GRADIENT FORMULAE AND SPDES

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Part 1. General results and definitions

1. Regularity, strong Feller and irreducibility

Let (P_t) be a Markov transition semigroup on Polish space $(E,\mathcal{B})$. Let $P_t(x,\Gamma)$ be the transition probabilities.

Definition 1. (P_t) is t_0-r*egular* if $P_t(x,\cdot) \ll P_s(y,\cdot)$ for all $t,s > t_0$ and $x,y \in E$.

(P_t) is stochastically continuous if

$$\lim_{t\downarrow 0} P_t(x,B(x,\varepsilon)) = 1, \qquad \forall\, x \in E,\ \varepsilon > 0.$$

or equivalently if

$$\lim_{t\to 0} P_t\psi(x) = \psi(x)$$

for all $x \in E$ and $\psi \in C_b(E), UC_b(E), \mathrm{Lip}(E)$.

Theorem 1. ***(Doob)*** *Let ν be an invariant measure for a stochastically continuous and t_0 regular (P_t). Then*

(i) *ν is the unique invariant measure,*
(ii) *ν is equivalent to all $P_t(x,\cdot)$, $t > t_0$ and $x \in E$,*
(iii) *ν is strongly mixing;*

$$\lim_{t\to\infty} \int_E \left| P_t\psi(x) - \int_E \psi(y)\nu(\mathrm{d}y)\right|^2 \nu(\mathrm{d}x) = 0,\ \forall\, \psi \in L^2(\nu)$$

and

$$\lim_{t\to\infty} P_t(x,\Gamma) = \nu(\Gamma), \qquad \forall\, \Gamma \in \mathcal{B}(E).$$

Definition 2. P_t is *strong Feller at* t_0 if $P_t \colon B_b(E) \mapsto C_b(E)$ for $t > t_0$.

P_t is *irreducible at* s_0 if $P_{s_0}(x,U) > 0$ for $x \in E$ and nonempty open $U \subset E$.

Theorem 2. ***(Khasminski)*** *If a stochastically continuous (P_t) is strong Feller at $t_0 >$ and irreducible at $s_0 > 0$, then is $t_0 + s_0$-regular.*

Part 2. **Bismut–Elworthy–Li formula**

Consider the Markov family $(X^x(t))$ defined on $E = \mathbb{R}^d$ by the equation

$$\mathrm{d}X(t) = b(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t), \qquad X(0) = x.$$

Let

$$P_t f(x) = \mathbb{E}\, f(X^x(t)), \qquad f \in B_b(\mathbb{R}^d),$$

be the transition semigroup.

Let $f \in C_b^2(\mathbb{R}^d)$. Let us fix a $t > 0$ and let us apply Itô's formula for the function $(s, x) \mapsto P_{t-s}f(x)$ and diffusion X. We obtain

$$\begin{aligned}
P_{t-t}f(X^x(t)) &= P_{t-0}f(x) \\
&\qquad + \int_0^t \left((\partial_s P_{t-s}f)(X^x(s)) + \mathcal{L}P_{t-s}f(X^x(s))\right) \mathrm{d}s \\
&\qquad + \int_0^t \langle (\nabla P_{t-s}f)(X^x(s)), \sigma(X^x(s))\mathrm{d}W(s)\rangle \\
&= P_t f(x) + \int_0^t \langle (\nabla P_{t-s}f)(X^x(s)), \sigma(X^x(s))\mathrm{d}W(s)\rangle.
\end{aligned}$$

Thus

$$f(X^x(t)) = P_t f(x) + \int_0^t \langle (\nabla P_{t-s}f)(X^x(s)), \sigma(X^x(s))\mathrm{d}W(s)\rangle.$$

Multiplying both sides by

$$\int_0^t \langle \nabla_x X^x(s)[h], \sigma(X^x(s))^{-1}\mathrm{d}W(s)\rangle$$

and taking the expectation we obtain

$$\begin{aligned}
&\mathbb{E}\left\{f(X^x(t))\int_0^t \langle \nabla_x X^x(s)[h], \sigma(X^x(s))^{-1}\mathrm{d}W(s)\rangle\right\}\\
&= \mathbb{E}\int_0^t \langle (\nabla P_{t-s}f)(X^x(s)), \nabla_x X^x(s)[h]\rangle\,\mathrm{d}s\\
&= \mathbb{E}\int_0^t \langle \nabla_x P_{t-s}f(X^x(s)), h\rangle \mathrm{d}s\\
&= \int_0^t \langle \nabla_x \mathbb{E} P_{t-s}f(X^x(s)), h\rangle \mathrm{d}s\\
&= \int_0^t \langle \nabla_x P_t f(x), h\rangle \mathrm{d}s\\
&= \int_0^t \langle \nabla_x P_t f(x), h\rangle \mathrm{d}s\\
&= t\langle \nabla_x P_t f(x), h\rangle.
\end{aligned}$$

Thus

$$\langle \nabla_x P_t f(x), h\rangle = \frac{1}{t}\mathbb{E}\left\{f(X^x(t))\int_0^t \langle \nabla_x X^x(s)[h], \sigma(X^x(s))^{-1}\mathrm{d}W(s)\rangle\right\}.$$

In conclusion we have

$$\|\nabla_x P_t f\|_\infty \le \frac{C}{\sqrt{t}}\|f\|_\infty,$$

C depends only on the Lipschitz constant of F and σ, and $\|\sigma^{-1}\|_\infty$.

1.1. **Typical example in infinite dimennsion.** As an example consider stochastic heat equation

$$\begin{aligned}
\mathrm{d}X(\xi,t) &= (\Delta_\xi X(\xi,t) + u(X(\xi,t)))\,\mathrm{d}t + v(X(\xi,t))\mathrm{d}W(\xi,t),\\
X(\xi,0) &= x(\xi),
\end{aligned}$$

considered on a bounded region $\mathcal{O} \subset \mathbb{R}^d$ with 0-Dirichlet boundary conditions.

We can write this in the form

$$\mathrm{d}X = F(X)\mathrm{d}t + \sigma(X)\mathrm{d}W$$

where $F = \Delta + F_0$, and

$$F_0(\psi)(\xi) = u(\psi(\xi)), \qquad \sigma(\psi)\phi\,[\xi] = v(\psi(\xi))\phi(\xi).$$

are the so-called Nemytskii type operators.

2. Jump case

$$\mathrm{d}X(t) = b(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t) + \int_{\mathbb{R}^d} \eta(X(s-), z)\overline{\pi}(\mathrm{d}s, \mathrm{d}z), \qquad X(0) = x.$$

where

$$\pi = \pi(\mathrm{d}z, \mathrm{d}t)$$

is a Poisson random measure on $\mathbb{R}^d \times [0, +\infty)$ with the intensity measure $\mu(\mathrm{d}z)\mathrm{d}t$. We assume that

$$\int_{\mathbb{R}^d} |z|^2 \wedge 1 \mu(\mathrm{d}z) < +\infty,$$

that is that μ is the intensity (Lévy) measure of the Lévy process

$$L(t) = \int_0^t \int_{\mathbb{R}^d} z\overline{\pi}(\mathrm{d}z, \mathrm{d}s), \qquad t \geq 0,$$

$$\overline{\pi}(\mathrm{d}z, \mathrm{d}t) = \chi_{\{|z|\leq 1\}}\widehat{\pi}(\mathrm{d}z, \mathrm{d}t) + \chi_{\{|z|>1\}}\pi(\mathrm{d}z, \mathrm{d}t),$$

$$\widehat{\pi}(\mathrm{d}z, \mathrm{d}t) = \pi(\mathrm{d}z, \mathrm{d}t) - \mu(\mathrm{d}z)\mathrm{d}t.$$

We can use the trick used for Gaussian noise to obtain

$$f(X^x(t)) = P_t f(x) + \int_0^t \langle (\nabla P_{t-s} f)(X^x(s)), \sigma(X^x(s))\mathrm{d}W(s)\rangle + \int_0^t (.......)\, \widehat{\pi}(\mathrm{d}s, \mathrm{d}z).$$

Thus, like in the Gaussian case

$$\langle \nabla_x P_t f(x), h\rangle = \frac{1}{t}\mathbb{E}\left\{ f(X^x(t)) \int_0^t \langle \nabla_x X^x(s)[h], \sigma(X^x(s))^{-1}\mathrm{d}W(s)\rangle \right\}.$$

- K.D. Elworthy and X.M. Li, J. Funct. Anal (1994),
- S. Peszat and J. Zabczyk, Ann. Probab. (1995),
- M. Davis and M. Johansson SPA (2004),
- E. Priola and J. Zabczyk, PTRF (2011).

Part 3. Gradient formulae for purely jump diffusions

3. The problem

Assume that $X^x = (X^x(t), t \geq 0)$ is a solution to SDE or SPDE driving by jump process. We assume that $X^x(0) = x$. We are interested in the formulae

$$\mathbb{E}\,(\nabla f)\,(X^x(t)) = \mathbb{E}\, f\,(X^x(t))\, Z(t, x),$$

$$\nabla_x\, E\, f\,(X^x(t)) = \mathbb{E}\, f\,(X^x(t))\, \tilde{Z}(t, x),$$

where $Z(t,x)$ and $\tilde{Z}(t,x)$ are integrable random variables (vectors in $\mathbb{R}^d$) independent of f.

The first formula ensures the absolute continuity of the law (distribution) of $X^x(t)$ with respect to Lebesque measure. It is a direct consequence of the following lemma due to Malliavin (see e.g. D. Nualart, The Mallivin Calculus and Related Topics, Springer 1996, p. 790).

Lemma 1. *Let ν be a finite measure on $\mathbb{R}^d$. Assume that there is a constant C such that for any $f \in C_b^1(\mathbb{R}^d)$,*

$$\left\| \int_{\mathbb{R}^d} \nabla f(x) \nu(\mathrm{d}x) \right\|_{\mathbb{R}^d} \leq C \|f\|_\infty,$$

then $\nu \ll \mathrm{d}x$. Moreover,

$$\frac{\mathrm{d}\nu}{\mathrm{d}x} \in L^{d/(d-1)}(\mathbb{R}^d) \qquad \text{if } d > 1.$$

Remark 1. $d = 1$, then the claim of the lemma is natural. Given $a < b$, and

$$f(x) = \begin{cases} 0, & \text{if } x < a, \\ \frac{x-a}{b-a} & \text{if } x \in [a,b], \\ 1 & \text{if } x > b \end{cases}$$

one obtains (see Nualart)

$$\nu[a,b] \leq C|b-a|.$$

The second formula

$$\nabla_x \mathbb{E}\, f(X^x(t)) = \nabla_x \mathbb{E}\, P_t f(x) = \mathbb{E}\, f(X^x(t)) \tilde{Z}(x,t)$$

provides the strong Feller property of the corresponding transition semigroup

$$P_t f(x) := \mathbb{E}\, f(X^x(t)), \qquad t \geq 0.$$

The plan is to outline two approaches, one based on the the Malliavin calculus for jump processes (due to J. Norris, R. Bass and M. Cranston), the second (due to A. Takeuchi) uses Bismut–Elworty–Li trick.

4. Basic notation

Let

$$\pi = \pi(\mathrm{d}z, \mathrm{d}t)$$

be a Poisson random measure on $\mathbb{R}^d \times [0,+\infty)$ with the intensity measure $\mu(\mathrm{d}z)\mathrm{d}t$ defined on a filtered probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t), \mathbb{P})$. Assume that $(\Omega, \mathfrak{F}, \mathbb{P})$ is complete,

$$\mathfrak{F}_t = \overline{\sigma\{\pi(\cdot, s) \colon s \leq t\}}.$$

Recall that

$$\pi \colon \Omega \ni \omega \mapsto \pi(\mathrm{d}z, \mathrm{d}t)(\omega) \in \mathcal{M}(\mathbb{Z}_+),$$

where $\mathcal{M}(\mathbb{Z}_+)$ is the space of all $\mathbb{Z}_+ \cup +\infty$-valued measures on $\mathbb{R}^d \times [0,+\infty)$. Moreover, for any disjoint $\Gamma_1, \Gamma_2 \in \mathcal{B}(\mathbb{R}^d \times [0,+\infty)$ the random variables $\pi(\Gamma_1)$, $\pi(\Gamma_2)$ are independent, and $\pi(\Gamma_i)$ has the Poisson distribution

$$\mathbb{P}\left(\pi(\Gamma_i) = k\right) = \mathrm{e}^{-\gamma_i} \frac{\gamma_i^k}{k!}, \qquad \gamma_i = (\mu \times \mathrm{d}t)(\Gamma_i).$$

We assume that

$$\int_{\mathbb{R}^d} |z|^2 \wedge 1 \mu(\mathrm{d}z) < +\infty,$$

that is that μ is the intensity (Lévy) measure of the Lévy process

$$L(t) = \int_0^t \int_{\mathbb{R}^d} z \overline{\pi}(\mathrm{d}z, \mathrm{d}s), \qquad t \geq 0,$$

$$\overline{\pi}(\mathrm{d}z, \mathrm{d}t) = \chi_{\{|z|\leq 1\}} \widehat{\pi}(\mathrm{d}z, \mathrm{d}t) + \chi_{\{|z|>1\}} \pi(\mathrm{d}z, \mathrm{d}t),$$
$$\widehat{\pi}(\mathrm{d}z, \mathrm{d}t) = \pi(\mathrm{d}z, \mathrm{d}t) - \mu(\mathrm{d}z)\mathrm{d}t.$$

We will deal with equation with additive noise

$$\mathrm{d}X(t) = F(X(t))\mathrm{d}t + \int_{\mathbb{R}^d} z \overline{\pi}(\mathrm{d}z, \mathrm{d}t) = F(X(t))\mathrm{d}t + \mathrm{d}L(t) \tag{1}$$

where $F \colon \mathbb{R}^d \mapsto \mathbb{R}^d$ is a regular (at least Lispchitz) mapping.

The methods work also for the case of general noise

$$\mathrm{d}X(t) = F(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W(t) + \int_{\mathbb{R}^d} \eta(X(t-), z)\overline{\pi}(\mathrm{d}z, \mathrm{d}t).$$

5. Malliavin calculus approach

My presentation is bases mostly on

- J.R. Norris, Integration by parts for jump processes, Séminaire de Probabilité XXII, pp. 271–315, Lecture Notes in Math. 1321, Springer 1988,
- Z. Dong and Y.L. Song, Derivatives of Jump processes and gradient estimates, preprint.

See also

- R.F. Bass, M. Cranston, The Malliavin calculus for pure jump process and applications to local time, Ann. Probab. 14 (1986), 490–532,
- J.M. Bismut, Calcul des variations stochastique et processus de sauts, Z. Wahrsch. Verw. Gebiete 63 (1983), 147–235,
- R. Léandre, Régularité de processus de sauts dégénéré, Ann. Inst. H. Poincaré Probab. Statist. 21 (1985), 125–146.

5.1. **Directional derivative of functional of π.** Let

$$V\colon \mathbb{R}^d \times [0,+\infty) \times \Omega \mapsto \mathbb{R}^d$$

be a predictable random field. Define the *perturbed random measure* π^V by

$$\int_0^{+\infty} \int_{\mathbb{R}^d} \psi(z,s)\pi^V(\mathrm{d}z,\mathrm{d}s) = \int_0^{+\infty} \psi(z+V(z,s),s)\pi(\mathrm{d}z,\mathrm{d}s).$$

Informally, if

$$\pi = \sum \delta_{(z_j,t_j)}$$

then

$$\pi^V = \sum \delta_{(z_j+V(z_j,t_j),t_j)}$$

Definition 3. ***(Bass)*** For a $p \geq 1$ a functional $\Psi = \Psi(\pi)$ is called to have an L^p-derivative in the direction of V if there is aa L^p-integrable random variable $D_V\Psi(\pi)$ such that

$$\lim_{\varepsilon\to 0} \mathbb{E}\left|\frac{\Psi(\pi^{\varepsilon V}) - \Psi(\pi)}{\varepsilon} - D_v\Psi(\pi)\right|^p = 0.$$

We take for granted the *chain role*

$$D_V f(\Psi(\pi)) = \langle (\nabla f)(\Psi(\pi)), D_V\psi(\pi)\rangle$$

for $f \in C_b^1(\mathbb{R}^m)$, $\Psi(\pi) = (\Psi_1(\pi),\ldots,\Psi_m(\pi))$.

Consider our SDE

$$\mathrm{d}X^x(t) = F(X^x(t))\mathrm{d}t + \int_{\mathbb{R}^d} z\overline{\pi}(\mathrm{d}z,\mathrm{d}t), \qquad X^x(0) = x.$$

Then

$$\begin{aligned} &\mathrm{d}D_V X^x(t) = \langle (\nabla F)(X^x(t)), D_V X^x(t)\rangle \mathrm{d}t + \int_{\mathbb{R}^d} V(z,t)\pi(\mathrm{d}z,\mathrm{d}t), \\ &D_V X^x(0) = 0. \end{aligned} \tag{2}$$

Obviously (2) is an inhomogeneous linear equation! Thus

$$D_V X^x(t) = \int_0^t \int_{\mathbb{R}^d} U(t,s,x)V(z,s)\pi(\mathrm{d}z,\mathrm{d}s),$$

where U is the fundamental solution to the homogeneous equation; that is

$$\frac{\partial U}{\partial t}(t,s,x) = (\nabla F)(X^x(t))U(t,s,x), \qquad U(s,s,x) = I.$$

Hence

$$U(t,s,x) = \nabla_x X^x(t)[\nabla_x X^x(s)]^{-1}.$$

5.2. **Integration by parts formula; existence of density.** Recall that we are looking for a integrable random fields $Z_1(t,x), \ldots Z_d(t,x)$ such that for any smooth $f\colon \mathbb{R}^d \mapsto \mathbb{R}$ we have

$$\mathbb{E}\,(\partial_j f)(X^x(t)) = \mathbb{E}\, f(X^x(t))\, Z_j(t,x), \qquad j = 1,\ldots,d.$$

Fix j, t and x. Let (e_j) be the canonical basis of $\mathbb{R}^d$.

First Step We will construct a random field $V = V_{j,t,x}(z,s)$ and a random variable $H_j(t,x)$ such that

$$(\partial_j f)(X^x(t)) = \langle (\nabla f)(X^x(t)), e_j \rangle = D_V\left[f(X^x(t))\right] H(t).$$

Since

$$D_V\left[f(X^x(t))\right] = (\nabla f)(X^x(t)) D_V X^x(t)$$

we need

$$D_V X^x(t) = e_j \frac{1}{H(t)}.$$

Taking into account the formula for $D_V X^x(t)$ we need

$$e_j \frac{1}{H(t)} = \int_0^t \int_{\mathbb{R}^d} U(t,s,x) V(z,s) \pi(\mathrm{d}z, \mathrm{d}s).$$

Thus a natural candidate for V is

$$V(z,s) = U^{-1}(t,s,x) e_j h(z)$$

where

$$h\colon \mathbb{R}^d \mapsto \mathbb{R}$$

is a deterministic (non-negative measurable) function. For example

$$|z|^2 \mathrm{e}^{-|z|^2}.$$

Then

$$H(t) = \left[\int_0^t \int_{\mathbb{R}^d} h(z) \pi(\mathrm{d}z, \mathrm{d}s)\right]^{-1}.$$

From now on given, $h\colon \mathbb{R}^d \mapsto [0,+\infty)$ we set

$$J_h(t) := \int_0^t \int_{\mathbb{R}^d} h(z) \pi(\mathrm{d}z, \mathrm{d}s).$$

With this notation

$$H(t) := \frac{1}{J_h(t)}, \qquad V(z,s) = U^{-1}(t,s,x)e_j h(z).$$

Second Step We have

$$\begin{aligned}\mathbb{E}\,(\partial_j f)(X^x(t)) &= \mathbb{E}\,D_V\left[f(X^x(t))\right]H(t)\\ &= \mathbb{E}\left\{D_V\left[f(X^x(t))H(t)\right] - F(X^x(t))D_V H(t)\right\}\\ &= \mathbb{E}\,f(X^x(t))\left[H(t)D_V^*\mathbf{1}(t) - D_V H(t)\right],\end{aligned}$$

where $D_V^*\mathbf{1}(t)$ is such a random variable that for any smooth functional Ψ such that $\Psi(\pi)$ is $\mathfrak{F}_t$-measurable,

$$\mathbb{E}\,D_V\Psi(\pi) = \mathbb{E}\,\Psi(\pi)D_V^*\mathbf{1}(t).$$

Summing up we have

$$Z_j(t,x) = \frac{D_V^*\mathbf{1}(t)}{J_h(t)} - D_V\frac{1}{J_h(t)} = \frac{J_h(t)D_V^*\mathbf{1}(t) - D_V J_h(t)}{J_h^2(t)},$$

where

$$\begin{aligned}V(z,s) &= \nabla x X^x(s)\left[\nabla_x X^x(t)\right]^{-1} e_j h(z),\\ D_V J_h(t) &= \int_0^t\int_{\mathbb{R}^d}\langle\nabla h(z), V(z,s)\rangle\pi(\mathrm{d}z,\mathrm{d}s).\end{aligned}$$

We will see that

$$\begin{aligned}D_V^*\mathbf{1}(t) &= -\int_0^t\int_{\mathbb{R}^d}\frac{\operatorname{div}\left(g(z)V(z,s)\right)}{g(z)}\widehat{\pi}(\mathrm{d}z,\mathrm{d}s),\\ g(z) &= \frac{\mathrm{d}\mu}{\mathrm{d}z}(z).\end{aligned}$$

Of course we need

- the existence of $D_V^*\mathbf{1}(t)$,
- the integrability of $Z_j(t,x)$, at least that

$$\mathbb{E}J_h(t)^{-p} < +\infty \qquad \text{for } p \geq 1 \text{ large enough.}$$

5.3. **Gradient estimates for the semigroup.** We have

$$\begin{aligned}\nabla_x P_t f(x) &= \nabla_x\mathbb{E}\,f(X^x(t))\\ &= \mathbb{E}\,\langle(\nabla f)(X^x(t)), \nabla_x X^x(t)\rangle\\ &= \mathbb{E}\,D_V\left[f(X^x(t))\right]H(t).\end{aligned}$$

Since

$$D_V\left[f(X^x(t))\right]H(t) = \langle(\nabla f)(X^x(t)), H(t)D_V X^x(t)\rangle$$

we need

$$H(t)D_V X^x(t) = \nabla_x X^x(t).$$

Taking into account the formula for $D_V X^x(t)$ we need

$$H(t)\int_0^t\int_{\mathbb{R}^d} \nabla_x X^x(t)\left[\nabla_x X^x(s)\right]^{-1} V(z,s)\pi(\mathrm{d}z,\mathrm{d}s) = \nabla_x X^x(t).$$

Hence

$$H(t)\int_0^t\int_{\mathbb{R}^d} \left[\nabla_x X^x(s)\right]^{-1} V(z,s)\pi(\mathrm{d}z,\mathrm{d}s) = I.$$

Therefore

$$V(z,s) = \nabla_x X^x(s)h(z)$$

$$H(t) = \frac{1}{J_h(t)} = \left[\int_0^t\int_{\mathbb{R}^d} h(z)\pi(\mathrm{d}z,\mathrm{d}s)\right]^{-1}$$

and

$$\tilde{Z}(t,x) = H(t)D_V^*\mathbf{1}(t) - D_V H(t).$$

The field V appearing in the first and second problem are different

- For the density
$$V(z,s) = \nabla_x X^x(t)\left[\nabla_x X^x(s)\right]^{-1} e_j h(z).$$
- For the gradient of the semigroup
$$V(z,s) = \nabla_x X^x(s)h(z).$$

5.4. **Integrability of $J_h(t)^{-p}$.** Let us recall that $h\colon \mathbb{R}^d \mapsto [0,+\infty)$ and that

$$J_h(t) := \int_0^t\int_{\mathbb{R}^d} h(z)\pi(\mathrm{d}z,\mathrm{d}s).$$

We knew that

$$\mathbb{E}\mathrm{e}^{-\beta J_h(t)} = \exp\left\{-t\int_{\mathbb{R}^d}\left(1-\mathrm{e}^{-\beta h(z)}\right)\mu(\mathrm{d}z)\right\}.$$

We have

$$x^{-p} = \frac{1}{\Gamma(p)}\int_0^{+\infty} \beta^{p-1}\mathrm{e}^{-\beta x}\mathrm{d}\beta.$$

Thus

$$\begin{aligned}\mathbb{E}\, J_h(t)^{-p} &= \frac{1}{\Gamma(p)}\int_0^{+\infty} \beta^{p-1}\mathbb{E}\,\mathrm{e}^{-\beta J_h(t)}\mathrm{d}\beta\\ &= \frac{1}{\Gamma(p)}\int_0^{+\infty} \beta^{p-1}\exp\left\{-t\int_{\mathbb{R}^d}\left(1-\mathrm{e}^{-\beta h(z)}\right)\mu(\mathrm{d}z)\right\}\mathrm{d}\beta.\end{aligned}$$

With this method on can obtain (see Norris)

Lemma 2. *1) If*

$$\frac{p}{t} < \lambda_h := \liminf_{\varepsilon\downarrow 0} \frac{\mu\{h \geq \varepsilon\}}{\log \frac{1}{\varepsilon}}$$

then $\mathbb{E}\, J_h(t)^{-p} < +\infty$.

2) If for a certain $\rho > 0$,

$$\liminf_{\varepsilon\downarrow 0} \frac{\mu\{h \geq \varepsilon\}}{\varepsilon^{-\rho}} > 0$$

then

$$\mathbb{E}\, J_h(t)^{-p} \leq C t^{-p/\rho}, \qquad p \geq 1,\ t \in (0,1].$$

5.5. **The adjoint operator.** We assume that $\mu(\mathrm{d}z) \ll \mathrm{d}z$. Let

$$g(z) = \frac{\mathrm{d}\mu}{\mathrm{d}z}(z),$$

$$\lambda^\varepsilon(z,t) := \det\left[I + \varepsilon \nabla_z V(z,t)\right] \frac{g(z + \varepsilon V(z,t))}{g(z)},$$

$$Z^\varepsilon(t) := \exp\left\{\int_0^t \int_{\mathbb{R}^d} \log \lambda^\varepsilon(z,s)\pi(\mathrm{d}z,\mathrm{d}s) - \int_0^t \int_{\mathbb{R}^d} \left[\lambda^\varepsilon(z,s) - 1\right]\mu(\mathrm{d}z)\mathrm{d}s\right\},$$

we need

$$\det\left[I + \varepsilon \nabla_z V(z,t)\right] > 0$$

and that the integrals are well defined.

Theorem 3. *Process* Z^ε *is a (local) martingale. If it is a martingale, then for any* $t > 0$*, under the measure*

$$\mathrm{d}\mathbb{P}^\varepsilon = Z^\varepsilon(t)\mathrm{d}\mathbb{P} \qquad \text{on } \mathfrak{F}_t$$

$\pi^{\varepsilon V}$ *is a Poisson random measure with intensity* $\mu(\mathrm{d}z)\mathrm{d}s$.

We need the Itô formula (see Applebaum)

Theorem 4. *Assume that*

$$\mathrm{d}Y(t) = F(t)\mathrm{d}t + \eta(z,t)\overline{\pi}(\mathrm{d}z,\mathrm{d}t),$$

Then

$$\begin{aligned}
&\mathrm{d}\psi(Y(t)) \\
&= \langle \nabla\psi(Y(t)), F(t)\rangle \mathrm{d}t + \int_{\mathbb{R}^d} \left[\psi(Y(t-) + \eta(z,t)) - \psi(Y(t-))\right]\overline{\pi}(\mathrm{d}z,\mathrm{d}t) \\
&\quad + \int_{\{|z|\leq 1\}} \left[\psi(Y(t-) + \eta(z,t)) - \psi(Y(t-)) - \langle \nabla\psi(Y(t-)), \eta(z,t)\rangle\right]\mu(\mathrm{d}z)\mathrm{d}t.
\end{aligned}$$

5.6. **Proof of the Girsanov type theorem (Theorem 3).** By the Itô formula

$$\mathrm{d}Z^{\varepsilon}(t) = \int_{\mathbb{R}^d} Z^{\varepsilon}(t-)\left[\lambda^{\varepsilon}(z,t)-1\right]\widehat{\pi}(\mathrm{d}z,\mathrm{d}t).$$

To show that under $\mathbb{P}^{\varepsilon}$, $\pi^{\varepsilon V}$ is a Poisson random measure with intensity measure $\mu(\mathrm{d}z)\mathrm{d}s$ we need to show that for any smooth non-negative test function

$$\varphi\colon \mathbb{R}^d \times [0,+\infty) \mapsto [0,+\infty)$$

we have

$$\mathbb{E}^{\mathbb{P}^{\varepsilon}} \exp\left\{\int_0^t \int_{\mathbb{R}^d} \varphi(z,s)\pi^{\varepsilon V}(\mathrm{d}z,\mathrm{d}s)\right\}$$
$$= \mathbb{E}^{\mathbb{P}} \exp\left\{\int_0^t \int_{\mathbb{R}^d} \varphi(z,s)\pi(\mathrm{d}z,\mathrm{d}s)\right\}.$$

We have

$$\mathbb{E}^{\mathbb{P}^{\varepsilon}} \exp\left\{\int_0^t \int_{\mathbb{R}^d} \varphi(z,s)\pi^{\varepsilon V}(\mathrm{d}z,\mathrm{d}s)\right\} = \mathbb{E}\, U^{\varepsilon}(t),$$

where

$$U^{\varepsilon}(t) = Z^{\varepsilon}(t)\exp\left\{\int_0^t \int_{\mathbb{R}^d} \varphi(z+\varepsilon V(z,s),s)\pi(\mathrm{d}z,\mathrm{d}s)\right\}.$$

We have (by Itô's)

$$\mathrm{d}U^{\varepsilon}(t) = \mathrm{d}\,(\text{martingale}) + \int_{\mathbb{R}^d} U^{\varepsilon}(t)\left[\mathrm{e}^{\varphi(z+\varepsilon V(z,t))}-1\right]\lambda^{\varepsilon}(z,t)\mu(\mathrm{d}z)\mathrm{d}t.$$

Thus

$$\mathbb{E}\, U^{\varepsilon}(t) = 1$$
$$+\,\mathbb{E}\int_0^t U^{\varepsilon}(s)\int_{\mathbb{R}^d}\left[\mathrm{e}^{\varphi(z+\varepsilon V(z,s),s)}-1\right] g(z+\varepsilon V(z,s))\det\nabla_z\left(z+\varepsilon V(z,s)\right)\mathrm{d}z\mathrm{d}s$$
$$= 1+\mathbb{E}\int_0^t U^{\varepsilon}(s)\int_{\mathbb{R}^d}\left[\mathrm{e}^{\varphi(z,s)}-1\right]\mu(\mathrm{d}z)\mathrm{d}s$$
$$= 1+\int_0^t \mathbb{E}\, U^{\varepsilon}(s)\int_{\mathbb{R}^d}\left[\mathrm{e}^{\varphi(z,s)}-1\right]\mu(\mathrm{d}z)\mathrm{d}s.$$

Thus $\mathbb{E}\, U^{\varepsilon}(t)$ does not depend on ε. Hence

$$\mathbb{E}\, U^{\varepsilon}(t) = \mathbb{E}\, U^{0}(t) = \mathbb{E}\, U(t).$$

5.7. **The existence od $D_v^*\mathbf{1}(t)$.** Let Φ be a smooth functional of π. Assume that $\Phi(\pi)$ is $\mathfrak{F}_t$-measurable. We have

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbb{E}\,\Phi(\pi^{\varepsilon V})Z^{\varepsilon}(t) = 0.$$

Thus

$$0 = \mathbb{E}\left[D_V\Phi(\pi)Z^0(t) + \Phi(\pi)R(t)\right] = \mathbb{E}\left[D_V\Phi(\pi) + \Phi(\pi)R(t)\right],$$

where

$$R(t) := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}Z^{\varepsilon}(t)|_{\varepsilon=0}.$$

Consequently

$$D_V^*\mathbf{1}(t) = -R(t).$$

We can compute $R(t)$! Namely, since

$$Z^{\varepsilon}(t) = \exp\left\{\int_0^t\int_{\mathbb{R}^d}\log\lambda^{\varepsilon}(z,s)\pi(\mathrm{d}z,\mathrm{d}s) - \int_0^t\int_{\mathbb{R}^d}\left(\lambda^{\varepsilon}(z,s)-1\right)\mu(\mathrm{d}z)\mathrm{d}s\right\},$$

we have

$$\begin{aligned}R(t) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}Z^{\varepsilon}(t)|_{\varepsilon=0}\\ &= Z^0(t)\left\{\int_0^t\int_{\mathbb{R}^d}\frac{\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\lambda^{\varepsilon}(z,s)}{\lambda^{\varepsilon}(z,s)}|_{\varepsilon=0}\pi(\mathrm{d}z,\mathrm{d}s) - \int_0^t\int_{\mathbb{R}^d}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\lambda^{\varepsilon}(z,s)|_{\varepsilon=0}\mu(\mathrm{d}z)\mathrm{d}s\right\}.\end{aligned}$$

Since $\lambda^0 = 1$ and

$$\lambda^{\varepsilon}(z,t) = \det\left(I + \varepsilon\nabla_z V(z,t)\right)\frac{g(z+\varepsilon V(z,t))}{g(z)}$$

we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\lambda^{\varepsilon}(z,t)|_{\varepsilon=0} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\det\left(I+\varepsilon\nabla_z V(z,t)\right)|_{\varepsilon=0} + \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{g(z+\varepsilon V(z,t))}{g(z)}|_{\varepsilon=0}\\ &= \nabla_z V(z,t) + \frac{\langle\nabla g(z), V(z,t)\rangle}{g(z)}\\ &= \frac{\operatorname{div}\left(g(z)V(z,t)\right)}{g(z)}.\end{aligned}$$

Consequently

$$D_V^*\mathbf{1}(t) = -\int_0^t\int_{\mathbb{R}^d}\frac{\operatorname{div}\left(g(z)V(z,s)\right)}{g(z)}\widehat{\pi}(\mathrm{d}z,\mathrm{d}s).$$

5.8. **Example (from the papers of Dong, and Norris.** Assume that $F \in C_b^1(\mathbb{R}^d)$. Let $S_0 \neq \emptyset$ be an open subset of $\mathcal{S}^{d-1}$, $\delta > 0$, $g \in C^1(\mathbb{R}^d)$, $g(0) > 0$. Assume that on the sector

$$\left\{z \in \mathbb{R}^{d-1} \colon 0 < |z| < \delta,\ \frac{z}{|z|} \in S_0\right\}$$

we have

$$\mu(\mathrm{d}z) = |z|^{-d} g(z)\mathrm{d}z.$$

Then the gradient formula golds for the semigroup holds for t large enough.

If

$$\mu(\mathrm{d}z) = |z|^{-d} \log\frac{1}{|z|} g(z)\mathrm{d}z,$$

then the gradient formula holds holds for all $t > 0$.

6. Takeuchi approach

The presentation is based on the paper

- A. Takeuchi, Bismut-Elworthy-Li-type formulae for stochastic differential equations with jumps, J. Teoret. Probab. 23 (2010), 576–604,

and also on the discussions with E. Priola and J. Zabczyk.

6.1. **Introduction.** The original Takeuchi paper deals with SDE

$$\mathrm{d}X(t) = F(\varepsilon, X(t))\mathrm{d}t + \sigma(\varepsilon, X(t))\mathrm{d}W(t) + \int_{\mathbb{R}^d} \eta(\varepsilon, z, X(t-))\overline{\pi}(\mathrm{d}z, \mathrm{d}t).$$

His goals were the formulae

$$\begin{aligned}
\nabla_x \mathbb{E}\, f(X^{x,\varepsilon}(t)) &= \mathbb{E}\, f(X^{x,\varepsilon}(t)) Z^1(t,\varepsilon),\\
\nabla_\varepsilon \mathbb{E}\, f(X^{x,\varepsilon}(t)) &= \mathbb{E}\, f(X^{x,\varepsilon}(t)) Z^2(t,\varepsilon),\\
\nabla_x\nabla_x \mathbb{E}\, f(X^{x,\varepsilon}(t)) &= \mathbb{E}\, f(X^{x,\varepsilon}(t)) Z^3(t,\varepsilon).
\end{aligned}$$

The forms of the fields Z are "explicite". Tools: BEL formula, Markov property, Girsanov (elementary).

6.2. **Simplification.** Based on the discussions with E. Priola and J. Zabczyk.

$$\mathrm{d}X(t) = F(X(t))\mathrm{d}t + \int_{\mathbb{R}^d} z\overline{\pi}(\mathrm{d}z, \mathrm{d}t), \qquad X(0) = x.$$

Let

$$h(z) = |z|^2 \mathrm{e}^{-|z|^2},$$

and

$$J(t) = J_h(t) = \int_0^t \int_{\mathbb{R}^d} h(z)\pi(\mathrm{d}z, \mathrm{d}s).$$

Step 1 Fix $t >$ and f. Let

$$u(s,x) = P_{t-s}f(x).$$

By Itô's formula we have

$$f(X^x(t)) = P_t f(x) + \int_0^t \int_{\mathbb{R}^d} \left[u(s, X^x(s-)+z) - u(s, X^x(s-))\right] \widehat{\pi}(\mathrm{d}z, \mathrm{d}s).$$

Step 2 Show that

$$\mathbb{E}\, f(X^x(t))J(t) = \mathbb{E} \int_0^t \int_{\mathbb{R}^d} u(s, X^x(s)+z)h(z)\mu(\mathrm{d}z)\mathrm{d}s.$$

To to this we multiply

$$f(X^x(t)) = P_t f(x) + \int_0^t \int_{\mathbb{R}^d} \left[u(s, X^x(s-)+z) - u(s, X^x(s-))\right] \widehat{\pi}(\mathrm{d}z, \mathrm{d}s).$$

by $J(t)$ and take the expectation. We have

$$\mathbb{E}\ f(X^x(t))J(t) = I_1 + I_2,$$

where

$$\begin{aligned}
I_1 &= \mathbb{E}\ f(X^x(t))\mathbb{E} \int_0^t \int_{\mathbb{R}^d} h(z)\pi(\mathrm{d}z.\mathrm{d}s),\\
I_2 &= \mathbb{E} \int_0^t \int_{\mathbb{R}^d} \left[u(s, X^x(s-)+z) - u(s, X^x(s-))\right] \widehat{\pi}(\mathrm{d}z, \mathrm{d}s) \int_0^t \int_{\mathbb{R}^d} h(z)\pi(\mathrm{d}z, \mathrm{d}s)\\
&= \mathbb{E} \int_0^t \int_{\mathbb{R}^d} \left[u(s, X^x(s-)+z) - u(s, X^x(s-))\right] \widehat{\pi}(\mathrm{d}z, \mathrm{d}s) \int_0^t \int_{\mathbb{R}^d} h(z)\widehat{\pi}(\mathrm{d}z, \mathrm{d}s)\\
&= \int_0^t \int_{\mathbb{R}^d} \mathbb{E} \left[u(s, X^x(s-)+z) - u(s, X^x(s-))\right] h(z)\mu(\mathrm{d}z)\mathrm{d}s.
\end{aligned}$$

Next

$$\begin{aligned}
I_1 &= \mathbb{E} \int_0^t \int_{\mathbb{R}^d} h(z)\mu(\mathrm{d}z)\mathbb{E}\left(f(X^x(t)|\mathfrak{F}_s\right)\mathrm{d}s\\
&= \mathbb{E} \int_0^t \int_{\mathbb{R}^d} h(z)\mu(\mathrm{d}z)P_{t-s}f(X^x(s))\mathrm{d}s\\
&= \mathbb{E} \int_0^t \int_{\mathbb{R}^d} h(z)\mu(\mathrm{d}z)u(s, X^x(s))\mathrm{d}s.
\end{aligned}$$

Step 3 (crucial) Let

$$D(t) := \int_0^t \int_{\mathbb{R}^d} \left\langle \nabla_x X^x(s), \frac{\nabla_z (h(z)g(z))}{g(z)} \right\rangle \widehat{\pi}(\mathrm{d}z, \mathrm{d}z)$$

where g is the density of μ with respect to Lebesgue measure.

We will show that

$$\mathbb{E}\, f(X^x(t))D(t) = -\nabla_x \mathbb{E}\, f(X^x(t))J(t).$$

To do this note that multiplying

$$f(X^x(t)) = P_t f(x) + \int_0^t \int_{\mathbb{R}^d} \left[u(s, X^x(s-) + z) - u(s, X^x(s-))\right] \widehat{\pi}(\mathrm{d}z, \mathrm{d}s)$$

by $D(t)$ and taking the expectation we obtain

$$\begin{aligned}
\mathbb{E}\, f(X^x(t))D(t) &= \mathbb{E} \int_0^t \int_{\mathbb{R}^d} \left[u(s, X^x(s-) + z) - u(s, X^x(s-))\right] \\
&\quad \times \left\langle \nabla_x X^x(s), \frac{\nabla_z (h(z)g(z))}{g(z)} \right\rangle g(z)\mathrm{d}z\mathrm{d}s \\
&= -\mathbb{E} \int_0^t \int_{\mathbb{R}^d} \nabla_x u(s, X^x(s-) + z) h(z) \mu(\mathrm{d}z)\mathrm{d}s \\
&= -\mathbb{E}\, f(X^x(t))J(t).
\end{aligned}$$

Step 4 From **Step 3** we have

$$\mathbb{E}\, f(X^x(t))D(t) = -\nabla_x \mathbb{E}\, f(X^x(t))J(t).$$

Obviously

$$\begin{aligned}
P_t f(x) = \mathbb{E}\, f(X^x(t)) &= \mathbb{E}\, f(X^x(t))J(t)\frac{1}{J(t)} \\
&= \int_0^{+\infty} \mathbb{E}\, \mathrm{e}^{-\beta J(t)} f(X^x(t))J(t)\mathrm{d}\beta \\
&= \int_0^{+\infty} N_\beta(t) \mathbb{E}\, \mathrm{e}^{-\beta J(t)} \frac{1}{N_\beta(t)} f(X^x(t))J(t)\mathrm{d}\beta.
\end{aligned}$$

Take

$$N_\beta(t) = \exp\left\{ -\int_0^t \int_{\mathbb{R}^d} \left(1 - \mathrm{e}^{-\beta h(z)}\right) \mu(\mathrm{d}z)\mathrm{d}s \right\}.$$

Theorem 5. ***(Girsanov)*** *We have*

$$\mathbb{E}\, \mathrm{e}^{-\beta J(t)} \frac{1}{N_\beta(t)} = 1,$$

and under the measure

$$d\mathbb{P}^\beta = e^{-\beta J(t)} \frac{1}{N_\beta(t)} d\mathbb{P}$$

π on $\mathfrak{F}_T$ is a Poisson random measure with intensity

$$e^{-\beta h(z)} \mu(dz) ds.$$

6.3. **Proof of the theorem.** We have

$$\begin{aligned}
\mathbb{E} e^{-\beta J(t)} &= \mathbb{E} \exp \left\{ -\beta \int_0^t \int_{\mathbb{R}^d} h(z) \pi(dz, ds) \right\} \\
&= \exp \left\{ - \int_0^t \int_{\mathbb{R}^d} \left(1 - e^{-\beta h(z)}\right) \mu(dz) \right\} \\
&= N_\beta(t).
\end{aligned}$$

Thus

$$d\mathbb{P}^\beta = e^{-\beta J(t)} \frac{1}{N_\beta(t)} d\mathbb{P}$$

is a probability measure.

To see that π is under $\mathbb{P}^\beta$ a Poisson random measure with intensity

$$e^{-\beta h(z)} \mu(dz) ds,$$

take a non-negative measurable function

$$\phi \colon \mathbb{R}^d \times [0, +\infty) \mapsto [0, +\infty).$$

We need to show that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^\beta} \exp \left\{ - \int_0^t \int_{\mathbb{R}^d} \phi(z, s) \pi(dz, ds) \right\} \\
&\quad = \exp \left\{ - \int_0^t \int_{\mathbb{R}^d} \left(1 - e^{-\phi(z,s)}\right) e^{-\beta h(z)} \mu(dz) ds \right\}.
\end{aligned}$$

To do this note that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^\beta} \exp \left\{ - \int_0^t \int_{\mathbb{R}^d} \phi(z, s) \pi(dz, ds) \right\} \\
&= \frac{1}{N_\beta(t)} \mathbb{E} \exp \left\{ - \int_0^t \int_{\mathbb{R}^d} \left(\phi(z, s) + \beta h(z)\right) \pi(dz, ds) \right\} \\
&= \exp \left\{ \int_0^t \int_{\mathbb{R}^d} \left(1 - e^{-\beta h(z)}\right) \mu(dz) ds \right\} \\
&\quad \times \exp \left\{ - \int_0^t \int_{\mathbb{R}^d} \left(1 - e^{-(\phi(z,s) + \beta h(z))}\right) \mu(dz) ds) \right\} \\
&= \exp \left\{ - \int_0^t \int_{\mathbb{R}^d} \left(e^{-\beta h(z)} - e^{-(\phi(z,s) + \beta h(z))}\right) \mu(dz) ds) \right\}.
\end{aligned}$$

7. Two Girsanov theorems

7.1. **On translation π^V of π by a field V.** We assume that $\mu(\mathrm{d}z) \ll \mathrm{d}z$ and that V is a predictable random field. Let

$$g(z) = \frac{\mathrm{d}\mu}{\mathrm{d}z}(z),$$

$$\lambda(z,t) := \det\left[I + \nabla_z V(z,t)\right] \frac{g(z+V(z,t))}{g(z)},$$

$$Z(t) := \exp\left\{\int_0^t \int_{\mathbb{R}^d} \log \lambda(z,s)\pi(\mathrm{d}z,\mathrm{d}s) - \int_0^t \int_{\mathbb{R}^d} \left[\lambda(z,s) - 1\right]\mu(\mathrm{d}z)\mathrm{d}s\right\},$$

we need

$$\det\left[I + \nabla_z V(z,t)\right] > 0$$

and that the integrals are well defined.

Theorem 6. *Process $Z = (Z(t))$ is a (local) martingale. If it is a martingale, then for any $t > 0$, under the measure*

$$\mathrm{d}\mathbb{P}^* = Z(t)\mathrm{d}\mathbb{P} \qquad \textit{on } \mathfrak{F}_t$$

π^V is a Poisson random measure with intensity $\mu(\mathrm{d}z)\mathrm{d}s$.

7.2. **On change of the intensity measure.** Let $h\colon \mathbb{R}^d \mapsto [0,+\infty)$ be measurable. Take

$$N(t) = \exp\left\{-\int_0^t \int_{\mathbb{R}^d} \left(1 - \mathrm{e}^{-h(z)}\right)\mu(\mathrm{d}z)\mathrm{d}s\right\}.$$

Theorem 7. *We have*

$$\mathbb{E}\,\mathrm{e}^{-J_h(t)}\frac{1}{N(t)} = 1,$$

and under the measure

$$\mathrm{d}\mathbb{P}^* = \mathrm{e}^{-J_h(t)}\frac{1}{N(t)}\mathrm{d}\mathbb{P}$$

π on $\mathfrak{F}_t$ is a Poisson random measure with intensity

$$\mathrm{e}^{-h(z)}\mu(\mathrm{d}z)\mathrm{d}s.$$

Part 4. The case of Lévy semigroup

Let $L = (L_t)$ be a Lévy process in a Hilbert space H with a generating triplet (m, Q, μ). Thus by the Lévy–Khinchin decomposition

$$L_t = mt + W_Q(t) + \int_{\{0<|z|_H\leq 1\}} z\widehat{\pi}(\mathrm{d}z, \mathrm{d}t) + \int_{\{|z|_H>1\}} z\pi(\mathrm{d}z, \mathrm{d}t), \qquad t \geq 0,$$

where $m \in H$, W_Q is a Wiener process in H with covariance operator Q, and π is a Poisson random measure on $H \times [0, +\infty)$ with intensity $\mu(\mathrm{d}z)\mathrm{d}t$. We call μ the *Lévy measure* of L. Recall that

$$\int_H 1 \wedge |z|_H^2 \mu(\mathrm{d}z) < \infty.$$

Consider the Markov family

$$L_t^x = x + L_t, \qquad t \geq 0,\ x \in H.$$

Its transition semigroup (P_t);

$$P_t f(x) = \mathbb{E}\, f(L_t^x), \qquad t \geq 0,\ x \in H,\ f \in B_b(H),$$

is C_0 on the space $UC_b(H)$ of uniformly continuous bounded functions on H. Moreover, the domain of its generator $\mathcal{L}$ contains the space $UC_b^2(H)$, and

$$\mathcal{L}f(x) = \langle Df(x), m\rangle_H + \frac{1}{2}\mathrm{Trace}\, QD^2 f(x)$$
$$+ \int_H \left(f(x+z) - f(x) - \mathbf{1}_{\{|z|_H\leq 1\}}\langle Df(x), z\rangle_H\right) \mu(\mathrm{d}z).$$

8. Basic facts

Assume that $H = \mathbb{R}^d$.

Theorem 8. *(Hawkes 1977, Edwards 1967) Suppose that the linear operator P has the form*

$$Pf(x) = \int_{\mathbb{R}^d} f(x+z)m(\mathrm{d}z)$$

for some finite measure m. Then the following are equivalent:

(i) *m is absolutely cont. with respect to Lebesgue measure,*
(ii) *$P(B_b(\mathbb{R}^d)) \subset C_b(\mathbb{R}^d)$,*
(iii) *$P(B_{b,c}(\mathbb{R}^d)) \subset C_b(\mathbb{R}^d)$.*

In the case of a Lévy semigroup

$$P_t f(x) = \int_{\mathbb{R}^d} f(x+z)\mathcal{L}(L_t)(\mathrm{d}z).$$

Only implication $(iii) \Longrightarrow (i)$ needs to be proven. If Γ is a bounded Borel set of Lebesgue measure 0, then

$$\langle P\mathbf{1}_\Gamma, \psi\rangle = \langle \mathbf{1}_\Gamma, P^*\psi\rangle = 0, \qquad \forall\, \psi \in B_b(\mathbb{R}^d).$$

Thus $P\mathbf{1}_\Gamma(x) = 0$ a.s. By the strong Feller property,

$$P\mathbf{1}_\Gamma(0) = 0 = m(\Gamma).$$

9. Absolute continuity

Theorem 9. *Let (L_t) be an $\mathbb{R}^d$-valued Lévy process with generating triplet (m, Q, μ). Then the following are equivalent:*

(i) *$\mathcal{L}(L_t)$ is continuous for every $t > 0$,*
(ii) *$\mathcal{L}(L_t)$ is continuous for a certain $t > 0$,*
(iii) *$Q \neq 0$ or $\mu(\mathbb{R}^d) = +\infty$.*

Theorem 10. ***(Tucker, Fisz and Varadarajan)*** *Let (L_t) be an $\mathbb{R}^d$-valued Lévy process with generating triplet (m, Q, μ). If μ is infinite and absolutely continuous with respect to Lebesgue measure, then $\mathcal{L}(L_t)$ is absolutely cont. for any $t > 0$.*

Let

$$\tilde{\mu}(B) = \int_B |z|^2 \wedge 1 \mu(\mathrm{d}z), \qquad B \in \mathcal{B}(\mathbb{R}^d).$$

Theorem 11. *Let (L_t) be an $\mathbb{R}^d$-valued Lévy process with generating triplet (m, Q, μ). If μ is infinite and $\tilde{\mu}^n$ is absolutely continuous with respect to Lebesgue measure, then $\mathcal{L}(L_t)$ is absolutely cont. for any $t > 0$.*

In general condition

$$\int_{B(0,1)} |z|^\lambda \mu(\mathrm{d}z) = +\infty,$$

for some $\lambda \in (0, 2)$ is not sufficient for absol. cont. (see Orey)

Theorem 12. ***(Picard)*** *Suppose that there is an $\alpha \in (0, 2)$ and a $c > 0$ such that for any $\rho \in (0, 1)$ and any unit vector v,*

$$\int_{\{|\langle z,v\rangle| \le \rho\}} |\langle z, v\rangle|^2 \mu(\mathrm{d}z) \ge c\rho^\alpha.$$

Then $\mathcal{L}(L_t)$ is absolutely cont. for any $t > 0$.

In the case of Picard

$$\left|\widehat{\mathcal{L}(L_t)}(y)\right| \le \exp\{-tC|y|^{2-\alpha}\} \quad \text{for } y\colon |y| \text{ large enough.}$$

10. Strong Feller

Assume that
$$\nu = \sum_n a_n \delta_{b_n}, \qquad \nu(\mathbb{R}^d) = \infty.$$
Then there are three possible cases

(i) $\mathcal{L}(L_t)$ is absolutely cont. for avert $t > 0$,
(ii) $\mathcal{L}(L_t)$ is absolutely cont. for all $t > t_0$ and continuous singular for $t \leq t_0$,
(iii) $\mathcal{L}(L_t)$ is continuous singular for all $t > 0$.

For gradient estimates (and consequently Strong Feller property) for diffusions driven by Lévy process see Bismut, Leandre, Norris, Takeuchi, Dong, X. Zhang.

11. The case of Lévy semigroup again

This part is based on the joint work with Zhao Dong (Beijing) and Lihu Xu (Macau). Given $q \in L^2(H, \mathcal{B}(H), \mu)$, define
$$\mathcal{D}_q := \left\{ f \in B_b(H) \colon \sup_{x \in H} \int_H |f(x+y) - f(x)|\, |q(y)| \mu(\mathrm{d}y) < \infty \right\}.$$
Next, let
$$A_q f(x) := \int_H [f(x+y) - f(x)]\, q(y) \mu(\mathrm{d}y) \qquad \text{for } f \in \mathcal{D}_q,\ x \in H.$$

Theorem 13. *Let $q \in L^2(H, \mathcal{B}(H), \mu)$. Then for all $t > 0$ and $f \in B_b(H)$, $P_t f \in \mathcal{D}_q$ and*
$$|A_q P_t f(x)|^2 \leq \frac{1}{t} P_t f^2(x) \int_H q^2(y) \mu(dy) \qquad \textit{for all } x \in H.$$

11.1. **Proof.** Assume that $f \in UC_b^2(H)$. By Itô's formula
$$\begin{aligned} f(L_t^x) - P_t f(x) &= -\int_0^t \mathcal{L} P_{t-s} f(L_s^x) \mathrm{d}s + \int_0^t P_{t-s} \mathcal{L} f(L_s^x) \mathrm{d}s \\ &\quad + \int_0^t \int_H [P_{t-s} f(L_s^x + y) - P_{t-s} f(L_s^x))]\, \widehat{\pi}(\mathrm{d}y, \mathrm{d}s) \\ &\quad + \int_0^t \langle DP_{t-s} f(L_s^x), \mathrm{d}W_Q(s) \rangle_H \\ &= \int_0^t \int_H [P_{t-s} f(X_s + y) - P_{t-s} f(X_s)]\, \widehat{\pi}(\mathrm{d}y, \mathrm{d}s) \\ &\quad + \int_0^t \langle DP_{t-s} f(L_s^x), \mathrm{d}W_Q(s) \rangle_H . \end{aligned}$$

Multiplying the both sides of by

$$\int_0^t \int_H q(y)\widehat{\pi}(\mathrm{d}y, \mathrm{d}s)$$

we further get

$$\begin{aligned}
&\mathbb{E}\left[f(L_t^x)\int_0^t \int_H q(y)\widehat{\pi}(\mathrm{d}y, \mathrm{d}s)\right] \\
&= \mathbb{E}\int_0^t \int_H \left[P_{t-s}f(L_s^x + y) - P_{t-s}f(L_s^x)\right] q(y)\mu(\mathrm{d}y)\mathrm{d}s \\
&= \int_0^t \int_H \left[P_s P_{t-s}f(x + y) - P_s P_{t-s}f(x)\right] q(y)\mu(\mathrm{d}y)\mathrm{d}s \\
&= t\int_H \left[P_t f(x+y) - P_t f(x)\right] q(y)\mu(\mathrm{d}y) = tA_q P_t f(x).
\end{aligned}$$

Thus, by the Hölder inequality and Itô isometry we obtain

$$\begin{aligned}
t\left|A_q P_t f(x)\right| &\le \left(\mathbb{E}\, f^2(L_t^x)\right)^{1/2} \left(\int_0^t \int_H q^2(y)\mu(\mathrm{d}y)\mathrm{d}s\right)^{1/2} \\
&\le \left(P_t f^2(x)\right)^{1/2} t^{1/2} \left(\int_H q^2(y)\mu(\mathrm{d}y)\right)^{1/2}.
\end{aligned}$$

Thus the desired estimate holds for any $f \in UC_b^2(H)$. Assume that $f \in B_b(H)$. Let $x \in H$. Then there is a sequence $(f_n) \subset UC_b^2(H)$ such that

$$\lim_{n\to\infty} P_t f_n^2(x) = P_t f^2(x),$$

and

$$\lim_{n\to\infty} P_t f_n(x+y) = P_t f(x+y) \quad \text{for } \mu \text{ almost all } y.$$

Consequently, the desired estimate for f follows from the Fatou lemma.

Corollary 1. *For arbitrary $f \in B_b(H)$ we have*

$$\int_H \left|P_t f(x+y) - P_t f(x)\right|^2 \mu(\mathrm{d}y) \le \frac{1}{t} P_t f^2(x), \quad x \in H,\ t > 0.$$

Since

$$\begin{aligned}
&\int_H \left|P_t f(x+y) - P_t f(x)\right|^2 \mu(\mathrm{d}y) \\
&\quad = \sup\left\{\left|A_q P_t f(x)\right|^2; \quad q\colon \int_H q^2(y)\mu(\mathrm{d}u) \le 1\right\}
\end{aligned}$$

we obtain

$$\int_H \left|P_t f(x+y) - P_t f(x)\right|^2 \mu(\mathrm{d}y) \le \frac{1}{t} P_t f^2(x).$$

Given $f \in B_b(H)$ we define the discrete difference operator $\nabla^n_{y_1,\ldots,y_n} f(x)$, $x, y_1, \ldots, y_n \in H$ putting

$$\nabla_y f(x) = f(x+y) - f(x),$$
$$\nabla^{n+1}_{y_1,\ldots,y_{n+1}} f(x) = \nabla_{y_{n+1}} \left(\nabla^n_{y_1,\ldots,y_n} f\right)(x).$$

Corollary 2. *For any $f \in B_b(H)$ and $n \in \mathbb{N}$,*

$$\sup_{x\in H} \int_H \cdots \int_H \left|\nabla_{y_1,\ldots,y_n} (P_t f)(x)\right|^2 \mu(\mathrm{d}y_1)\ldots\mu(\mathrm{d}y_n) \le \left(\frac{n}{t}\right)^n \|f\|^2_\infty.$$

It is enough to show the desired estimate for $f \in UC^2_b(H)$. Let $q_1, \ldots, q_n \in L^2(H, \mathcal{B}(H), \mu)$. Note that the operators A_{q_j} and P_s commute. Thus by the theorem

$$\begin{aligned}\|A_{q_1}\ldots A_{q_n} P_t f\|^2_\infty &= \sup_{x\in H} |A_{q_1}\ldots A_{q_n} P_t f(x)|^2 \\ &= \sup_{x\in H} \left|(A_{q_1} P_{t/n})\ldots(A_{q_n} P_{t/n}) f(x)\right|^2 \\ &\le \frac{n}{t} \|(A_{q_2} P_{t/n})\ldots(A_{q_n} P_{t/n}) f\|^2_\infty.\end{aligned}$$

12. Lévy process and its transition semigroup

Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space.

Definition 4. A stochastic process L with values in H is *Lévy* if:

(i) $L(0) = 0$,
(ii) L has stationary independent increments,
(iii) L is stochastically continuous.

Let L be a Lévy process and let ν_t be the law of $L(t)$. Then:

(i') $\nu_0 = \delta_0$,
(ii') $\nu_{t+s} = \nu_t * \mu_s$, $t, s \ge 0$,
(iii') $\nu_t(\{x\colon |x|_H \ge r\}) = \mathbb{P}(|L(t)|_H \ge r) \to 0$ as $t \downarrow 0$ for any $r > 0$.

Clearly (iii') can be stated equivalently that ν_t converges weakly to δ_0 as $t \downarrow 0$.

Definition 5. The family of probability measures satisfying (i') to (iii') is called *convolution semigroup of measures* or *infinitely divisible family.* Sometimes each ν_t is called *infinitely divisible measure.*

Any Lévy process is Markov with transition probability $P_t(x,\Gamma) = \nu_t(\Gamma - x)$. The corresponding semigroup is given by

$$P_t\psi(x) = \int_H \psi(x+y)\mu_t(\mathrm{d}y).$$

Theorem 14. *Every Lévy process has a càdlàg modification. This modification is a Lévy process.*

The theorem follows from the following general result of Kinney. Here $B(x,r)$ denotes the closed ball of radius r with centre at x and $B^c(x,r)$ denotes its complement.

Theorem 15. ***(Kinney 1953)*** *Assume that X is a Markov process with transition probabilities $P_t(x,\mathrm{d}y)$, $x \in H$, $t \geq 0$. If*

$$\lim_{t\downarrow 0} \sup_{x\in H} P_t(x, B^c(x,r)) = 0, \qquad \forall\, r > 0,$$

then X has a càdlàg modification in H.

Let us now apply the Kinney theorem to the Lévy process. Let $r > 0$. By (iii') we have

$$\lim_{t\downarrow 0} \sup_{x\in H} P_t(x, B^c(x,r)) = \lim_{t\downarrow 0} \sup_{x\in H} \nu_t(B^c(x,r)-x) = \lim_{t\downarrow 0} \nu_t(B^c(0,r)) = 0.$$

Note that the Kinney theorem cannot be applied to the family given by the generalised Ornstein–Uhlenbeck equation

$$\mathrm{d}X = AX\mathrm{d}t + \mathrm{d}L, \qquad X(0) = x,$$

even in finite dimensional case. For

$$\lim_{t\downarrow 0} \sup_{x\in H} P_t(x, B^c(x,r)) = \lim_{t\downarrow 0} \sup_{x\in H} \mathbb{P}\left(\left| \mathrm{e}^{At}x - x + \int_0^t \mathrm{e}^{A(t-s)}\mathrm{d}L(s) \right| \geq r \right).$$

Clearly the right hand side of the identity above equals $+\infty$ unless $A = 0$.

12.1. **Semigroups.** Let (ν_t) be a convolution semigroup of measures on H. Let $C_b(H)$ and $UC_b(H)$ be the spaces of bounded continuous and bounded uniformly continuous functions on H equipped with the supremum norm $\|\cdot\|_\infty$.

Theorem 16. ***(Tessitore and Zabczyk 2001)*** *Transition semigroup (P_t) is C_0 on $C_b(H)$ if and only if (ν_t) corresponds to a compound Poisson process or $\nu_t \equiv \delta_0$.*

Proof. If part is simple. Namely if (ν_t) corresponds to a compound Poisson process L, then denoting by τ_n its consecutive jump times we have

$$\begin{aligned} P_t\psi(x) &= \sum_{n=0}^{\infty} \mathbb{P}\left(t \in [\tau_n, \tau_{n+1})\right) \mathbb{E}\psi(L(\tau_n)) \\ &= \mathrm{e}^{-\alpha t} \sum_{n=0}^{\infty} \frac{(\alpha t)^n}{n!} \mathbb{E}\psi(L(\tau_n)). \end{aligned}$$

Consequently

$$\limsup_{t\downarrow 0} \|P_t\psi - \psi\|_\infty \le \limsup_{t\downarrow 0} \mathrm{e}^{-\alpha t} \sum_{n=1}^{\infty} \frac{(\alpha t)^n}{n!} \|\psi\|_\infty = 0.$$

Assume now that neither $\nu_t \equiv \delta_0$ nor (ν_t) corresponds to a compound Poissson process. Let $\varepsilon \in (0,1)$. Then there is a sequence $t_n \downarrow 0$ such that

$$\nu_{t_n}\left(|x|_H \le \frac{1}{2}\right) = \mathbb{P}\left(|L(t_n)|_H \le \frac{1}{2}\right) \ge 1 - \varepsilon$$

and (at this moment we use the assumption that L is not a compound Poisson nor $L \equiv 0$) such that for each n we can find an $0 < r_n < \frac{1}{2}$ such that

$$\nu_{t_n}\left(|x|_H \le r_n\right) = \mathbb{P}\left(|L(t_n)|_H \le r_n\right) \le \varepsilon.$$

Let (x_n) be a sequence of elements of H such that $|x_n - x_m|_H \ge 1$ if $n \ne m$. Let $\psi_n \in C_b(H)$ be such that $0 \le \psi_n(x) \le 1$ for all x, $\psi_n(x_n) = 1$ and $\psi_n(x) = 0$ if $|x - x_n|_H \ge r_n$. Let $\psi = \sum \psi_n$. Then $\psi \in C_b(H)$, $0 \le \psi(x) \le 1$ for all x, and $\psi(x_n) = 1$ or all n. We have

$$\|P_{t_n}\psi - \psi\|_\infty \ge |P_{t_n}\psi(x_n) - \psi(x_n)| = |1 - P_{t_n}\psi(x_n)|.$$

Since

$$\begin{aligned} P_{t_n}\psi(x_n) =& P_{t_n}\psi_n(x_n) + \sum_{m\ne n} P_{t_n}\psi_m(x_n) \\ & \le \nu_{t_n}\left(|x|_H \le r_n\right) + \nu_{t_n}\left(|x|_H > \frac{1}{2}\right) \le 2\varepsilon, \end{aligned}$$

we have $\|P_{t_n}\psi - \psi\|_\infty \ge 1 - 2\varepsilon$. □

Definition 6. *A semigroup (P_t) of continuous linear operators on $UC_b(H)$ is* translation invariant *if for all $a \in H$, $t \ge 0$, and $\psi \in UC_b(H)$, $P_t\tau_a\psi = \tau_a P_t\psi$, where τ_a is the translation on a vector a; $\tau_a\psi(x) = \psi(x+a)$.*

For a simple proof of the following result we refer the reader to Peszat and Zabczyk.

Theorem 17. *(i) The transition semigroup of a Lévy process is C_0 on $UC_b(H)$. (ii) A Markov transition semigroup on $UC_b(H)$ is translation invariant if and only it is the transition semigroup of a Lévy process on H.*

13. Lévy–Khinchin decomposition

The so-called Lévy–Khinchin decomposition and Lévy–Khinchin formula play fundamental roles in the theory of Lévy processes. Let L be a Lévy process taking values in a Hilbert space H. Taking if necessary a modification we may assume that L is càdlàg, see Theorem 14. Define $\Delta L(s) := L(s) - L(s-)$,

$$\mu(A) := \mathbb{E} \sum_{s \le 1} \chi_A(\Delta L(s)), \qquad A \in \mathcal{B}(H \setminus \{0\}).$$

Next, given $A \in \mathcal{B}(H)$ such that $\operatorname{dist}(A, \{0\}) > 0$ write

$$L_A(t) := \sum_{s \le t} \chi_A(\Delta L(s)) \Delta L(s), \qquad t \ge 0.$$

Note that due to the fact that L has càdlàg trajectories in H, $\chi_A(\Delta L(s)) \neq 0$ only for a finite number of $s \le t$. For a proof of the following result we refer the reader to e.g. Gikmann–Skorokood book.

Theorem 18. ***(Lévy–Khinchin)*** *(i) μ is a measure satisfying*

$$\int_H |y|_H^2 \wedge 1 \mu(\mathrm{d}y) < \infty.$$

(ii) For any $A \in \mathcal{B}(H)$ such that $\operatorname{dist}(A, \{0\}) > 0$, L_A is a compound Poisson process with jump intensity measure μ_A being equal to μ restricted to A.
(iii) For an arbitrary sequence (r_n) degreasing to 0,

$$L(t) = at + W(t) + \sum_{n=1}^{\infty} \left(L_{A_n}(t) - t \int_{A_n} y \mu(\mathrm{d}y) \right) + L_{A_0}(t), \tag{3}$$

where the series converges $\mathbb{P}$-a.s. uniformly in t on any bounded interval $[0, T]$, $a \in H$, W is a Wiener process in H, $A_0 = \{|x|_H > r_0\}$, and $A_n = \{r_n < |x|_H \le r_{n-1}\}$. All components are independent, and W does not depend on the choice of (r_n).

Sketch of the proof. First assume that L is a Lévy process in H with continuous trajectories. Can we show that $L(t) = at + W(t)$, where $a \in H$ and W is a Wiener process? To do this observe that L is square integrable. This follows from the following result, whose relatively easy proof can be found in the book of Protter (see also the book of Peszat and Zabczyk).

Theorem 19. ***(Kruglov 1972)*** *Assume that L is a càdlàg Lévy process in a Banach space E with jumps bounded by a fixed constant $C > 0$;*

that is there is a $C > 0$ such that $|\Delta L(t)|_E \le C$ for all $t > 0$. Then there is a constant $\beta > 0$ such that

$$\mathbb{E} e^{\beta |L(t)|_E} < \infty, \qquad \forall\, t \ge 0. \tag{4}$$

Remark 2. De Acosta showed that under the hypothesis of Kruglov's theorem, (4) holds for any $\beta \ge 0$.

Going back to the proof of the Lévy–Khinchin theorem, we see that any continuous Lévy process L is in particular square integrable. Then, since L has stationary and independent increments, the function $f\colon [0,\infty) \mapsto H$ given by $f(t) = \mathbb{E}\, L(t)$, $t \ge 0$, satisfies $f(t+s) = f(t) + f(s)$. Since, by the Fubini theorem, f is measurable, $f(t) = f(1)t$, $t \ge 0$. Therefore $\widehat{L}(t) := L(t) - t\mathbb{E}\, L(1)$, $t \ge 0$, is a square integrable martingale in H. Let Q be the covariance operator of $\widehat{L}(1)$:

$$\langle Q\psi, \phi\rangle_H = \mathbb{E}\langle \widehat{L}(1), \psi\rangle_H \langle \widehat{L}(1), \phi\rangle_H, \qquad \psi, \phi \in H.$$

Then for all $\psi, \phi \in H$,

$$\langle \widehat{L}(t), \psi\rangle_H \langle \widehat{L}(t), \phi\rangle_H - t\langle Q\psi, \phi\rangle_H, \qquad t \ge 0,$$

is a martingale. Therefore, by the Lévy characterisation $\widehat{L}$ is a Wiener process with covariance Q.

In the second part of the proof we would like to subtract from L its jumps. Note that if $A \in \mathcal{B}(H)$ is such that $\operatorname{dist}(A, \{0\}) > 0$, then

$$\Delta(L - L_A)(t) \notin A, \qquad \forall\, t \ge 0.$$

In particular $L - L_{A_0}$ does not have jumps of the size bigger than r_0. One can show that L_A is a Lévy process. It is piecewise constant as it has isolated jumps. Therefore L_A is a compound Poisson process. The intensity of L_A is

$$\mu_A(\Gamma) = \mathbb{E} \sum_{s \le 1} \chi_\Gamma(\Delta L_A(s)) = \mathbb{E} \sum_{s \le 1} \chi_{\Gamma \cap A}(\Delta L(s)), \qquad \Gamma \in \mathcal{B}(H).$$

Therefore μ_A is the restriction of μ to A. Moreover, it can be shown that $L - L_A$ and L_A are independent. Therefore we may expect that

$$L - L_{A_0} - \sum_{n=1}^{\infty} L_{A_n}$$

is a continuous Lévy process. We need however to prove the convergence of the series. It terns out, that the sum

$$\sum_{n=1}^{\infty} \left(L_{A_n}(t) - t \int_{A_n} y \mu(\mathrm{d}y) \right), \qquad t \ge 0,$$

converges in H, $\mathbb{P}$-a.s. uniformly in t from any bounded interval! Indeed, let

$$M_n(t) = L_{A_n}(t) - t\int_{A_n} y\mu(\mathrm{d}y).$$

Then, M_n is a square integrable martingale (also a Lévy process), and

$$\mathbb{E}\,|M_n(t)|_H^2 = \int_A |y|_H^2\mu(\mathrm{d}y).$$

The proof of this is not difficult as each L_A is a compound Poisson process. The convergence follows from the Doob maximal inequality for submartingales

$$r\mathbb{P}\left(\sup_{0\le t\le T}\sum_{n=N}^K |M_n(t)|_H^2 \ge r\right) \le \mathbb{E}\sum_{n=N}^K |M_n(T)|_H^2 = \int_{\bigcup_{n=N}^K A_n} |y|_H^2\mu(\mathrm{d}y).$$

From this we obtain the convergence in probability uniform in $t\in[0,T]$. The convergence $\mathbb{P}$-a.s. follows from the following result:

Theorem 20. ***(Itô–Nisio 1968)*** *If X_n, $n\in\mathbb{N}$, are independent random vectors in a not necessarily separable Banach space E, then the convergence of $\sum_{n=1}^\infty X_n$ in probability and $\mathbb{P}$-a.s. are equivalent.*

In fact we apply the Itô–Nisio theorem to $X(n) = (M_n(t); t\in[0,T])$ and $E = D([0,T];H)$, $D([0,T];H)$ is the space of all càdlàg H-valued mappings. The space E is equipped with the supremum norm. E is then complete but not separable!

13.1. **Poisson random measure.** Define

$$\pi([0,t]\times A) := \sum_{0\le s\le t}\chi_A(\Delta L(s)).$$

Then π is a Poisson random measure with intensity measure $\mathrm{d}t\mu(\mathrm{d}z)$, and

$$L_A(t) = \sum_{0\le s\le t}\chi_A(\Delta L(s))\Delta L(s) = \int_0^t\int_A z\pi(\mathrm{d}s,\mathrm{d}z).$$

Therefore we arrive at the following representation formula:

$$L(t) = at + W(t) + \int_0^t\int_H z\tilde{\pi}(\mathrm{d}s,\mathrm{d}z),$$

where

$$\tilde{\pi}(\mathrm{d}s,\mathrm{d}z) := \pi(\mathrm{d}s,\mathrm{d}z)|_{[0,\infty)\times A_0} + \left(\pi(\mathrm{d}s,\mathrm{d}z) - \mathrm{d}s\mu(\mathrm{d}x)\right)|_{[0,\infty)\times(H\setminus A_0)}.$$

13.2. **Generator of a Lévy process.** Using either Itô formula ore direct calculation as in Peszat and Zabczyk book on obtains the following result.

Theorem 21. *Assume that A is the generator of the transition semigroup on $UC_b(H)$ of a Lévy process L with the Lévy–Khinchin decomposition* (3). *Then $UC_b^2(H) \subset Dom\, A$, and*

$$A\psi(x) = \langle a, D\psi(x)\rangle_H + \frac{1}{2}QD^2\psi(x) + \int_H \big(\psi(x+y) - \psi(x) - \chi_{\{|y|_H \le 1\}}(y)\langle D\psi(x), y\rangle_H\big)\, \mu(\mathrm{d}y).$$

Remark 3. In 1973 Nemirovskii and Semenov showed that $UC_b^2(H)$ is dense in $UC_b(H)$ if and only if H is finite dimensional. Therefore, in infinite dimensional case the theorem above gives the description of the generator on a non dense subset of its domain!

14. Some basic facts about asymptotic stability

We present some abstract results concerning the asymptotic stability of a transition semigroup $(P_t)_{t\ge 0}$ on the Polish space E. We assume that the semigroup is Feller, i.e. $P_t(C_b(E)) \subset C_b(E)$ for all $t \ge 0$ and that the corresponding Markov family is stochastically continuous.

Denote by $\mathrm{Lip}_b\,(E)$ the space of all bounded Lipschitz continuous functions on E equipped with the norm

$$\|\psi\|_L := \sup_x |\psi(x)| + \sup_{x\ne y} \frac{|\psi(x) - \psi(y)|}{\rho(x,y)}.$$

Definition 7. Semigroup $(P_t)_{t\ge 0}$ satisfies the *e–property* if for any bounded, Lipschitz function ψ, the family of functions $(P_t\psi)_{t\ge 0}$ is asymptotically equicontinuous at every point of E, i.e. for any $z \in E$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_{t\to+\infty} \sup_{v\in B(z,\delta)} |P_t\psi(v) - P_t\psi(z)| < \varepsilon. \tag{5}$$

One can show that if E is a separable Banach space, then $(P_t)_{t\ge 0}$ satisfies the e-property if (1) holds for any bounded, continuously differentiable function ψ with a bounded Fréchet derivative.

Definition 8. A semigroup $(P_t)_{t\ge 0}$ is said to be *asymptotically stable* if there exists a unique invariant measure $\nu_* \in \mathcal{M}_1(E)$ such that $P_t^*\nu$ converges weakly to ν_* as $t \to +\infty$ for every $\nu \in \mathcal{M}_1(E)$.

Observe that the measure ν_* from the definition of asymptotic stability has to be ergodic.

Definition 9. We say that a semigroup $(P_t)_{t\geq 0}$ satisfies *exponential e-property* if there exist non-negative $\mathcal{C}, L\colon H\times H \to [0,+\infty)$ and $\gamma>0$ such that

1) $\mathcal{C}$ is bounded on balls, i.e. for any $R>0$ we have

$$\sup_{|u_0|,|u_1|\leq R} \mathcal{C}(u_0,u_1) < +\infty,$$

2) for any $u_0\in H$ we have $L(u_0,u_1)\to 0$, as $|u_1-u_0|\to 0$,
3) for any $\psi\in C_b^1(H)$

$$|P_t\psi(u_0)-P_t\psi(u_1)| \leq \left[\mathcal{C}(u_0,u_1)\|\psi\|_\infty + e^{-\gamma t}\|D\psi\|_\infty\right] L(u_0,u_1)$$

for all $u_0,u_1\in H$ and $t\geq 1$.

In case $L(u_0,u_1)=|u_0-u_1|$ the above definition agrees with a version of the asymptotic strong Feller property introduced by Heirer and Mattingly. The e-property holds for a semigroup $(P_t)_{t\geq 0}$ that is asymptotically stable, provided we control the rate of convergence to the invariant measure in the Wasserstein metric. Indeed, suppose that there exists an invariant measure $\nu_*\in\mathcal{M}_1(E)$ such that

$$d_1(P_t^*\delta_x,\nu_*)\leq R(x,t), \quad \forall\, x\in E,\, t\geq 0, \tag{6}$$

where $R\colon H\times[0,+\infty)\to[0,+\infty)$ is such that

$$\lim_{\delta\to 0+}\limsup_{t\to+\infty}\sup_{y\in B(x,\delta)} R(y,t)=0, \quad \forall\, x\in E \tag{7}$$

and

$$d_1(\mu,\nu) := \sup_{\|\psi\|_L\leq 1} \left|\langle\psi,\mu\rangle - \langle\psi,\nu\rangle\right|, \quad \mu,\nu\in\mathcal{M}_1(E)$$

is the Wasserstein metric.

Then, obviously for any $v\in E$ and $\varepsilon>0$ one can choose $\delta>0$ such that

$$|P_t\psi(v)-P_t\psi(z)|\leq \|\psi\|_L[R(v,t)+R(z,t)]<\varepsilon,$$
$$\forall\, z\in B(v,\delta), \psi\in \mathrm{Lip}_b(E),$$

provided that t is sufficiently large.

Definition 10. A semigroup $(P_t)_{t\geq 0}$ is called *averagely bounded in probability* if for any $\varepsilon, r>0$ there is an $R>0$ such that for every $u_0\in B(0,r)$:

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T P_s(u_0,B(0,R))\mathrm{d}s \geq 1-\varepsilon. \tag{8}$$

Usually the verification of condition (4) requires a construction of a suitable Lyapunov function. In case of a locally compact state space E it guarantees the existence of an invariant measure for the corresponding Feller semigroup. This needs not be the case for a general Polish space, see e.g. Vrkoc. Furthermore, if the semigroup is asymptotically stable then it can be easily argued that (4) is in force, with $R > 0$ that can be chosen independently of r.

Definition 11. A semigroup $(P_t)_{t\geq 0}$ is *concentrating* at $v \in E$ if for any $\varepsilon, r > 0$ there exists an $\alpha > 0$ such that for any $u_0, u_1 \in B(0, r)$:

$$P_t(u_i, B(v, \varepsilon)) \geq \alpha \quad \text{for } i = 0, 1 \text{ and some } t > 0. \tag{9}$$

It is easy to observe that condition (5) holds for an arbitrary point from the support of the invariant measure for a semigroup that is asymptotically stable.

Summarizing, both the average boundedness in probability and concentration condition are necessary for the asymptotic stability of a semigroup, while the e-property has to be satisfied for any such semigroup with the rate of convergence towards the equilibrium described by (2) and (3).

The following theorem summarizes the results of Lasota, Szarek, Komorowski, Peszat, Slaczka, Kapica, Bessaih (see Semigroup Forum 2014).

Theorem 22. *Suppose that a semigroup satisfies the e–property. Then, the following are true:*

(i) *any two distinct invariant and ergodic measures for the semigroup have disjoint supports.*
(ii) *if, in addition, the semigroup is averagely bounded in probability and concentrating at some point then it is asymptotically stable.*

15. Stochastic integration

15.1. **With respect to a square integrable Lévy martingale.** In this and next sections U, H, and V are real separable Hilbert spaces. We denote by $L(U, H)$ the space of all bounded linear operators form U into H, and by $L_{(HS)}(U, H)$ its subspace of Hilbert–Schmidt operators. Recall that $\alpha \in L(U, H)$ belongs to $L_{(HS)}(U, H)$ if

$$\|\alpha\|^2_{L_{(HS)}(U,H)} := \sum_{k=1}^{\infty} |\alpha e_k|^2_H < \infty$$

for any, or equivalently for some orthonormal basis (e_k) of U.

Assume that L is a square integrable Lévy process (large jumps removed) taking values U. Then

$$M(t) = L(t) - t\mathbb{E}\, L(1), \qquad t \geq 0,$$

is a square integrable martingale. Let Q be the covariance operator of $L(1)$. Let

$$\psi = \sum_k \alpha_k \chi_{(t_k, t_{k+1}]}$$

be a simple function; α_k are $L(U,H)$-valued random variables, $\alpha_k(u)$ is $\mathcal{F}_{t_k}$ measurable for any $u \in U$. We define

$$\int_0^t \psi(s)\mathrm{d}M(s) := \sum_k \alpha_k \left(M(t \wedge t_{k+1}) - M(t \wedge t_k) \right).$$

Then after simple calculation we have

$$\mathbb{E} \left| \int_0^t \psi(s)\mathrm{d}M(s) \right|_H^2 = \int_0^t \mathbb{E} \| \psi(s) Q^{1/2} \|^2_{L_{(HS)}(U,H)} \mathrm{d}s.$$

Let $\mathcal{H} = Q^{1/2}(U)$ be the image of $Q^{1/2}$. On $\mathcal{H}$ we consider the scalar product inherited from U by $Q^{1/2}$. We call $\mathcal{H}$ the *Reproducing Kernel Hilbert Space* of L. We extend the integral to the completion of the class of simple function with respect to the family of semi-norms

$$|||\psi|||_T := \sqrt{\int_0^T \mathbb{E} \| \psi(s) Q^{1/2} \|^2_{L_{(HS)}(U,H)} \mathrm{d}s}, \qquad T > 0.$$

Thus the space of integrands is the space of all predictable square integrable random processes

$$\psi \colon \Omega \times [0,\infty) \mapsto L_{(HS)}(\mathcal{H}, H).$$

satisfying $|||\psi|||_T < \infty$ for any $T > 0$.

The isometry formula holds

$$\mathbb{E} \left| \int_0^t \psi(s)\mathrm{d}M(s) \right|_H^2 = \int_0^t \mathbb{E} \| \psi \|^2_{L_{(HS)}(\mathcal{H},H)} \mathrm{d}s.$$

15.2. **Existence and uniqueness to SPDE.** Assume that a Hilbert space H is continuously imbedded into a Hilbert space V. Consider SPDE

$$\mathrm{d}X = (AX + F(X))\,\mathrm{d}t + B(X)\mathrm{d}M, \qquad X(0) = x \in H, \tag{10}$$

where $(A, D(A))$ generates a C_0-semigroup S on H, $F \colon H \mapsto V$, and for any $x \in H$, $B(x)$ is a linear operator (not necessarily bounded) from $\mathcal{H}$ to H. We have the following simple existence result.

Theorem 23. *Assume that for any $t > 0$, the semigroup $S(t)$ has a (unique) extension to a bounded linear map from V into H, and that*

$$|S(t)(F(x) - F(y))|_H \le b(t)|x - y|_H,$$
$$\|S(t)\,(B(x) - B(y))\,\|_{L_{(HS)}(\mathcal{H},H)} \le a(t)|x - y|_H$$

and

$$|S(t)F(x)|_H \le b(t)\,(1 + |x|_H)\,,$$
$$\|S(t)B(x)\|_{L_{(HS)}(\mathcal{H},H)} \le a(t)\,(1 + |x|_H)\,,$$

where

$$\int_0^T \big(b(t) + a^2(t)\big)\,\mathrm{d}t < \infty, \qquad \forall\, T > 0.$$

Then there is a unique adapted process u such

$$\sup_{0\le t\le T} \mathbb{E}\,|X(t)|_H^2 < \infty, \qquad \forall\, T > 0,$$

and for all $t \ge 0$,

$$X(t) = S(t)x + \int_0^t S(t-s)F(X(s))\mathrm{d}s + \int_0^t S(t-s)B(X(s))\mathrm{d}M(s), \qquad \mathbb{P}\text{-}a.s.$$

Sketch of the proof. Let us fix a finite time horizon $T > 0$. Let $\mathcal{X}_T$ be the space of all square-integrable adapted processes $X\colon \Omega \times [0,T] \mapsto H$ such

$$[0,T] \ni t \to \mathbb{E}\,|X(t)|_H^2 \in \mathbb{R}$$

is continuous. On $\mathcal{X}_T$ consider the family of equivalent norms

$$\|X\|_\beta := \sup_{0\le t\le T} \mathrm{e}^{-\beta t}\sqrt{\mathbb{E}|X(t)|_H^2}, \qquad \beta > 0.$$

Then $\mathcal{X}_T$ equipped with $\|\cdot\|_\beta$ is a Banach space. Consider the mapping

$$\Psi(X)(t) = S(t)u_0 + \int_0^t S(t-s)F(X(s))\mathrm{d}s + \int_0^t S(t-s)B(X(s))\mathrm{d}M(s).$$

Then $\Psi\colon \mathcal{X}_T \mapsto \mathcal{X}_T$. Moreover, for β large enough Ψ is a contraction. Thus the desired conclusion follows from the Banach fixed point theorem. □

15.3. **Typical example.** As an example consider stochastic heat equation

$$\mathrm{d}X = (\Delta X + f(X))\,\mathrm{d}t + b(X)\mathrm{d}M, \qquad u(0) = u_0,$$

considered on a bounded region $\mathcal{O} \subset \mathbb{R}^d$ with 0-Dirichlet boundary conditions. Assume that the RKHS $\mathcal{H}$ of M is a subset of $H = L^2(\mathcal{O})$, and $f, b\colon \mathbb{R} \mapsto \mathbb{R}$. Then we are in the framework of equation (10), with A being the Laplace operator on $H = L^2(\mathcal{O})$ with the Dirichlet boundary conditions, and F and B of the Nemytskii type operators

$$F(\psi)(x) = f(\psi(x)), \quad B(\psi)[\phi](x) = b(\psi(x))\phi(x),$$

for $\psi \in L^2(\mathcal{O})$, $\phi \in \mathcal{H}$, $x \in \mathcal{O}$.

Note the if $f\colon \mathbb{R} \mapsto \mathbb{R}$ is Lipschitz then the corresponding $F\colon L^2(\mathcal{O}) \mapsto L^2(\mathcal{O})$ is Lipchitz as well. As far as B is concerned, then $B(u)$ is a bounded linear operator from $L^2(\mathcal{O})$ to $L^2(\mathcal{O})$ if and only if $b(u) \in L^\infty(\mathcal{O})$. Therefore B is an $L(L^2(\mathcal{O}), L^2(\mathcal{O}))$-valued if and only if b is bounded. Assume now that b is bounded. Note that

$$B\colon L^2(\mathcal{O}) \mapsto L(L^2(\mathcal{O}), L^2(\mathcal{O}))$$

is continuous if and only if b is constant. For

$$\begin{aligned}\|B(u) - B(v)\|^2_{L(L^2(\mathcal{O}),L^2(\mathcal{O}))} &= \sup_{|\psi|_{L^2(\mathcal{O})}\le 1} \int_{\mathcal{O}} \left(b(u(x)) - b(v(x))\right)^2 \psi^2(x)\mathrm{d}x \\ &= \|b(u) - b(v)\|^2_\infty.\end{aligned}$$

Let $a_1 \neq a_2 \in \mathbb{R}$ and let $\mathcal{O}_\varepsilon$ be a subset of $\mathcal{O}$ of Lebesgue measure ε. Take $u_\varepsilon(x) = a_1\chi_{\mathcal{O}_\varepsilon}(x)$ and $v_\varepsilon(x) = a_2\chi_{\mathcal{O}_\varepsilon}(x)$ for $x \in \mathcal{O}$. Then $\|b(u_\varepsilon) - b(v_\varepsilon)\|_\infty = |b(a_1) - b(a_2)|$. On the other hand

$$|u_\varepsilon - v_\varepsilon|_{L^2(\mathcal{O})} = |a_1 - a_2|\sqrt{\varepsilon}.$$

Note that $B(u)$ is Hilbert–Schmidt if and only if $b \equiv 0$.

Let G be the Green kernel. Then

$$\begin{aligned}&\|S(t)\left(B(u) - B(v)\right)\|_{L(L^2(\mathcal{O}),L^2(\mathcal{O}))} \\ &= \sup_{|\psi\|_{L^2(\mathcal{O})}\le 1} \int_{\mathcal{O}} \psi(x) S(t)\left(B(u) - B(v)\right)(x)\mathrm{d}x \\ &= \left|S(t)\left(B(u) - B(v)\right)\right|_{L^\infty(\mathcal{O})} \\ &= \sup_{x\in\mathcal{O}} \int_{\mathcal{O}} G(t,x,y)|b(u(y)) - b(v(y))|\mathrm{d}y \\ &\le |b(u) - b(v)|_{L^2(\mathcal{O})} \sup_{x\in\mathcal{O}} \left(\int_{\mathcal{O}} G^2(t,x,y)\mathrm{d}y\right)^{1/2}.\end{aligned}$$

Recall that d is the dimension of the domain $\mathcal{O}$. Taking into account the Arronson estimates for the Green kernel, see Arronson [2], Eidelman [8], Solonnikov [27] and [28],

$$G(t,x,y) \le C_1 t^{-d/2} \exp\left\{-C_2\frac{|x-y|^2}{t}\right\}$$

we arrive at the estimate

$$\sup_{x\in\mathcal{O}}\left(\int_{\mathcal{O}} G^2(t,x,y)\mathrm{d}y\right)^{1/2} \le C_3 t^{-d/4}.$$

On the other hand

$$\begin{aligned}
&\|S(t)\left(B(u)-B(v)\right)\|^2_{L_{(HS)}(L^2(\mathcal{O}),L^2(\mathcal{O}))}\\
&\quad = \int_{\mathcal{O}}\int_{\mathcal{O}} G^2(t,x,y)|b(u(y))-b(v(y))|^2\mathrm{d}y\mathrm{d}x\\
&\quad \le |b(u)-b(v)|^2_{L^2(\mathcal{O})}\sup_{y\in\mathcal{O}}\int_{\mathcal{O}} G^2(t,x,y)\mathrm{d}x \le C_3 t^{-d/2}\left|b(u)-b(v)\right|^2_{L^2(\mathcal{O})}.
\end{aligned}$$

Therefore, if $d=1$, then the existence of the solution follows from Theorem 23.

For $d=1$ on can also use the following arguments, let (e_k) be the orthonormal basis of $L^2(\mathcal{O})$ of eigenvectors of Δ and let $(-\lambda_k)$ be the corresponding sequence of eigenvalues. Then

$$\begin{aligned}
&\|S(t)\left(B(u)-B(v)\right)\|^2_{L_{(HS)}(L^2(\mathcal{O}),L^2(\mathcal{O}))}\\
&= \sum_{k,j}\langle S(t)\left((b(u)-b(v))e_k\right), e_j\rangle^2_{L^2(\mathcal{O})} = \sum_{k,j}\langle (b(u)-b(v))e_k, S(t)e_j\rangle^2_{L^2(\mathcal{O})}\\
&= \sum_{k,j}\mathrm{e}^{-2\lambda_j t}\langle (b(u)-b(v))e_k, e_j\rangle^2_{L^2(\mathcal{O})} = \sum_j \mathrm{e}^{-2\lambda_j t}\left|(b(u)-b(v))e_j\right|^2_{L^2(\mathcal{O})}\\
&\le \sum_j \mathrm{e}^{-2\lambda_j t}\left|b(u)-b(v)\right|^2_{L^2(\mathcal{O})}.
\end{aligned}$$

Since λ_j is of order j^2, there is a constant C such that $\sum_j \mathrm{e}^{-2\lambda_j t} \le Ct^{-1/2}$.

16. Predictability

It is know that if we integrate with respect to a Wiener process, then it is enough to assume that the integrand is measurable, adapted and locally square integrable with respect to time with probability 1. The following examples shows also that in general the integrand should be predictable or the stochastic integration differs from the Lebesgue–Stieltjes integral in the case of the integration with respect to a process with bounded variation.

Example 1. Let Π be a Poisson process with intensity λ. Let τ be the moment of the first jump of Π. Then $\chi_{[0,\tau)}$ is a measurable adapted process. We note that $\chi_{[0,\tau)}$ is not predictable. Clearly a predictable process is $\chi_{[0,\tau]}$. Note that $\chi_{[0,\tau]}$ is a modification of $\chi_{[0,\tau)}$.

Let $\widehat{\Pi}$ be the compensated process. Then, if we treat the integral as the Lebesgue–Stieltjes integral with respect to a process $\widehat{\Pi}$ with bounded variation, then

$$X(t) := \int_0^t \chi_{[0,\tau)}(s)\mathrm{d}\widehat{\Pi}(s) = -\lambda t \wedge \tau + \int_0^t \chi_{[0,\tau)}(s)\mathrm{d}\Pi(s) = -\lambda t \wedge \tau.$$

Note that X is not a martingale, nor a local martingale. It has decreasing trajectories. On the other hand, the process

$$Y(t) := \int_0^t \chi_{[0,\tau]}(s)\mathrm{d}\widehat{\Pi}(s) = -\lambda t \wedge \tau + \int_0^t \chi_{[0,\tau]}(s)\mathrm{d}\Pi(s) = -\lambda t \wedge \tau + \chi_{\{t\geq\tau\}}$$

is a martingale.

Obviously if X is càdlàg and adapted, then $X(t-)$, $t \geq 0$, is predictable. Unfortunately, in important cases X does not have a càdlàg modification. It can be mean square continuous, that is

$$\lim_{s\uparrow t} \mathbb{E}\, |X(t) - X(s)|_H^2 = 0, \qquad \forall\, t \geq 0.$$

Then there is its predictable modification due to the following general result (see Gikhmann and Skorokhod [9] or Peszat and Zabczyk [22], Prop. 3.21).

Theorem 24. *Any measurable stochastically continuous adapted process has a predictable modification.*

The problem of predictability of integrands is treated in more details by Albeverio, Mandrekar, and Rüdiger [1] and by Mandrekar and Rüdiger [15], [16], and [17].

17. Poisson random measures

Let $(E, \mathcal{E})$ be a measurable space. Let π be the Poisson random measure on $[0,\infty) \times E$ with the intensity measure $\mathrm{d}t\mu(\mathrm{d}z)$, and let $\widehat{\pi}(\mathrm{d}t, \mathrm{d}\xi) := \pi(\mathrm{d}t, \mathrm{d}\xi) - \mu(\mathrm{d}\xi)\mathrm{d}t$ be the compensated measure. We would like to integrate with respect to π a random field $X(t,\xi)$, $t \geq 0$, $\xi \in E$. Here $X(t,\xi)$ can be real valued or taking values in a Banach space V. Define the filtration

$$\mathcal{F}_t := \sigma\left(\pi([0,s] \times A) \colon 0 \leq s \leq t,\ A \in \mathcal{E}\right), \qquad t \geq 0.$$

In the first step we integrate *simple fields*; that is the fields of the form

$$X = \sum_{j=1}^{K} X_j \chi_{(t_j, t_{j+1}]} \chi_{A_j},$$

where $K \in \mathbb{N}$, $\mu(A_j) < \infty$, X_j are bounded and X_j is $\mathcal{F}_{t_j}$-measurable. Namely we write

$$I_t^{\pi}(X) := \int_0^t \int_E X(s,\xi)\pi(\mathrm{d}s, \mathrm{d}\xi) = \sum_{j=1}^{K} X_j \pi\left((t_j \wedge t, t_{j+1} \wedge t] \times A_j\right).$$

In the same way we define $I_t^{\widehat{\pi}}(X)$. Observe, that in the sum on above, X_j does not depend on the random variable $\pi\left((t_j \wedge t, t_{j+1} \wedge t] \times A_j\right)$ having the Poisson distribution with intensity $\mu(A_j)\left(t \wedge t_{j+1} - t \wedge t_j\right)$. Therefore

$$\begin{aligned} \mathbb{E} I_t^{\pi}(X) &= \sum_{j=1}^{K} \mathbb{E}\, X_j \mu(A_j)\left(t \wedge t_{j+1} - t \wedge t_j\right) \\ &= \mathbb{E} \int_0^t \int_E X(s,\xi)\mathrm{d}s\mu(\mathrm{d}\xi). \end{aligned}$$

Next since each X_j is bounded $I_t^{\pi}(X)$ has all moments finite. Obviously

$$\mathbb{E}\left|I_t^{\pi}(X)\right|_V \leq \mathbb{E} I_t^{\pi}\left(|X|_V\right).$$

Assume now that the integrand is real-valued.

Lemma 3. *For any simple real-valued field X, the process $I_t^{\widehat{\pi}}(X)$, $t \geq 0$, is a square integrable real valued martingale with the quadratic variation*

$$\left[I^{\widehat{\pi}}(X), I^{\widehat{\pi}}(X)\right]_t = I_t^{\pi}(X^2), \qquad t \geq 0.$$

We have now the following result of Saint Loubert Bié. It plays a fundamental role in the L^p-theory of SPDEs with Lévy noise, see e.g. [22] and the original paper by Saint Loubert Bié [26].

Lemma 4. *Let $p \in [1,2]$. Then there is a constant C_p such that for arbitrary simple field X and $T > 0$,*

$$\mathbb{E} \sup_{0 \leq t \leq T} \left|I_t^{\widehat{\pi}}(X)\right|^p \leq C_p \mathbb{E} \int_0^T \int_E |X(t,\xi)|^p \,\mathrm{d}t\mu(\mathrm{d}\xi).$$

Proof. By the Burkholder–Davis–Gundy inequality

$$\mathbb{E} \sup_{0 \leq t \leq T} \left|I_t^{\widehat{\pi}}(X)\right|^p \leq C_p \mathbb{E}\left[I^{\widehat{\pi}}(X), I^{\widehat{\pi}}(X)\right]_T^{p/2} = C_p \mathbb{E}\left(I_T^{\pi}(X^2)\right)^{p/2}.$$

Now

$$I_T^{\pi}(X^2) = \sum_{j=1}^{K} X_j^2 \pi\left((t_j \wedge t, t_{j+1} \wedge t] \times A_j\right).$$

But $\pi\left((t_j \wedge t, t_{j+1} \wedge t] \times A_j\right)$ are non-negative integers! Therefore since $p/2 \le 1$,

$$\left(\sum_{j=1}^{K} X_j^2 \pi\left((t_j \wedge t, t_{j+1} \wedge t] \times A_j\right)\right)^{p/2} \le \sum_{j=1}^{K} |X_j|^p \pi\left((t_j \wedge t, t_{j+1} \wedge t] \times A_j\right).$$

Hence $(I_T^{\pi}(X^2))^{p/2} \le I_T^{\pi}(|X|^p)$, and consequently

$$\mathbb{E}\left(I_T^{\pi}(X^2)\right)^{p/2} \le \mathbb{E} I_T^{\pi}(|X|^p) = \mathbb{E}\int_0^t \int_E |X(s,\xi)|^p \mathrm{d}s\mu(\mathrm{d}\xi).$$

□

Having defined the stochastic integral of a simple field we would like to extend it to a more general class of random fields. Namely, given $T < \infty$, we denote by $\mathcal{P}_{[0,T]}$ the σ-field of predictable sets in $[0,T] \times \Omega$. Define

$$\mathcal{L}^p_{\mu,T} := L^p\left([0,T] \times \Omega \times E, \mathcal{P}_{[0,T]} \otimes \mathcal{E}, \mathrm{d}t\mathbb{P}\mu\right).$$

The space $\mathcal{L}^p_{\mu,T}$ is equipped with the norm

$$\|X\|_{\mathcal{L}^p_{\mu,T}} = \left(\int_0^T \int_E \mathbb{E}\,|X(s,\xi)|^p \mathrm{d}s\mu(\mathrm{d}\xi)\right)^{1/p}.$$

The simple fields are dense in $\mathcal{L}^p_{\mu,T}$, yielding the following consequence of Lemmas 3 and 4.

Theorem 25. (1) *For $p \in [1,2]$ and $t \in [0,T]$ there is a unique extension of the stochastic integral $I_t^{\widehat{\pi}}$ to a bounded linear operator, denoted also by $I_t^{\widehat{\pi}}$, from $\mathcal{L}^p_{\mu,t}$ into $L^p(\Omega, \mathcal{F}_t, \mathbb{P})$.*

(2) *There is a unique extension of the mapping $\mathcal{L}_0 \ni X \mapsto I_t^{\pi}(X) \in L^1(\Omega, \mathcal{F}_t, \mathbb{P})$ to a bounded linear operator from $\mathcal{L}^1_{\mu,t}$ into $L^1(\Omega, \mathcal{F}_t, \mathbb{P})$. The value of this operator at X is given by*

$$\int_0^t \int_E X(s,\xi)\pi(\mathrm{d}s, \mathrm{d}\xi),$$

or by $I_t^{\pi}(X)$.

(3) *For $X \in \mathcal{L}^1_{\mu,T}$ and $0 \le s \le t \le T$,*

$$\mathbb{E}\left|I_t^{\widehat{\pi}}(X) - I_s^{\widehat{\pi}}(X)\right| \le c_1 \int_s^t \int_E \mathbb{E}\,|X(r,\xi)| \mathrm{d}r\mu(\mathrm{d}\xi)$$

and

$$\mathbb{E}\,|I^{\pi}_t(X) - I^{\pi}_s(X)| \le \int_s^t \int_E \mathbb{E}\,|X(r,\xi)|\mathrm{d}r\mu(\mathrm{d}\xi).$$

Hence the processes $I^{\widehat{\pi}}(X)$ and $I^{\pi}(X)$ admit predictable modifications.

(4) *If $X \in \mathcal{L}^2_{\mu,T}$ then $\big(I^{\widehat{\pi}}_t(X),\, t\in[0,T]\big)$ is a square integrable martingale. Moreover, for $X, Y \in \mathcal{L}^2_{\mu,T}$ and $t\in[0,T]$, $[I^{\widehat{\pi}}(X), I^{\widehat{\pi}}(Y)]_t = I^{\pi}_t(XY)$.*

As for the case of simple fields, we write $\int_0^t\int_E X(s,\xi)\widehat{\pi}(\mathrm{d}s,\mathrm{d}\xi)$ instead of $I^{\widehat{\pi}}_t(X)$.

17.1. **Example of equations.** Consider the following heat equation on a bounded region $\mathcal{O}\subset\mathbb{R}^d$;

$$\mathrm{d}X(t,\xi) = \Delta X(t,\xi)\mathrm{d}t + \int_S b(X(t,\xi),\sigma)\widehat{\pi}(\mathrm{d}t,\mathrm{d}\xi,\mathrm{d}\sigma), \qquad u(0,x)=u_0(x),$$

with homogeneous Dirichlet or Neumann boundary conditions. In the equation π is a Poisson random measure on $[0,\infty)\times\mathcal{O}\times S$, with intensity measure $\mathrm{d}t\mathrm{d}x\nu(\mathrm{d}\sigma)$, σ is a measure on a measurable space $(S,\mathcal{S})$. Equations of this type were investigated in e.g. [19, 20, 26, 22]. The mild formulation of our problem is

$$X(t,\xi) = \int_{\mathcal{O}} G(t,\xi,y)X_0(y)\mathrm{d}y + \int_0^t\int_{\mathcal{O}}\int_S G(t-s,\xi,y)b(X(s,y),\sigma)\widehat{\pi}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}\sigma).$$

A much simple problem is when the random Poisson measure does not depend on space variable ξ;

$$\mathrm{d}X(t,\xi) = \Delta X(t,\xi)\mathrm{d}t + \int_S b(X(t,\xi),\sigma)\widehat{\pi}(\mathrm{d}t,\mathrm{d}\sigma), \qquad X(0,\xi)=X_0(\xi).$$

Its mild form is

$$X(t,\xi) = \int_{\mathcal{O}} G(t,\xi,y)X_0(y)\mathrm{d}y + \int_0^t\int_{\mathcal{O}}\int_S G(t-s,\xi,y)b(X(s,y),\sigma)\widehat{\pi}(\mathrm{d}s,\mathrm{d}\sigma).$$

Then, roughly speaking

$$\int_0^t\int_{\mathcal{O}}\int_S G(t-s,\cdot,y)b(X(s,y),\sigma)\widehat{\pi}(\mathrm{d}s,\mathrm{d}\sigma) = \int_0^t S(t-s)\mathrm{d}M(s),$$

where

$$M(s) = \int_0^t \int_S b(u(s),\sigma)\widehat{\pi}(\mathrm{d}s,\mathrm{d}\sigma), \qquad t \geq 0,$$

is a martingale. It turns out that in the first case the solution does not have a càdlàg modification in $L^2(\mathcal{O})$ whereas in the second case it does, see Section 19.

18. Impulsive white noise

Let $\mathcal{O}$ be an open not necessarily bounded domain in $\mathbb{R}^d$ (possibly $\mathcal{O} = \mathbb{R}^d$). Let π be a Poisson random measure on $[0,\infty) \times \mathcal{O} \times \mathbb{R}$ with intensity of jump measure $\mathrm{d}t\mathrm{d}x\nu(\mathrm{d}\sigma)$. Assume that $\int_{\mathbb{R}} \sigma^2 \wedge 1\nu(\mathrm{d}\sigma) < \infty$. Consider the distributions-valued process

$$Z(t) = \int_0^t \int_{\{|\sigma|<R\}} \sigma\widehat{\pi}(\mathrm{d}s\mathrm{d}x\mathrm{d}\sigma) + \int_0^t \int_{\{|\sigma|\geq R\}} \sigma\pi(\mathrm{d}s\mathrm{d}x\mathrm{d}\sigma).$$

Taking into account the representation

$$\pi(\mathrm{d}s\mathrm{d}x\mathrm{d}\sigma) = \sum \delta_{\tau_k,x_k,\sigma_k},$$

we obtain the following a bit formal expression for Z;

$$Z(t) = \left[\sum_{|\sigma_k|<R,\tau_k\leq t} \sigma_k\delta_{\tau_k,x_k} - t\int_{|\sigma|<R} \sigma\mathrm{d}x\nu(\mathrm{d}\sigma)\right] + \sum_{|\sigma_k|\geq R,T_k\leq t} \sigma_k\delta_{\tau_k,x_k}.$$

Intuitively, at random points (τ_k, x_k) at time and space Z gives random impulses of random size σ_k.

Remark 4. One can show that

$$M(t) = \int_0^t \int_{\{|\sigma|<R\}} \sigma\widehat{\pi}(\mathrm{d}s\mathrm{d}x\mathrm{d}\sigma)$$

is a square integrable martingale in a sufficiently large space, and that its RKHS equals

$$\mathcal{H} = L^2(\mathcal{O},\mathcal{B}(\mathcal{O}),a_R\mathrm{d}x), \qquad a_R := \int_{\{|\sigma|<R\}} \sigma^2\nu(\mathrm{d}\sigma).$$

Thus, in particular, M takes values in any Hilbert space V such that the embedding $\mathcal{H} \hookrightarrow V$ is Hilbert–Schmidt.

Remark 5. The jump measure μ of Z is the image of the measure $\mathrm{d}x\nu(\mathrm{d}\sigma)$ under the transformation $\mathcal{O} \times \mathbb{R} \ni (x,\sigma) \mapsto \sigma\delta_x \in \mathcal{D}(\mathcal{O})$, where $\mathcal{D}(\mathcal{O})$ is the space of distribution on $\mathcal{O}$.

Therefore our definition is the following.

Definition 12. *Impulsive cylindrical (or white) noise with intensity of jumps measure* $\mathrm{d}x\nu(\mathrm{d}\sigma)$ is the Lévy process on the space of distributions with the Lévy measure μ being the image of $\mathrm{d}x\nu(\mathrm{d}\sigma)$ under the transformation $(x,\sigma) \mapsto \sigma\delta_x$.

Remark 6. Impulsive cylindrical process L takes values in a Hilbert space U provided that $\int_U |u|_U^2 \wedge 1\mu(\mathrm{d}u) < \infty$. Let $U = H^{-\alpha}$ be the Sobolev space of order $-\alpha$ with an $\alpha > d/2$. Then, by Sobolev embedding, $C := \sup_{x\in\mathcal{O}} |\delta_x|_{H^{-\alpha}} < \infty$. Therefore

$$\begin{aligned}\int_{H^{-\alpha}} |u|^2_{H^{-\alpha}} \wedge 1\mu(\mathrm{d}u) &= \int_{\mathcal{O}}\int_{\mathbb{R}} \sigma^2 |\delta_x|_{H^{-\alpha}} \mathrm{d}x\nu(\mathrm{d}\sigma)\\ &\le C\int_{\mathbb{R}} \sigma^2 \wedge 1\nu(\mathrm{d}\sigma) < \infty.\end{aligned}$$

Consequently, L takes values in $H^{-\alpha}$. For more details on SPDEs driven by impulsive cylindrical process we refer the reader to Mueller [19], Mytnik [20], or [22].

19. Regularity of stochastic convolution

We start this section with results on the lack of a càdlàg modification for SPDEs driven by a process whose jump measure is not supported on the state space. Then we present different tools useful for study regularity of stochastic convolutions.

19.1. **Lack of càdlàg modification.** As the following example shows in some cases the solution to linear stochastic evolution equation does not have a càdlàg modification.

Example 2. Let U and H be Hilbert spaces such that

(i) H is densely embedded into U.

(ii) One has

$$\int_0^T \|S(s)\|^2_{L_{(HS)}(H,H)}\, ds < \infty, \qquad \forall T > 0.$$

(iii) For any $t > 0$, $S(t)$ has a continuous extension to an operator $S(t) \in L(U,H)$.

(iv) For any $u \in U \setminus H$, $\lim_{t\downarrow 0} |S(t)u|_H = \infty$.

Let Z be a square integrable mean zero random variable in U with RKHS H, and let L be a compound Poisson process with Lévy measure ν which is the distribution of Z. Then

$$X(t) = \int_0^t S(t-s)\mathrm{d}L(s) = \sum_{\tau_n < t} S(t-\tau_n)Z_n,$$

where τ_n are the jump times of L and Z_j are independent copies of Z. Then, by (ii), $\sup_{t\leq T}\mathbb{E}\,|X(t)|_H^2<\infty$ but

$$\lim_{t\downarrow\tau_n}|X(t)|_H=\lim_{t\downarrow\tau_n}|S(t-\tau_n)Z_n|_H=\infty,$$

since Z_n take values in $U\setminus H$.

Explicitly, take $H=L^2(0,1)$, $U=W_0^{-1,2}(0,1)$, S the heat semigroup generated by the Laplace operator with Dirichlet boundary conditions, and $Z=\eta\delta_\xi$, where $\xi\in(0,1)$, and η is a mean zero random variable.

The following have ben proven by Brzezniak and Zabczyk [3], and Peszat and Zabczyk [22].

Theorem 26. *If the jump measure of the noise is not supported on E then the stochastic convolution does not have càdlàg trajectories in E.*

19.2. **Factorisation.** Stochastic integral with respect to the square integrable martingale as a square integrable martingale has a càdlàg modification. This is not always true for stochastic convolution processes

$$X(t):=\int_0^t S(t-s)\Psi(s)\mathrm{d}M(s),\qquad t\geq 0,$$

where the integrand depends on t.

One way to show the continuity of its trajectories is to use the so-called *Da Prato–Kwapień–Zabczyk factorisation*, see the original paper by Da Prato, Kwapień, and Zabczyk [6], or [22],

$$X(t)=\Gamma(1)I_\alpha(X_\alpha)(t),$$

where

$$X_\alpha(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}S(t-s)\psi(s)\mathrm{d}M(s),\qquad t\geq 0,$$

and I_α is the *fractional derivative* operator given by

$$I_\alpha\psi(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}S(t-s)\psi(s)\mathrm{d}s,$$

and Γ is the Euler Γ-function. It is easy to show that

$$I_\alpha\in L(L^p(0,T;H),C([0,T];H))$$

provided that $1/q<\alpha<1$.

For the Wiener integral it is usually not hard to show that X_α has trajectories in $L^q(0,T;H)$ with some $1/q<\alpha<1$. Therefore the continuity of trajectories of X follows. However, for discontinuous Lévy process, X_α does not have trajectories in $L^q(0,T;H)$ with any $1/q<\alpha<1$. This can been seen as a consequence of the Bichteler–Jacod estimate (see e.g. [18]).

19.3. **Kotelenez regularity result.** Kotelenez [14] proved the regularity of stochastic convolution

$$\int_0^t S(t-s)\mathrm{d}M(s), \qquad t \geq 0,$$

driven by an arbitrary square integrable martingale in H for a generalized contraction semigroup S. Recall that for any C_0-semigroup S there are constants $\beta > 0$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\|_{L(H,H)} \leq \beta \mathrm{e}^{\omega t}, \qquad t \geq 0. \tag{11}$$

If (11) holds with $\beta = 1$, then S is a *generalized contraction semigroup.* If moreover, $\omega \leq 0$, then S is a *contraction semigroup.*

We outline here the proof of Kotelenez result due to Hausenblas and Seidler [11]. Their method is based on the Nagy dilation theorem.

Theorem 27. ***(Nagy)*** *If S is a C_0-semigroup of contractions on H, then there is a Hilbert space $\tilde{H}$ containing H and a unitary group R on $\tilde{H}$ such that $S = PR$, where $P \in L(\tilde{H}, H)$ is a projection.*

Proof of the Kotelenez regularity result. If S is a generalized contraction semigroup then for ω large enough $\mathrm{e}^{-\omega t}S(t)$, $t \geq 0$, is a semigroup of contractions. Then by the Nagy theorem

$$\mathrm{e}^{-\omega t}S(t) = PR(t), \qquad t \geq 0,$$

where $P \in L(\tilde{H}, H)$ and R is a unitary C_0-group. Hence

$$\begin{aligned}\int_0^t S(t-s)\mathrm{d}M(s) &= \int_0^t \mathrm{e}^{\omega(t-s)}\mathrm{e}^{-\omega(t-s)}S(t-s)\mathrm{d}M(s) \\ &= \int_0^t \mathrm{e}^{\omega(t-s)}PR(t-s)\mathrm{d}M(s) \\ &= \mathrm{e}^{\omega t}PR(t)\int_0^t R(-s)\mathrm{d}M(s).\end{aligned}$$

Since

$$Y(t) := \int_0^t R(-s)\mathrm{d}M(s), \qquad t \geq 0,$$

is a square integrable martingale, Y has càdlàg trajectories in $\tilde{H}$ and consequently X has càdlàg trajectories in H as R is strongly continuous.

19.4. **Criterion for the absence of discontinuities of the second kind.** The following criterion of the Chentsov type (see [4]), follows from a certain more general result (see Gikhman and Skorokhod [9], Chapter 3). For its proof we refer the reader to Gikhman and Skorokhod [9], or [23].

Let $\xi = (\xi(t),\, t \in [0,T])$ be a separable process taking values in a metric space (U,ρ). We extent ξ on $\mathbb{R}$ putting $\xi(t) = \xi(0)$ for $t < 0$ and $\xi(t) = \xi(T)$ for $t \geq T$.

Theorem 28. *Assume that there are $p, r, K > 0$ such that for all $t \in [0,T]$ and $h > 0$,*

$$\mathbb{E}\left[\rho\left(\xi(t),\xi(t-h)\right)\rho\left(\xi(t),\xi(t+h)\right)\right]^p \leq K h^{1+r}. \tag{12}$$

Then with probability 1, ξ has no discontinuities of the second kind. Moreover, for any $1 \leq q < 2p$,

$$\mathbb{E} \sup_{t,s\in[0,T]} \left(\rho(\xi(t),\xi(s))\right)^q \leq (2G)^q \mathbb{E}\left(\rho(\xi(T),\xi(0))\right)^q + R, \tag{13}$$

where $0 < r' < r$,

$$G = \sum_{n=1}^{\infty} (T2^{-n})^{r'/(2p)} < \infty, \tag{14}$$

and $R := 1 + \frac{q}{2p-q} \frac{K(2G)^{2p} T^{1+r-r'}}{1-2^{r'-r}}$.

The criterion yields the following result (see [23]) on the existence of a càdlàg in H solution to linear equation with the noise taking values in a bigger space $U \hookleftarrow H$. For more specific examples we refer the reader to [23].

Theorem 29. *Let X be the solution to the following linear equation*

$$\mathrm{d}X = AX\mathrm{d}t + \mathrm{d}Z,$$

where A is the generator of an exponentially stable analytic semigroup S on a Hilbert space H and Z is a pure jump Lévy process taking values in a Hilbert space $U = H_{-\rho}$ for a certain $\rho < 1/2$. Assume that the Lévy measure ν of Z satisfies $\nu(H_{-\rho} \setminus H) = 0$ and that

$$\int_H \left(|z|_{-\rho}^2 + |z|_{\varepsilon}^4\right)\nu(\mathrm{d}z) < \infty$$

for a certain $\varepsilon > 0$. Then X has a càdlàg modification in H and

$$\mathbb{E} \sup_{0\leq t\leq T} |X(t)|_H^q < \infty, \qquad \forall\, T < \infty,\ \forall\, q \in [1,4).$$

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