STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH LÉVY NOISE (A FEW ASPECTS)

SZYMON PESZAT, JAGIELLONIAN UNIVERSITY AND POLISH ACADEMY OF SCIENCES

Abstract. The course is concerned with the following topics:

- Examples of equations. I will be motivated by the development of the theory as well as applications of SPDEs in modeling. At this point I will be also concerned with different concepts of solutions and their regularity, and their Markov property. Finally we will say something about motivation for the study of SPDEs.
- Stochastic integration in Hilbert spaces (or more general infinite dimensional spaces), Lévy processes in Hilbert spaces. As examples we will consider the so-called cylindrical processes and an impulsive Lévy noise.
- Basic existence result, time regularity.
- Long time behaviour of solutions (in particular we will be concerned with the existence and uniqueness of invariant measures).

1. Examples of equations

1.1. Transport equation.

$$
\begin{aligned}
\partial_t u(t,x) &= \partial_x u(t,x) + F(t,x,u(t,x)) + G_1(t,x,u(t,x))\partial_t M(t,x)\\
&\qquad + G_2(t,x,u(t,x))\partial_t \Pi(t,x), \qquad x\in\mathbb{R},\ t>0,\\
u(0,x) &= u_0(x), \qquad x\in\mathbb{R}.
\end{aligned}
\tag{1}
$$

In (1), $M = M(t), t \geq 0$, is a Lévy (square integrable) martingale and Π is a compound Poisson process, both taking values in some function space. Obviously, as for SDE, (1) should be understood as (symbolic) differential form of some integral (in t) equation. The question is about differentiability in x.

The first idea, would be to consider (1) as equation on the state space C^1. This would require stochastic integration in the Banach space C^1. Unfortunately it is not possible! Namely, let W be a real-valued standard Brownian motion. It turns out that one can find a deterministic mapping $\xi \in C([0,1]; C^1([0,1]))$ such that the random field

$$\int_0^t \xi(s)(x)\mathrm{d}W(s), \qquad x \in [0,1],\ t \in [0,1],$$

cannot be modified into a C^1-valued process. We can integrate in Hilbert spaces (in a similar way as in Euclidean spaces) but only in some Banach spaces (like L^p, $W^{s,p}$ but not C or C^k).

In conclusion, contrary to elementary theory of PDEs, we are forced to forget about strong solution, and therefore we have to find a way how to treat the linear part $\partial_x u$ of the equation. To to this we introduce the concept of *weak solution*[1] Consider first linear (deterministic) equation

$$\begin{aligned} \partial_t u(t,x) &= \partial_x u(t,x), \qquad t>0,\ x \in \mathbb{R}, \\ u(0,x) &= u_0(x), \qquad x \in \mathbb{R}. \end{aligned} \tag{2}$$

Clearly, if u_0 is differentiable, then the solution is given by the formula

$$u(t,x) = u_0(t+x). \tag{3}$$

Obviously, the right hand side of (3) is well defined even if $u_0 \in L^2_{\mathrm{loc}}(\mathbb{R})$. Moreover, it is easy to check that for any test function $\phi \in C^1_0(\mathbb{R})$,

$$\begin{aligned} \langle u(t), \phi\rangle &= \int_{\mathbb{R}} u(t,x)\phi(x)\mathrm{d}x \\ &= \langle u_0, \phi\rangle - \int_0^t \langle u(s), \phi'\rangle \mathrm{d}s. \end{aligned}$$

Thus in the terminology of PDEs, u is a weak solution to (2).

Assume that $\Pi = 0$, $G = G_1$. Our goal is to invent a reasonable concept of a weak solution to (1), show its existence and uniqueness and finally study its regularity. Roughly (as we have not define the stochastic integral yet) a weak solution to (1) is a predictable process u taking values in L^2_{loc} such that for any $\phi \in C^1_0(\mathbb{R})$,

$$\begin{aligned} \langle u(t), \phi\rangle = \langle u_0, \phi\rangle &+ \int_0^t \left\{ -\langle u(s), \phi'\rangle + \langle F(s,u(s)), \phi\rangle \right\} \mathrm{d}s \\ &+ \int_0^t \langle G(s,u(s))\mathrm{d}M(s), \phi\rangle. \end{aligned}$$

Let us give the sense of the stochastic term in the most natural way possible. Assume that M is of the form

$$M(t,x) = \sum_k M_k(t) e_k(x),$$

[1] Here weak is in PDEs sense.

where the sum is finite or infinite, M_k are (possibly correlated) real-valued Lévy martingales and e_k are given functions of space variable x. Then

$$\int_0^t \langle G(s,u(s))\mathrm{d}M(s),\phi\rangle = \sum_k \int_0^t \langle G(s,u(s))e_k,\phi\rangle \mathrm{d}M_k(s).$$

It is not so obvious however how to show the existence of a solution (in principle compactness method should work (see e.g. Rozovskii (1990), Viot (1974)). This problem is much simpler in the case of the so-called *mild solution*. We will show later that the motions of weak and mild solutions are equivalent.

The concept of a mild-solution goes back to the well known *variation of constants* or *Duhamel* formula. Namely, given a matrix $A\in M(n\times n)$ consider semilinear equation

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = Au(t) + F(t,u(t)), \qquad u(0)=u_0.$$

Then, u solves the integral equation

$$u(t) = \mathrm{e}^{At}u_0 + \int_0^t \mathrm{e}^{A(t-s)}F(s,u(s))\mathrm{d}s.$$

Therefore, there is a hope that the (weak) solution to (1) satisfies the integral equation

$$u(t) = \mathrm{e}^{t\partial_x}u_0 + \int_0^t \mathrm{e}^{(t-s)\partial_x}F(s,u(s))\mathrm{d}s + \int_0^t \mathrm{e}^{(t-s)\partial x}G(s,u(s))\mathrm{d}M(s),$$

where for a given function w of x-variable

$$\mathrm{e}^{t\partial_x}w(x) = w(t+x),$$

is a solution to the linear equation

$$\begin{aligned} \partial_t u(t,x) &= \partial_x u(t,x), \qquad t>0,\ x\in\mathbb{R},\\ u(0,x) &= w(x), \qquad x\in\mathbb{R}. \end{aligned} \tag{4}$$

For more precise analysis we need to fix a state space E (Hilbert or nice Banach) for our problem. Then our equation will define a Markov process in E. For the transport equation as the state space we can take $L^2 := L^2(\mathbb{R})$ or weighted space $L^2_\rho := L^2(\mathbb{R},\rho(x)\mathrm{d}x)$. The linear problem should be well-posed on E. Formally $(\mathrm{e}^{t\partial_x})$ should for a *strongly continuous semigroup*, that is

- $\forall\, t\geq 0$, $\mathrm{e}^{t\partial_x}$ should be a bounded (i.e. continuous) linear operator on E,
- $\mathrm{e}^{0\partial_x} = \mathrm{I}$,

- $\forall\, t, s \geq 0$,
$$e^{t\partial_x} e^{s\partial_x} = e^{(t+s)\partial_x},$$
- $\forall\, \psi \in E$,
$$\lim_{t\downarrow 0} \|e^{t\partial_x}\psi - \psi\|_E = 0.$$

For more information on C_0-semigroups see Appendix A, and such textbooks as for example Pazy (1983) or Engel and Nagel (2000), Davis (1980), or Lunardi (1995).

One problem should be pointed out: the integrals appearing in the definition are of the convolution type. In particular the stochastic component has the form
$$\int_0^t e^{(t-s)\partial_x} G(s, u(s)) \mathrm{d}M(s).$$
In the case of the transport equation the semigroup is in fact a group
$$e^{(t-s)\partial_x} = e^{t\partial_x} e^{-s\partial_x}.$$
Hence
$$\int_0^t e^{(t-s)\partial_x} \sigma(s, u(s)) \mathrm{d}M(s) = e^{t\partial_x} \int_0^t e^{-s\partial_x} \sigma(s, u(s)) \mathrm{d}M(s).$$
But in general (see for example the heat equation) a linear part of the equation generates only a semigroup!

1.2. **Heat equation.** Heat equation can be considered on the whole space $\mathbb{R}^d$ or on a bounded domain $\mathcal{O} \subset \mathbb{R}^d$. We will consider here the heat equation on the interval $[0, 1]$;
$$\begin{aligned} &\partial_t u(t,x) = \partial^2_{xx} u(t,x) + F(t,x,u(t,x)) + G(t,x,u(t,x))\partial_t M(t,x), \\ &\qquad\qquad\qquad t > 0,\ x \in (0,1), \\ &u(0,x) = u_0(x), \qquad x \in (0,1). \end{aligned} \tag{5}$$
Heat equation on a bounded domain should be considered with boundary conditions. In our case we should add to (5) either Dirichlet or Neumann or mixed boundary conditions. We will consider the equation with homogeneous boundary conditions. Thus in the first two cases it is assumed that
$$u(t,x) = 0 \qquad \text{for } x \in \partial\mathcal{O} = \{0,1\},$$
or
$$\partial_{\mathbf{n}} u(t,x) = 0 \qquad \text{for } x \in \partial\mathcal{O} = \{0,1\}.$$
What are the weak and mild formulation of the problem. In the weak formulation we cannot take the space $C_0^2(0,1)$ of compactly supported

C^2-functions as the set of test functions because we will not see the boundary conditions. Taking into account the following calculations

$$\int_0^1 \psi(x)v''(x)\mathrm{d}x = \{\psi(1)v'(1) - \psi(0)v'(0)\} - \int_0^1 \psi'(x)v'(x)\mathrm{d}x$$

$$= \{\psi(1)v'(1) - \psi(0)v'(0) - \psi'(1)v(1) + \psi'(0)v(0)\} + \int_0^1 \psi''(x)v(x)\mathrm{d}x,$$

we can find that the right answers are:

- In case of the Dirichlet boundary conditions the set of test functions is
$$\left\{\psi \in C^2([0,1])\colon \psi(0) = 0 = \psi(1)\right\}.$$
- In case of the Neumann boundary conditions the set of test functions is
$$\left\{\psi \in C^2([0,1])\colon \psi'(0) = 0 = \psi'(1)\right\}.$$

Then the integral condition in both cases looks like

$$\langle u(t), \psi\rangle = \langle u_0, \psi\rangle + \int_0^t \{\langle u(s), \psi''\rangle + \langle F(s, u(s)), \psi\rangle\}\,\mathrm{d}s$$
$$+ \int_0^t \langle G(s, u(s))\mathrm{d}M(s), \psi\rangle.$$

In the formula above

$$\langle v, \psi\rangle = \int_0^1 v(x)\psi(x)\mathrm{d}x,$$

and

$$\int_0^t \langle G(s, u(s))\mathrm{d}M(s), \psi\rangle = \sum_k \int_0^t \langle G(s, u(s))f_k, \psi\rangle \mathrm{d}M_k(s),$$

if M has the representation

$$M(t) = \sum_k M_k(t) f_k,$$

M_k are real-valued martingales, f_k functions on $[0,1]$.

We will consider the heat equation on the state space $E = L^2(0,1)$. To find a mild formulation of the heat equation we need to know the semigroup generated by the linear part of the equation. To do this let $(A_D, \mathrm{Dom}(A_D))$ and $(A_N, \mathrm{Dom}(A_N))$ be the Laplace operators with Dirichlet and Neumann boundary conditions, respectively. More precisely

$$A_D\psi = \psi'' \qquad \text{for any } \psi \in \mathrm{Dom}(A_D)$$

and

$$A_N\psi = \psi'' \qquad \text{for any } \psi \in \mathrm{Dom}(A_N).$$

However

$$\mathrm{Dom}(A_D) = W^{2,2}(0,1) \cap W^{1,2}_0(0,1)$$

and

$$\mathrm{Dom}(A_N) = \left\{\psi \in W^{2,2}(0,1)\colon \partial_{\mathbf{n}}\psi(0) = 0 = \partial_{\mathbf{n}}\psi(0)\right\}.$$

Then the semigroups $(\mathrm{e}^{A_D t})$ and $(\mathrm{e}^{A_N t})$ generated on $L^2(0,1)$ by the operators $(A_D, \mathrm{Dom}(A_D))$ and $(A_N, \mathrm{Dom}(A_N))$ are different. Moreover, for $\psi \in L^2(0,1)$,

$$\mathrm{e}^{A_D t}\psi = \sum_{k=1}^{\infty} \mathrm{e}^{-\pi^2 k^2 t}\langle \psi, e_k\rangle e_k,$$

where $e_k(x) = \sqrt{2}\sin(\pi k x)$, and

$$\mathrm{e}^{A_N t}\psi = \sum_{k=0}^{\infty} \mathrm{e}^{-\pi^2 k^2 t}\langle \psi, f_k\rangle f_k,$$

where $f_k(x) = \sqrt{2}\cos(\pi k x)$.

The mild version of (5) is

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(s,u(s))\mathrm{d}s + \int_0^t S(t-s)G(s,u(s))\mathrm{d}M(s),$$

where $S$is the semigroup generated by linear part of the equation; that is $S(t) = \mathrm{e}^{A_D t}$ or $S(t) = \mathrm{e}^{A_N t}$.

1.3. **Heath–Jarrow–Morton.** Let $P(t,T)$ be the price at time t of a bond giving 1 at time $T \geq t$. Define the *forward rate* f by the relation

$$P(t,T) = \mathrm{e}^{-\int_t^T f(t,\theta)\mathrm{d}\theta}.$$

On can model f in the following way

$$\mathrm{d}f(t,\theta) = \alpha(t,\theta)\mathrm{d}t + \langle \sigma(t,\theta), \mathrm{d}L(t)\rangle_U,$$

where L is a Lévy process taking values in a Hilbert space U.

Using the so-called *Musiela parametrization*

$$r(t)(x) = f(t, x{+}t), \quad a(t)(x) = \alpha(t, x{+}t), \quad b(t)u(x) = \langle \sigma(t, x{+}t), u\rangle_U,$$

we obtain the following expression for the evolution of the curve $r(t)$, $t \geq 0$,

$$r(t) = S(t)r(0) + \int_0^t S(t-s)a(s)\mathrm{d}s + \int_0^t S(t-s)b(s)\mathrm{d}L(s),$$

where

$$S(t)\psi(x) = \psi(x+t).$$

Thus formally S is the semigroup generated by the derivative operator, and r is the so-called *mild solution* to the following stochastic partial differential equation

$$\mathrm{d}r = \left(\frac{\mathrm{d}}{\mathrm{d}x}r + a\right)\mathrm{d}t + b\mathrm{d}L.$$

In some models b and a depend on r and sometimes also on t, and thus r is a solution to the nonlinear stochastic partial differential equation

$$\mathrm{d}r = \left(\frac{\mathrm{d}}{\mathrm{d}x}r + a(t,r)\right)\mathrm{d}t + b(t,r)\mathrm{d}L.$$

One can show that the model is arbitrage-free in the strong sense, that is the original measure is a martingale measure, if and only if

$$a(t)(x) = \frac{\mathrm{d}}{\mathrm{d}x}J\left(\int_0^x b(t)(y)\mathrm{d}y\right),$$

where J is the *Laplace exponent* of L;

$$\begin{aligned} J(v) = &-\langle a, v\rangle_U + \tfrac{1}{2}\langle Qv, v\rangle_U \\ &+ \int_U \left(\mathrm{e}^{-\langle v,u\rangle_U} - 1 + \chi_{\{|u|_U<1\}}\langle v, u\rangle_U\right)\nu(\mathrm{d}y), \end{aligned}$$

and ν is the *Lévy measure* of L.

Note that if L is a real-valued Wiener process, then the non-arbitrage conditions reads

$$a(t)(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{2}\left(\int_0^x b(t)(y)\mathrm{d}y\right)^2 = b(t)(x)\int_0^t b(t)(y)\mathrm{d}y.$$

For more details see Björk, DiMassi, Kabanov and Runggaldier (1997), Björk and Christiansen (1999), Björk, Kabanov and Runggaldier (1997), Eberlein and Raible (1999), Filipović and Tappe (2006), Jakubowski and Zabczyk (2004), Peszat, Rusinek, and Zabczyk (2007), and Rusinek (2006a), (2006b).

1.4. **Lifts of diffusions.** Consider an ordinary stochastic differential equation

$$\mathrm{d}y = f(y)dt + g(y)\mathrm{d}M(s), \qquad y(0) = x \in \mathbb{R}^d, \tag{6}$$

where M is an $\mathbb{R}^m$-valued martingale, $f\colon \mathbb{R}^d \mapsto \mathbb{R}^d$ and $g\colon \mathbb{R}^d \mapsto M(d\times m)$.

Let $D \subset \mathbb{R}^d$ be a closed set and let E be a Hilbert space of (some) functions from D into $\mathbb{R}^d$. Then the following infinite dimensional equation

$$\mathrm{d}X = F(X)\mathrm{d}t + G(X)\mathrm{d}M, \qquad X(0) = \mathrm{I},$$

in which $F(X)(x) = f(X(x))$, $G(X)(x) = g(X(x))$ describes the dynamics of the flow defined by (6). From this observation on can derive basic properties of the flow, for more details see Brzezniak and Elworthy (1996), and Carverhill and Elworthy (1983).

1.5. **Other motivations.** Other motivations come from the filtering (see e.g. Pardoux (1979), Rozovskii (1990)) theory of delay equations (see e.g. Chojnowska-Michalik (1978)), theory of particle systems, super-processes and cathalytic branching processes (see works by Dalang, Dowson, Mytnik, Muller, Perkins, Tribe and others).

1.6. **Construction of Markov processes.** Let (X_n) be a Markov chain (discrete time) on a complete separable metric space E. Then it admits the representation

$$X_{n+1} = F(X_n, \xi_{n+1}),$$

where (ξ_n) is a discrete time white noise (can be chosen as a sequence of independent uniformly distributed random variables on $[0,1]$).

In continuous time case we can expect that any (reasonable) Markov process X on a Hilbert space E is a solution to the stochastic equation

$$\mathrm{d}X = F(t,X)\mathrm{d}t + G(t,X)\mathrm{d}L,$$

where L is a Lévy process. This statement is basically true if the state space of X is finite dimensional, see Itô (1951) and Stroock (2003), and for the form of the generator Courrège (1965/66).

1.7. **Statistical Mechanics.** In statistical mechanics often one has a probability measure on $\mathbb{M} = \mathbb{R}^{\mathbb{Z}^d}$ or $\mathbb{M} = L^2(E,\mathcal{E},\mathrm{d}m)^{\mathbb{Z}^d}$. In order to simulate the values (characteristics) of μ one considers a Markov process X on $\mathbb{M}$ such that μ is invariant and ergodic for X. Typically

$$\mathrm{d}X_l = \left(\sum a_{kl}X_k + f_l(X_l)\right)\mathrm{d}t + \sigma_l \mathrm{d}L_l, \qquad l \in \mathbb{Z}^d.$$

Here $A = (a_{kl})$ is the infinite dimensional matrix of global interactions, and (f_l) are local interactions.

2. Form of the equation and equivalence of mild and weak solutions

We will be concerned with equations of the form

$$\mathrm{d}X = (AX + F(t,X))\,\mathrm{d}t + G(t,X)\mathrm{d}L,$$

where A generates a C_0-semigroup S on a Hilbert space E, and L is a Lévy process taking values in a Hilbert space U, $F\colon [0,\infty)\times E_0 \mapsto E$, and $G\colon [0,\infty)\times E_0 \mapsto L(U,E)$, E_0 is a dense subspace of E.

Definition 1. A predictable process X taking values in E_0 is a mild solution if it solves the stochastic integral equation

$$X(t) = S(t)X(0) + \int_0^t S(t-s)F(s, X(s))\mathrm{d}s + \int_0^t S(t-s)G(s, X(s))\mathrm{d}L(s),$$

the identity for $\mathrm{d}t\mathrm{d}\mathbb{P}$ almost all t and ω.

Definition 2. A predictable process X is a *weak solution* if for any $\psi \in \mathrm{Dom}\,(A^*)$, for any $t \geq 0$, $\mathbb{P}$-a.s.

$$\langle X(t), \psi\rangle_E = \langle X(0), \psi\rangle_E + \int_0^t \{\langle X(s), A^*\psi\rangle_E + \langle F(s, X(s)), \psi\rangle_E\}\,\mathrm{d}s + \int_0^t \langle G^*(s, X(s))\psi, \mathrm{d}L(s)\rangle_E.$$

We have the following general result.

Theorem 1. *The concepts of weak and mild solutions are equivalent; that is a modification of a mild solution is a weak solution, and a modification of a weak solution is a mild solution.*

Proof. We give only a sketch of the proof. For more details see e.g. Da Prato and Zabczyk (1992), or Peszat and Zabczyk (2007). First we show that a weak solution is mild. We will do this in two steps:
Step 1 Show that for any $z \in C^1([0,\infty); \mathrm{Dom}\,(A^*))$,

$$\begin{aligned}\langle X(t), z(t)\rangle_E &= \langle X(0), z(0)\rangle_E \\ &\quad + \int_0^t \{\langle X(s), A^*z(s) + z'(s)\rangle_E + \langle F(s, X(s)), z(s)\rangle_E\}\,\mathrm{d}s \\ &\quad + \int_0^t \langle G^*(s, X(s))z(s), \mathrm{d}L(s)\rangle_E.\end{aligned} \tag{7}$$

One can do this proving (7) first for z of the form $z(t) = \psi g(t)$, $g \in C^1([0,\infty); \mathbb{R})$, $\psi \in \mathrm{Dom}\,(A^*)$, and then by an approximation argument.
Step 2 Let us fix a $t > 0$. Let $z(s) = S^*(t-s)\psi$, where $\psi \in \mathrm{Dom}\,(A^*)$. Then $z'(s) = -A^*z(s)$. Hence, applying (7) we obtain

$$\langle X(t), z(t)\rangle_E = \langle X(0), z(0)\rangle_E + \int_0^t \langle F(s, X(s)), z(s)\rangle_E \mathrm{d}s + \int_0^t \langle G^*(s, X(s))z(s), \mathrm{d}L(s)\rangle_E,$$

which leads immediately to

$$\langle X(t),\psi\rangle_E = \langle S(t)X(0),\psi\rangle_E + \int_0^t \langle S(t-s)F(s,X(s)),\psi\rangle_E \mathrm{d}s$$
$$+ \int_0^t \langle S(t-s)G(s,X(s))\mathrm{d}L(s),\psi\rangle_E,$$

and completes the proof of the implication as $\mathrm{Dom}\,(A^*)$ is dense in E.

A proof that any mild solution is weak requires a stochastic Fubini theorem. Assume for simplicity that $F\equiv 0$ and that $X(0)=0$. Then

$$X(t) = \int_0^t S(t-s)G(s,X(s))\mathrm{d}L(s),$$

and for any $\psi \in \mathrm{Dom}\,(A^*)$, we have

$$\begin{aligned}
\int_0^t \langle A^*\psi, X(s)\rangle_E \mathrm{d}s &= \int_0^t \int_0^t \chi_{[0,s]}(r)\langle A^*\psi, S(s-r)G(r,X(r))\mathrm{d}L(r)\rangle_E \mathrm{d}s \\
&= \int_0^t \left\langle \int_r^t S^*(s-r)A^*\psi \mathrm{d}s, G(r,X(r))\mathrm{d}L(r)\right\rangle_E \\
&= \int_0^t \left\langle \int_r^t \frac{\mathrm{d}}{\mathrm{d}s} S^*(s-r)\psi \mathrm{d}s, G(r,X(r))\mathrm{d}L(r)\right\rangle_E \\
&= \int_0^t \langle (S^*(t-r)-I)\psi, G(r,X(r))\mathrm{d}L(r)\rangle_E \\
&= \int_0^t \langle \psi, S(t-r)G(r,X(r))\mathrm{d}L(r)\rangle_E \\
&\quad - \int_0^t \langle \psi, G(r,X(r))\mathrm{d}L(r)\rangle_E \\
&= \langle \psi, X(t)\rangle_E - \int_0^t \langle \psi, G(r,X(r))\mathrm{d}L(r)\rangle_E .
\end{aligned}$$

Hence we obtain the desired identity.

$$\langle \psi, X(t)\rangle_E = \int_0^t \langle A^*\psi, X(s)\rangle_E \mathrm{d}s + \int_0^t \langle \psi, G(r,X(r))\mathrm{d}L(r)\rangle_E .$$

□

Provided uniqueness, SPDE defines a Markov familly. Usually the existence follows from the Banach fixed point argument. Then we get the continuous (in L^2) dependence of the solution on the initial data. Hence we have Feller, and consequently strong Markov property!

3. Lévy processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition 3. A process $L\colon \Omega \times [0,\infty) \mapsto U$ is *Lévy* if and only if

(i) $L(0) = 0$,
(ii) L is stochastically continuous,
(iii) L has independent stationary increments.

Remark 1. Any Lévy process admits a càdlàg modification. Recall that "càdlàg" means right continuous and having left limits.

Example 1. Any Wiener process W in U is Lévy. Moreover,

$$\mathbb{E}\, \mathrm{e}^{\mathrm{i}\langle W(t),u\rangle_U} = \mathrm{e}^{-\frac{t}{2}\langle Qu,u\rangle_U}, \qquad u \in U,\, t \geq 0,$$

where Q is the covariance of W.

Example 2. Any *compound Poison* process on U is Lévy. Let ν be a finite Borel measure on U. Recall that compound Poisson process with *jump* or equivalently *Lévy measure* ν is given by

$$L(t) := \sum_{j=1}^{\Pi(t)} X_j,$$

where Π is a *Poisson* process with *intensity* $\lambda = \nu(U) < \infty$[2], and X_j are independent identically distributed random variable (briefly i.i.d) with the distribution

$$\mathbb{P}\left(X_j \in \Gamma\right) = \frac{\nu(\Gamma)}{\nu(U)}, \qquad \Gamma \in \mathcal{B}(U).$$

It is easy to show that

$$\mathbb{E}\, \mathrm{e}^{\mathrm{i}\langle L(t),u\rangle_U} = \mathrm{e}^{-t\Psi(u)},$$

where

$$\Psi(u) := \int_U \left(1 - \mathrm{e}^{\mathrm{i}\langle u,v\rangle_U}\right) \nu(\mathrm{d}v).$$

Moreover, L is integrable if and only if

$$\int_U |u|_U \nu(\mathrm{d}u) < \infty,$$

and if this is a case, then

$$\mathbb{E}\, L(t) = t \int_U u\nu(\mathrm{d}u).$$

[2] $\mathbb{P}(\Pi(t) = k) = \mathrm{e}^{-\lambda t}\frac{(\lambda t)^k}{k!}$, it is a Lévy process with jump measure $\lambda\delta_1$.

L is square integrable if and only if

$$\int_U |v|_U^2 \nu(\mathrm{d}v) < \infty,$$

and if this is a case, then

$$\mathbb{E}\, \langle L(t) - \mathbb{E}\, L(t), u\rangle_U \, \langle L(t) - \mathbb{E}\, L(t), v\rangle_U = \int_U \langle z, u\rangle_U \langle z, v\rangle_U \nu(\mathrm{d}z).$$

Remark 2. The concept of a compound Poisson process is crucial for understanding the characterization of an arbitrary Lévy process. The Lévy–Khinchin theorem says that an arbitrary Lévy process is a sum of a Wiener process, an uniform movement and a "compound Poisson process L with infinite jump measure". In order to construct a process L on U with infinite (but σ-finite) jump measure we can divide U into a countable sum $U = \bigcup U_n$ of measurable sets U_n such that $\nu(U_n) = 1$ and $U_n \cap U_m = \emptyset$ for $n \neq m$. Then one may try to write $L = \sum L_n$, here L_n are independent compound Poisson processes each with Lévy measure ν_n being restriction of ν to U_n. The problem is with convergence. Usually the series does not converges in any reasonable sense!. The idea is to write

$$U = U_0 \cup U_0^c,$$

where $U_0 = \bigcup_{n\in I} U_n$ is such that

$$\int_{U_0} |u|_U^2 \nu(\mathrm{d}u) < \infty, \qquad \int_{U_n} |u|_U \nu(\mathrm{d}u) < \infty, \qquad \forall\, n \in I,$$

and I^c is finite. We can define then the Lévy process with intensity ν as

$$\sum_{n\in I} (L_n(t) - \mathbb{E}\, L_n(t)) + \sum_{n\notin I} L_n(t).$$

The first sum is a sum of square integrable martingales with

$$\sum_{n\in I} \mathbb{E}\, |L_n(t) - \mathbb{E}\, L_n(t)|_U^2 = \int_{U_0} |u|_U^2 \nu(\mathrm{d}u) < \infty.$$

Therefore it converges in probability (and $\mathbb{P}$-a.s) uniformly in t from any bounded interval, due to the Doob submartingale inequality [3] yielding

$$\mathbb{P}\left(\sup_{t\in[0,T]}\sum_{n\le N, n\in I}|L_n(t)-\mathbb{E}\,L_n(t)|_U^2\ge r\right)\le \frac{\int_{\bigcup_{n\le N}U_n}|u|_U^2\nu(\mathrm{d}u)}{r}.$$

4. Lévy–Khinchin formula and Lévy–Khinchin decomposition

In the theory of Lévy processes the Lévy–Khinchin formula and the Lévy–Khinchin decomposition are crucial.

Theorem 2. ***(Lévy–Khinchin formula)*** *Given $a\in U$, $Q\in L_1^+(U)$ and a non-negative Borel measure ν on U satisfying $\mu(\{0\})=0$ and*

$$\int_U |u|_U^2\wedge 1\nu(\mathrm{d}u)<\infty, \tag{8}$$

there is a Lévy process L such that

$$\mathbb{E}\,\mathrm{e}^{\mathrm{i}\langle L(t),u\rangle_U}=\mathrm{e}^{-t\Psi(u)}, \tag{9}$$

where

$$\begin{aligned}\Psi(u):=&-\mathrm{i}\langle a,u\rangle_U+\tfrac{1}{2}\langle Qu,u\rangle_U\\ &+\int_U\left(1-\mathrm{e}^{\mathrm{i}\langle u,v\rangle_U}+\chi_{\{|v|_U\le 1\}}(v)\mathrm{i}\langle u,v\rangle_U\right)\nu(\mathrm{d}v).\end{aligned} \tag{10}$$

Conversely, for each Lévy process L there are $a\in U$, $Q\in L_1^+(U)$, and a Borel measure ν, $\nu(\{0\})=0$, satisfying (8) *such that* (9) *holds with Ψ given by* (10).

Remark 3. Measure ν is called *Lévy measure* or *jump measure* of L. One can show that for $\Gamma\in U\setminus\{0\}$,

$$\nu(\Gamma)=\mathbb{E}\,\pi(1,\Gamma)=\mathbb{E}\sum_{t\le 1}\chi_\Gamma\left(\Delta L(t)\right),$$

where $\Delta L(t):=L(t)-L(t-)$. Therefore, $\nu(\Gamma)$ is the expected number of jumps of L up to time 1 from the set Γ.

---

[3] If X is a right continuous submartingale (that is $\mathbb{E}(X(t)|\mathcal{F}_s)\ge X(S)$), then

$$r\,\mathbb{P}\left(\sup_{t\in[0,T]}X(t)\ge r\right)\le\mathbb{E}\,X^+(T).$$

Theorem 3. ***(Lévy–Khinchin decomposition)*** *Let $R > 0$. Any Lévy process L taking values in U can be written in the form*

$$L(t) = a_R t + W(t) + M_R(t) + L_R(t), \tag{11}$$

where $a_r \in U$, W is a Wiener process in U, M_R is Lévy process being a square integrable martingale (with respect to filtration generated by L), and L_R is a compound poisson process. Moreover,

(a) *M_R, L_R and W are independent,*
(b) *W does not depend on R,*
(c) *if ν is the Lévy measure of L, then M_R has the Lévy measure $\chi_{\{|u\|_U<R\}}(u)\nu(\mathrm{d}u)$, and L_R has the Lévy measure $\chi_{\{|u|_U\geq R\}}(u)\nu(\mathrm{d}u)$,*
(d) *the covariance operator [4] of M_R is given by*

$$\langle Q_R u, v\rangle_U = \int_{\{|z|_U\leq R\}} \langle u, z\rangle_U \langle v, z\rangle_U \nu(\mathrm{d}z).$$

For details see Parthasarathy (1967), Linde (1986), Peszat and Zabczyk (2007), and for finite dimensional case Sato (1999), Applebaum (2004).

5. Stochastic integration with respect to Lévy processes

Our goal is to develop a theory of integration of operator-valued processes

$$\psi\colon [0,\infty) \times \Omega \mapsto L(U, H)$$

with respect to a Lévy process L taking values in U. By Lévy–Khinchin decomposition theorem

$$L(t) = a_R t + W(t) + M_R(t) + L_R(t).$$

where $a_R \in U$, W is a Wiener process in U, M_R is a square integrable Lévy martingale, and L_R is a compound Poisson process. Since L_R has trajectories of bounded variation (in fact there are piecewise constant) we can integrate with respect to L_R pathwise. Thus

$$\mathrm{d}L_R(t) = \sum_j Y_j \delta_{\tau_j}(\mathrm{d}t),$$

where τ_j are moments of jumps and Y_j are the sizes of jumps.

Clearly, integration with respect to $a_R t$ is the deterministic integration with respect to Lebesgue measure $\mathrm{d}t$. Therefore, what is left is the theory of integration with respect to square integrable martingales W (with continuous paths) and M_R (with discontinuous paths).

[4] As $M_R(0) = 0$, M_R is mean zero.

Assume that M is a square integrable mean zero martingale with respect to filtration $(\mathcal{F}_t)$ taking values in U having stationary independent increments. Let Q be its covariance. Then it is easy to see that

$$|M(t)|_U^2 - \operatorname{Tr} Qt$$

and

$$M(t) \otimes M(t) - Qt$$

are martingales, respectively real- and $L_1^+(U)$-valued. In the general theory of stochastic integration (see e.g. Metivier (1982)) this mean that the *angle bracket* $\langle M, M\rangle$ and *operator angle bracket* $\langle\langle M, M\rangle\rangle$ of M equal $\operatorname{Tr} Q\, t$ and Qt, respectively.

Let

$$\psi = \sum_k \alpha_k \chi_{(t_k, t_{k+1}]},$$

where α_j are $L(U, H)$-valued random variables, be a simple function. We assume that ψ is predictable in the sense, that for any j and any $u \in U$, $\alpha_j u$ is $\mathcal{F}_{t_j}$-measurable H-valued random variable. Write

$$\int_0^t \psi(s)\mathrm{d}M(s) := \sum_j \alpha_j \left(M(t_{j+1} \wedge t) - M(t_j \wedge t)\right).$$

Then, by simple calculation, one can show that

$$\int_0^t \psi(s)\mathrm{d}M(s), \qquad t \geq 0,$$

is a square integrable H-valued martingale and that

$$\mathbb{E}\left|\int_0^t \psi(s)\mathrm{d}M(s)\right|_H^2 = \int_0^t \mathbb{E}\,\|\psi(s)Q^{1/2}\|^2_{L_{(HS)}(U,H)}\mathrm{d}s, \qquad t \geq 0. \tag{12}$$

This isometry gives us clue how to extend the integral. First of all note that $\psi(s)$ does not have to be bounded operator. What is really necessarily is the fact that $\psi(s)Q^{1/2}$ is Hilbert–Schmidt for almost all s.

Let $\mathcal{H} := Q^{1/2}(U)$. We equip $\mathcal{H}$ with the scalar product induced from U by $Q^{1/2}$, that is

$$\langle u, v\rangle_{\mathcal{H}} := \left\langle Q^{-1/2}u, Q^{-1/2}v\right\rangle_U,$$

where $Q^{-1/2}$ is the pseudo-inverse od $Q^{1/2}$,

$$Q^{-1/2}u = z \Leftrightarrow Q^{1/2}z = u, \quad \text{and} \quad |z|_U = \inf\left\{|y|_U \colon Q^{1/2}y = u\right\}.$$

We will call $\mathcal{H}$ the *Reproducing Hilbert Kernel Space* of M, (shortly *RKHS*), and M a *cylindrical martingale* in $\mathcal{H}$.

Remark 4. Generally M does not take values in $\mathcal{H}$ unless $\dim \mathcal{H} < \infty$. It does, however, in any Hilbert space V such that the embedding $\mathcal{H} \hookrightarrow V$ is Hilbert–Schmidt.

Remark 5. RKHS and Lévy measure are intrinsic characteristics of Lévy processes, that is if L takes values in U and V is a Hilbert space containing U then the RHS and the Lévy measure do not depend n the choice of U or V.

It follows from (12), that the class of admissible integrants equals

$$L^2_M := L^2(\Omega \times [0,\infty), \mathcal{P}, \mathrm{d}\mathbb{P}\mathrm{d}t; L_{(HS)}(\mathcal{H}, H)),$$

where $\mathcal{P}$ is the σ-field of predictable sets. Moreover, for any $\psi \in L^2_M$,

$$\int_0^t \psi(s)\mathrm{d}M(s), \qquad t \geq 0,$$

is a square integrable H-valued martingale and that

$$\mathbb{E}\left|\int_0^t \psi(s)\mathrm{d}M(s)\right|^2_H = \int_0^t \mathbb{E}\,\|\psi(s)\|^2_{L_{(HS)}(\mathcal{H},H)}\mathrm{d}s, \qquad t \geq 0. \tag{13}$$

For more information on stochastic integration in Hilbert spaces see Metivier (1982), Peszat and Zabczyk (2007).

Theorem 4. *Let L be a Lévy process with the exponent Ψ;*

$$\mathbb{E}\,\mathrm{e}^{\mathrm{i}\langle L(t),u\rangle_U} = \mathrm{e}^{-t\Psi(u)}.$$

Let $F\colon [0,T] \mapsto U = U^*$. Then

$$\mathbb{E}\mathrm{e}^{\mathrm{i}\int_0^T F(s)\mathrm{d}L(s)} = \mathrm{e}^{-\int_0^T \Psi(F(s))\mathrm{d}s},$$

provided that the integrals are well defined in the Riemman sense.

Proof. Let (t^n_j) be the partition of $[0,T]$. We have

$$\begin{aligned}
\mathbb{E}\,\mathrm{e}^{\mathrm{i}\int_0^T F(s)\mathrm{d}L(s)} &= \lim_{n\to\infty} \mathbb{E}\,\mathrm{e}^{\mathrm{i}\sum_j \langle F(s^n_j), L(s^n_{j+1}) - L(s^n_j)\rangle_U} \\
&= \lim_n \prod_j \mathbb{E}\,\mathrm{e}^{\mathrm{i}\langle F(s^n_j), L(s^n_{j+1} - s^n_j)\rangle_U} \\
&= \lim_n \prod_j \mathrm{e}^{-(s^n_{j+1} - s^n_j)\Psi(F(s^n_j))} \\
&= \lim_n \mathrm{e}^{-\sum_j \Psi(F(s^n_j))(s^n_{j+1} - s^n_j)} \\
&= \mathrm{e}^{-\int_0^T \Psi(F(s))\mathrm{d}s}.
\end{aligned}$$

□

Example 3. Let

$$Z(t) = \int_0^t \mathrm{e}^{-s} \mathrm{d}\pi(s),$$

where π is a Poisson process with intensity 1. Then

$$\mathbb{E}\, \mathrm{e}^{ixZ(t)} = \mathrm{e}^{-\int_0^t \Psi(x\mathrm{e}^{-s})\mathrm{d}s},$$

where

$$\Psi(x) = 1 - \mathrm{e}^{\mathrm{i}x} = \int_{\mathbb{R}} \left(1 - \mathrm{e}^{\mathrm{i}xy}\right) \delta_1(\mathrm{d}y).$$

Therefore, since

$$\mathrm{e}^{-s}\delta_1(\Gamma) = \delta_1(\mathrm{e}^{s}\Gamma) = \delta_{\mathrm{e}^{-s}}(\Gamma),$$

we have

$$\begin{aligned}\int_0^t \Psi(x\mathrm{e}^{-s})\mathrm{d}s &= \int_{\mathbb{R}} \left(1 - \mathrm{e}^{\mathrm{i}xy}\right) \int_0^t \mathrm{e}^{-s}\delta_1 \mathrm{d}s(\mathrm{d}y) \\ &= \int_{\mathbb{R}} \left(1 - \mathrm{e}^{\mathrm{i}xy}\right) \int_0^t \delta_{\mathrm{e}^{-s}}(\mathrm{d}y)\mathrm{d}s.\end{aligned}$$

Note that

$$\int_{\mathbb{R}} f(y) \int_0^t \delta_{\mathrm{e}^{-s}} \mathrm{d}s\mathrm{d}y = \int_0^t f(\mathrm{e}^{-s})\mathrm{d}s.$$

Changing the variables $r = \mathrm{e}^{-s}$, $\mathrm{d}r = -r\mathrm{d}s$, we obtain

$$\int_0^t f(\mathrm{e}^{-s})\mathrm{d}s = \int_{\mathrm{e}^{-t}}^1 f(r)\frac{\mathrm{d}r}{r}.$$

Thus the corresponding jump measure equals

$$\chi_{(\mathrm{e}^{-t},1]}(r)\frac{\mathrm{d}r}{r}.$$

For the complete exposition of the theory of integration with respect to infinite-dimensional martingale see Metivier (1982), or Peszat and Zabczyk (2007).

6. In general integrants have to be predictable

It is know that if we integrate with respect to a Wiener process, then it is enough to assume that the integrant is measurable, adapted and with probability 1, locally square integrable with respect to time. The following examples shows that in general the integrant should be also predictable.

Example 4. Let Π be a Poisson process with intensity λ. Let τ be the moment of the first jump of Π. Then $\chi_{[0,\tau)}$ is a measurable adapted process. We note that $\chi_{[0,\tau)}$ is not predictable. Clearly a predictable process is $\chi_{[0,\tau]}$. Note that $\chi_{[0,\tau]}$ is a modifcation of $\chi_{[0,\tau)}$.

Let $\widehat{\Pi}$ be the compensated process. Then

$$X(t) := \int_0^t \chi_{[0,\tau)}(s)\mathrm{d}\widehat{\Pi}(s) = -\lambda t \wedge \tau + \int_0^t \chi_{[0,\tau)}(s)\mathrm{d}\Pi(s) = -\lambda t \wedge \tau.$$

Clearly, X is not a martingale, nor a local martingale. It has decreasing trajectories. On the other hand, the process

$$Y(t) := \int_0^t \chi_{[0,\tau]}(s)\mathrm{d}\widehat{\Pi}(s) = -\lambda t\wedge\tau + \int_0^t \chi_{[0,\tau]}(s)\mathrm{d}\Pi(s) = -\lambda t\wedge\tau + \chi_{\{t\geq\tau\}}$$

is a martingale.

Obviously if X is càdlàg and adapted, then $X(t-)$, $t \geq 0$, is predictable. Unfortunately, in important cases X does not have a càdlàg modification (see Section 11.1). It can be mean square continuous, that is

$$\lim_{s\uparrow t} \mathbb{E}\,|X(t) - X(s)|_E^2 = 0, \qquad \forall\, t \geq 0.$$

Then there is its predictable modification due to the following general result (see Gikhmann and Skorokhod (1980) or Peszat and Zabczyk (2007), Prop. 3.21).

Theorem 5. *Any measurable stochastically continuous adapted process has a predictable modification.*

7. Poisson random measures

Let $(E,\mathcal{E})$ be a measurable space, and let $\mathcal{P}_{\overline{\mathbb{Z}_+}}([0,\infty)\times E)$ be the space of all measures on $[0,\infty)\times E$ with values in $\overline{\mathbb{Z}_+} := \{0,1,\ldots\}\cup\{\infty\}$. We consider on $\mathcal{P}_{\overline{\mathbb{Z}_+}}([0,\infty)\times E)$ the σ-field generated by mappings

$$\mathcal{P}_{\overline{\mathbb{Z}_+}}([0,\infty)\times E) \ni \rho \mapsto \rho(\Gamma) \in \overline{\mathbb{Z}_+}, \qquad \Gamma \in \mathcal{B}([0,\infty))\times\mathcal{E}.$$

Let μ be a σ-finite measure on $(E,\mathcal{E})$.

Definition 4. A *Poisson random measure* on $[0\infty)\times E$ with *intensity measure* $\lambda(\mathrm{d}t\mathrm{d}\xi) := \mathrm{d}t\mu(\mathrm{d}\xi)$ is a random element π in $\mathcal{P}_{\overline{\mathbb{Z}_+}}([0,\infty)\times E)$ such that

(i) for every $\Gamma \in \mathcal{B}([0,\infty))\times\mathcal{E}$, the random variable $\pi(\Gamma)$ has the Poisson distribution

$$\mathbb{P}\left(\pi(\Gamma) = k\right) = \mathrm{e}^{-\lambda(\Gamma)}\frac{(\lambda(\Gamma)t)^k}{k!},$$

(ii) for any disjoin $\Gamma_i, \dots \Gamma_k \in \mathcal{B}([0,\infty)) \times \mathcal{E}$, the random variables $\pi(\Gamma_1), \dots, \pi(\Gamma_k)$ are independent.

Definition 5. Let π be a Poisson random measure with intensity measure $\mathrm{d}t\mu(\mathrm{d}\xi)$. We call

$$\widehat{\pi}(\mathrm{d}t\mathrm{d}\xi) := \pi(\mathrm{d}t\mathrm{d}\xi) - \mathrm{d}t\mu(\mathrm{d}\xi)$$

the *compensated Poisson random measure.*

7.1. **Construction of a Poisson random measure.** Assume that μ is finite. Then the Poisson random measure with intensity measure $\mathrm{d}t\mu(\mathrm{d}\xi)$ can be written as follows

$$\pi(\mathrm{d}t\mathrm{d}\xi) = \sum_j \delta_{\tau_j,\xi_j},$$

where τ_j are the moments of jumps of a Poisson process with intensity $\mu(E)$ and ξ_j are independent random variables with distribution

$$\mathbb{P}\left(\xi_j \in \Gamma\right) = \frac{\mu(\Gamma)}{\mu(E)}.$$

In general $E = \bigcup E_n$, where E_n are disjoint and of finite measure, and $\pi = \sum \pi_n$, where π_n are independent Poisson random measures with intensity measures $\mathrm{d}t\chi_{E_n}(\xi)\mu(\mathrm{d}\xi)$.

7.2. **Stochastic integration (deterministic integrands).** We will introduce here the stochastic integral with respect to a Poisson random measure of deterministic mappings. Namely, let $f\colon [0,\infty) \times E \mapsto U$ be simple, that is

$$f = \sum_k u_k \chi_{(t_j,t_{j+1}]}\chi_{E_j},$$

where $u_j \in U$, $0 \le t_j \le t_{j+1}$ and $E_j \in \mathcal{E}$. Define

$$\int_0^t \int_E f(s,\xi)\pi(\mathrm{d}t\mathrm{d}\xi) = \sum_k u_k \pi((t_j \wedge t, t_{j+1} \wedge t] \times E_j).$$

Then by simple calculation one can show that

$$\mathbb{E}\left|\int_0^t \int_E f(s,\xi)\pi(\mathrm{d}s\mathrm{d}\xi)\right|_U = \int_0^t \int_E |f(s,\xi)|_U \mathrm{d}t\mu(\mathrm{d}\xi), \tag{14}$$

and that

$$\mathbb{E}\left|\int_0^t \int_E f(s,\xi)\widehat{\pi}(\mathrm{d}s\mathrm{d}\xi)\right|_U^2 = \int_0^t \int_E |f(s,\xi)|_U^2 \mathrm{d}t\mu(\mathrm{d}\xi). \tag{15}$$

Therefore we can extend the stochastic integral with respect to π or $\widehat{\pi}$ to the class of measurable integrable (square integrable functions).

8. Poisson random measure corresponding to Lévy process

A Lévy process L on U with Lévy measure ν, defines Poisson random measure

$$\pi((s,t]\times\Gamma) := \sum_{r\in(s,t]} \chi_\Gamma(\Delta L(r)),$$

where $\Delta L(r) := L(r) - L(r-)$. The measure (defined first on cylinders $(s,t]\times\Gamma$ has an extension to Poisson random measure on $[0,\infty)\times U$ with the intensity measure $\mathrm{d}t\nu(\mathrm{d}u)$.

The Lévy–Khinchin decomposition can be written in the following form:

Theorem 6. ***(Lévy–Khinchin decomposition)*** *Let $R>0$. Any Lévy process L taking values in U can be written in the form*

(16)

$$L(t) = a_R t + W(t) + \int_0^t\int_{\{|v|_U<R\}} v\widehat{\pi}(\mathrm{d}s\mathrm{d}v) + \int_0^t\int_{\{|v|_U\geq R\}} v\pi(\mathrm{d}s\mathrm{d}v).$$

9. Processes on l^2

Assume that

$$U = l^2 := \left\{(x_k)\in\mathbb{R}^{\mathbb{N}}\colon \sum_k x_k^2 < \infty\right\}.$$

Let (Z_k) be a sequence of real-valued Lévy processes each with Lévy measure ν_k, and let (λ_k) be a sequence of non-negative real numbers.

9.1. **Square integrable case.** Assume that each Z_k is square integrable, mean zero, normalized $\mathbb{E}\, Z_k^2(1) = 1$, and that Z_k are uncorrelated. Since Z_k are mean zero, $\lambda_k Z_k$ are martingales. Note that $Z := (\lambda_k Z_k)$ is a square integrable in l^2 if and only if

$$\sum_k \lambda_k^2 < \infty.$$

Let $\{e_k\}$ be the canonical basis of l^2. The covariance operator Q is given by

$$\langle Qe_j, e_k\rangle_{l^2} = \lambda_j\lambda_k\,\mathbb{E}\, Z_j(1)Z_k(1) = \lambda_k^2\delta_{k,j}.$$

Thus $Qe_k = \lambda_k e_k$. The RKHS of Z is equal to

$$\mathcal{H} = Q^{1/2}(U) = \left\{(x_k)\colon \sum_k \frac{x_k^2}{\lambda_k^2} < \infty\right\}$$

with the scalar product

$$\langle (x_k),(y_k)\rangle_{\mathcal{H}} = \sum_k \frac{x_k y_k}{\lambda_k^2},$$

where we adopt the convention that $0/0 = 0$.

9.2. **Jump measure.** If Z_j are independent, then the jump measure ν of Z is concentrated on axes, and

$$\nu\left\{(x_j)\colon x_j \in \Gamma, x_k = 0,\ j \neq k\right\} = \nu(\lambda_k \Gamma)$$

for any $k \in \mathbb{N}$ and any $\Gamma \in \mathcal{B}(\mathbb{R})$. Moreover, Z takes values in l^2 if and only if

$$\int_{l^2} |x|_{l^2}^2 \nu(\mathrm{d}x) < \infty,$$

or equivalently if

$$\sum_k \int_{\mathbb{R}} |\lambda_k x_k|^2 \wedge 1 \nu_k(\mathrm{d}x_k) < \infty,$$

which can be written equivalently as follows

$$\sum_k \left\{ \int_{(-\lambda_k^{-1},\lambda_k^{-1})} x_k^2 \nu_k(\mathrm{d}x_k) + \nu_k\{\mathbb{R}\setminus(-\lambda_k^{-1},\lambda_k^{-1})\} \right\} < \infty.$$

10. Impulsive white noise

Let $\mathcal{O}$ be an open not necessarily bounded domain in $\mathbb{R}^d$ (possibly $\mathcal{O} = \mathbb{R}^d$). Let π be a Poisson random measure on $[0,\infty)\times\mathcal{O}\times\mathbb{R}$ with intensity of jump measure $\mathrm{d}t\mathrm{d}x\nu(\mathrm{d}\sigma)$. Assume that

$$\int_{\mathbb{R}} \sigma^2 \wedge 1 \nu(\mathrm{d}\sigma) < \infty.$$

Consider the distributions-valued process

$$Z(t) = \int_0^t \int_{\{|\sigma|<R\}} \sigma \widehat{\pi}(\mathrm{d}s\mathrm{d}x\mathrm{d}\sigma) + \int_0^t \int_{\{|\sigma|\geq R\}} \sigma \pi(\mathrm{d}s\mathrm{d}x\mathrm{d}\sigma).$$

Taking into account the representation

$$\pi(\mathrm{d}s\mathrm{d}x\mathrm{d}\sigma) = \sum \delta_{\tau_k,x_k,\sigma_k},$$

we obtain the following a bit formal expression for Z

$$Z(t) = \left[\sum_{|\sigma_k|<R,\tau_k\leq t} \sigma_k \delta_{\tau_k,x_k} - t\int_{|\sigma|<R} \sigma \mathrm{d}x\nu(\mathrm{d}\sigma) \right] + \sum_{|\sigma_k|\geq R, T_k\leq t} \sigma_k \delta_{\tau_k,x_k}.$$

Intuitively, at random points (τ_k, x_k) at time and space Z gives random impulses of random size σ_k.

Remark 6. One can show that

$$M(t) = \int_0^t \int_{\{|\sigma|<R\}} \sigma\widehat{\pi}(\mathrm{d}s\mathrm{d}x\mathrm{d}\sigma)$$

is a square integrable martingale in a sufficiently large space, and that its RKHS equals

$$\mathcal{H} = L^2(\mathcal{O}, \mathcal{B}(\mathcal{O}), a_R\mathrm{d}x), \qquad a_R := \int_{\{|\sigma|<R\}} \sigma^2\nu(\mathrm{d}\sigma).$$

Thus, in particular, M takes values in any Hilbert space V such that the embedding $\mathcal{H} \hookrightarrow V$ is Hilbert–Schmidt.

Remark 7. The jump measure μ of Z is the image of the measure $\mathrm{d}x\nu(\mathrm{d}\sigma)$ under the transformation

$$\mathcal{O} \times \mathbb{R} \ni (x, \sigma) \mapsto \sigma\delta_x \in \mathcal{D}(\mathcal{O}),$$

where $\mathcal{D}(\mathcal{O})$ is the space of distribution on $\mathcal{O}$.

Therefore our definition is the following.

Definition 6. *Impulsive cylindrical (or white) noise with intensity of jumps measure* $\mathrm{d}x\lambda(\mathrm{d}\sigma)$ is the Lévy process on the space of distributions with the Lévy measure ν being the image of $\mathrm{d}x\lambda(\mathrm{d}\sigma)$ under the transformation $(x, \sigma) \mapsto \sigma\delta_x$.

Remark 8. Impulsive cylindrical process L takes values in a Hilbert space U provided that

$$\int_U |u|_U^2 \wedge 1\nu(\mathrm{d}u) < \infty.$$

Let $U = H^{-\alpha}$ be the Sobolev space of order $-\alpha$ with $\alpha > d/2$. Then, by Sobolev embedding,

$$C := \sup_{x\in\mathcal{O}} |\delta_x|_{H^{-\alpha}} < \infty.$$

Therefore

$$\begin{aligned}\int_{H^{-\alpha}} |u|_{H^{-\alpha}}^2 \wedge 1\lambda(\mathrm{d}u) &= \int_{\mathcal{O}} \int_{\mathbb{R}} \sigma^2|\delta_x|_{H^{-\alpha}}\mathrm{d}x\nu(\mathrm{d}\sigma)\\ &\leq C \int_{\mathbb{R}} \sigma^2 \wedge 1\lambda(\mathrm{d}\sigma) < \infty.\end{aligned}$$

Consequently, L takes values in $H^{-\alpha}$.

11. Regularity of stochastic convolution

We start this section with results on the lack of a càdlàg modification for SPDEs driven by a process whose jump measure is not supported on the state space. Then we present different tools useful for study regularity of stochastic convolutions.

11.1. **Lack of càdlàg modification.** As the following example shows in some cases the solution to linear stochastic evolution equation does not have a cádlág modification.

Example 5. Let U and H be Hilbert spaces such that

(i) H is densely embedded into U.
(ii) One has

$$\int_0^T \|S(s)\|^2_{L_{(HS)}(H,H)}\, ds < \infty. \qquad \forall\, T > 0.$$

(iii) For any $t > 0$, $S(t)$ has a continuous extension to an operator $S(t) \in L(U, H)$.
(iv) For any $u \in U \setminus H$,

$$\lim_{t \uparrow 0} |S(t)u|_H = \infty.$$

Let Z be a square integrable mean zero random variable in U with RKHS H, and let L be a compound Poisson process with Lévy measure ν which is the distribution of Z. Then

$$X(t) = \int_0^t S(t-s)\mathrm{d}L(s) = \sum_{\tau_n < t} S(t-\tau_n) Z_n,$$

where τ_n are the jump times of L and Z_j are independent copies of Z. Then, by (ii),

$$\sup_{t \le T} \mathbb{E}\, |X(t)|^2_H < \infty$$

but

$$\lim_{t\uparrow\tau_n} |X(t)|_H = \lim_{t\uparrow\tau_n} |S(t-\tau_n)Z_n|_H = \infty,$$

since Z_n take values in $U \setminus H$.

Explicitly, take $H = L^2(0,1)$, $U = W_0^{-1,2}(0,1)$, S the heat semigroup generated by the Laplace operator with Dirichlet boundary conditions, and $Z = \eta\delta_\xi$, where $\xi \in (0,1)$, and η is a mean zero random variable.

We have the following have ben proven by Brzezniak and Zabczyk (2009), and Peszat and Zabczyk (2007).

Theorem 7. *If the jump measure of the noise is not supported on E then the stochastic convolution does not have càdlàg trajectories in E.*

11.2. **Factorization.** Stochastic integral with respect to the square integrable martingale as a square integrable martingale has a càdlàg modification. This is not always true for stochastic convolution processes

$$X(t) := \int_0^t S(t-s)\Psi(s)\mathrm{d}M(s), \qquad t \geq 0.$$

One way to show the continuity of trajectories is to use the so-called *factorization*

$$X(t) = \Gamma(1) I_\alpha(X_\alpha)(t),$$

where

$$X_\alpha(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} S(t-s)\psi(s)\mathrm{d}M(s), \qquad t \geq 0,$$

and I_α is the *fractional derivative* operator given by

$$I_\alpha \psi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} S(t-s)\psi(s)\mathrm{d}s,$$

and Γ is the Euler Γ-function. It is easy to show that

$$I_\alpha \in L(L^p(0,T;H), C([0,T];H))$$

provided that $1/q < \alpha < 1$.

For the Wiener integral it is usually not hard to show that X_α has trajectories in $L^q(0,T;H)$ with some $1/q < \alpha < 1$. Therefore the continuity of trajectories of X follows. For the Lévy process this is usually not true.

11.3. **Kotelenez.** Kotelenez (1982) proved the regularity of stochastic convolution

$$\int_0^t S(t-s)\mathrm{d}M(s), \qquad t \geq 0,$$

driven an arbitrary square integrable martingale in H for contraction semigroups S, that is under assumption that

$$\|S(t)\|_{L(H,H)} \leq e^{\omega t}, \qquad t \geq 0.$$

We outline here some different proof due to Hausenblas and Seidler (2001) and (2006). Their method is based on the Nagy dilation theorem (see Davies (1980) or Nagy and Foias (1970)).

Theorem 8. ***(Nagy)*** *If S is a C_0-semigroup of contractions on H, then there is a Hilbert space $\tilde{H}$ containing H and a unitary group R on $\tilde{H}$ such that $S = PR$, where $P \in L(\tilde{H}, H)$ is a projection.*

If $S(t)=PR(t)$, $t\geq 0$, where $P\in L(\tilde H,H)$ and R is a group, then

$$\int_0^t S(t-s)\mathrm{d}M(s)=P\int_0^t R(t-s)\mathrm{d}M(s)=PR(t)\int_0^t R(-s)\mathrm{d}M(s).$$

Since

$$Y(t):=\int_0^t R(-s)\mathrm{d}M(s),\qquad t\geq 0,$$

is a square integrable martingale, Y has càdlàg trajectories in $\tilde H$ and consequently X has càdlàg trajectories in H as R is strongly continuous.

11.4. **General criterion for the absence of discontinuities of the second kind.** We recall here some basic facts from Gikhman and Skorokhod (1980).

Let $\xi=(\xi(t),\, t\in[0,T])$ be a separable process taking values in a metric space (U,ρ). We extent ξ in $\mathbb{R}$ putting $\xi(t)=\xi(0)$ for $t<0$ and $\xi(t)=\xi(T)$ for $t\geq T$. The following result holds (see Gikhman and Skorokhod (1980), Lemma 3 and Theorem 1 of Chapter 3).

Theorem 9. *Assume that there are an increasing function $g\colon(0,\infty)\mapsto(0,\infty)$ and a function $q\colon(0,\infty)\times(0,\infty)\mapsto(0,\infty)$ such that for all $C,h>0$,*

$$\begin{aligned}\mathbb{P}\left\{[\rho(\xi(t),\xi(t-h))>Cg(h)]\cap[\rho(\xi(t),\xi(t+h))>Cg(h)]\right\}\\ \leq q(C,h),\end{aligned}\tag{17}$$

and

$$G:=\sum_n g(T2^{-n})<\infty,\qquad Q(C):=\sum_n 2^n q(C,T2^{-n})<\infty.\tag{18}$$

Then with probability 1, ξ has no discontinuities of the second kind, and for all $N>0$,

$$\mathbb{P}\left\{\sup_{t,s\in[0,T]}\rho(\xi(t),\xi(s))>N\right\}\leq\mathbb{P}\left\{\rho(\xi(0),\xi(T))>\frac{N}{2G}\right\}+Q\left(\frac{N}{2G}\right).$$

Remark 9. Assume that there are $p,r,K>0$ such that for all $t\in[0,T]$ and $h>0$,

$$\mathbb{E}\left[\rho(\xi(t),\xi(t-h))\rho(\xi(t),\xi(t+h)\right]^p\leq Kh^{1+r}.\tag{19}$$

Let $0<r'<r$. Then (17) and (18) hold with

$$g(h):=h^{r'/(2p)}\qquad\text{and}\qquad q(C,h):=\frac{K}{C^{2p}}h^{1+r-r'}.$$

For, by Chebyshev's inequality

$$
\begin{aligned}
&\mathbb{P}\left\{[\rho(\xi(t),\xi(t-h)) > Cg(h)] \cap [\rho(\xi(t),\xi(t+h)) > Cg(h)]\right\}\\
&\quad\le \mathbb{P}\left\{\rho(\xi(t),\xi(t-h))\rho(\xi(t),\xi(t+h)) > C^2g^2(h)\right\}\\
&\quad\le \frac{Kh^{1+r}}{C^{2p}g^{2p}(h)} = \frac{Kh^{1+r-r'}}{C^{2p}}.
\end{aligned}
$$

Note that in this case $G = \sum_n (T2^{-n})^{r'/(2p)} < \infty$, and

$$
\begin{aligned}
Q\left(\frac{N}{2G}\right) &= \sum_n 2^n \frac{K(2G)^{2p}}{N^{2p}}(T2^{-n})^{1+r-r'}\\
&= \frac{K(2G)^{2p}}{N^{2p}} T^{1+r-r'} \sum_n 2^{-n(r-r')} \qquad (20)\\
&= \frac{K(2G)^{2p}T^{1+r-r'}}{1-2^{r'-r}} N^{-2p}.
\end{aligned}
$$

Remark 10. Let $q \ge 1$. Assume (17) and (18). Since

$$
\mathbb{E} \sup_{t,s\in[0,T]} (\rho(\xi(t),\xi(s)))^q = q\int_0^\infty \mathbb{P}\left\{\sup_{t,s\in[0,T]} \rho(\xi(t),\xi(s)) \ge N\right\} N^{q-1}\mathrm{d}N,
$$

Theorem 9 yields

$$
\begin{aligned}
&\mathbb{E} \sup_{t,s\in[0,T]} (\rho(\xi(t),\xi(s)))^q\\
&\qquad\le (2G)^q \mathbb{E}\left(\rho(\xi(T),\xi(0)\right)^q + 1 + q\int_1^\infty Q\left(\frac{N}{2G}\right) N^{q-1}\mathrm{d}N.
\end{aligned}
$$

Combining Theorem 9 with Remarks 9 and 10, and identity (20), we obtain the following result.

Corollary 1. *Assume* (19)*. Then with probability* 1*,* ξ *has no discontinuities of the second kind and for any* $1 \le q < 2p$*,*

$$
\begin{aligned}
&\mathbb{E} \sup_{t,s\in[0,T]} (\rho(\xi(t),\xi(s)))^q\\
&\qquad\le (2G)^q \mathbb{E}\left(\rho(\xi(T),\xi(0)\right)^q + 1 + \frac{q}{2p-q}\frac{K(2G)^{2p}T^{1+r-r'}}{1-2^{r'-r}}.
\end{aligned}
$$

12. Stationary solutions to linear problem

12.1. **Wiener case.** We are concerned with stationary solutions to the following Ornstein–Uhlenbeck equation

$$
\mathrm{d}X = AX\mathrm{d}t + B\mathrm{d}W, \qquad (21)
$$

where A generates a C_0 semigroup S on H, $B \in L(U, H)$ and W is a cylindrical Wiener process on U. Let

$$Q_t := \int_0^t S(t-s)BB^*S^*(t-s)\mathrm{d}s.$$

Assume that $X(0)$ is a random element in H, Note that

$$X(t) = S(t)X(0) + \int_0^t S(t-s)B\mathrm{d}W(s)$$

takes values in H if and only if

$$\int_0^t \|S(t-s)N\|^2_{L_{(HS)}(U,H)}\mathrm{d}s < \infty.$$

or equivalently if and only if $Q_t \in L_1^+(H)$ for any or all $t > 0$.

Note that $\int_0^t S(t-s)B\mathrm{d}W(S)$ has the distribution $N(0, Q_t)$.

Definition 7. We call probability measure m on H an *invariant* or *stationary* measure (solution) to (21) if and only if

$$m = S(t)m * N(0, Q_t), \qquad \forall\, t \geq 0.$$

Theorem 10. ***(Zabczyk (1985))*** *The following conditions are equivalent*

(i) *there is a stationary solution to* (21)*,*
(ii) *one has*

$$\sup_{t\geq 0} \operatorname{Tr} Q_t < \infty.$$

Moreover, if (i) or (ii) holds, then any invariant measure m is of the form

$$m = \beta * N(0, Q_\infty),$$

where $Q_\infty = \lim_{t\to\infty} Q_t$, and β is a probability measure on H which is invariant for the semigroup S, that is $S(t)\beta = \beta$ for $t \geq 0$.

For the proof we need the Bochner theorem.

Theorem 11. ***(Bochner)*** *Let $\phi\colon U \mapsto \mathbb{C}$. Then the following conditions are equivalent:*

(i) $\phi = \widehat{\eta}$ *for some probability measure η on U,*
(ii) $\phi(0) = 0$, *ϕ is positive-definite, that is*

$$\sum_{j,k} \phi(u_j - u_i)\lambda_j\overline{\lambda_j} \geq 0.$$

and S-continuous, that is for any $\varepsilon > 0$ there is $S_\varepsilon \in L_1^+(U)$ such that

$$\operatorname{Re}\phi(u) \geq 1 - \varepsilon, \qquad \forall\, u \in U\colon\ \langle u, S_\varepsilon u\rangle_U \leq 1.$$

12.2. **Proof of Theorem 10.** Let m be an invariant measure. Then

$$m = S(t)m * \gamma_t,$$

where $\gamma + t = N(0, Q_t)$. Thus

$$\widehat{m}(u) = \widehat{m}(S^*(t)u)\,\mathrm{e}^{-\frac{1}{2}\langle Q_t u, u\rangle_U}.$$

Since

$$\operatorname{Re} \mathrm{e}^{\frac{1}{2}\langle Q_t u, u\rangle_U} \widehat{m}(u) \leq \operatorname{Re} \widehat{m}(u) \leq 1,$$

we have

$$\mathrm{e}^{\frac{1}{2}\langle Q_t u, u\rangle_U} \leq \frac{1}{\operatorname{Re} \widehat{m}(u)},$$

and consequently

$$\langle Q_t u, u\rangle_U \leq 2 \log \frac{1}{\operatorname{Re} \widehat{m}(u)}, \qquad \forall\, t \geq 0.$$

By S-continuity, there is an $S \in L_1^+(U)$ such that

$$\operatorname{Re} \widehat{\mu}(u) \geq \frac{1}{2}, \qquad \forall\, u\colon\ \langle Su, u\rangle_U \leq 1.$$

Thus

$$\langle Q_t u, u\rangle_U \leq 2 \log 2, \qquad \forall\, u\colon\ \langle Su, u\rangle_U \leq 1.$$

Consequently,

$$0 \leq Q_t \leq 2 \log 2 S,$$

and hence

$$\sup_{t \geq 0} \operatorname{Tr} Q_t \leq 2 \log 2 \operatorname{Tr} S < \infty.$$

Hence $(i) \Rightarrow (ii)$. To see that $(ii) \Rightarrow (i)$ note that

$$\operatorname{Tr} Q_t = \int_0^t \|S(s)B\|^2_{L_{(HS)}(U,H)} \mathrm{d}s.$$

For, let $\{e_n\}$ be an othonormal basis on H. We have

$$\begin{aligned}
\sum_n \langle Q_t e_n, e_n\rangle_U &= \sum_n \left\langle \int_0^t S(s)BB^*S^*(s)\mathrm{d}s e_n, e_n \right\rangle_U \\
&= \sum_n \int_0^t \langle B^*S^*(s)e_n, B^*S^*(s)e_n\rangle_U \\
&= \int_0^t \|B^*S^*(s)\|^2_{L_{(HS)}(H,U)} \mathrm{d}s.
\end{aligned}$$

Thus, as $\|T\|_{L_{(HS)}(U,H)} = \|T^*\|_{L_{(HS)}(H,U)}$ we obtain that

$$Q_\infty = \int_0^\infty S(s)BB^*S^*(s)\mathrm{d}s$$

is nuclear. Therefore $N(0, Q_\infty)$ is invariant measure as the limit of distributions of the solution $X(t)$ starting from 0.

We are showing now that if m is an invariant measure, then

$$m = \beta * N(0, Q_\infty)$$

where β is probability measure invariant for S. To see this note that

$$\widehat{m}(u) = \widehat{m}(S^*(t)u)\mathrm{e}^{-\frac{1}{2}\langle Q_t u, u\rangle_U}.$$

By (ii),

$$\phi(u) := \lim_{t\to\infty} \widehat{m}(S^*(t)u) = \widehat{m}(u)\mathrm{e}^{\frac{1}{2}\langle Q_\infty u, u\rangle_U}.$$

It is a characteristic functional of a probability measure β by the Bochner theorem. The fact that β is invariant with respect to S is obvious. □

12.3. **Examples of non-uniqueness of invariant measure for linear equation.**

Example 6. Let $U = L^2(0,\infty)$, let $S(t)u(x) = \mathrm{e}^t u(t+x)$, $u \in U$, $x, t \geq 0$. It is easy to check that S is a C_0-semigroup on U. Let

$$\phi(x) = \mathrm{e}^{-kx} \quad \text{for } x \in [k, k+1).$$

Then $S(1)\phi = \phi$. Thus if q has the uniform distribution on $[0,1]$, and β is the distribution of $S(q)\psi$. Then β is invariant for S. Let A be the generator of S. Consider the equation

$$\mathrm{d}X = AX\mathrm{d}t + f\mathrm{d}W,$$

where W is a real-valued Wiener process and

$$f(x) = \mathrm{e}^{-x^2}.$$

Then, with $Ba = fa$, $B \in L(\mathbb{R}, U)$,

$$\sup_{t\geq 0} \mathrm{Tr} \int_0^t S(s)BB^*S^*(s)\mathrm{d}s < \infty.$$

There are at least two (in fact infinitely many) stationary solutions.

Example 7. Consider the system

$$\mathrm{d}x = 0\mathrm{d}t,$$
$$\mathrm{d}y = -\frac{1}{2}y\mathrm{d}t + \mathrm{d}W,$$

Let

$$\mu = N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right),$$

and let κ be a probability measure on $\mathbb{R}\times\{0\}$. Then $\kappa*\mu$ is an invariant measure.

12.4. **Lévy case.** Let L be a Lévy process in U with Lévy measure ν, and let A be a generator of a C_0-semigroup S on U. Consider the following Ornstein–Uhlenbeck equation

$$\mathrm{d}X = AX\mathrm{d}t + \mathrm{d}L. \tag{22}$$

Recall that for any independent of L random variable $X(0)$ in U, the solution starting from $X(0)$ is given by

$$X(t) = S(t)X(0) + \int_0^t S(t-s)\mathrm{d}L(s).$$

Assume that

$$L(t) = at + W(t) + M(t) + L_0(t),$$

where $a \in U$, W is a Wiener process in U with the (nuclear) covariance Q^W, and $M + L_0$ is a pure jump process with Lévy measure ν.

Recall that if m is an invariant (or stationary) measure for the equation, then for any $t > 0$ the distribution of $X(t)$ is equal to m, provided that the distribution of $X(0)$ is m.

Proposition 1. *If there is a stationary solution then*

$$\sup_{t\geq 0} \mathrm{Tr} \int_0^t S(s)Q^W S^*(s)\mathrm{d}s < \infty.$$

Proof. Let

$$Q_t := \int_0^t S(s)Q^W S^*(s)\mathrm{d}s, \qquad t \geq 0.$$

We have

$$m = S(t)m * N(0, Q_t) * \gamma_t, \qquad \forall\, t \geq 0,$$

where γ_t is the distribution of

$$\int_0^t S(t-s)\mathrm{d}(as + M(s) + L_0(s)).$$

Passing to the characteristic functionals we obtain

$$\widehat{m}(u) = \widehat{m}(S^*(t)u)\mathrm{e}^{-\frac{1}{2}\langle Q_t u, u\rangle_U}\widehat{\gamma}_t(u), \qquad u \in U.$$

Thus

$$\langle Q_t u, u\rangle_U \leq 2\log\frac{1}{\mathrm{Re}\,\widehat{m}(u)}$$

as

$$\mathrm{Re}\,\widehat{m}(S^*(t)u)\widehat{\gamma}_t(u) \leq 1.$$

By Bochner theorem, there is a trace class operator $S \in L_1^+(U)$ such that

$$\operatorname{Re}\widehat{m}(x) \geq \frac{1}{2}, \qquad \forall\, u\colon\ \langle Su, u\rangle_U \leq 1.$$

Then

$$\langle Q_t u, u\rangle_U \leq 2\log 2, \qquad \forall\, u\colon\ \langle Su, u\rangle_U \leq 1.$$

Consequently

$$Q_t \leq 2\log 2S, \qquad \forall\, t \geq 0.$$

□

From now on assume that $W = 0$ and $a = 0$. Thus we are concerned with the pure jump case.

Proposition 2. ***(Chojnowska-Michalik (1987))*** *If γ_t converges weakly to a probability measure γ, then there is an invariant measure m for* (22). *Moreover, any invariant measure m is of the form*

$$m = \beta * \gamma, \tag{23}$$

where β is any invariant measure for S.

Proof. If γ is the weak limit of γ_t, then by the Krylov–Bogolyubov theorem γ is invariant. What is left is to show that any invariant measure has the form (23). To do this assume that m is an invariant measure. Then

$$m = S(t)m * \gamma_t, \qquad \forall\, t \geq 0,$$

and consequently

$$\widehat{m}(u) = \widehat{m}(S(t)u)\widehat{\gamma}_t(u), \qquad \forall\, t \geq 0,\ \forall\, u \in U.$$

Since $\widehat{\gamma}_t(u) \to \widehat{\gamma}(u)$, there is a limit

$$\psi(u) := \lim_{t\to\infty} \widehat{m}(S^*(t)u) = \frac{\widehat{m}(u)}{\widehat{\gamma}(u)}.$$

We have to show that ψ is a characteristic function of a probability measure β. Then the invariance of β with respect to S is obvious since

$$\psi(S^*(t)u) = \psi(u), \qquad \forall\, t \geq 0,\ \forall\, u \in U.$$

To to this we can use the following general result whose proof can be shown in Parthasarathy (1967), Th 2.1 , p. 58.

Theorem 12. *Let (λ_n), (μ_n) and (ν_n) be sequences of measures on a Polish space $\mathcal{X}$. Assume that*

$$\lambda_n = \mu_n * \nu_n, \qquad \forall\, n.$$

Then, if (λ_n) and (ν_n) are relatively weakly compact, then (μ_n) is relatively weakly compact.

We apply the theorem to the case of $\mathcal{X} = U$, $\lambda_n = m$, $\mu_n = S(t_n)m$ and $\nu_n = \gamma_{t_n}$ where $t_n \uparrow \infty$. By the theorem there is a sequence $t_n \uparrow \infty$ such that $S(t_n)m$ converges weakly as $n \uparrow \infty$ to a certain measure β. Therefore $\psi = \widehat{\beta}$ s required. $\square$

Assume that the semigroup S generated by A is *exponentially stable*, that is there are C and $\omega > 0$ such that

$$\|S(t)\|_{L(U,U)} \leq C\mathrm{e}^{-\omega t}, \qquad \forall\, t \geq 0.$$

Theorem 13. ***(Chojnowska-Michalik (1987)*** *If S is exponentially stable then the following conditions are equivalent:*

(i) *there is an invariant measure for* (22)*.*

(ii) *the distributions γ_t of*

$$\int_0^t S(t-s)\mathrm{d}L(s)$$

converge weakly as $t \uparrow \infty$.

(iii) *The jump measure ν of L satisfies*

$$\int_U \log^+ |u|_U \nu(\mathrm{d}u) < \infty.$$

Moreover, the measure if exists must me unique and is equal $\gamma = \lim_{t\to\infty} \gamma_t$.

Proof. The implication $(ii) \Rightarrow (i)$ follows from Proposition 2. In order to show the uniqueness note that by Proposition 2, any invariant measure m must by of the form $m = \beta * \gamma$, where β is invariant for S. Therefore it s enough to note that any invariant for S probability measure must by equal to δ_0. To see this note that

$$\widehat{\beta}(u) = \lim_{t\to\infty} \widehat{m}(S^*(t)u).$$

Since $S^*(t)u \to 0$ as $t \to \infty$, we have $\widehat{\beta}(u) = \widehat{m}(0) = 1$, and hence $\beta = \delta_0$.

In order to show $(i) \Rightarrow (ii)$, note that if m is an invariant measure then

$$m = S(t)m * \gamma_t.$$

Since $S(t)m$ converges weakly to δ_0 we conclude by Theorem 12, that γ_t converges.

The proof of the equivalence of (ii) and (iii) is the most difficult part. We will present a sketch of the proof of $(iii) \Rightarrow (ii)$. The strategy is the following. Let

$$Z(t) = \int_0^t S(t-s)\mathrm{d}L(s).$$

We will show that for each t, $Z(t)$ has the same distribution as a certain Lévy process $L^t(1)$, with the Lévy exponent

$$\Psi_t(u) = -\mathrm{i}a_t u + \int_U \left(1 - \mathrm{e}^{\mathrm{i}\langle u,v\rangle_U} + \chi_{\{|v|_U<1\}}(v)\mathrm{i}\langle u,v\rangle_U\right)\nu_t(\mathrm{d}v).$$

where $a_t \in U$ and ν_t is a Lévy measure. Using stability of S we will show that a_t converges to certain $\tilde{a}$. We will show that ν_t converges to a certain $\tilde{\nu}$ satisfying

$$\int_U |u|_U^2 \wedge \tilde{\gamma}(\mathrm{d}u) < \infty,$$

and hence $\Psi_t(u)$ converges to

$$\Psi(u) = -\mathrm{i}\tilde{a}u + \int_U \left(1 - \mathrm{e}^{\mathrm{i}\langle u,v\rangle_U} + \chi_{\{|v|_U<1\}}(v)\mathrm{i}\langle u,v\rangle_U\right)\tilde{\nu}(\mathrm{d}v).$$

To calculate (a_t, ν_t) we need to calculate the characteristic functional of $Z(t)$. To this end we use Theorem 4. Let $B = \{v \in U\colon |v|_U < 1\}$. We have

$$\mathbb{E}\,\mathrm{e}^{\mathrm{i}\langle Z(t),u\rangle_U} = \mathrm{e}^{-\Psi_t(u)},$$

where

$$\begin{aligned}
\Psi_t(u) &= \int_0^t \Psi(S^*(t-s)u)\mathrm{d}s = \int_0^t \Psi(S^*(s)u)\mathrm{d}s \\
&= -\mathrm{i}\int_0^t \langle a, S^*(s)u\rangle_U \mathrm{d}s \\
&\quad + \int_0^t \int_U \left(1 - \mathrm{e}^{\mathrm{i}\langle S^*(s)u,v\rangle_U} - \chi_B(v)\mathrm{i}\langle S^*(s)u, v\rangle_U\right)\nu(\mathrm{d}v) \\
&= -\mathrm{i}\left\langle u, \int_0^t S(s)u\mathrm{d}s\right\rangle_U \\
&\quad + \mathrm{i}\left\langle a, \int_0^t \int_U \left[\chi_B(v)S(s)v - \chi_B(S(s)v)\right]\nu(\mathrm{d}v)\mathrm{d}s\right\rangle_U \\
&\quad + \int_0^t \int_U \left(1 - \mathrm{e}^{\mathrm{i}\langle S^*(s)u,v\rangle_U} - \chi_B(s(s)v)\mathrm{i}\langle u, S(s)v\rangle_U\right)\nu(\mathrm{d}v).
\end{aligned}$$

Therefore,

$$\nu_t = \int_0^t S(s)\nu\mathrm{d}s,$$

and we have to show that

$$\int_U |u|_U^2 \wedge 1 \int_0^\infty S(s)\mu(\mathrm{d}u)\mathrm{d}s < \infty.$$

Additionally have to show that

$$I_t := \int_0^t \int_U \left[\chi_B(v)S(s)v - \chi_B(S(s)v)\right] \nu(\mathrm{d}v)\mathrm{d}s$$

converges. Using more detailed calculations one can shows that the above claims are equivalent to

$$\int_U \log^+ |u|_U \nu(\mathrm{d}u).$$

We will do some calculations for the convergence of I_t. We have $I_t = I_t^1 - I_t^2$, where

$$I_t^1 = \int_0^t \int_{\{|u|_U \le 1, |S(s)u|_U > 1\}} S(s)u\nu(\mathrm{d}u)\mathrm{d}s$$

and

$$I_t^1 = \int_0^t \int_{\{|u|_U > 1, |S(s)u|_U \le 1\}} S(s)u\nu(\mathrm{d}u)\mathrm{d}s.$$

We have

$$\begin{aligned} |I_t^1|_U &\le \int_0^t \int_{\{|u|_U \le 1\}} |S(s)u|_U^2 \nu(\mathrm{d}u)\mathrm{d}s \\ &\le \int_0^\infty \int_{\{|u|_U \le 1\}} |u|_U^2 \nu(\mathrm{d}u) C^2 \mathrm{e}^{-2\omega s}\mathrm{d}s < \infty. \end{aligned}$$

Clearly

$$\begin{aligned} |I_t^2|_U &\le \int_0^\infty \int_{\{|u|_U > 1, |S(s)u|_U \le 1\}} |S(s)u|_U \nu(\mathrm{d}u)\mathrm{d}s \\ &\le \int_0^\infty \int_{\{|u|_U \in (1, \mathrm{e}^{\omega t/2})\}} |S(s)u|_U \nu(\mathrm{d}u)\mathrm{d}s + \int_0^\infty \int_{\{|u|_U \ge \mathrm{e}^{\omega s/2}\}} \nu(\mathrm{d}u)\mathrm{d}s. \end{aligned}$$

We have

$$\begin{aligned} \int_0^\infty \int_{|u|_U \ge \mathrm{e}^{\omega s/2}\}} \nu(\mathrm{d}u)\mathrm{d}s &= \int_0^\infty \int_{\{\frac{2}{\omega} \log |u|_U \ge t\}} \nu(\mathrm{d}u)\mathrm{d}s \\ &= \frac{2}{\omega} \int_U \log^+ |u|_U \nu(\mathrm{d}u). \end{aligned}$$

Finally,

$$
\begin{aligned}
&\int_0^\infty \int_{\{1<|u|_U<\mathrm{e}^{\omega s/2}\}} |S(s)u|_U \nu(\mathrm{d}u)\mathrm{d}s \\
&\le C \int_0^\infty \int_{\{1<|u|_U<\mathrm{e}^{\omega s/2}\}} |u|_U \mathrm{e}^{-\omega s} \nu(\mathrm{d}u)\mathrm{d}s \\
&\le C \int_0^\infty \int_{\{1<|u|_U<\mathrm{e}^{\omega s/2}\}} \mathrm{e}^{-\frac{\omega s}{2}} \nu(\mathrm{d}u)\mathrm{d}s \\
&\le C \int_0^\infty \mathrm{e}^{-\frac{\omega s}{2}} \mathrm{d}s \nu\{|u|_U > 1\} < \infty.
\end{aligned}
$$

For more details see Chojnowska-Michalik (1987) and Applebaum (2006). Finite dimensional case is treated also in Sato (1999). □

Appendix A. Hilbert–Schmidt and nuclear operators

In what follows U and H are real separable Hilbert spaces. The space of all bounded (i.e. continuous) linear operators from U to H is denoted by $L(U,H)$. A bounded linear operator $T\colon U \mapsto H$ is *Hilbert–Schmidt* if and only if

$$\sum_k |Te_k|_H^2 < \infty$$

for every or equivalently for some orthonormal basis $\{e_k\}$ of U. The space of Hilbert–Schmidt operators from U to H is denoted by $L_{(HS)}(U,H)$. It is a Hilbert space with the norm

$$\|T\|_{L_{(HS)}(U,H)} := \left(\sum_k |Te_k|_H^2\right)^{1/2}.$$

Given $u \in U$ and $h \in H$ we denote by $u \otimes h$ a linear operator from U to H given by

$$u \otimes h(v) = \langle u, v\rangle_U\, h, \qquad v \in U.$$

A bounded linear operator $T\colon U \mapsto H$ is *trace class* or *nuclear* if and only if it can be written in the form

$$T = \sum_k u_k \otimes h_k,$$

where $\{u_k\} \subset U$, $\{h_k\} \subset H$ and

$$\sum_k |u_k|_U |h_k|_H < \infty.$$

The space $L_1(U,H)$ of all nuclear operators is equipped with the norm

$$\|T\|_{L_1(U,H)} := \inf\left\{\sum_k |u_k|_U |h_k|_H \colon T = \sum_k u_k\otimes h_k\right\}.$$

Given $T\in L_1(U) := L_1(U,U)$ we denote by $\operatorname{Tr} T$ the *trace* of T,

$$\operatorname{Tr} T := \sum_k \langle Te_k, e_k\rangle_U,$$

where $\{e_k\}$ is an orthonormal basis of U. It is known that $\operatorname{Tr} T$ does not depend on the choice of the orthonormal basis $\{e_k\}$ of U.

Let $L_1^+(U)$ be the class of all nuclear operators $T\in L_1(U)$ such that $T = T^*\geq 0$.

Remark 11. Let X be a square integrable (i.e. $\mathbb{E}\,|X|_U^2<\infty$) random vector in U. Then its *covariance operator* Q defined by

$$\langle Qu, v\rangle_U := \mathbb{E}\,\langle X-\mathbb{E}\,X, u\rangle_U\langle X-\mathbb{E}\,X, v\rangle_U, \qquad u,v\in U,$$

belongs to $L_1^+(U)$. Moreover,

$$\operatorname{Tr} Q = \mathbb{E}\,|X-\mathbb{E}\,X|_U^2.$$

Example 8. Let $U = H = L^2(E,\mathcal{E},\gamma)$. Then the operator

$$T\psi(x) = \int_E G(x,y)\psi(y)\gamma(\mathrm{d}y)$$

is Hilbert–Schmidt if and only if

$$\int_E\int_E G^2(x,y)\gamma(\mathrm{d}x)\gamma(\mathrm{d}y)<\infty.$$

Moreover,

$$\|T\|^2_{L_{(HS)}(U,U)} = \int_E\int_E G^2(x,y)\gamma(\mathrm{d}x)\gamma(\mathrm{d}y)<\infty.$$

Appendix B. C_0-semigroups

Definition 8. Let $(E,\|\cdot\|_E)$ be a Banach space. A family $S(t)$, $t\geq 0$, of bounded linear operators from E into E is a *strongly continuous* (C_0 in short) *semigroup* if

- $S(0) = I$, (I stands for the identity operator on E),
- $S(t+s) = S(t)S(s)$ for all $t,s\geq 0$,
- $\|S(t)\psi-\psi\|_E\to 0$ as $t\downarrow 0$, for every $\psi\in E$.

The second property listed above is the *semigroup property*, and the last one is the strong *continuity property.* For the proof of the following theorems see e.g. Pazy (1983)

Theorem 14. *Let $S = (S(t))$ be a C_0-emigroup on E. Then there are constants $\gamma \geq 0$ and $M \geq 1$ such that*

$$\|S(t)\|_{L(E,E)} \leq M\mathrm{e}^{\gamma t}, \qquad \forall\, t \geq 0.$$

Definition 9. If S is a C_0-semigroup such that there is a $\gamma \geq 0$ such that $\|S(t)\|_{L(E,E)} \leq \mathrm{e}^{\gamma t}$ for $t \geq 0$, then S is called C_0-semigroup of *(generalized) contractions.*

Definition 10. Let S be a C_0-semigroup. A linear operator A defined by

$$\mathrm{Dom}\,(A) = \left\{\psi \in E\colon \lim_{t\downarrow 0} \frac{1}{t}\,(S(t)\psi - \psi) \text{ exists}\right\}$$

and

$$A\psi = \lim_{t\downarrow 0} \frac{1}{t}\,(S(t)\psi - \psi) \quad \text{for } \psi \in \mathrm{Dom}\,(A)$$

is the *infinitesimal generator* of the semigroup.

Theorem 15. *The generator of a C_0-semigroup is densely defined and closed.*

Appendix C. Itô–Nisio theorem

The following result due to Itô and Nisio is concerned with different types of convergence of a series of independent random variables taking values in b separable Banach space E. For its proof we refer the reader to Kwapień and Woyczyński (1992) or the original paper of Itô and Nisio (1968). In its formulation $S_n := X_1 + \ldots + X_n$ and $\mathcal{L}(S_n)$ denotes the distribution of S_n.

Theorem 16. *Let (X_k) be a sequence of independent random variables taking values in E. Then the following conditions are equivalent:*

(i) *the series $\sum_k X_k$ converges a.s.,*
(ii) *the series $\sum_k X_k$ converges in probability,*
(iii) *distributions $\mathcal{L}(S_n)$ converge weakly.*

It additionally (X_k) are symmetric, then conditions $(i)-(iii)$ are equivalent to each of the following condition:

(iv) *the sequence $(\mathcal{L}(S_n))$ is relatively weakly compact,*
(v) *there exists a random variable S with values in E, and a familly $D \subset E^*$, separating points of E, such that for each $x^* \in D$, series $\sum_k x^*(X_k)$ converges a.s. to $x^*(S)$,*
(vi) *there exists a probability measure μ on E, and a familly $D \subset E^*$, separating points of E, such that for each $x^* \in D$, series $\sum_k x^*(X_k)$ converges in distribution to $x^*(\mu)$.*

The following result whose proof can be found in (**?**), Corollary 2.2.1 is of particular interest.

Theorem 17. *Let $p > 0$, and let X_k are independent random elements in E. Then the series $\sum_k X_k$ converges a.s. if and only if for some $a > 0$, or equivalently for any $a > 0$, the following conditions are satisfied:*

(i) *the series* $\sum_k \mathbb{P}(\|X_k\|_E > a) < \infty$,

(ii) *the series* $\sum_k X_k \chi_{\{\|X_k\|_E \leq a\}}$ *converges in p-th mean.*

In the case of Hilbert space-valued random elements the theorem above yields the so-called Three Series Theorem

Theorem 18. *Let X_k are independent random elements in a Hilbert space E. Then the series $\sum_k X_k$ converges a.s. if and only if for some $a > 0$, or equivalently for any $a > 0$, the following three series converge:*

(i) $\sum_k \mathbb{P}(\|X_k\|_E > a)$,

(ii) $\sum_k \mathbb{E}\, X_k \chi_{\{\|X_k\|_E \leq a\}}$,

(iii) $\sum_k \mathrm{Var}\left(X_k \chi_{\{\|X_k\|_E \leq a\}}\right)$.

References

[1] Albeverio, S. and Rüdiger, B. (2005). Stochastic integrals and the Lévy-Itô decomposition on separable Banach spaces, *Stoch. Anal. Appl.* **23**, 217–253.

[2] Applebaum, D. (2004). *Lévy Processes and Stochastic Calculus* (Cambridge, Cambridge University Press,).

[3] Applebaum, D. (2006). Martingale-valued measures, Ornstein–Uhlenbeck processes with jumps and operator self-decomposability in Hilbert space, *Séminaire de Probabilité*, Lecture Notes in Mathematics, Vol. 1897, (Berlin, Springer), pp. 173–198.

[4] Björk, T. and Christensen, B.J. (1999). Interest rate dynamics and consistent forward rate curves, *Math. Finance* **9**, 323–348.

[5] Björk, T., DiMassi, G., Kabanov, Y. and Runggaldier, W. (1997). Towards a general theory of bound markets, *Finance Stoch.* **1**, 141–174.

[6] Björk, T., Kabanov, Y. and Runggaldier, W. (1997). Bond market structure in the presence of marked point process, *Math. Finance* **7**, 211–239.

[7] Brzezniak, Z. and Elworthy, K. D. (1996). Stochastic flows of diffeomorphisms, *Stochastic analysis and applications (Powys, 1995)*, pp.107–138, World Sci. Publ., River Edge, NJ.

[8] Carverhill, A.P. and Elworthy, D. (1983). Flow of stochastic dynamical systems: the Functional Analysis approach, *Z. Wahrsch. Verw. Gebiete*, **65**, 245–267.

[9] Chojnowska-Michalik, A. (1978). Representation theorem for general stochastic delay equations, *Bull. Acad. Pol. Sci. Ser. Sci. Math.* **26**, 634–641.

[10] Chojnowska-Michalik, A. (1987). On processes of Ornstein–Uhlenbeck type in Hilbert spaces, *Stochastics* **21**, 251–286.

[11] Courrège, Ph. (1965/66). Sur la forme integro-différentielle des opérateurs de C_K^∞ dans C_0 satisfaisant au principe du maximum, in *Sém. Théorie du Potentiel*, Exposé 2.
[12] Da Prato, G., Kwapień, S. and Zabczyk, J. (1987). Regularity of solutions of linear stochastic equations in Hilbert spaces, *Stochastics* **23**, 1–23.
[13] Da Prato, G. and Zabczyk, J. (1992a). *Stochastic Equations in Infinite Dimensions* (Cambridge, Cambridge University Press).
[14] Da Prato, G. and Zabczyk, J. (1996). *Ergodicity for Infinite Dimensional Systems* (Cambridge, Cambridge University Press).
[15] Eberlein, E. and Raible, S. (1999). Term structure models driven by general Lévy processes, *Math. Finance* **9**, 31–53.
[16] Engel, K-J., and Nagel, R. (2000). *One-Parameter Semigroups for Liner Evolution Equations* (Springer).
[17] Filipović, D. (2001). *Consistency problems for Heath-Jarrow-Morton interest rate models*, Lecture Notes in Mathematics, Vol. 1760 (Berlin, Springer).
[18] Filipović, D. and Tappe, S. (2006). Existence of Lévy term structure models, submitted.
[19] Gikhman, I.I. and Skorokhod, A.V. (1980). *The Theory of Stochastic Processes I*, (Springer).
[20] Hausenblas, E. and Seidler, J. (2001). A note on maximal inequality for stochastic convolutions, *Czechoslovak Math. J.* **51**, 785–790.
[21] Hausenblas, E. and Seidler, J. (2006). Maximal inequalities and exponential integrability for stochastic convolutions driven by martingales, preprint.
[22] Heath, D., Jarrow, R. and Morton, A. (1992). Bond pricing and the term structure of interest rates: A new methodology, *Econometrica* **61**, 77–105.
[23] Ichikawa, A. (1986). Some inequalities for martingales and stochastic convolutions, *Stochastic Anal. Appl.* **4**, 329–339.
[24] Itô, K. (1951). On stochastic differential equations, *Mem. Amer. Math. Soc.* **4**, 1–51.
[25] Itô, K. (1984). *Foundations of Stochastic Differential Equations in Infinite Dimensional Spaces*, (Philadelphia, SIAM,).
[26] Itô, K. and Nisio, M. (1968). On the convergence of sums of independent Banach space valued random variables, *Osaka Math. J.* **5**, 35–48.
[27] Jakubowski, J. and Zabczyk, J. (2004). HJM condition for models with Lèvy noise, *IM PAN Preprint 651*, (Warsaw).
[28] Jakubowski, J., M. and Zabczyk, J. (2007). Exponential moments for HJM models with jumps, *Finance Stochastics*, to appear.
[29] Knoche, C. (2004). SPDEs in infinite dimension with Poisson noise, *C. R. Acad. Sci. Paris, Ser. I* **339**, 647–652.
[30] Kotelenez, P. (1982). A submartingale type inequality with applications to stochastic evolution equations, *Stochastics* **8**, 139–151.
[31] Kotelenez, P. (1984). A stopped Doob inequality for stochastic convolution integrals and stochastic evolution equations, *Stochastic Anal. Appl.* **2**, 245–265.
[32] Kotelenez, P. (1987). A maximal inequality for stochastic convolution integrals on Hilbert space and space-time regularity of linear stochastic partial differential equations, *Stochastics* **21**, 345–458.

[33] Kwapień, S. and Woyczyński, W.A. (1992). *Random Series and Stochastic Integrals: Single and Multiple*, (Birkhäuser, Boston, Basel, Berlin).

[34] Lasota, A. and Szarek, T. (2006). Lower bound technique in the theory of a stochastric differential equation, *J. Differential Equations* **231**, 513–533.

[35] Linde, W. (1986). *Probability in Banach Spaces - Stable and Infinitely Divisible Distributions*, (Wiley, New York).

[36] Lunardi, A. (1995). *Analytic Semigroup and Optimal Regularity of Parabolic Problems*, (Birkhäuser).

[37] Metivier, M. (1982). *Semimartingales: A Course on Stochastic Processes*, (De Gruyer, Berlin).

[38] Mueller, C. (1998). The heat equation with Lévy noise, *Stochastic Process. Appl.* **74**, 67–82.

[39] Mytnik, L. (2002). Stochastic partial differential equation driven by stable noise, *Probab. Theory Relat. Fields* **123**, 157–201.

[40] Nagy, S. and Foiaş, C. (1970). *Harmonic Analysis of Operators on Hilbert Space*, (North-Holland, Amsterdam).

[41] Pardoux, E. (1979). Stochastic partial differential equations and filtering of diffusion processes, *Stochastics* **3**, 127–67.

[42] Parthasarathy, K.R. (1967). *Probability Measures on Metric Spaces*, (Academic Press, New York).

[43] Pazy, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations*, (Springer, Berlin).

[44] Peszat, S., Rusinek, A., and Zabczyk, J. (2007). On a stochastic partial differential equation of bond market, submitted

[45] Peszat, S. and Zabczyk, J. (2007). Stochastic Partial Differential Equations with Lévy Noise, (Cambridge Univ. Press, Cambridge).

[46] Peszat, S. and Zabczyk, J. (2006). Stochastic heat and wave equations driven by an impulsive noise, in *Stochastic Partial Differential Equations and Applications- VII*, eds. G. Da Prato and L. Tubaro, Lect. Notes Pure Appl. Math., 245, (Chapman & Hall/CRC, Boca Raton), 229–242.

[47] Rozovskii B.L. (1990). Stochastic Evolution Equations. Linear Theory and Applications to Non-linear Filtering, (Kluwer).

[48] Rusinek, A. (2006a). Invariant measures for a class of stochastic evolution equations, Preprint IMPAN 667, Warszawa.

[49] Rusinek, A. (2006b). Invariant measures for forward rate HJM model with Lévy noise, Preprint IMPAN 669.

[50] Saint Loubert Bié, E. (1998). Étude d'une EDPS conduite par un bruit poissonien, *Probab. Theory Related Fields* **111**, 287–321.

[51] Sato, K.I. (1999). *Lévy Processes and Infinite Divisible Distributions*, (Cambridge University Press, Cambridge).

[52] Stroock, D.W. (2003). *Markov Processes from K. Itô's Perspective*, (Princeton University Press, Princeton).

[53] Tehranchi, M. (2005). A note on invariant measures for HJM models, *Finance Stochastics* **9**, 387–398.

[54] Viot M. (1974). Solution en lois d'une équation aux dérivées partielles non linéaires, *Thèse*, Université Pierre et Marie Curie, Paris.

[55] Walsh, J.B. (1986). An introduction to stochastic partial differential equations, in *École d'été de probabilités de Saint-Flour XIV - 1984*, Lecture Notes in Math., 1180, Springer, Berlin New York, 265–439.

[56] Yosida, K. (1965). *Functional Analysis*, Grundlehren Math. Wiss., Vol. 123, (Springer, Berlin).

[57] Zabczyk, J. (1983). Stationary distributions for linear equations driven by general noise, *Bull. Polish Acad. Sci.* **31**, 197–209.

[58] Zabczyk, J. (1889). Symmetric solutions of semi-linear stochastic equations, in *Stochastic Partial Differential Equations and Applications, II (Trento, 1988)*, Lecture Notes in Math., 1390, (Springer, Berlin), 237–256.