Chapter 9
Utility maximization, martingale optimality and backward stochastic differential equations (BSDE)

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1 Utility maximization: the financial market model

(N. El Karoui, R. Rouge '00; J. Sekine '02; J. Cvitanic, J. Karatzas '92)

$(\Omega, \mathcal{F}, P)$ probability space with m-dim. Brownian motion $W = (W^1, \cdots, W^m)$, $(\mathcal{F}_t)_{t\geq 0}$ filtration of W, T finite time horizon

price processes

trivial bond $S^0 \equiv 1$; stocks $1 \leq i \leq d$

$$\begin{aligned} \frac{dS_t^i}{S_t^i} &= \sum_{j=1}^m \sigma_t^{ij} dW_t^j + b_t^i dt \\ &= \sigma_t^i dW(t) + b_t^i dt \\ &= \sigma_t^i [dW(t) + \theta_t dt], \quad t \in [0,T] \end{aligned}$$

$$\theta = \sigma^* [\sigma\sigma^*]^{-1} b, \quad \underline{\varepsilon} I^d \leq [\sigma\sigma^*] \leq \overline{\varepsilon} I^d \quad \text{with} \quad \underline{\varepsilon} > 0$$

1 Utility maximization: the financial market model

investment strategies with non-convex constraints

(N. El Karoui, R. Rouge '00 for convex constraints)

$\tilde{C} \subset \mathbb{R}^d$ closed (e. g. $\tilde{C} = \mathbb{Z}^d$); $\tilde{A}$: strategies $\pi = (\pi^1, \cdots, \pi^d)$ s. th.
$\pi \in \tilde{C}$ $\quad P \otimes \lambda$-a.s. $\quad$ (λ Lebesgue measure)
$\{\exp(-\alpha \int_0^\tau \pi_s \frac{dS_s}{S_s}) : \tau$ stopping time in $[0, T]\}$ uniformly integrable

wealth process

$$X_t^\pi = x + \sum_{i=1}^d \int_0^t \pi_s^i \frac{dS_s^i}{S_s^i} = x + \int_0^t \pi_s \sigma_s [dW(s) + \theta_s ds]$$

preferences of small agent measured by **exponential utility** from terminal wealth

utility function

$$U(x) = -\exp(-\alpha x), \quad x \in \mathbb{R}.$$

1 Utility maximization: liabilities

F $\mathcal{F}_T$-measurable liability at time T.

Example 1:
Insurer sells weather derivative, strike price K, strike time T:

T_i minimum temperature on day i at specific location
heating degree days $HDD_i = \max(0, 18 - T_i)$
cumulative heating degree days $cHDD_t = \sum_{i=1}^{30} HDD_{t-i}$.

insurer has obligation to pay

$$F = (cHDD_T - K)^+.$$

Market incompleteness: e.g. S^i driven by $W^j, j \leq k < m$

1 Utility optimization: the optimization problem

First formulation:

Find

$$V(x) = \sup_{\pi \in \tilde{\mathcal{A}}} E(U(X_T^{\pi} - F)) = \sup_{\pi \in \tilde{\mathcal{A}}} E(U(x + \int_0^T \pi_s \sigma_s [dW(s) + \theta_s ds] - F)).$$

For simplicity:

$$\begin{aligned} p &= \pi\sigma, \\ C &= \tilde{C}\sigma, \\ \mathcal{A} &= \tilde{\mathcal{A}}\sigma. \end{aligned}$$

$$X_t^p = x + \int_0^t p_s [dW(s) + \theta_s ds], \quad t \in [0, T]$$

Second formulation:

Find

$$V(x) = \sup_{p \in \mathcal{A}} E(U(X_T^p - F)) = \sup_{p \in \mathcal{A}} E(-\exp(-\alpha(x + \int_0^T p_s [dW(s) + \theta_s ds] - F))).$$

1 Utility optimization: a solution method based on BSDE

Idea: Construct family of processes $R^{(p)}$ such that

form 1

$$\begin{array}{lcl} R_0^{(p)} & = & \text{constant, indep. of } p, \\ R_T^{(p)} & = & -\exp(-\alpha(X_T^p - F)), \\ R^{(p)} & & \text{supermartingale,} \quad p \in \mathcal{A}, \\ R^{(p^*)} & & \text{martingale, for (exactly) one} \quad p^* \in \mathcal{A}. \end{array}$$

Then

$$\begin{aligned} E(-\exp(-\alpha[X_T^p - F])) &= E(R_T^{(p)}) \\ &\leq E(R_0^p) \\ &= V(x) \\ &= E(R_0^{(p^*)}) \\ &= E(-\exp(-\alpha[X_T^{(p^*)} - F])). \end{aligned}$$

Hence p^* optimal strategy.

1 Utility optimization: a solution method based on BSDE

Introduction of BSDE into problem: find generator f of BSDE

$$Y_t = Y_0 + \int_0^t Z_s dW(s) + \int_0^t f(s, Z_s) ds, \quad Y_T = F,$$

$$\text{eq. } Y_t = F - \int_t^T Z_s dW(s) - \int_t^T f(s, Z_s) ds, \quad \text{s. th. with}$$

$$R_t^{(p)} = -\exp(-\alpha[X_t^p - Y_t]), \quad t \in [0, T], \quad \text{we have}$$

form 2

$$R_0^{(p)} = -\exp(-\alpha(x - Y_0)) = \text{constant}, \qquad \text{(fulfilled)}$$

$$R_T^{(p)} = -\exp(-\alpha(X_T^p - F)) \qquad \text{(fulfilled)}$$

$R^{(p)}$ supermartingale, $p \in \mathcal{A}$,
$R^{(p^*)}$ martingale, for (exactly) one $p^* \in \mathcal{A}$.

This gives solution of valuation problem.

1 Utility optimization: construction of generator of BSDE

How to determine f:
Suppose f generator of BSDE. Then

$$\begin{aligned} R_t^{(p)} &= -\exp(-\alpha[X_t^p - Y_t]) \\ &= -\exp(-\alpha[x - Y_0]) \cdot \exp(-\alpha[\int_0^t (p_s - Z_s)dW(s) - \int_0^t [f(s, Z_s) - p_s\theta_s]ds]) \\ &= \exp(-\alpha[x - Y_0]) \cdot \exp(-\alpha \int_0^t (p_s - Z_s)dW(s) - \frac{\alpha^2}{2} \int_0^t (p_s - Z_s)^2 ds) \\ &\quad \cdot - \exp(\int_0^t [\alpha f(s, Z_s) - \alpha p_s\theta_s + \frac{\alpha^2}{2}(p_s - Z_s)^2]ds) \\ &= M_t^{(p)} \cdot A_t^{(p)}, \end{aligned}$$

with $M^{(p)}$ nonnegative martingale. $R^{(p)}$ satisfies **(form 2)** iff for

$$q(\cdot, p, z) = f(\cdot, z) - p\theta + \frac{\alpha}{2}(p - z)^2, \quad p \in \mathcal{A}, z \in \mathbb{R},$$

we have

1 Utility optimization: construction of generator of BSDE

form 3
$$\begin{array}{rcll} q(\cdot,p,z) & \geq & 0, & p \in \mathcal{A} \quad \text{(supermartingale cond.)} \\ q(\cdot,p^*,z) & = & 0, & \text{for (exactly) one} \quad p^* \in \mathcal{A} \quad \text{(martingale cond.)}. \end{array}$$

Now

$$\begin{aligned} q(\cdot,p,z) &= f(\cdot,z) - p\theta + \frac{\alpha}{2}(p-z)^2 \\ &= f(\cdot,z) + \frac{\alpha}{2}(p-z)^2 - (p-z)\cdot\theta + \frac{1}{2\alpha}\theta^2 \quad - z\theta - \frac{1}{2\alpha}\theta^2 \\ &= f(\cdot,z) + \frac{\alpha}{2}[p - (z + \frac{1}{\alpha}\theta)]^2 \quad - z\theta - \frac{1}{2\alpha}\theta^2. \end{aligned}$$

Under **non-convex constraint** $p \in C$:

$$[p - (z + \frac{1}{\alpha}\theta)]^2 \geq d^2(C, z + \frac{1}{\alpha}\theta).$$

with **equality** for at least one possible choice of p^* due to **closedness** of C.
Hence **(form 3)** is solved by the choice

1 Utility optimization: construction of generator of BSDE

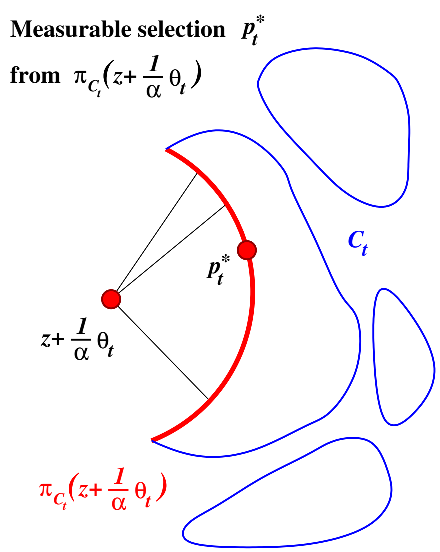
form 4

$$f(\cdot, z) = -\frac{\alpha}{2}d^2(C, z + \frac{1}{\alpha}\theta) + z \cdot \theta + \frac{1}{2\alpha}\theta^2 \quad \text{(supermartingale cond.)}$$
$$p^* \quad \text{s. th.} \quad d(C, z + \frac{1}{\alpha}\theta) = d(p^*, z + \frac{1}{\alpha}\theta) \quad \text{(martingale cond.)}.$$

Problem: Let

$$\pi_C(v) = \{p \in \mathbb{R}^d : d(C, v) = d(p, v)\}.$$

Find measurable selection p^*_t from $\pi_{C_t}(Z_t + \frac{1}{\alpha}\theta_t)$. Solved by classical **measurable selection method**.



1 Utility optimization: main result

Thm 1

(Y, Z) unique solution of BSDE

$$Y_t = F - \int_t^T Z_s dW(s) - \int_t^T f(s, Z_s) ds, \quad t \in [0, T],$$

with

$$f(t, Z_t) = -\frac{\alpha}{2} d^2(C_t, Z_t + \frac{1}{\alpha}\theta_t) + Z_t \cdot \theta_t + \frac{1}{2\alpha}\theta_t^2.$$

Then value function of utility optimization problem under constraint $p \in \mathcal{A}$ given by

$$V(x) = -\exp(-\alpha[x - Y_0]).$$

There exists an optimal trading strategy $p^* \in \mathcal{A}$ such that

$$p_t^* \in \Pi_{C_t}(Z_t + \frac{1}{\alpha}\theta_t), \quad t \in [0, T].$$

1 Utility optimization: methods of proof of main result

Proof:

- existence and uniqueness for BSDE locally Lipschitz in z (M. Kobylanski '00)
- measurable selection theorem for $\Pi_{C_t}(Z_t + \frac{1}{\alpha}\theta_t)$
- BMO properties of the martingales $\int Z_s dW(s), \int p^*_s dW(s)$ for uniform integrability of exponentials (regularity of coefficients)•

Aim: existence and uniqueness for BSDE with Lipschitz drivers, for simplicity in dimension 1.