 7. the claim follows from

$$\begin{aligned}\inf_{f\in G(\eta,\delta)\cap\mathcal{H}_1}\frac{1}{2}\int_0^1\dot{f}(s)^2\,ds &= \inf\Big\{\sum_{p\geq 0,0\leq m\leq 2^p}\frac{\eta_{pm}^2}{2c_{pm}(\alpha)^2},\ \eta_{pm}\in]\xi_{pm}-\delta,\xi_{pm}+\delta[\Big\}\\ &= \sum_{(p,m)\in\Lambda_1}\frac{\bar{\xi}_{pm}^2}{2c_{pm}(\alpha)^2}.\end{aligned}$$

□

Section 5 gives us a recipe for deriving an LDP for Brownian motion on Hölder space. According to Corollary 5.11, we just have to prove exponential tightness for $(\mu_\epsilon)_{\epsilon>0}$.

Theorem 6.4. *Let* $0 < \beta < \frac{1}{2}$. *Then* $(\mu_\varepsilon)_{\epsilon>0}$ *is exponentially tight on the topological space* $\mathcal{C}_0^\beta([0,1])$. *More precisely, for* $\delta > 0$ *and* $0 < \beta < \alpha < \frac{1}{2}$, $K_{\alpha,\delta} = \overline{T_{\alpha,0}^{-1}(B_\delta^\infty(0))}$ *is compact in* $\mathcal{C}_0^\beta([0,1])$, *and we have*

$$\limsup_{\epsilon\to 0} \epsilon \ln \mu_\varepsilon(K_{\alpha,\delta}^c) \leq -\frac{\delta^2}{2}. \tag{27}$$

Proof. 1. **Show:** $K_{\alpha,\delta}$ is compact in $C_0^\beta([0,1])$.

Recall that

$$\overline{B_\delta^\infty(0)}) = \prod_{n=0}^{\infty}[-\delta,\delta].$$

Moreover, for $0 < \beta < \alpha < \frac{1}{2}$ we have

$$\begin{aligned} T_{\beta,0}(T_{\alpha,0}^{-1}(\overline{B_\delta^\infty(0)})) &= \prod_{p\geq 0, 0\leq m\leq 2^p}^{\infty} [-\frac{c_{pm}(\beta)}{c_{pm}(\alpha)}\delta, \frac{c_{pm}(\beta)}{c_{pm}(\alpha)}\delta] \\ &= \prod_{p\geq 0, 0\leq m\leq 2^p} [-2^{(p+1)(\beta-\alpha)}, 2^{(p+1)(\beta-\alpha)}]. \end{aligned}$$

$\prod_{p\geq 0,0\leq m\leq 2^p}^{\infty}[-2^{(p+1)(\beta-\alpha)}, 2^{(p+1)(\beta-\alpha)}]$ is compact, since it is complete and totally bounded in l^∞.

Since $T_{\beta,0}$ is an isomorphism, also $K_{\alpha,\delta} = \overline{T_{\alpha,0}^{-1}(B_\delta^\infty(0))}$ is compact in $C_0^\beta([0,1])$.

2. **Show:** $\limsup_{\epsilon\to 0} \epsilon \ln \mu_\epsilon(K_{\alpha,\delta}^c) \leq -\frac{\delta^2}{2}$.

Recall that (see part 1. of proof of Theorem 6.3)

$$(\sqrt{\varepsilon}B)^{-1}[K_{\alpha,\delta}]^c = \bigcup_{p\geq 0, 0\leq m\leq 2^p} \left\{\sqrt{\varepsilon} c_{pm}(\alpha) Z_{pm} \in [-\delta,\delta]^c\right\}.$$

Hence for $\epsilon < \delta^2$ (see proof of Theorem 5.6)

$$\begin{aligned} \mu_\epsilon(K_{\alpha,\delta}^c) &\leq \sum_{p\geq 0, 0\leq m\leq 2^p} \mathbb{P}(Z_{pm} \notin [-\frac{\delta}{\sqrt{\epsilon}c_{pm}(\alpha)}, \frac{\delta}{\sqrt{\epsilon}c_{pm}(\alpha)}]) \\ &\leq \sqrt{\frac{2}{\pi}} \sum_{p\geq 0, 0\leq m\leq 2^p} \exp(-\frac{\delta^2}{2\epsilon c_{pm}(\alpha)^2}). \end{aligned}$$

Observe that $c_{pm}(\alpha)$ strictly decreases to 0, starting at $c_{00}(\alpha) = 1$. Hence by monotone convergence using the proof of Lemma 5.9

$$\begin{aligned} \limsup_{\epsilon\to 0} \epsilon \ln \sqrt{\frac{2}{\pi}} \sum_{p\geq 0, 0\leq m\leq 2^p} \exp(-\frac{\delta^2}{2\epsilon c_{pm}(\alpha)^2}) &= \sup_{p\geq 0, 0\leq m\leq 2^p} -\frac{\delta^2}{2c_{pm}(\alpha)^2} \\ &= -\frac{\delta^2}{2}. \end{aligned}$$

This implies (27) and the proof is complete. □

We are ready to state the main result of this section, which is a version of Schilder's Theorem with respect to a finer topology.

Theorem 6.5. (Baldi-Roynette) *Let $0 < \alpha < \frac{1}{2}$. For $\epsilon > 0$ let μ_ε be the law of $\sqrt{\epsilon}B$ on the topological space $\mathcal{C}_0^\alpha([0,1])$. Then $(\mu_\varepsilon)_{\epsilon>0}$ satisfies a large deviations principle with good rate function*

$$I(f) = \begin{cases} \frac{1}{2}\int_0^1 (\dot{f})^2(u)du, & \text{if } f \in \mathcal{H}_1, \\ \infty, & \text{if } f \text{ continuous}, \notin \mathcal{H}_1. \end{cases} \tag{28}$$

Proof. Combine Theorem 6.3 and Theorem 6.4 in Corollary 5.11. □

To obtain the classical result of Schilder's from the LDP w.r.t. the finer topologies in Theorem 6.5, we have to apply the contraction principle in the form of Theorem 5.12.

Theorem 6.6. (Schilder) *For $\epsilon > 0$ let μ_ε be the law of $\sqrt{\epsilon}B$ on $(\mathcal{C}([0,1]), ||\cdot||_\infty)$. Then $(\mu_\varepsilon)_{\epsilon>0}$ satisfies an LDP with the rate function of Theorem 6.5.*

Proof. According to Theorem 6.5, $(\mu_\varepsilon)_{\epsilon>0}$ satisfies an LDP on the space $(\mathcal{C}_0^\alpha([0,1]), \|\cdot\|_\alpha)$ with rate function I. Since the Hölder topology is finer than the uniform topology, Theorem 5.12 implies that $(\mu_\varepsilon)_{\epsilon>0}$ satisfies an LDP on $(\mathcal{C}_0^\alpha([0,1]), \|\cdot\|_\infty)$ with rate function I.

Finally observe that the LDP is preserved under the identity map from $(\mathcal{C}_0^\alpha([0,1]), \|\cdot\|_\infty)$ to $(\mathcal{C}([0,1]), \|\cdot\|_\infty)$, again by Theorem 5.12. □

7 Paradifferential calculus and Young integration

In this section we develop the basic tools for our Young and rough path integrals in terms of Schauder functions. We shall formally decompose the integral into three components that will be seen to possess quite different regularity. In these terms, we shall derive Young's integral.

We first slightly change notation. We want to get rid of the factor $2^{-p/2}$ in (4), and only retain the regularity parameter α. For $p \geq 0$ and $0 \leq m \leq 2^p$ we define the rescaled functions

$$\chi_{pm} := 2^{\frac{p}{2}} H_{pm} \qquad \text{and} \qquad \varphi_{pm} := 2^{\frac{p}{2}} G_{pm},$$

as well as $\varphi_{-10} := G_{-10} \equiv 1$, and $\langle \chi_{-10}, df \rangle = f(0)$.

Then we have for $p \in \mathbb{N}$ and $1 \leq m \leq 2^p$

$$\|\varphi_{pm}\|_\infty = \varphi_{pm}(t^1_{pm}) = 2^{\frac{p}{2}} \int_{t^0_{pm}}^{t^1_{pm}} 2^{\frac{p}{2}} \mathrm{d}s = 2^p \left(\frac{2m-1}{2^{p+1}} - \frac{2m-2}{2^{p+1}} \right) = \frac{1}{2},$$

so that $\|\varphi_{pm}\|_\infty \leq 1$ for all p, m.

The expansion of f in terms of (φ_{pm}) is given by

$$f_k = \sum_{p=0}^{k} \sum_{m=0}^{2^p} \langle H_{pm}, df \rangle G_{pm} = \sum_{p=0}^{k} \sum_{m=0}^{2^p} 2^{-p} \langle \chi_{pm}, df \rangle \varphi_{pm} = \sum_{p=0}^{k} \sum_{m=0}^{2^p} f_{pm} \varphi_{pm},$$

where $f_{-10} := f(1)$, and $f_{00} := f(1) - f(0)$ and for $p \in \mathbb{N}$ and $m \geq 1$

$$\begin{aligned} f_{pm} &:= 2^{-p} \langle \chi_{pm}, \mathrm{d}f \rangle \\ &= 2f\left(t^1_{pm}\right) - f\left(t^0_{pm}\right) - f\left(t^2_{pm}\right) \\ &= f_{t^0_{pm}, t^1_{pm}} - f_{t^1_{pm}, t^2_{pm}} = 2^{-\frac{p}{2}} \langle H_{pm}, df \rangle. \end{aligned}$$

We write $\langle \chi_{pm}, \mathrm{d}f \rangle := 2^p f_{pm}$ for all values of (p, m), despite not having defined χ_{-10}.

Definition 7.1. For $\alpha > 0$ and $f \colon [0,1] \to \mathbb{R}^d$ the norm $\|\cdot\|_\alpha$ is defined as

$$\|f\|_\alpha := \sup_{pm} 2^{p\alpha} |f_{pm}| = \sup_{pm} 2^{(\alpha - \frac{1}{2})p} |\langle H_{pm}, df \rangle|,$$

and we write

$$\mathcal{C}^\alpha := \mathcal{C}^\alpha(\mathbb{R}^d) := \left\{ f \in C([0,1], \mathbb{R}^d) : \|f\|_\alpha < \infty \right\}.$$

According to Lemma 3.1, we may indeed use the same name for the sequence space norm as before for the function space norm. And the old space $\mathcal{C}^\alpha$ is identical with the one just defined. For $\alpha \in (0,1)$, we have $\mathcal{C}^\alpha = C^\alpha([0,1], \mathbb{R}^d)$.

Littlewood-Paley notation. We will employ notation inspired from Littlewood-Paley theory. For $p \geq -1$ and $f \in C([0,1])$ we define

$$\Delta_p f := \sum_{m=0}^{2^p} f_{pm}\varphi_{pm} \qquad \text{and} \qquad S_p f := \sum_{q \leq p} \Delta_q f.$$

We will occasionally refer to $(\Delta_p f)_{p\geq 0}$ as the *Schauder blocks* of f. Note that

$$\mathcal{C}^\alpha = \{f \in C([0,1],\mathbb{R}^d) : \|(2^{p\alpha}\|\Delta_p f\|_\infty)_{p\geq 0}\|_{\ell^\infty} < \infty\}.$$

Let us now describe the integral algebraically in terms of Schauder blocks.

The canonical decomposition of the integral We identify three components of the integral that behave quite differently. This will be our starting point towards an extension of the integral beyond the Young regime. If $v \in \mathcal{C}^\alpha$ and $w \in \mathcal{C}^\beta$, we formally write

$$\begin{aligned}
\int_0^\cdot v dw &= \sum_{p,q} \int_0^\cdot \Delta_p v d\Delta_q w \\
&= \sum_{p<q} \int_0^\cdot \Delta_p v d\Delta_q w + \sum_{p\geq q} \int_0^\cdot \Delta_p v d\Delta_q w \\
&= \sum_{q} \int_0^\cdot S_{q-1} v d\Delta_q w + \sum_{p} \int_0^\cdot \Delta_p v d\Delta_p w + \sum_{p} \int_0^\cdot \Delta_p v dS_{p-1} w.
\end{aligned}$$

Integration by parts gives

$$\begin{aligned}
\sum_q \int_0^\cdot S_{q-1} v d\Delta_q w &= \sum_q S_{q-1} v \Delta_q w - \sum_q \int_0^\cdot \Delta_q w dS_{q-1} v \\
&= \pi_<(v,w) - \sum_q \int_0^\cdot \Delta_q w dS_{q-1} v.
\end{aligned}$$

We call

$$\pi_<(f,g) = \sum_q S_{q-1} v \Delta_q w$$

the *Bony paraproduct* of v and w.

We further define the *antisymmetric Lévy area* of v and w by

$$L(v,w) = \sum_p \int_0^\cdot (\Delta_p v dS_{p-1} w - \Delta_p w dS_{p-1} v),$$

and the *symmetric part* of v and w as

$$S(v,w)=\sum_p \int_0^\cdot \Delta_p v d\Delta_p w = c+\frac{1}{2}\sum_p \Delta_p v\Delta_p w.$$

We consequently obtain the following formal decomposition

$$\int_0^\cdot v dw = \pi_<(v,w)+S(v,w)+L(v,w).$$

We will now investigate the regularity of the three canonical components of the integral separately, starting with the paraproduct.

7.1 The paraproduct in terms of Schauder functions

Our calculus is in terms of Schauder functions. Paradifferential calculus is usually formulated in terms of Littlewood-Paley blocks and was initiated by Bony [Bon81]. For a gentle introduction see [BCD11].

We will need to study the regularity of $\sum_{p,m} u_{pm}\varphi_{pm}$, where u_{pm} are functions and not constant coefficients. For this purpose we define the following space of sequences of functions.

Definition 7.2. If $(u_{pm} : p \geq -1, 0 \leq m \leq 2^p)$ is a family of affine functions of the form $u_{pm} : [t^0_{pm}, t^2_{pm}] \to \mathbb{R}^d$, we set for $\alpha > 0$

$$\|(u_{pm})\|_{\mathcal{A}^\alpha} := \sup_{p,m} 2^{p\alpha}\|u_{pm}\|_\infty,$$

where $\|u_{pm}\|_\infty := \max_{t\in[t^0_{pm},t^2_{pm}]} |u_{pm}(t)|$.

The space $\mathcal{A}^\alpha := \mathcal{A}^\alpha(\mathbb{R}^d)$ is then defined as

$$\mathcal{A}^\alpha := \{(u_{pm})_{p\geq -1, 0\leq m\leq 2^p} : u_{pm} \in C([t^0_{pm}, t^2_{pm}], \mathbb{R}^d) \text{ is affine and } \|(u_{pm})\|_{\mathcal{A}^\alpha} < \infty\}.$$

Before proving a regularity estimate for affine expansions, let us establish an auxiliary result.

Lemma 7.3. *Let $s < t$ and let $f : [s,t] \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)$ and $g : [s,t] \to \mathbb{R}^d$ be affine functions. Then for all $r \in (s,t)$ and for all $h > 0$ with $r-h \in [s,t]$ and $r+h \in [s,t]$ we have*

$$|(fg)_{r-h,r} - (fg)_{r,r+h}| \leq 8|t-s|^{-2}h^2\|f\|_\infty\|g\|_\infty. \qquad (29)$$

Proof. For $f(r) = a_1 + (r-s)b_1$ and $g(r) = a_2 + (r-s)b_2$ we have

$$\begin{aligned}
& |(fg)_{r-h,r} - (fg)_{r,r+h}| \\
= \; & |2f(r)g(r) - f(r-h)g(r-h) - f(r+h)g(r+h)| \\
= \; & |2a_1b_1 + 2(r-s)a_1b_2 + 2(r-s)b_1a_2 + 2(r-s)^2b_1b_2 \\
& -a_1a_2 - (r-s-h)a_1b_2 - (r-s-h)b_1a_2 - (r-s-h)^2b_1b_2 \\
& -a_1a_2 - (r-s+h)a_1b_2 - (r-s+h)a_2b_1 - (r-s+h)^2b_1b_2| \\
= \; & 2h^2|b_1b_2|.
\end{aligned}$$

Now $f_{s,t} = b_1(t-s)$ so that $|b_1| \leq 2|t-s|^{-1}\|f\|_\infty$, and similarly for b_2. This implies the desired inequality. □

We now prove a regularity estimate for affine expansions.

Lemma 7.4. *Let $\alpha \in (0,2)$ and let $(u_{pm}) \in \mathcal{A}^\alpha$. Then $\sum_{p,m} u_{pm}\varphi_{pm} \in \mathcal{C}^\alpha$, and*

$$\Big\|\sum_{p,m} u_{pm}\varphi_{pm}\Big\|_\alpha \lesssim \|(u_{pm})\|_{\mathcal{A}^\alpha}.$$

Proof. We have

$$\begin{aligned}
\Big\|\sum_{p,m} u_{pm}\varphi_{pm}\Big\|_\alpha &= \sup_{qn} 2^{q\alpha}\, \Big|\Big(\sum_{p,m} u_{pm}\varphi_{pm}\Big)_{qn}\Big| \\
&= \sup_{qn} 2^{q\alpha}\, 2^{-q}|\langle \chi_{qn}, \mathrm{d}(\sum_{p,m} u_{pm}\varphi_{pm})\rangle| \\
&= \sup_{qn} 2^{q\alpha}\, 2^{-q}|\sum_{p,m}\langle \chi_{qn}, \mathrm{d}(u_{pm}\varphi_{pm})\rangle|.
\end{aligned}$$

We need to examine the coefficients in the last line, and calculate contributions for different p. Omitting trivial cases, we let $q \geq 0$ and $1 \leq n \leq 2^q$.

Case 1: $p > q$

In this case $\varphi_{pm}(t^i_{qn}) = 0$ for $i = 0, 1, 2$ and for all m, and therefore

$$\sum_{1\leq m\leq 2^p} \Big\langle \chi_{qn}, \mathrm{d}\Big(u_{pm}\varphi_{pm}\Big)\Big\rangle = 0.$$

Case 2: $p < q$

In this case, there is at most one m_0 with $\langle \chi_{qn}, \mathrm{d}(u_{pm_0}\varphi_{pm_0})\rangle \neq 0$. The support of χ_{qn} is then contained in $[t^0_{pm_0}, t^1_{pm_0}]$ or in $[t^1_{pm_0}, t^2_{pm_0}]$ and u_{pm_0}

and φ_{pm_0} are affine on these intervals. So Lemma 7.3 yields, with $|t-s| = 2^{-p}, h = 2^{-q}$

$$\begin{aligned}\sum_m |2^{-q}\langle \chi_{qn}, \mathrm{d}(u_{pm}\varphi_{pm})\rangle| &= \sum_m |(u_{pm}\varphi_{pm})_{t^0_{qn},t^1_{qn}} - (u_{pm}\varphi_{pm})_{t^1_{qn},t^2_{qn}}| \\ &\lesssim 2^{2p}2^{-2q}\|u_{pm}\|_\infty \|\varphi_{pm}\|_\infty \\ &\lesssim 2^{p(2-\alpha)-2q}\|(u_{pm})\|_{\mathcal{A}^\alpha}.\end{aligned}$$

Case 3: $p = q$

Here we have $\varphi_{qn}(t^0_{qn}) = \varphi_{qn}(t^2_{qn}) = 0$ and $\varphi_{qn}(t^1_{qn}) = 1/2$, and thus

$$\begin{aligned}\sum_m |2^{-q}\langle \chi_{qn}, \mathrm{d}(u_{qm}\varphi_{qm})\rangle| &= \left|(u_{qn}\varphi_{qn})_{t^0_{qn},t^1_{qn}} - (u_{qn}\varphi_{qn})_{t^1_{qn},t^2_{qn}}\right| \\ &= |u(t^1_{qn})| \lesssim 2^{-\alpha q}\|(u_{pm})\|_{\mathcal{A}^\alpha}.\end{aligned}$$

Combining these estimates and using that $\alpha < 2$, we obtain

$$2^{-q}\Big|\Big\langle \chi_{qn}, \mathrm{d}\Big(\sum_{pm} u_{pm}\varphi_{pm}\Big)\Big\rangle\Big| \lesssim \sum_{p\le q} 2^{p(2-\alpha)-2q}\|(u_{pm})\|_{\mathcal{A}^\alpha} \simeq 2^{-\alpha q}\|(u_{pm})\|_{\mathcal{A}^\alpha},$$

which completes the proof. □

We shall now apply the results obtained to paraproducts.

Lemma 7.5. *Let $\beta \in (0,2)$, let $v \in C([0,1], \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n))$, and $w \in \mathcal{C}^\beta(\mathbb{R}^d)$. Then*

$$\pi_<(v,w) := \sum_{p=0}^\infty S_{p-1}v\Delta_p w \in \mathcal{C}^\beta(\mathbb{R}^n) \quad \textit{and} \quad \|\pi_<(v,w)\|_\beta \lesssim \|v\|_\infty \|w\|_\beta. \tag{30}$$

Proof. We have $\pi_<(v,w) = \sum_{p,m} u_{pm}\varphi_{pm}$ with $u_{pm} = (S_{p-1}v)|_{[t^0_{pm},t^2_{pm}]} w_{pm}$. For every (p,m), the function $(S_{p-1}v)|_{[t^0_{pm},t^2_{pm}]}$ is the linear interpolation of v between t^0_{pm} and t^2_{pm} (see Lemma 3.1). As

$$\|(S_{p-1}v)|_{[t^0_{pm},t^2_{pm}]} w_{pm}\|_\infty \le 2^{-p\beta}\|v\|_\infty \|w\|_\beta,$$

setting $u_{pm} = S_{p-1}v|_{[t^0_{pm},t^2_{pm}]} w_{pm}$, we obtain

$$\|(u_{pm})\|_{\mathcal{A}_\beta} \le \|v\|_\infty \|w\|_\beta.$$

Hence the statement follows from Lemma 7.4. □

7.2 Regularity of the other components of the integral: the Young case

We show that the symmetric term is regular for any combination of α, β, and the Lévy area in case $\alpha + \beta > 1$.

In a first step, we have to estimate the Schauder coefficients of the iterated integrals of Schauder functions arising in our terms.

Lemma 7.6. *1. Let $p > q \geq 0$. Then for all m, n*

$$\langle \varphi_{pm}, \chi_{qn} \rangle = 2^{-p-2} \chi_{qn}(t_{pm}^0), \text{ and } |\langle \varphi_{pm}, \chi_{qn} \rangle| \leq 2^{p+q-2(p \vee q)-2} \tag{31}$$

2. If $p = q$, then $\langle \varphi_{pm}, \chi_{pn} \rangle = 0$, except if $p = q = 0$, in which case the integral is bounded by 1.

3. Let $0 \leq p < q$. Then for all (m, n)

$$\langle \varphi_{pm}, \chi_{qn} \rangle = -2^{-q-2} \chi_{pm}\left(t_{qn}^0\right), \text{ and } |\langle \varphi_{pm}, \chi_{qn} \rangle| \leq 2^{p+q-2(p \vee q)-2}. \tag{32}$$

If $p = -1$, then the integral is bounded by 1.

Proof. 1. Let $p > q \geq 0$. Since $\chi_{qn} \equiv \chi_{qn}(t_{pm}^0)$ on the support of φ_{pm}, we have

$$\int_0^1 \varphi_{pm}(s) \mathrm{d}\varphi_{qn}(s) = \chi_{qn}(t_{pm}^0) \int_0^1 \varphi_{pm}(s) \mathrm{d}s = \chi_{qn}(t_{pm}^0) 2^{-p-2}.$$

2. If $0 \leq p < q$, then integration by parts and (31) imply (32).

3. The other cases are easy. □

Next we estimate the coefficients of iterated integrals in the Schauder basis.

Lemma 7.7. *Let $i, p \geq -1$, $q \geq 0$, $0 \leq j \leq 2^i$, $0 \leq m \leq 2^p$, $0 \leq n \leq 2^q$. Then*

$$2^{-i} \Big| \Big\langle \chi_{ij}, \mathrm{d}\Big(\int_0^{\cdot} \varphi_{pm} \chi_{qn} \mathrm{d}s \Big) \Big\rangle \Big| \leq 2^{-2(i \vee p \vee q)+p+q}, \tag{33}$$

except if $p < q = i$. In this case we only have the worse estimate

$$2^{-i} \Big| \Big\langle \chi_{ij}, \mathrm{d}\Big(\int_0^{\cdot} \varphi_{pm} \chi_{qn} \mathrm{d}s \Big) \Big\rangle \Big| \leq 1. \tag{34}$$

Proof. 0. We have $\langle \chi_{-10}, \mathrm{d}(\int_0^{\cdot} \varphi_{pm} \chi_{qn} \mathrm{d}s) \rangle = 0$ for all (p, m) and (q, n).

1. Let $i \geq 0$.
Case 1: $i < q$.

In this case χ_{ij} is constant on the support of χ_{qn}. Therefore Lemma 7.6 gives

$$2^{-i}\,|\langle\chi_{ij},\varphi_{pm}\chi_{qn}\rangle| \le |\langle\varphi_{pm},\chi_{qn}\rangle| \le 2^{p+q-2(p\vee q)} = 2^{-2(i\vee p\vee q)+p+q}.$$

Case 2: $i > q$.

Here χ_{qn} is constant on the support of χ_{ij}, and therefore another application of Lemma 7.6 implies that

$$2^{-i}\,|\langle\chi_{ij},\varphi_{pm}\chi_{qn}\rangle| = 2^{q-i}|\langle\varphi_{pm},\chi_{ij}\rangle| \le 2^{q-i}2^{p+i-2(p\vee i)} = 2^{-2(i\vee p\vee q)+p+q}.$$

Case 3: $i = q \ge p$.

Here

$$2^{-i}\,|\langle\chi_{ij},\varphi_{pm}\chi_{qn}\rangle| \le 2^{i}\int_{t^0_{ij}}^{t^2_{ij}}\varphi_{pm}(s)\mathrm{d}s \le \|\varphi_{pm}\|_\infty \le 1.$$

Case 4: $i = q < p$

In this case χ_{ij} is constant on the support of $\phi_{pm}\chi_{qn}$. So we can argue as in Case 1. □

Corollary 7.8. *Let $i, p \ge -1$ and $q \ge 0$. Let $v \in C([0,1],\mathcal{L}(\mathbb{R}^d,\mathbb{R}^n))$ and $w \in C([0,1],\mathbb{R}^d)$. Then*

$$\Big\|\Delta_i\Big(\int_0^\cdot \Delta_p v(s)\mathrm{d}\Delta_q w(s)\Big)\Big\|_\infty \lesssim 2^{-(i\vee p\vee q)-i+p+q}\|\Delta_p v\|_\infty\|\Delta_q w\|_\infty, \tag{35}$$

except if $i = q > p$. In this case we only have the worse estimate

$$\Big\|\Delta_i\Big(\int_0^\cdot \Delta_p v(s)\mathrm{d}\Delta_q w(s)\Big)\Big\|_\infty \lesssim \|\Delta_p v\|_\infty\|\Delta_q w\|_\infty. \tag{36}$$

Proof. The case $i = -1$ is easy, so let $i \ge 0$. We have

$$\Delta_i\Big(\int_0^\cdot \Delta_p v(s)\mathrm{d}\Delta_q w(s)\Big) = \sum_{j,m,n} v_{pm}w_{qn}\langle 2^{-i}\chi_{ij},\varphi_{pm}\chi_{qn}\rangle\varphi_{ij}.$$

For fixed j, there are at most $2^{(i\vee p\vee q)-i}$ non-vanishing terms in the double sum. Hence, we obtain from Lemma 7.7 that

$$\begin{aligned}\Big\|\sum_{m,n} v_{pm}w_{qn}\langle 2^{-i}\chi_{ij},\varphi_{pm}\chi_{qn}\rangle\varphi_{ij}\Big\|_\infty &\lesssim 2^{(i\vee p\vee q)-i}\|\Delta_p v\|_\infty\|\Delta_q w\|_\infty \\ &\qquad (2^{-2(i\vee p\vee q)+p+q} + \mathbf{1}_{i=q>p}) \\ &= (2^{-(i\vee p\vee q)-i+p+q} + \mathbf{1}_{i=q>p}) \\ &\qquad \|\Delta_p v\|_\infty\|\Delta_q w\|_\infty.\end{aligned}$$

□

Corollary 7.9. *Let $i, p, q \geq -1$. Let $v \in C([0,1], \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n))$ and $w \in C([0,1], \mathbb{R}^d)$.*

1. Then for $p \vee q \leq i$ we have

$$\|\Delta_i (\Delta_p v \Delta_q w)\|_\infty \lesssim 2^{-(i \vee p \vee q) - i + p + q} \|\Delta_p v\|_\infty \|\Delta_q w\|_\infty, \tag{37}$$

except if $i = q > p$ or $i = p > q$, in which case we only have the worse estimate

$$\|\Delta_i(\Delta_p v \Delta_q w)\|_\infty \lesssim \|\Delta_p v\|_\infty \|\Delta_q w\|_\infty. \tag{38}$$

2. If $p \vee q > i$, then $\Delta_i(\Delta_p v \Delta_q w) \equiv 0$.

Proof. 0. The case $p = -1$ or $q = -1$ is easy.

1. In case $p \vee q \leq i$, we apply integration by parts and note that the estimates (35) and (36) are symmetric in p and q.

2. If for example $p > i$, then $\Delta_p v(t^k_{ij}) = 0$ for all k, j, which implies that $\Delta_i(\Delta_p v \Delta_q w) = 0$. □

On the basis of the auxiliary results just proved, let us now come to the description of the regularity of L and S.

Lemma 7.10. *Let $\alpha, \beta \in (0,1)$ be such that $\alpha + \beta > 1$. Then L is a bounded bilinear operator from $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{C}^{\alpha+\beta}$.*

Proof. We only argue for $\sum_p \int_0^\cdot \Delta_p v \mathrm{d} S_{p-1} w$ The term $-\int_0^\cdot \Delta_p w \mathrm{d}(S_{p-1} v)$ can be treated with the same arguments. Corollary 7.8 (more precisely (35)) implies that for $i \geq 0$

$$\begin{aligned}
\Big\|\Delta_i\Big(\sum_p \int_0^\cdot \Delta_p v \mathrm{d} S_{p-1} w\Big)\Big\|_\infty &= \Big\|\sum_p \Delta_i\Big(\int_0^\cdot \Delta_p v \mathrm{d} S_{p-1} w\Big)\Big\|_\infty \\
&\leq \sum_{p \leq i} \sum_{q<p} \Big\|\Delta_i\Big(\int_0^\cdot \Delta_p v \mathrm{d}\Delta_q w\Big)\Big\|_\infty \\
&\qquad + \sum_{p > i} \sum_{q<p} \Big\|\Delta_i\Big(\int_0^\cdot \Delta_p v \mathrm{d}\Delta_q w\Big)\Big\|_\infty \\
&\leq \Big(\sum_{p \leq i} \sum_{q<p} 2^{-2i+p+q} 2^{-p\alpha} \|v\|_\alpha 2^{-q\beta} \|w\|_\beta \\
&\qquad + \sum_{p > i} \sum_{q<p} 2^{-i+q} 2^{-p\alpha} \|v\|_\alpha 2^{-q\beta} \|w\|_\beta\Big) \\
&\lesssim_{\alpha+\beta} 2^{-i(\alpha+\beta)} \|v\|_\alpha \|w\|_\beta,
\end{aligned}$$

where we used $1 - \alpha > 0$ and $1 - \beta > 0$. For the second series we also used that $\alpha + \beta > 1$. □

Unlike the Lévy area L, the symmetric part S is always well defined. It is also smooth.

Lemma 7.11. *Let $\alpha, \beta \in (0,1)$. Then S is a bounded bilinear operator from $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{C}^{\alpha+\beta}$.*

Proof. This is shown using the same arguments as in the proof of Lemma 7.10. □

In conclusion, the integral consists of three components. The Lévy area $L(v,w)$ is only defined if $\alpha + \beta > 1$, but then it is smooth. The symmetric part $S(v,w)$ is always defined and smooth. And the paraproduct $\pi_<(v,w)$ is always defined, but it is rougher than the other components. To summarize:

Theorem 7.12 (Young's integral)**.** *Let $\alpha, \beta \in (0,1)$ be such that $\alpha+\beta > 1$, and let $v \in \mathcal{C}^\alpha$ and $w \in \mathcal{C}^\beta$. Then the integral*

$$I(v, \mathrm{d}w) := \sum_{p,q} \int_0^\cdot \Delta_p v \mathrm{d}\Delta_q w = L(v,w) + S(v,w) + \pi_<(v,w) \in \mathcal{C}^\beta$$

satisfies $\|I(v,\mathrm{d}w)\|_\beta \lesssim \|v\|_\alpha \|w\|_\beta$ and

$$\|I(v,\mathrm{d}w) - \pi_<(v,w)\|_{\alpha+\beta} \lesssim \|v\|_\alpha \|w\|_\beta. \tag{39}$$

8 Paracontrolled paths, pathwise integration beyond Young

In this section we construct a rough path integral in terms of Schauder functions.

Let us first motivate by an example (see [IP15]) what might be missing for two functions that serve as integrand and integrator in a rough integral, in case for the Hölder coefficients we have the inequality $\alpha + \beta \leq 1$. Since we use trigonometric functions instead of Haar and Schauder functions we shall briefly switch the domain from $[0,1]$ to $[-1,1]$.

Example 8.1. Let us consider for $m \in \mathbb{N}$ the functions $(f^m, g^m)\colon [-1,1] \to \mathbb{R}^2$ with components given by

$$f^m(t) := \sum_{k=1}^m a_k \sin(2^k \pi t) \quad \text{and} \qquad g^m(t) := \sum_{k=1}^m a_k \cos(2^k \pi t), \quad t \in [-1,1],$$

where $a_k := 2^{-\alpha k}$ and $\alpha \in [0,1]$. Set $f := \lim_{m\to\infty} f^m, g := \lim_{m\to\infty} g^m$.

1. **Show:** These functions are α-Hölder continuous uniformly in m.

Indeed, let $s, t \in [-1, 1]$ and choose $k \in \mathbb{N}$ such that $2^{-k-1} \leq |s-t| \leq 2^{-k}$. Then

$$\begin{aligned}
|f^m(t) - f^m(s)| &= \left| \sum_{l=1}^{m} a_l 2 \cos(2^{l-1}\pi(s+t)) \sin(2^{l-1}\pi(s-t)) \right| \\
&\leq 2 \sum_{l=1}^{k} |a_l| |\sin(2^{l-1}\pi(s-t))| + 2 \sum_{l=k+1}^{\infty} |a_l| \\
&\leq 2 \sum_{l=1}^{k} |a_l| 2^{l-1}\pi |s-t| + 2 \sum_{l=k+1}^{\infty} |a_l| \\
&\leq \sum_{l=1}^{k} 2^{l-\alpha l} \pi |s-t| + 2^{-\alpha(k+1)+1} \frac{1}{1-2^{-\alpha}} \\
&\leq \frac{2^{(k+1)(1-\alpha)} - 1}{2^{1-\alpha} - 1} \pi |s-t| + \frac{2^{1-\alpha}}{1-2^{-\alpha}} |s-t|^\alpha \\
&\leq \frac{2^{(k+1)(1-\alpha)} - 1}{2^{1-\alpha} - 1} \pi 2^{-k(1-\alpha)} |s-t|^\alpha + \frac{2^{1-\alpha}}{1-2^{-\alpha}} |s-t|^\alpha \\
&\leq C |s-t|^\alpha
\end{aligned}$$

for some constant $C > 0$ independent of $m \in \mathbb{N}$. Analogously, we can get the uniform α-Hölder continuity of g^m.

It can be seen with the same estimate that (f^m) converges uniformly to f, (g^m) to g and thus also in α-Hölder topology.

2. **Show:** f and g are not β-Hölder continuous for every $\beta > \alpha$.

In order to see this, choose $s = 0$ and $t = t_n = 2^{-n}$ for $n \in \mathbb{N}$ and observe that

$$\frac{|f(t_n) - f(0)|}{|t_n - 0|^\beta} = \sum_{k=1}^{n-1} 2^{-\alpha k + \beta n} \sin(2^{k-n}\pi) \geq 2^{(\beta-\alpha)n+\alpha},$$

which obviously tends to infinity with n.

3. **Show:** f possesses no *fractional Taylor expansion* up to first order with respect to g and vice versa.

We will name this expansion a *control relationship* between f and g. So the example will show that neither f is controlled by g nor vice versa.

For this purpose, note that for $-1 \le s \le t \le 1$, and $0 \ne f_s^g \in \mathbb{R}$

$$
\begin{aligned}
&|f_{s,t} - f_s^g g_{s,t}| \\
&\quad = \Big| \sum_{k=1}^{\infty} a_k [(\sin(2^k \pi t) - \sin(2^k \pi s)) - f_s^g (\cos(2^k \pi t) - \cos(2^k \pi s))] \Big| \\
&\quad = \Big| 2 \sum_{k=1}^{\infty} a_k \big[\sin(2^{k-1}\pi(s-t)) \cos(2^{k-1}\pi(s+t)) \\
&\qquad\qquad + f_s^g \sin(2^{k-1}\pi(s+t)) \sin(2^{k-1}\pi(s-t)) \big] \Big| \\
&\quad = \Big| 2 \sum_{k=1}^{\infty} a_k \sin(2^{k-1}\pi(s-t)) \\
&\qquad\qquad \sqrt{1 + (f_s^g)^2} \sin(2^{k-1}\pi(s+t) + \arctan((f_s^g)^{-1})) \Big| .
\end{aligned}
$$

Let us now investigate Hölder regularity at $s = 0$. First, assume $f_0^g > 0$, and take $t = 2^{-n}$ to obtain

$$
\begin{aligned}
&\frac{|f_{0,2^{-n}} - f_0^g g_{0,2^{-n}}|}{2^{-\beta n}} \\
= \quad & 2^{\beta n} \Big| 2 \sum_{k=1}^{n} a_k \sin(2^{k-1-n}\pi) \sqrt{1 + (f_0^g)^2} \sin(2^{k-1-n}\pi + \arctan((f_0^g)^{-1})) \Big| \\
& \ge 2^{(\beta - \alpha) n} \sin\big(\frac{\pi}{2} + \arctan((f_0^g)^{-1}) \big).
\end{aligned}
$$

For $f_0^g < 0$ the same estimates work for $t_n = -2^{-n}$ instead.

Therefore, Hölder regularity at 0 is not better than α. So f is not controlled by g for $\frac{1}{2} > \alpha$. Switching the roles of f and g with similar arguments leads to the same conclusion.

4. **Show:** $(\int_{-1}^{1} f^m(s) dg^m(s))_{m \in \mathbb{N}}$ does not converge as $m \to \infty$.

In fact, for $m \in \mathbb{N}$ we have

$$
\begin{aligned}
&\int_{-1}^{1} f^m(s) \mathrm{d} g^m(s) \\
&\qquad = - \sum_{k,l=1}^{m} a_k a_l \int_{-1}^{1} \sin(2^k \pi s) \sin(2^l \pi s) \; 2^l \pi \mathrm{d}s \\
&\qquad = - \sum_{k,l=1}^{m} a_k a_l 2^l \pi \int_{-1}^{1} \frac{1}{2} \big(\cos((2^k - 2^l)\pi s) - \cos((2^k + 2^l)\pi s) \big) \mathrm{d}s \\
&\qquad = - \sum_{k=1}^{m} a_k^2 2^k \pi = - \sum_{k=1}^{m} 2^{(1-2\alpha)k} \pi .
\end{aligned}
$$

This evidently does not converge as $m \to \infty$ for $\alpha < \frac{1}{2}$.

So missing *control* between the integrand and the integrator leads to defective integrals.

8.1 Paracontrolled paths

For going beyond the Young limit in the theory of integration of rough paths, according to the example just given the concept of *control* will play the essential role.

In fact, we will assume that integrand and integrator are controlled by a joint rough function for which we know that the three terms obtained in the decomposition given in Section 7.2 make sense. The notion of control of rough functions generalizes the approximation by linear or quadratic terms in Taylor's formula of differential calculus into the domain of fractional approximation orders.

We use *control by paraproducts*, and obtain the following notion of paracontrolled paths.

Definition 8.2. Let $\alpha > 0$ and $v \in \mathcal{C}^\alpha(\mathbb{R}^d)$. We define

$$\begin{aligned}\mathcal{D}_v^\alpha \; := \; & \mathcal{D}_v^\alpha(\mathbb{R}^n) := \{(f, f^v) \in \mathcal{C}^\alpha(\mathbb{R}^n) \times \mathcal{C}^\alpha(\mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)) : \\ & f^\sharp = f - \pi_<(f^v, v) \in \mathcal{C}^{2\alpha}(\mathbb{R}^n)\}.\end{aligned}$$

If $(f, f^v) \in \mathcal{D}_v^\alpha$, then f is called *paracontrolled* by v. The function f^v is called the *derivative* of f with respect to v. Abusing notation, we write $f \in \mathcal{D}_v^\alpha$ if it is clear from the context what the derivative f^v is supposed to be.

We equip $\mathcal{D}_v^\alpha$ with the norm

$$\|f\|_{v,\alpha} := \|f^v\|_\alpha + \|f^\sharp\|_{2\alpha}.$$

If $v \in \mathcal{C}^\alpha$ and $(\tilde{f}, \tilde{f}^{\tilde{v}}) \in \mathcal{D}_{\tilde{v}}^\alpha$, then we also write

$$d_{\mathcal{D}^\alpha}(f, \tilde{f}) := \|f^v - \tilde{f}^{\tilde{v}}\|_\alpha + \|f^\sharp - \tilde{f}^\sharp\|_{2\alpha}.$$

Example 8.3. Let $2\alpha > 1$ and $v \in \mathcal{C}^\alpha$, $w \in \mathcal{C}^\alpha$. Then by (39), the Young integral $I(v, \mathrm{d}w)$ is in $\mathcal{D}_w^\alpha$, with derivative v.

Let us first motivate how paracontrol can be used in order to obtain a rough path integral. We shall see that we always need the knowledge of the existence of Lévy's area of a reference function v that controls the functions for which we want to define integrals. In the simplest case we are in the following setting. Let $\alpha \in (1/3, 1)$, $v \in \mathcal{C}^\alpha(\mathbb{R}^d)$ and assume that the Lévy area

$$L(v, v) := \lim_{N \to \infty} \big(L(S_N v^k, S_N v^\ell)\big)_{1 \le k \le d, 1 \le \ell \le d}$$

exists uniformly and that

$$\sup_N \|L(S_N v, S_N v)\|_{2\alpha} < \infty.$$

Let $f \in \mathcal{D}_v^\alpha(\mathcal{L}(\mathbb{R}^d, \mathbb{R}^n))$. We want to show that $I(S_N f, \mathrm{d}S_N v)$ converges in $\mathcal{C}^{\alpha-\varepsilon}$ for all $\varepsilon > 0$. For this purpose, we employ paracontrol to write the following decomposition of the integral. For $N \in \mathbb{N}$

$$\begin{aligned} I(S_N f, \mathrm{d}S_N v) &= S(S_N f, S_N v) + \pi_<(S_N f, S_N v) + L(S_N f^\sharp, S_N v) \\ &\quad + [L(S_N \pi_<(f^v, v), S_N v) - I(f^v, \mathrm{d}L(S_N v, S_N v))] \\ &\quad + I(f^v, \mathrm{d}L(S_N v, S_N v)). \end{aligned}$$

It then follows from the theory of the Young integral and previous results that all terms converge, except the *commutator*

$$L(S_N \pi_<(f^v, v), S_N v) - I(f^v, \mathrm{d}L(S_N v, S_N v)).$$

The commutator has to be considered separately. This is done in the following subsection.

8.2 A basic commutator estimate

Here we prove the commutator estimate which is the main ingredient in the construction of the integral $I(f, \mathrm{d}g)$, where f and g are paracontrolled by v, and where we assume that the Lévy area of the control $L(v, v)$ exists.

Propostion 8.4. *Let $\alpha \in (0,1)$, and assume that $2\alpha < 1 < 3\alpha$. Let $f, v, w \in \mathcal{C}^\alpha$. Then the* commutator

$$\begin{aligned} C(f, v, w) &:= L(\pi_<(f, v), w) - I(f, \mathrm{d}L(v, w)) \qquad (40) \\ &:= \lim_{N\to\infty} [L(S_N(\pi_<(f, v)), S_N w) - I(f, \mathrm{d}L(S_N v, S_N w))] \\ &= \lim_{N\to\infty} \sum_{p\le N} \sum_{q<p} \Bigg[\int_0^\cdot \Delta_p(\pi_<(f, v))(s) \mathrm{d}\Delta_q w(s) - \int_0^\cdot f(s) \Delta_p v(s) \mathrm{d}\Delta_q w(s) \\ &\quad - \Big(\int_0^\cdot \Delta_p w(s) \mathrm{d}(\Delta_q(\pi_<(f, v)))(s) - \int_0^\cdot f(s) \Delta_p w(s) \mathrm{d}(\Delta_q v)(s) \Big) \Bigg] \end{aligned}$$

exists, with convergence in $\mathcal{C}^{3\alpha-\varepsilon}$ for all $\varepsilon > 0$. Moreover,

$$\|C(f, v, w)\|_{3\alpha} \lesssim \|f\|_\alpha \|v\|_\alpha \|w\|_\alpha.$$

Proof. 1. **Show:** The sequence

$$X_N := \sum_{p\le N} \sum_{q<p} \Big[\int_0^\cdot \Delta_p(\pi_<(f, v))(s) \mathrm{d}\Delta_q w(s) - \int_0^\cdot f(s) \Delta_p v(s) \mathrm{d}\Delta_q w(s) \Big] \qquad (41)$$

converges uniformly w.r.t $||\cdot||_\infty$. (The second difference in (40) can be handled using the same arguments.)

Note that for $N \in \mathbb{N}$

$$X_N - X_{N-1} \tag{42}$$
$$= \sum_{q<N} \left[\int_0^\cdot \Delta_N(\pi_<(f,v))(s)\mathrm{d}\Delta_q w(s) - \int_0^\cdot f(s)\Delta_N v(s)\mathrm{d}\Delta_q w(s) \right].$$

We further calculate the two expressions in (42). For $q < N$ we have by the definition of the paraproduct, by uniform convergence of the Schauder development and since $\Delta_q w$ is of bounded variation

$$\begin{aligned}
&\int_0^\cdot \Delta_N(\pi_<(f,v))(s)\mathrm{d}\Delta_q w(s) \\
&\quad = \sum_{j\in\mathbb{N}} \sum_{i<j} \int_0^\cdot \Delta_N(\Delta_i f \Delta_j v)(s)\mathrm{d}\Delta_q w(s) \\
&\quad = \sum_{j\le N} \sum_{i<j} \int_0^\cdot \Delta_N(\Delta_i f \Delta_j v)(s)\mathrm{d}\Delta_q w(s).
\end{aligned}$$

For the last equation we also use Corollary 7.9.

For the second term we use a similar reasoning to get for $q < N$

$$\begin{aligned}
&\int_0^\cdot f(s)\Delta_N v(s)\mathrm{d}\Delta_q w(s) \\
&\quad = \sum_{i\in\mathbb{N}} \int_0^\cdot (\Delta_i f\, \Delta_N v)(s)\mathrm{d}\Delta_q w(s) \\
&\quad = \sum_{j\in\mathbb{N}} \sum_{i\le j} \int_0^\cdot \Delta_j(\Delta_i f \Delta_N v)(s) d\Delta_q w(s) \\
&\quad = \sum_{j\ge N} \sum_{i\le j} \int_0^\cdot \Delta_j(\Delta_i f \Delta_N v)(s)\mathrm{d}\Delta_q w(s).
\end{aligned}$$

In summary we have

$$X_N - X_{N-1} \tag{43}$$
$$\begin{aligned}
= \sum_{q<N} \Bigg[&\sum_{j\le N} \sum_{i<j} \int_0^\cdot \Delta_N(\Delta_i f \Delta_j v)(s)\mathrm{d}\Delta_q w(s) \\
&- \sum_{j\ge N} \sum_{i\le j} \int_0^\cdot \Delta_j(\Delta_i f \Delta_N v)(s)\mathrm{d}\Delta_q w(s) \Bigg].
\end{aligned}$$

The terms in (43) for $j = N$ cancel, except the one with $j = i = N$ in the second term. These cancellations are crucial, since they eliminate terms for which we only have the worse estimate (38) in Corollary 7.9. We obtain

$$\begin{aligned} X_N - X_{N-1} \\ &= \sum_{q<N}\sum_{j<N}\sum_{i<j}\int_0^{\cdot} \Delta_N(\Delta_i f \Delta_j v)(s)\mathrm{d}\Delta_q w(s) \quad (44)\\ &\quad - \sum_{q<N}\int_0^{\cdot} \Delta_N(\Delta_N f \Delta_N v)(s)\mathrm{d}\Delta_q w(s) \\ &\quad - \sum_{q<N}\sum_{j>N}\sum_{i<j}\int_0^{\cdot} \Delta_j(\Delta_i f \Delta_N v)(s)\mathrm{d}\Delta_q w(s) \\ &\quad - \sum_{q<N}\sum_{j>N}\int_0^{\cdot} \Delta_j(\Delta_j f \Delta_N v)(s)\mathrm{d}\Delta_q w(s). \end{aligned}$$

Here the first expression is the remainder of the first term of the previous expression after cancellation, while expressions 3 and 4 come from the negative second term in the original expression, and represent the conditions $j > N, i < j$ (expression 3) and $j > N, i = j$ (expression 4). Note that $\|\partial_t \Delta_q w\|_\infty \lesssim 2^q \|\Delta_q w\|_\infty$. Hence, an application of Corollary 7.9, where we use (37) for the first three terms and (38) for the fourth term, yields

$$\begin{aligned} \|X_N - X_{N-1}\|_\infty \lesssim \\ &[\sum_{q<N}\sum_{j<N}\sum_{i<j} 2^{-2N+i+j}2^q ||\Delta_i f||_\infty ||\Delta_j v||_\infty ||\Delta_q w||_\infty \\ &+ \sum_{q<N} 2^q ||\Delta_N f||_\infty ||\Delta_N v||_\infty ||\Delta_q w||_\infty \\ &+ \sum_{q<N}\sum_{j>N}\sum_{i<j} 2^{-2j+i+N}2^q ||\Delta_i f||_\infty ||\Delta_N v||_\infty ||\Delta_q w||_\infty \\ &+ \sum_{q<N}\sum_{j>N} 2^q ||\Delta_j f||_\infty ||\Delta_N v||_\infty ||\Delta_q w||_\infty] \\ \lesssim \|f\|_\alpha \|v\|_\alpha \|w\|_\alpha \\ &[\sum_{q<N}\sum_{j<N}\sum_{i<j} 2^{-2N+i+j}2^{-i\alpha}2^{-j\alpha}2^{q(1-\alpha)} \\ &+ \sum_{q<N} 2^{-2N\alpha}2^{q(1-\alpha)} \\ &+ \sum_{q<N}\sum_{j>N}\sum_{i<j} 2^{-2j+i+N}2^{-i\alpha}2^{-N\alpha}2^{q(1-\alpha)} \\ &+ \sum_{q<N}\sum_{j>N} 2^{-j\alpha}2^{-N\alpha}2^{q(1-\alpha)}] \\ \lesssim \|f\|_\alpha \|v\|_\alpha \|w\|_\alpha 2^{-N(3\alpha-1)}, \quad (45) \end{aligned}$$

This yields uniform convergence of (X_N), since $3\alpha > 1$.

2. **Show:** $\|X_N\|_{3\alpha} \lesssim \|f\|_\alpha \|v\|_\alpha \|w\|_\alpha$ for all N.

Similarly to (44) we obtain for $n \in \mathbb{N}$

$$\Delta_n X_N = \sum_{p\le N}\sum_{q<p} \Delta_n \Bigg[\sum_{j<p}\sum_{i<j} \int_0^\cdot \Delta_p(\Delta_i f \Delta_j v)(s) \mathrm{d}\Delta_q w(s) - \int_0^\cdot \Delta_p(\Delta_p f \Delta_p v)(s)\mathrm{d}\Delta_q w(s) - \sum_{j>p}\sum_{i\le j} \int_0^\cdot \Delta_j(\Delta_i f\Delta_p v)(s)\mathrm{d}\Delta_q w(s)\Bigg],$$

and therefore by Corollary 7.8

$$\|\Delta_n X_N\|_\infty \lesssim \sum_p \sum_{q<p}\Bigg[\sum_{j<p}\sum_{i<j} 2^{-(n\vee p)-n+p+q}\|\Delta_p(\Delta_i f\Delta_j v)\|_\infty \|\Delta_q w\|_\infty + 2^{-(n\vee p)-n+p+q}\|\Delta_p(\Delta_p f\Delta_p v)\|_\infty\|\Delta_q w\|_\infty + \sum_{j>p}\sum_{i\le j} 2^{-(n\vee j)-n+j+q}\|\Delta_j(\Delta_i f\Delta_p v)\|_\infty\|\Delta_q w\|_\infty\Bigg].$$

Now we apply Corollary 7.9, where for the last term we distinguish the cases $i<j$ and $i=j$. Using that $1-\alpha>0$, we get

$$\begin{aligned}\|\Delta_n X_N\|_\infty &\lesssim \|f\|_\alpha\|v\|_\alpha\|w\|_\alpha \\ &\qquad \sum_p 2^{p(1-\alpha)}\Bigg[\sum_{j<p}\sum_{i<j} 2^{-(n\vee p)-n+p}2^{-2p}2^{i(1-\alpha)}2^{j(1-\alpha)} \\ &\qquad + 2^{-(n\vee p)-n+p}2^{-p\alpha}2^{-p\alpha} \\ &\qquad + \sum_{j>p}\sum_{i<j} 2^{-(n\vee j)-n+j}2^{-2j+i(1-\alpha)+p(1-\alpha)} \\ &\qquad + \sum_{j>p} 2^{-(n\vee j)-n+j}2^{-j\alpha-p\alpha}\Bigg] \\ &\lesssim \|f\|_\alpha\|v\|_\alpha\|w\|_\alpha 2^{-n(3\alpha)},\end{aligned}$$

where we used both that $3\alpha>1$ and that $2\alpha<1$. This implies the desired boundedness.

3. From 1. in combination with 2. the desired result follows, since bounded sets in $\mathcal{C}^{3\alpha}$ are relatively compact in $\mathcal{C}^{3\alpha-\varepsilon}$ for $\varepsilon>0$ (see Theorem 6.4). □

Remark 8.5. If $2\alpha = 1$, we can apply Proposition 8.4 with $\alpha-\varepsilon$ to obtain that $C(f,v,w)\in\mathcal{C}^{3\alpha-\varepsilon}$ for every sufficiently small $\varepsilon>0$. If $2\alpha>1$, then we are in the Young setting and there is no need to introduce the commutator.

8.3 Pathwise integration for paracontrolled paths

In this section we apply the commutator estimate to obtain the rough path integral under the assumption that the Lévy area exists for a given reference path.

Theorem 8.6. *Let* $\alpha \in (1/3, 1)$*, and assume that* $2\alpha \neq 1$*. Let* $v \in \mathcal{C}^\alpha(\mathbb{R}^d)$ *and assume that the Lévy area*

$$L(v, v) := \lim_{N \to \infty} \big(L(S_N v^k, S_N v^\ell)\big)_{1 \le k \le d, 1 \le \ell \le d}$$

converges uniformly and that

$$\sup_N \|L(S_N v, S_N v)\|_{2\alpha} < \infty.$$

Let $f \in \mathcal{D}_v^\alpha(\mathcal{L}(\mathbb{R}^d, \mathbb{R}^m))$*. Then* $(I(S_N f, \mathrm{d}S_N v))_{N \in \mathbb{N}}$ *converges in* $\mathcal{C}^{\alpha - \varepsilon}$ *for all* $\varepsilon > 0$*.*

Denoting the limit by $I(f, \mathrm{d}v)$*, we have*

$$\|I(f, \mathrm{d}v)\|_\alpha \lesssim \|f\|_{v,\alpha}\big(\|v\|_\alpha + \|v\|_\alpha^2 + \|L(v, v)\|_{2\alpha}\big).$$

Moreover, $I(f, \mathrm{d}v) \in \mathcal{D}_v^\alpha$ *with derivative* f *and*

$$\|I(f, \mathrm{d}v)\|_{v,\alpha} \lesssim \|f\|_{v,\alpha}\big(1 + \|v\|_\alpha^2 + \|L(v, v)\|_{2\alpha}\big).$$

Proof. If $2\alpha > 1$, everything follows from the Young case, Theorem 7.12.

So let $2\alpha < 1$. Recall the decomposition for $N \in \mathbb{N}$

$$\begin{aligned} I(S_N f, \mathrm{d}S_N v) = {} & S(S_N f, S_N v) + \pi_<(S_N f, S_N v) + L(S_N f^\sharp, S_N v) \\ & + [L(S_N \pi_<(f^v, v), S_N v) - I(f^v, \mathrm{d}L(S_N v, S_N v))] \\ & + I(f^v, \mathrm{d}L(S_N v, S_N v)). \end{aligned}$$

Convergence then follows from Proposition 8.4 and Theorem 7.12. The limit is given by

$$I(f, \mathrm{d}v) = S(f, v) + \pi_<(f, v) + L(f^\sharp, v) + C(f^v, v, v) + I(f^v, \mathrm{d}L(v, v)),$$

from where we easily deduce the claimed bounds. □

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