Equilibrium with Anticipation in a Financial Market

Guillaume Lasserre[1]

LPMA Université Paris 7 and LFA CREST

The purpose of this paper is to study the existence of an equilibrium when a strategic agent anticipates a release of information at some future date. We use the notion of equilibrium introduced by Kyle (1985) and the notion of anticipation given by Baudoin (2002). We give sufficient conditions to construct an equilibrium and we establish an explicit equilibrium when the strategic agent is risk neutral.

Key Words: Weak Information, Equilibrium theory, Stochastic Control, Pricing in continuous time

1. Introduction

The main purpose of this paper is to construct a model of equilibrium in continuous time, where an agent anticipates a release of information at some future date. First, we give an appropriate framework using the results of the paper of Baudoin (2002). Secondly, we give sufficient conditions to build an equilibrium. Thirdly, we construct an explicit equilibrium when the strategic agent is risk neutral.
Many ways to study asymmetric information have been developed. Since the models of Grossman-Stiglitz (1980), Kyle (1985), Glosten-Milgrom (1985) and the development of the theory of enlargement of filtrations in the 80's, a large literature has tackled this subject. Most of the papers which have focused on the influence of private information on financial market are based on a strong assumption. They assume that an agent (or a class of agents) receives some precise information about the future, like the knowledge of the liquidative value of a risky asset at some future date. Unfortunately, this kind of information appears to be too singular to properly explain, on its own, the microstructure of financial markets. Actually, the strategic agent may know that there will be a public release of information, but without observing it when he trades. Thus, his natural behaviour is to anticipate this information.
Back (1992) and Cho (1997) have studied models of equilibrium with asymmetric information in continuous time. They assume that there exists an agent, called the insider, who perfectly knows the liquidative value of a risky asset at the end of the period of trade. They show the existence of an equilibrium which consists on a pricing rule for the risky asset and a strategy of the insider which maximizes his expected utility. Lasserre (2003) extends those models in a multivariate framework and he allows the insider to have precise information about the value of the global demand of the risky asset at the end of the period of trade.
Baudoin (2002) develops a notion of weak information or anticipation. A strategic agent anticipates the law of a random variable Y which will be realized at some future date T. Baudoin shows the existence of a *minimal probability* under which Y follows the law chosen by the agent. Moreover, he solves problems of portfolio optimization under this probability measure, for various usual utility functions, when the price process is exogeneously given. Baudoin and Nguyen Ngoc (2003) solve

[1]The author would like to thank his supervisor H. Pham and K. Back for useful comments and discussions.

problems of optimization with anticipation when the price is driven by a Brownian motion or a Levy process. They also extend the paper of Baudoin (2002) in modelling a weak information flow as the dynamic correction of the weak information. Our model, in continuous time, remains close to the ones exposed by Back (1992), Cho (1997) and Lasserre (2003). The main difference with the cited papers is the fact that the strategic agent does not have precise information on the future liquidative value of a risky asset. Actually, he makes an anticipation on its law. Moreover, we still allow him to influence the asset price through his demand in the risky asset. Hence, an equilibrium will be a price process fixed by the market maker and an optimal strategy, solution of a problem of portfolio optimization, for the strategic agent. From a mathematical point of view, the private information is not expressed anymore by a difference of filtrations between agents, but by different probabilities allocated to agents. It turns out that we are able to find sufficient conditions for the existence of an equilibrium using Hamilton Jacobi Bellman equation. These conditions are similar to the ones given in Lasserre (2003) excepted fro the strategy of the strategic agent. Then, we construct an equilibrium when the strategic agent is risk neutral and when his anticipation of the liquidative value of the risky asset is lognormal. Actually, we remark that this equilibrium is not unique which is a simple consequence of the fact that the strategic agent is risk neutral.
The plan of the paper is as follows. We expose the model in Section 2. We give sufficient conditions to build an equilibrium in Section 3. Section 4 focuses on the case of a risk neutral strategic agent for which we construct an explicit equilibrium. The proofs are in the Appendix.

2. The model

The model is close to the one exposed in Lasserre (2003). The market is made by one risk-free asset and one risky asset. We assume without loss of generality that the risk-free rate is zero. Moreover, the different assets can be indefinitely divisible and we assume that these assets are continuously traded. Finally, we assume that there is no constraint on the assets such as transaction costs.
We denote by T the announcement date of a public release of information that will affect the value of the asset, and by $t = 0$ the present date. At time T, the liquidative value of the risky asset will be given by the random variable V.

We consider a centralized market, where a class of agents being able to observe every trade, organizes the market, one usually calls them market makers. Actually, we can consider that there is only one reasonable market maker putting up all the prices with respect to the information he knows. We also assume that this market maker is risk neutral, this can be justified by the fact that he has a diversified portfolio and that the inside information is about idiosyncratic risk.
At the date $t = 0$, a unique agent, different from the market maker, anticipates the fact that V will follow a given law ν, we call him the strategic agent. He tries to use his anticipation to maximize his utility. The market maker also knows that there will be a release of information at T and that there is only one strategic agent. The strategic agent will have to hide his demand in order not to reveal his anticipation.

The third type of agents are the noise traders who trade for liquidity reasons. As in Lasserre (2003), their demand is given exogeneously and one agent represents all the class.

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space. For all agents, the value of the risky asset at time T is the random variable V. The demand in number of shares of the strategic agent (respectively the noise trader) is denoted by X (respec. Z). According to Baudoin (2002), we assume that the minimal probability $\mathbb{P}^\nu$ exists, where $\mathbb{P}^\nu$ is defined by

$$\text{For } A \in \mathcal{F}_T, \quad \mathbb{P}^\nu(A) = \int \mathbb{P}(A \mid V = v)\nu(dv)$$

The law of V under $\mathbb{P}^\nu$ is ν and the law of any events conditionally to V is the same under $\mathbb{P}$ or $\mathbb{P}^\nu$. To use most of the results of Baudoin (2002), let us assume that $\nu \ll \mathbb{P}_V$ and $\xi(V) = \frac{d\nu}{d\mathbb{P}_V}$ admits a continuously differentiable version with bounded partial derivative. For more details on the assumptions needed to have the existence of the minimal probability $\mathbb{P}^\nu$, we refer to Baudoin (2002). In fact, $\mathbb{P}^\nu$ appears to be the right probability measure associated to the strategic agent to manage problems of optimization with an anticipation. We recall, that under $\mathbb{P}^\nu$, the law of V is exactly ν.

We assume that the noise trader's demand has the following form :

$$\forall t \in [0, T] \quad dZ_t = \sigma dB_t \tag{2.1}$$

where $(B_t)_{t\in[0,T]}$ is a $(\mathbb{P}, \mathbb{F})$-Brownian motion and σ a positive constant.

Using the same kind of arguments as in Lasserre (2003), we can say that the information of the strategic agent is the filtration generated by the Brownian motion. Thanks to Baudoin (2002), we know that there exists a process (μ_t^ν) such that $(\widetilde{B}_t) = (B_t - \int_0^t \mu_s^\nu ds)$ is a $(\mathbb{P}^\nu, \mathbb{F})$-Brownian motion. More precisely, there exists a measurable function F such that $\mu_t^\nu = F(t, (B_s)_{0\le s\le t})$. Thus, we define the class of strategy we want to consider as follows.

Definition 2.1. Let $\mathcal{X}$ be the class of continuous semi martingales which can be written in the following form

$$\forall t \in [0, T], \quad X_t = \int_0^t \alpha_s ds + \int_0^t \beta_s d\widetilde{B}_s \tag{2.2}$$

where α and β are $(\mathbb{P}^\nu, \mathcal{F})$-adapted processes.

Definition 2.2. We call cumulative demand the process defined by :

$$\forall t \in [0, T] \quad Y_t = X_t + Z_t \tag{2.3}$$

Keeping the framework used by Kyle 1985, we say that the information of the market maker at time t is $\mathcal{F}_t^Y = \sigma(Y_s, 0 \le s \le t)$. The probability associated to the market maker is the natural one, that is the one of the market $\mathbb{P}$.

An equilibrium is given by a pricing rule defined by the market maker and an optimal strategy for the strategic agent. However, as the framework is not strictly identical to Lasserre (2003), we define the equilibrium we are looking for.

Definition 2.3. A pricing rule is a couple (H, λ) defined by

$$H : [0, T] \times \mathbb{R} \to \mathbb{R}$$

where H is of class $C^{1,2}([0, T] \times \mathbb{R})$, $\forall t \in [0, T] \quad y \to H(t, y)$ is a strictly increasing C^1-diffeomorphism and $\lambda \in C^\infty([0, T[, \mathbb{R}_+^*)$.

Moreover, we assume that $\mathbb{E}\left(\int_0^T \left(\frac{\partial H}{\partial y}(Z_s)\right)^2 ds\right) < +\infty$.

We denote by $\mathcal{H}$ the set of all couples (H, λ) having the previous properties.

Definition 2.4. An admissible price is a price process such that $S_t = H(t, \Psi_t)$ where Ψ_t is defined by :

$$\Psi_t = \int_0^t \lambda(s)\, dY_s \text{ and } (H, \lambda) \in \mathcal{H}. \tag{2.4}$$

Definition 2.5. We say that the admissible price vector $S = (S_t, t \in [0, T])$ is rational if

$$\forall t \in [0, T] \quad S_t = \mathbb{E}\left[V \mid \mathcal{F}_t^Y\right] \tag{2.5}$$

where the expectation is taken under the probability $\mathbb{P}$.

We denote by W_{T+} the final wealth of the strategic agent after the release of information V. It is a function of the agent's demand $X = (X_t)_{t\in[0,T]}$ and of the price $S = (S_t)_{t\in[0,T]}$. We assume that the strategic agent has a utility function $U(.)$ in the Von Neumann-Morgenstein sense. He wants to maximize the expectation of the utility of his final wealth $\mathbb{E}^\nu[U(W_{T+}(S, X))]$ knowing his anticipation. $\mathbb{E}^\nu(.)$ means that we take expectation under the probability $\mathbb{P}^\nu$.

Definition 2.6. Let $\mathcal{H}$ be a class of pricing rules. Given a pricing rule (H, λ), we say that a strategy X^* is H-optimal on the class $\mathcal{X}$ if :

$$\forall X \in \mathcal{X} \quad \mathbb{E}^\nu[U(W_{T^+}(H, X))] \leq \mathbb{E}^\nu[U(W_{T^+}(H, X^*))]$$

Definition 2.7. We say that (H^*, λ^*, X^*, F) is an equilibrium on the space $(\mathcal{H},\mathcal{X})$ if it satisfies :
(i) The market efficiency condition : $H^*(t, \Psi^*)$ is a rational price for a given strategy of the insider X^*, with $Y^* = X^* + Z$ and $\Psi_t^* = \int_0^t \lambda^*(s)\, dY_s^*$.
(ii) The insider's optimality condition : X^* is an H^*-optimal strategy for the given pricing rule (H^*, λ^*).
(iii) The Markovian anticipation consistency condition:

$$\forall t \in [0, T[, \quad \mu_t^\nu = F(t, \Psi_t^*) \tag{2.6}$$

where $F \in C^1([0, T[\times\mathbb{R})$.

Remark 2.8. Conditions (i) and (ii) already exist in Lasserre (2003). However, the drift of change of probability μ^ν only depends on the anticipation of the strategic agent and on the Brownian motion $(B_t)_t$. The kind of equilibrium we are looking for has to be explained only by the observation of the modified cumulative demand Ψ. Condition (iii) of the definition of the equilibrium corresponds to this fact. To sum up, we search for equilibria which are consistent with the fact that Ψ is the explanatory variable of our model.

3. Characterization of the equilibrium

We are going to use the dynamic programming and Hamilton-Jacobi-Bellman equation to find necessary conditions for the existence of an equilibrium. It turns out that the state variables for the strategic agent are the modified cumulative demand Ψ and his wealth $\widetilde{W}$. Using the argument of Lasserre (2003), we know that it is easier to deal with a process W which is continuous on $[0, T]$.

$$dW_t = \left(\alpha_t(V - H) - \lambda_t\beta_t(\sigma + \beta_t)\frac{\partial H}{\partial \psi}\right) dt + \beta_t(V - H)d\widetilde{B}_t \tag{3.1}$$

However, in our model, the strategic agent does not know the realization of the random variable V. In fact, we are going to consider two different problems.
Let us define the function j and J as follows.

$$\begin{array}{rcl} j(t,\psi,w) & = & \mathbb{E}^{\nu}(U(W_T) \mid \Psi_t = \psi, W_t = w) \\ J(t,\psi,w,v) & = & \mathbb{E}^{\nu}(U(W_T) \mid \Psi_t = \psi, W_t = w, V = v) \end{array}$$

We remark that we have the following relation

$$\forall (t,\psi,w) \in [0,T] \times \mathbb{R}^2, \quad j(t,\psi,w) = \mathbb{E}^{\nu}\left[J(t,\psi,w,V)\right] \tag{3.2}$$

Our goal is to find conditions on J and to report them on j. In fact, we can first show a result which was announced in Back (1992) and Lasserre (2003). We give the proof since it differs from the one of Lasserre (2003).

Proposition 3.1. *Let (H,λ) be a pricing rule and we assume that the value function J is smooth. Among the class $\mathcal{X}$, the semimartingales which have a martingale part cannot satisfy the insider optimality condition.*

Now, we give sufficient conditions to construct an equilibrium. This constitutes one of our main results.

Theorem 3.2. *Let $(H,\lambda) \in \mathcal{H}$, $X_t^* \in \mathcal{X}$, $F \in C^1([0,T]\times\mathbb{R})$ and $J \in C^{1,3,3}\left([0,T],\mathbb{R},\mathbb{R}\right)$ such that J is strictly increasing in w, $\mathbb{E}\left(\int_0^T \left(\frac{\partial J}{\partial w}\right)^2 (V-H)^2 ds\right) < +\infty$ and $X_t^* = \int_0^t \alpha_s^* ds$ satisfy the conditions :*
(i) $\lambda \frac{\partial J}{\partial \psi} + \frac{\partial J}{\partial w}(v-H) = 0$
(ii) $\frac{\partial J}{\partial t} + \lambda \frac{\partial J}{\partial \psi}\sigma \mu_t^{\nu} + \frac{1}{2}\lambda^2\sigma^2 \frac{\partial^2 J}{\partial \psi^2} = 0$
(iii) $J(T,\psi,w,v) \geq U(w) \quad \forall \psi, \forall w, \forall v$ and $J(T,\psi,w,v) = U(w)$ if $H(T,\psi) = v$.
(iv) $\alpha_t^* = \sigma \mu_t^{\nu} = \sigma F(t,\Psi_t^*)$ where $\Psi_t^* = \int_0^t \lambda(\alpha_s^* ds + dZ_s)$.
(v) $H(T,\Psi_T^*) = V$.
Then (H,λ,X^*,F) is an equilibrium.

Remark 3.3. We recall that this theorem gives sufficient conditions for an equilibrium. In particular, condition (v) seems to be very strong and not necessary. In the case of strong information as in Part A and B, this condition is really meaningful. An intuitive condition will be that the price at time T has to follow the law ν under $\mathbb{P}^{\nu}$. In the case of weak information, condition (v) can be seen as a strong efficiency of the market.

In order to prove this theorem, we need two intermediate results which proofs are in the Appendix. The first lemma gives a relationship between the Bellman equation and a partial differential equation satisfies by prices. The second lemma gives a relationship between the prices and the strategy of the strategic agent.

Lemma 3.4. *Let us assume that J, λ, F and H are smooth and satisfy the following system :*

$$\left\{ \begin{array}{rcll} \lambda \frac{\partial J}{\partial \psi} + \frac{\partial J}{\partial w}(v-H) & = & 0 & \quad (I) \\ \frac{\partial J}{\partial t} + \lambda \frac{\partial J}{\partial \psi}\sigma F(t,\psi) + \frac{1}{2}\lambda^2\sigma^2 \frac{\partial^2 J}{\partial \psi^2} & = & 0 & \quad (II) \end{array} \right.$$

then, (H,λ) is a solution of

$$\frac{\partial H}{\partial t} + \lambda \sigma F(t,\psi)\frac{\partial H}{\partial \psi} + \frac{1}{2}\lambda^2\sigma^2 \frac{\partial^2 H}{\partial \psi^2} = 0 \tag{3.3}$$

Lemma 3.5. *If J, λ, F and H are smooth and satisfy the system $(I)-(II)$, then the two following assertions are equivalent:*
(i) The admissible price process $S_t = H(t,\Psi_t)$ is a $(\mathbb{P},\mathcal{F}_t)$- local martingale.
(ii) $\forall t \in [0,T[$, $\alpha_t = \sigma F(t,\Psi_t)$.

Proof of Theorem 3.2: The sketch of the proof is exactly the same as in Lasserre (2003). If condition (i) and (ii) of the theorem are satisfied, then using Lemma 3.5 and condition (iv), we have that the price is a local martingale. Moreover, we know from condition (v) that $S_T = H(T,\Psi_T)$, hence, $(S_t)_{t\in[0,T]}$ is a rational price. It remains to show that X is H-optimal. We use exactly the same kind of arguments as in Lasserre (2003). We first consider a strategy which have the following form : $dX_t = \alpha_t dt + \beta_t d\widetilde{B}_t$. We apply Ito's formula to the function J

$$
\begin{aligned}
J\left(T,\Psi_T,W_T,v\right) &= J\left(t,\Psi_t,W_t,v\right) + \int_t^T \left(L^{\alpha,\beta}_{\Psi,W,V} J + \frac{\partial J}{\partial s}\right) ds \\
&\quad + \int_t^T \left(\left(\frac{\partial J}{\partial \psi}\right)\lambda\left(\beta+\sigma\right) + \frac{\partial J}{\partial w}\left(v-H\right)\beta\right) d\widetilde{B}_s
\end{aligned}
$$

Using conditions (i) and (ii), we can simplify this expression to

$$
\begin{aligned}
J\left(T,\Psi_T,W_T,v\right) &= J\left(t,\Psi_t,W_t,v\right) - \frac{1}{2}\int_t^T \left(\lambda \frac{\partial J}{\partial w}\frac{\partial H}{\partial \psi}\beta^2\right) ds \\
&\quad + \int_t^T \frac{\partial J}{\partial w}\left(v-H\right)\sigma d\widetilde{B}_s
\end{aligned}
$$

Let us define $\zeta_t = -\frac{1}{2}\int_t^T tr\left(\lambda \frac{\partial J}{\partial w}\frac{\partial H}{\partial \psi}\beta^2\right) ds$. We want to show that ζ is a non positive process. This is clearly the case since $\frac{\partial J}{\partial \widetilde{w}} > 0$, $\lambda > 0$ and $\frac{\partial H}{\partial \psi}\beta^2$ is also positive. So, we can say that

$$
J\left(T,\Psi_T,W_T,v\right) \le J\left(t,\Psi_t,W_t,v\right) + \int_t^T \frac{\partial J}{\partial w}\left(v-H\right)\sigma d\widetilde{B}_s
$$

We have the fact that $J\left(T,\Psi_T,W_T,v\right) \ge U\left(W_T\right)$ from the assumptions of the theorem so we have the following inequality

$$
U\left(W_T\right) \le J\left(t,\Psi_t,W_t,v\right) + \int_t^T \frac{\partial J}{\partial w}\left(v-H\right)\sigma d\widetilde{B}_s
$$

We take conditional expectation with respect to $\mathbb{P}^\nu$ on both side, we get :

$$
\mathbb{E}^\nu\left(U(W_T) \mid \Psi_t = \psi, W_t = w, V = v\right) \le J\left(t,\psi,w,v\right)
$$

The second term of the right hand side has vanished since we have assumed that $\mathbb{E}^\nu\left(\int_0^T \left(\frac{\partial J}{\partial w}\right)^2 \left(v-H\right)^2 ds\right) < +\infty$. Hence using (3.2), we get

$$
\mathbb{E}^\nu\left(U(W_T) \mid \Psi_t = \psi, W_t = w\right) \le j\left(t,\psi,w\right)
$$

Actually, the process $\left(\int_t^u \frac{\partial J}{\partial w}\left(v-H\right)\sigma d\widetilde{B}_s\right)_{t\le u\le T}$ is a $\mathcal{F}_u$-martingale. So, we have showed that the supremum we are looking for is bounded by our function j. Let us show that for a particular strategy, the supremum is attained.

We consider the strategy given by the theorem, hence we look to $dX_t = \alpha_t^* dt$. The same kind of computation leads to

$$J\left(T, \Psi_T^*, W_T, v\right) = J\left(t, \Psi_t^*, W_t, v\right) + \int_t^T \frac{\partial J}{\partial w}\left(v - H\right)\sigma d\widetilde{B}_s$$

since there is no martingale part in the strategy. But for our particular strategy, we have $H\left(T, \Psi_T^*\right) = V$ so we can say that $J\left(T, \Psi_T^*, W_T, v\right) = U\left(W_T\right)$. By taking the conditional expectation of the previous equality, we get

$$\mathbb{E}^\nu\left(U\left(W_T\right) \mid \Psi_t^* = \psi, W_t = w, V = v\right) = J\left(t, \psi, w, v\right)$$

We use the same argument to show that $\left(\int_t^u \frac{\partial J}{\partial w}\left(v - H\right)\sigma d\widetilde{B}_s\right)_{t \le u \le T}$ is a $\mathcal{F}_u$-martingale. Using one more time, equation (3.2), we get that

$$\mathbb{E}^\nu\left(U(W_T) \mid \Psi_t^* = \psi, W_t = w\right) = j\left(t, \psi, w\right)$$

So if we take this particular strategy the supremum is attained and it is exactly $j\left(t, \psi, w\right)$. We are able to conclude from this that α^* is H-optimal.
Finally, we have showed that (H, λ, α^*) is an equilibrium. $\square$

It follows that if conditions (i)-(v) are satisfied, then

$$d\Psi_t = \lambda\sigma F(t, \Psi_t)dt + \lambda\sigma dB_t$$

Hence, $(\Psi_t)_t$ is a Markov process. Thus, if $S_t = H(t, \Psi_t)$, $(S_t)_t$ is also Markovian. Thanks to Baudoin (2002), we know that if the anticipation is the endpoint of a Markov process, from condition (v), we have a closed form for the drift of change of probability. If $P_t(x, dy) = p_t(x, y)dy$ denotes the semi group of $(S_t)_t$, then :

$$F(t, \Psi_t) = \frac{\int_{\mathbb{R}} \partial_x p_{T-t}(S_t, y)\widetilde{\nu}(dy)}{\int_{\mathbb{R}} p_{T-t}(S_t, y)\widetilde{\nu}(dy)}$$

where $\widetilde{\nu}(dy) = \frac{\nu(dy)}{p_T(S_0, y)}$. We define the function ϕ as follows :

$$\phi(t, x) = \int_{\mathbb{R}} p_{T-t}(x, y)\widetilde{\nu}(dy)$$

Hence Proposition 37 of Baudoin (2002) says that ϕ satisfies the following partial differential equation:

$$\phi_t' + \frac{1}{2}\lambda^2\sigma^2 H_\psi^{'}(t, H^{-1}(t, x))\phi_{xx}^{''} = 0$$

associated with the limit condition $\phi(T, x) = \xi(x)$ where ξ is the density of ν with respect to S_T. We recall that at the equilibrium, $dS_t = \lambda\sigma H_\psi^{'}(t, \Psi_t)dB_t$ and $\Psi_t = H^{-1}(t, S_t)$.
However, Baudoin (2002) gives an other way to compute the drift of change of probability. We recall that $\xi = \frac{d\nu}{d\mathbb{P}_V}$ and that ξ admits a continuously differentiable version with bounded partial derivative. Thanks to Baudoin (2002), we have a representation of F using Malliavin calculus :

$$F(t, \Psi_t) = \mathbb{E}^\nu\left(\frac{\xi'(S_T)}{\xi(S_T)} D_t S_T \mid S_t\right)$$

where D_tS_T is the Malliavin derivative of S_T. To compute this derivative, we recall that $S_T = H(T, \Psi_T)$ and $d\Psi_t = \lambda_t \sigma F(t, \Psi_t)dt + \lambda_t \sigma dB_t$. Hence, we get $D_tS_T = H^{'}_{\psi}(T, \Psi_T)D_t\Psi_t$. We can compute $D_t\Psi_T$ (see Nualart (1995)) :

$$D_t\Psi_T = \lambda_t \sigma \exp(\int_t^T \sigma \lambda_s \frac{\partial F}{\partial \psi}(s, \Psi_s)ds)$$

Finally, we have a representation of the drift μ_t^ν

(3.4)

$$F(t, \psi) = \mathbb{E}^\nu \left(\frac{\xi'(H(T, \Psi_T))}{\xi(H(T, \Psi_T))} H^{'}_{\psi}(T, \Psi_T)\lambda_t \sigma \exp(\int_t^T \sigma \lambda_s \frac{\partial F}{\partial \psi}(s, \Psi_s)ds) \mid \Psi_t = \psi \right)$$

4. The risk neutral case

In this section, we assume that the utility function is $U(x) = x$, we are then looking for a particular function J which has the form $J(t, \psi, w, v) = w + \widehat{J}(t, \psi, v)$. Hence, it appears that $\frac{\partial J}{\partial w} = 1$ and that $\frac{\partial^2 J}{\partial w^2} = 0$. Thus using (A.15), we get that

$$\frac{\lambda_t'}{\lambda_t} = \lambda_t \sigma \frac{\partial F}{\partial \psi}(t, \psi) \tag{4.1}$$

As we have assumed that λ depends only on time, we have:

$$F(t, \psi) = A(t) + \frac{\lambda_t'}{\sigma \lambda_t^2}\psi \tag{4.2}$$

where A depends only on time. Thus, in our model, F has to be linear in ψ. This leads to the following equalities:

$$\begin{cases} F(t, \psi) = A(t) + \frac{\lambda_t'}{\sigma \lambda_t^2}\psi \\ \frac{\partial H}{\partial t} + (\lambda \sigma A(t) + \frac{\lambda_t'}{\lambda_t}\psi)\frac{\partial H}{\partial \psi} + \frac{1}{2}\lambda^2\sigma^2\frac{\partial^2 H}{\partial \psi^2} = 0 \\ \alpha_t = \sigma A(t) + \frac{\lambda_t'}{\lambda_t^2}\Psi_t \\ \frac{\phi_x^{'}(t, H(t, \psi)}{\phi(t, H(t, \psi))} = A(t) + \frac{\lambda_t'}{\sigma \lambda_t^2}\psi \\ \phi(T, x) = \xi(x) \end{cases}$$

From this, we are able to compute the dynamics of Ψ_t :

$$\begin{aligned} d\Psi_t &= (\lambda_t \sigma A(t) + \frac{\lambda_t'}{\lambda_t}\Psi_t)dt + \sigma \lambda_t dB_t \\ &= 2(\lambda_t \sigma A(t) + \frac{\lambda_t'}{\lambda_t}\Psi_t)dt + \sigma \lambda_t d\widetilde{B}_t \end{aligned}$$

It turns out that $(\Psi_t)_t$ is an Ornstein Ulhenbeck process under $\mathbb{P}$, but also under $\mathbb{P}^\nu$. In particular, $(\Psi_t)_t$ is Gaussian. An easy computation tells us that

$$\begin{aligned} \Psi_t &\sim \mathcal{N}\left(\sigma \lambda_t \int_0^t A(s)ds, \sigma^2\lambda_t^2 t\right) \text{ under } \mathbb{P} \\ \Psi_t &\sim \mathcal{N}\left(2\sigma \lambda_t^2 \int_0^t A(s)\lambda_s^{-1}ds, \sigma^2\lambda_t^4 \int_0^t \lambda_s^{-2}ds\right) \text{ under } \mathbb{P}^\nu \end{aligned}$$

For simplicity, we use the following notations :

$$\begin{aligned} m &= 2\sigma \lambda_T^2 \int_0^T A(s)\lambda_s^{-1}ds & \Sigma^2 &= \sigma^2 \lambda_T^4 \int_0^T \lambda_s^{-2}ds \\ m' &= \sigma \lambda_T \int_0^T A(s)ds & \Sigma'^2 &= \sigma^2 \lambda_T^2 T \end{aligned}$$

This implies that S_T has the same kind of law under $\mathbb{P}$ and $\mathbb{P}^\nu$ since the form $S_T = H(T, \Psi_T)$ does not depend on the probability measure we use. If we assume that $H(T, .)$ is bijective, it is easy to see that an anticipation on S_T is equivalent to an anticipation on Ψ_T. Hence, our modeling and hypotheses force the aggregate demand to follow a Gaussian anticipation for Ψ_T at the equilibrium.
We have the following results :

$$\begin{cases} S_T \sim \nu \text{ under } \mathbb{P}^\nu \\ \Psi_T = H^{-1}(T, S_T) \sim \hat{\nu} \text{ under } \mathbb{P}^\nu \\ \Psi_T \sim \mathcal{N}(m, \Sigma) \text{ under } \mathbb{P}^\nu \\ \Psi_T \sim \mathcal{N}(m', \Sigma') \text{ under } \mathbb{P} \end{cases}$$

where $\hat{\nu}$ is the measure obtained by the transfer of ν using $H^{-1}(T, .)$

Remark 4.1. It turns out, that if we are looking for (J, λ, H, α) that satisfies conditions (i)-(v) of Theorem 3.2, then Ψ_T is Gaussian under $\mathbb{P}$ and $\mathbb{P}^\nu$. Hence, if condition (v) is satisfied then the law of $V = H(T, \Psi_T)$ under $\mathbb{P}$ is of the same type as ν. Hence, if, for example, the true law of V is Gaussian and the anticipation of the strategic agent is lognormal, Theorem 3.2 will be not helpful to provide us an equilibrium.

Lemma 4.2. *If conditions of Theorem 3.2 are satisfied and if F has the form given in (4.2), then*

(1) $H(T, \psi) = G^{-1}\left(c + \Phi(\frac{\psi - m}{\Sigma})\right)$ *where $c \in \mathbb{R}$ and G is the cumulative function associated to the density $g = \frac{d\nu}{d\mathbb{P}^\nu}$.*
(2) $\frac{1}{\Sigma}\phi(\frac{y-m}{\Sigma}) = \frac{1}{\Sigma'}\widehat{\xi}(y)\phi(\frac{y-m'}{\Sigma'})$ *where $\widehat{\xi} = \frac{d\widehat{\nu}}{d\mathbb{P}^\nu}$*

Proof: Let χ be a bounded and smooth function and g the density of ν with respect to $\mathbb{P}^\nu$. Let us denote by φ the Gaussian density, by $h(.)$ the function $H(T, .)$. We have

$$\mathbb{E}^\nu(\chi(\Psi_T)) = \frac{1}{\Sigma}\int_{\mathbb{R}} \chi(y)\varphi(\frac{y-m}{\Sigma})dy$$

since $\Psi_T \sim \mathcal{N}(m, \Sigma)$ under $\mathbb{P}^\nu$. However, we have

$$\begin{aligned} \mathbb{E}^\nu(\chi(\Psi_T)) &= \mathbb{E}^\nu(\chi(h^{-1}(S_T))) \\ &= \int_{\mathbb{R}} \chi(h^{-1}(x))g(x)dx \\ &= \int_{\mathbb{R}} \chi(y)g(h(y))h'(y)dy \end{aligned}$$

From this, we can say that a.e.

$$\sqrt{2\pi}\Sigma g(h(y))h'(y) = e^{-\frac{(y-m)^2}{2\Sigma^2}} \tag{4.3}$$

If we denote by Φ (resp. G) the cumulative function associated to φ (resp. g), and if we assume that $\lim_{x \to +\infty} h(x)$ exists, then we have

$$h(y) = G^{-1}\left(c + \Phi(\frac{y-m}{\Sigma})\right) \tag{4.4}$$

Let χ be a bounded and smooth function and ξ the density of ν with respect to $\mathbb{P}_{\Psi_T}$.

$$\mathbb{E}^\nu(\chi(\Psi_T)) = \frac{1}{\Sigma}\int \chi(y)\phi(\frac{y-m}{\Sigma})dy$$

$$\begin{aligned}\mathbb{E}^{\nu}(\chi(\Psi_T)) &= \mathbb{E}(\chi(\Psi_T)\xi(\Psi_T)) \\ &= \frac{1}{\Sigma'}\int \chi(y)\xi(y)\phi(\frac{y-m'}{\Sigma'})dy\end{aligned}$$

Thus, we get

$$\frac{1}{\Sigma}\phi(\frac{y-m}{\Sigma}) = \frac{1}{\Sigma'}\xi(y)\phi(\frac{y-m'}{\Sigma'})$$

□

Lemma 4.3. *If conditions of theorem 3.2 are satisfied, if F has the form given in (4.2) and if ν is absolutely continuous with respect to the Lebesgues measure, then (A,λ) is a solution of:*

$$\lambda'_t = \frac{\sigma^2\lambda_T^3(\Sigma^2-\Sigma'^2)}{\Sigma'^2\Sigma^2} \tag{4.5}$$

$$A(t) = \frac{\lambda_T\sigma}{\Sigma^2\Sigma'^2}(\Sigma^2(m-m') - (\Sigma^2-\Sigma'^2)\frac{\lambda_T^2}{\lambda_t^2}m_t) \tag{4.6}$$

Proof: Since $S_T = h(\Psi_T)$ and since $h(.) = H(T,.)$ is bijective from Definition 2.3, we have

$$(T,S_T,\nu) \Leftrightarrow (T,\Psi_T,\widehat{\nu})$$

where $\widehat{\nu}(dx) = g(h(x))h'(x)dx$ if $\nu(dx) = g(x)dx$. Knowing this, we compute the drift of change of probability given by (3.4).

$$\begin{aligned}F(t,\psi) &= \mathbb{E}^{\nu}(\frac{\xi'(\Psi_T)}{\xi(\Psi_T)}D_t\Psi_T \mid \Psi_t=\psi) \\ &= \mathbb{E}^{\nu}\left(\frac{\xi'(\Psi_T)}{\xi(\Psi_T)}\lambda_t\sigma\exp(\int_t^T\sigma\lambda_s\frac{\partial F}{\partial\psi}(s,\Psi_s)ds) \mid \Psi_t=\psi\right) \\ &= \mathbb{E}^{\nu}\left(\frac{\xi'(\Psi_T)}{\xi(\Psi_T)}\lambda_t\sigma\frac{\lambda_T}{\lambda_t} \mid \Psi_t=\psi\right) \qquad (4.7)\end{aligned}$$

where $\widehat{\xi}(\psi) = \frac{d\widehat{\nu}}{d\mathbb{P}_{\Psi_T}}(\psi)$. Indeed, $\int_t^T\sigma\lambda_s\frac{\partial F}{\partial\psi}(s,\Psi_s)ds = \int_t^T\sigma\lambda_s(\frac{\lambda'_s}{\sigma\lambda_s^2})ds$, hence we have $\exp(\int_t^T\sigma\lambda_s\frac{\partial F}{\partial\psi}(s,\Psi_s)ds) = \exp(\ln(\frac{\lambda_T}{\lambda_t}))$.
However, we can compute $\widehat{\xi}$:

$$\begin{aligned}\widehat{\xi}(y) &= \frac{d\widehat{v}}{dx}(y)\frac{dx}{d\mathbb{P}_{\Psi_T}}(y) \\ &= \frac{\Sigma'}{\Sigma}\frac{\varphi(\frac{y-m}{\Sigma})}{\varphi(\frac{y-m'}{\Sigma'})}\end{aligned}$$

Hence, we get

$$\widehat{\xi}'(y) = \frac{\Sigma'}{\Sigma}(-\frac{y-m}{\Sigma^2}+\frac{y-m'}{\Sigma'^2})\frac{\varphi(\frac{y-m}{\Sigma})}{\varphi(\frac{y-m'}{\Sigma'})}$$

and finally

$$\frac{\widehat{\xi}'}{\widehat{\xi}}(y) = \frac{1}{\Sigma^2\Sigma'^2}(y(\Sigma^2-\Sigma'^2)+\Sigma'^2m-\Sigma^2m') \tag{4.8}$$

Thus, using (4.7) and (4.8), we get

$$F(t,\psi)=\sigma\lambda_T\left(\frac{(\Sigma^2-\Sigma'^2)}{\Sigma^2\Sigma'^2}\mathbb{E}^\nu(\Psi_T\mid\Psi_t=\psi)+(\frac{m}{\Sigma^2}-\frac{m'}{\Sigma'^2})\right) \tag{4.9}$$

We need to compute the conditional expectation. But, Ψ_T is the end point of an Ornstein Ulhenbeck process, hence

$$\begin{aligned}
\mathbb{E}^\nu(\Psi_T\mid\Psi_t) &= 2\sigma\lambda_T^2\int_0^T A(s)\lambda_s^{-1}ds+\mathbb{E}^\nu(\sigma\lambda_T^2\int_0^T\lambda_s^{-1}d\widetilde{B}_s\mid\Psi_t)\\
&= m+\mathbb{E}^\nu(\sigma\lambda_T^2\int_0^t\lambda_s^{-1}d\widetilde{B}_s+\sigma\lambda_T^2\int_t^T\lambda_s^{-1}d\widetilde{B}_s\mid\Psi_t)\\
&= m+\sigma\lambda_T^2\int_0^t\lambda_s^{-1}d\widetilde{B}_s\\
&= m+\frac{\lambda_T^2}{\lambda_t^2}(\sigma\lambda_t^2\int_0^t\lambda_s^{-1}d\widetilde{B}_s+2\sigma\lambda_t^2\int_0^t A(s)\lambda_s^{-1}ds-2\sigma\lambda_t^2\int_0^t A(s)\lambda_s^{-1}ds)\\
&= m-\frac{\lambda_T^2}{\lambda_t^2}m_t+\frac{\lambda_T^2}{\lambda_t^2}\Psi_t
\end{aligned}$$

where $m_t=2\sigma\lambda_t^2\int_0^t A(s)\lambda_s^{-1}ds$. So, using (4.9) and (4.2), we get $\forall\psi\in\mathbb{R}$:

$$A(t)+\frac{\lambda_t'}{\sigma\lambda_t^2}\psi=\frac{\lambda_T\sigma}{\Sigma^2\Sigma'^2}\left[m\Sigma^2-m\Sigma'^2+(\Sigma^2-\Sigma'^2)(\frac{\lambda_T^2}{\lambda_t^2}\psi-\frac{\lambda_T^2}{\lambda_t^2}m_t)+\Sigma'^2m-\Sigma^2m'\right]$$

A simple identification finishes the proof. □

Remark 4.4. We recall that the quantities m, m', Σ and Σ' depend on λ and A, hence (4.5) and (4.6) are not some classical ordinary classical equation. However, we can exhibit a trivial solution for $(\lambda_t,A(t))$ which is $(C,0)$ and which correspond to an absence of anticipation (since $F(t,\psi)\equiv 0$).

Lemma 4.5. *The solutions of the differential system (4.5)-(4.6) are*

$$\left\{(\lambda_t,A(t))=(at+b,-\frac{2aC}{(at+b)^2}),\quad(a,b,C)\in\mathbb{R}^3\right\}$$

Proof: For λ, we can formulate the equation (4.5) as follows :

$$\lambda_t'=C(T,\lambda) \tag{4.10}$$

Hence, we get

$$\lambda_t-\lambda_0=C(T,\lambda)t$$

Hence, we have the general form for λ which is $\lambda_t=at+b$. It remains to show that a solution of this form is compatible with equation (4.10).
Indeed, we have $C(T,\lambda)=\frac{\lambda_T^2\int_0^T\lambda_t^{-2}dt-T}{T\lambda_T\int_0^T\lambda_t^{-2}dt}$. Plugging the linear form of λ with $\lambda_t'=a$ and $\int_0^T\lambda_t^{-2}dt=\frac{1}{a}[\frac{1}{b}-\frac{1}{aT+b}]$, we get

$$\begin{aligned}
C(T,\lambda) &= \frac{(aT+b)^2[a(aT+b)-ab]-a^2b(aT+b)T}{(aT+b)T[a(aT+b)-ab]}\\
&= \frac{a^2T(aT+b)-a^2bT}{a^2T^2}\\
&= \frac{a^3T^2}{a^2T^2}=a
\end{aligned}$$

Hence, we obtain $\lambda'_t = C(T, \lambda)$, thus the linear form for λ is a solution of (4.10). If you are able to find a solution for λ, then we can solve (4.6). In fact, A satisfies an integro differential equation :

$$A(t) + 2a \int_0^t A(s)\lambda_s^{-1} ds = K(a, b, A, T) \tag{4.11}$$

where $K(a, b, A, T)$ can be computed knowing that $\lambda_t = at + b$ and using the form of Σ, Σ', m and m'. After some computation, we get

$$K(a, b, A, T) = \frac{1}{T}(2(aT + b) \int_0^T A(s)\lambda_s^{-1} ds - \int_0^T A(s)ds)$$

Considering $B(t) = \int_0^t A(s)\lambda_s^{-1} ds$, we have to solve an ordinary differential equation

$$2aB(t) + (at + b)B'(t) = K(a, b, A, T) \tag{4.12}$$

The homogeneous equation associated to (4.12) can be written :

$$2aB(t) + (at + b)B'(t) = 0 \tag{4.13}$$

A general solution of (4.13) is $B_h(t) = \frac{C}{(at+b)^2}$. It turns out that we do not need to use the constant variation method, since (4.12) has an obvious solution which is $B_p(t) = \frac{K(a,b,A,T)}{2a}$. Hence a general solution to (4.12) is

$$B(t) = \frac{C}{(at + b)^2} + \frac{K(a, b, A, T)}{2a} \tag{4.14}$$

We recall that $A(t) = \lambda_t B'(t)$, hence we get $A(t) = -\frac{2aC}{(at+b)^2}$. Now, we have to check that this solution is consistent with (4.11).
Let us first compute the left hand side of (4.11) using our previous solution for A

$$\begin{aligned} 2a \int_0^t A(s)\lambda_s^{-1} ds &= -4a^2 C \int_0^t \frac{ds}{(as + b)^3} \\ &= 2aC[\frac{1}{(at + b)^2} - \frac{1}{b^2}] \\ &= -\frac{2aC}{b^2} - A(t) \end{aligned}$$

Now, we compute $K(a, b, A, T)$

$$\begin{aligned} K(a, b, A, T) &= \frac{1}{T}(2(aT + b) \int_0^T A(s)\lambda_s^{-1} ds - \int_0^T A(s)ds) \\ &= \frac{1}{T}(2(aT + b) \int_0^T \frac{-2aC}{(at + b)^3} ds + \int_0^T \frac{2aC}{(at + b)^2} ds) \\ &= \frac{1}{T}[\frac{2C}{aT + b} - \frac{2(aT + b)C}{b^2} + \frac{2C}{b} - \frac{2C}{aT + b}] \\ &= -\frac{2aC}{b^2} \end{aligned}$$

Hence A is a solution of (4.11). □

Corollary 4.6. *If (λ, A) has the form given in Lemma 4.5, then*

$$\begin{aligned} m &= -\frac{2\sigma C a T}{b^2}(2b + aT) & \Sigma^2 &= \frac{\sigma^2 T}{b}(aT + b)^3 \\ m' &= -\frac{2\sigma C a T}{b} & \Sigma'^2 &= \sigma^2 T(aT + b)^2 \end{aligned}$$

Recalling Remark 3.3, we are now able to give an explicit equilibrium up to the resolution of a non linear system which will be solved in a example.

Proposition 4.7. *Let us assume that there exists $\widehat{h}(.) \in C^1(\mathbb{R})$ and $\widehat{h}$ bijective such that $V = \widehat{h}(\Theta)$ where $\Theta \sim \mathcal{N}(\widehat{m}, \widehat{\Sigma})$ under $\mathbb{P}$ and such that ν is the law of a variable which can be written $\widehat{h}(\widetilde{\Theta})$ where $\widetilde{\Theta} \sim \mathcal{N}(\widetilde{m}, \widetilde{\Sigma})$.*
Let us assume that there exist a solution $(a, b, C) \in \mathbb{R} \times \mathbb{R}_+ \times R$ with $b > -aT$ and $h(.) \in C^1(\mathbb{R})$ and bijective to the system

$$\begin{cases} \Sigma g(h(y))h'(y) &= \varphi(\frac{y-m}{\Sigma}) \qquad \forall y \in \mathbb{R} \\ h(m' + \Sigma' y) &= \widehat{h}(\widehat{m} + \widehat{\Sigma} y) \qquad \forall y \in \mathbb{R} \end{cases}$$

where g is the density of ν with respect to the Lebesgues measure and where (m, m', Σ, Σ') has the form given in Corollary 4.6.

If we define X, λ, H and J as follows
$\forall t \in [0, T] \quad X_t = \int_0^t \frac{a}{(as+b)^2}(-2\sigma C + \Psi_s)ds$
$\forall t \in [0, T] \quad \lambda_t = at + b$
$\forall t \in [0, T], \forall \psi \in \mathbb{R} \quad F(t, \psi) = \frac{a}{\sigma(at+b)^2}(-2\sigma C + \psi)$
$J(t, \psi, w, v) = w + \mathbb{E}^{\nu}_{t,\psi}\left[J(T, \psi + \Psi_T - \Psi_t) \mid V = v\right]$
$J(T, \psi, w, v) = w + \int_{h^{-1}(v)}^{\psi} \frac{1}{aT+b}(h(y) - v)dy$
$H(t, \psi) = \mathbb{E}(h(\Psi_T) \mid \Psi_t = \psi)$
then (H, λ, X, F) is an equilibrium.

Proof: We just need to check that the assumptions of Theorem 3.2 are satisfied. The proof is straightforward using Lemma 4.2 and Lemma 4.3. □

Example 4.8. We take $\widehat{h}(.) = \exp(.)$. Thus, le liquidative value of the risky asset has a lognormal law, and the anticipation of the strategic agent is also lognormal. First, we have to prove the existence of h. We know that $\nu(dx) = \mathbf{1}_{\mathbb{R}_+}(x)\frac{1}{\sqrt{2\pi}\widetilde{\Sigma}x}e^{-\frac{(\ln(x)-\widetilde{m})^2}{2\widetilde{\Sigma}^2}}dx$ and we are going to use (4.3) to compute the form of h. We need to solve the following equation

$$\frac{1}{\widetilde{\Sigma}}e^{-\frac{(\ln(h(y))-\widetilde{m})^2}{2\widetilde{\Sigma}^2}}\frac{h'(y)}{h(y)} = \frac{1}{\Sigma}e^{-\frac{(y-m)^2}{2\Sigma^2}}$$

This leads to $h'(y) = \frac{\widetilde{\Sigma}}{\Sigma}h(y)e^{\frac{(\ln(h(y))-\widetilde{m})^2}{2\widetilde{\Sigma}^2}}e^{-\frac{(y-m)^2}{2\Sigma^2}}$. Thus, we have an equation of the form $h'(y) = \Pi(y, h(y))$ where Π is locally Lipschitz. Hence, thanks to Cauchy Lipschitz theorem, given an initial condition, we have an unique solution to this equation. Moreover, we are looking for a increasing function h. Integrating between u and x, we get

$$\int_u^x \frac{h'(y)}{\widetilde{\Sigma}h(y)}\phi(\frac{ln(h(y)) - \widetilde{m}}{\widetilde{\Sigma}})dy = \int_u^x \frac{1}{\Sigma}\phi(\frac{y-m}{\Sigma})dy$$

Using a change of variable, we get

$$\int_{\ln(h(u))}^{\ln(h(x))} \frac{1}{\widetilde{\Sigma}}\phi(\frac{y-\widetilde{m}}{\widetilde{\Sigma}})dy = \int_u^x \frac{1}{\Sigma}\phi(\frac{y-m}{\Sigma})dy \tag{4.15}$$

Let us assume that $\lim_{u\to+\infty} h(u) = c > 0$. It turns out that we obtain :

$$\int_{\ln(c)}^{\ln(h(x))} \frac{1}{\widetilde{\Sigma}}\phi(\frac{y-\widetilde{m}}{\widetilde{\Sigma}})dy = \int_{-\infty}^x \frac{1}{\Sigma}\phi(\frac{y-m}{\Sigma})dy$$

Hence, we remark that $\lim\limits_{x\to+\infty} RHS(x) = 1$ and $\lim\limits_{x\to+\infty} LHS(x) = l < 1$ which is not possible, hence $\lim\limits_{u\to+\infty} h(u) = 0$. Now, using a linear change of variable in both integrals, it is easy to see that (4.15) implies that $\frac{\ln(h(x))-\widetilde{m}}{\widetilde{\Sigma}} = \frac{x-m}{\Sigma}$. Hence we have the form of the function h

$$h(x) = e^{\widetilde{m}+\frac{\widetilde{\Sigma}}{\Sigma}(x-m)} \tag{4.16}$$

Now, we have to find appropriate (a, b, C) to solve

$$h(m' + \Sigma' y) = \widehat{h}(\widehat{m} + \widehat{\Sigma} y)$$

Remembering that $\widehat{h}(.)$ is an exponential and using (4.16), we immediately get $\widetilde{m} + \frac{\widetilde{\Sigma}}{\Sigma}(m' - m) = \widehat{m}$ and $\frac{\Sigma'\widetilde{\Sigma}}{\Sigma} = \widehat{\Sigma}$. Using Corollary 4.6, we get $\frac{\sigma^2 T(aT+b)^2}{\sigma^2 T(aT+b)^3} = (\frac{\widehat{\Sigma}}{\widetilde{\Sigma}})^2$ and $\frac{2Ca}{b\sigma(aT+b)^2} = \frac{\widehat{m}-\widetilde{m}}{\widetilde{\Sigma}}$. The first equation becomes $a = \frac{\widetilde{\Sigma}^2-\widehat{\Sigma}^2}{\widehat{\Sigma}^2 T} b$. We plug this in the second equation, and we get $C = \frac{\widehat{\Sigma}\sqrt{T}}{2(\widetilde{\Sigma}^2-\widehat{\Sigma}^2)}(\widehat{m} - \widetilde{m})b$.

We have to check that $aT + b > 0$. Using the equation between a and b, we get $aT + b = \frac{b\widetilde{\Sigma}^2}{\widehat{\Sigma}^2}$. Hence $aT + b > 0$ if and only if $b > 0$.

We write the function h in term of b, since $\Sigma = \frac{\widetilde{\Sigma}^3}{\widehat{\Sigma}^3}\sigma\sqrt{T}b$ and $m = \frac{(\widetilde{m}-\widehat{m})(\widetilde{\Sigma}^2+\widehat{\Sigma}^2)}{\widehat{\Sigma}^3}\sigma\sqrt{T}b$.

Thus, using the form given by (4.16), we get

$$\forall x \in \mathbb{R} \quad h(x) = \exp\left(\frac{\widehat{\Sigma}^3}{\sigma\widetilde{\Sigma}^2\sqrt{T}b}x + \frac{\widehat{\Sigma}^2 + \widetilde{\Sigma}^2}{\widetilde{\Sigma}^2}\widehat{m} - \frac{\widehat{\Sigma}^2}{\widetilde{\Sigma}^2}\widetilde{m}\right)$$

We have found a solution (a, b, C, h) to the system of Proposition 4.7, hence we have proved the existence of an equilibrium in our example. We can remark that the equilibrium is not unique since b can be any positive real number. In particular, m and Σ are linear in b, hence the strategic agent can drive the cumulative demand to $+\infty$. This is possible since the strategic agent is risk neutral. However, he can choose a non degenerated equilibrium by choosing a finite b.

5. Conclusion

We have seen that, in the context of weak information, we are able to prove the existence of an equilibrium when the strategic agent is risk neutral, using the sufficient conditions of Theorem 3.2. However, we have remarked that condition (v) of this theorem appears to be too strong. Indeed, the equilibrium which we have exposed implies that the anticipation of the strategic agent is not so far from the true law of the random variable V. Thus, an interesting extension of our model will be to find some weaker condition for the terminal value of the price process, like an equality in law, for example, in order to find equilibria in which the strategic agent makes, perhaps wrong, anticipation on the law of V and not only on the parameters of a given law.

Appendix

Proof of Proposition 3.1: The state variable associated to J is (Ψ, W, V) where

$$d\Psi_t = \lambda_t(\alpha_t + \sigma\mu_t^{\nu})dt + \lambda_t(\sigma + \beta_t)d\widetilde{B}_t, \quad \Psi_0 = 0 \tag{A.1}$$

$$dW_t = \left(\alpha_t(V - H) - \lambda_t\beta_t(\sigma + \beta_t)\frac{\partial H}{\partial \psi}\right)dt + \beta_t(V - H)d\widetilde{B}_t, \quad W_0 = 0 \tag{A.2}$$

(A.3) $$dV_t = 0, \quad V_0 = V$$

Hence, the Bellman equation associated to the value function J is

(A.4) $$\frac{\partial J}{\partial t} + \sup_{(\alpha,\beta)\in\mathbb{R}^2} \mathcal{L}^{\alpha,\beta}_{\Psi,W,V} J \leq 0$$

where $\mathcal{L}^{\alpha,\beta}_{\Psi,W,V}$ is the infinitesimal generator associated to the state variable (Ψ, W, V). Using (A.1) and (A.2), we have :

$$\begin{aligned}\mathcal{L}^{\alpha,\beta}_{\Psi,W,V} J &= \lambda\frac{\partial J}{\partial \psi}(\alpha + \sigma\mu^\nu_t) + \frac{\partial J}{\partial w}(\alpha(v-H) - \lambda\beta(\beta+\sigma)\frac{\partial H}{\partial \psi}) \\ &\quad + \frac{1}{2}\lambda^2(\beta+\sigma)^2\frac{\partial^2 J}{\partial \psi^2} + \frac{1}{2}\beta^2(v-H)^2\frac{\partial^2 J}{\partial w^2} + \lambda\beta(\beta+\sigma)(v-H)\frac{\partial^2 J}{\partial \psi \partial w}\end{aligned}$$

Hence, we get

(A.5) $$A + \sup_{(\alpha,\beta)\in\mathbb{R}^2}(\alpha B + \beta C + \beta^2 D) \leq 0$$

where

$$\begin{aligned}A &= \frac{\partial J}{\partial t} + \lambda\frac{\partial J}{\partial \psi}\sigma\mu^\nu_t + \frac{1}{2}\lambda^2\sigma^2\frac{\partial^2 J}{\partial \psi^2} \\ B &= \lambda\frac{\partial J}{\partial \psi} + \frac{\partial J}{\partial w}(v-H) \\ C &= -\lambda\sigma\frac{\partial J}{\partial w}\frac{\partial H}{\partial \psi} + \lambda^2\sigma\frac{\partial^2 J}{\partial \psi^2} + \lambda\sigma(v-H)\frac{\partial^2 J}{\partial \psi \partial w} \\ D &= -\lambda\frac{\partial J}{\partial w}\frac{\partial H}{\partial \psi} + \frac{1}{2}\lambda^2\frac{\partial^2 J}{\partial \psi^2} + \frac{1}{2}(v-H)^2\frac{\partial^2 J}{\partial w^2} + \lambda(v-H)\frac{\partial^2 J}{\partial \psi \partial w}\end{aligned}$$

The problem is not degenerated if $B = 0$ since the control α may be unbounded. A quick computation gives $\frac{\partial B}{\partial \psi} = \lambda\sigma C$, hence if $B = 0$, then $C = 0$. Finally, we can rewrite D using the fact that $C = 0$:

$$D = -\frac{1}{2}\lambda^2\frac{\partial^2 J}{\partial \psi^2} + \frac{1}{2}(v-H)^2\frac{\partial^2 J}{\partial w^2}$$

It follows that if $D \leq 0$, then the supremum in β is attained for $\beta = 0$, and then from (A.5), we get $A \leq 0$ i.e. :

$$\frac{\partial J}{\partial t} + \lambda\frac{\partial J}{\partial \psi}\sigma\mu^\nu_t + \frac{1}{2}\lambda^2\sigma^2\frac{\partial^2 J}{\partial \psi^2} \leq 0$$

If $D > 0$, then the problem is degenerated since the supremum is $+\infty$. □

Proof of Lemma 3.4: Let us differentiate (I) with respect to ψ :

$$\lambda\frac{\partial^2 J}{\partial \psi^2} + \frac{\partial^2 J}{\partial w \partial \psi}(v-H) - \frac{\partial J}{\partial w}\frac{\partial H}{\partial \psi} = 0$$

We differentiate the previous equation one more time with respect to ψ :

(A.6) $$\lambda\frac{\partial^3 J}{\partial \psi^3} + \frac{\partial^3 J}{\partial w \partial \psi^2}(v-H) - 2\frac{\partial^2 J}{\partial w \partial \psi}\frac{\partial H}{\partial \psi} - \frac{\partial J}{\partial w}\frac{\partial^2 H}{\partial \psi^2} = 0$$

Let us differentiate (II) with respect to ψ :

$$\frac{\partial^2 J}{\partial t \partial \psi} + \lambda \frac{\partial^2 J}{\partial \psi^2} \sigma \mu_t^\nu + \sigma \lambda \frac{\partial J}{\partial \psi} \frac{\mu_t^\nu}{\partial \psi} + \frac{1}{2} \lambda^2 \sigma^2 \frac{\partial^3 J}{\partial \psi^3} = 0 \tag{A.7}$$

Combining (A.6) and (A.7), we get

$$\begin{aligned} 0 &= \frac{\partial^2 J}{\partial t \partial \psi} + \lambda \frac{\partial^2 J}{\partial \psi^2} \sigma \mu_t^\nu + \sigma \lambda \frac{\partial J}{\partial \psi} \frac{\mu_t^\nu}{\partial \psi} - \frac{1}{2} \lambda \sigma^2 \frac{\partial^3 J}{\partial w \partial \psi^2} (v - H) \\ &\quad + \lambda \sigma^2 \frac{\partial^2 J}{\partial w \partial \psi} \frac{\partial H}{\partial \psi} + \frac{1}{2} \lambda \sigma^2 \frac{\partial J}{\partial w} \frac{\partial^2 H}{\partial \psi^2} \end{aligned} \tag{A.8}$$

Let us now differentiate (I) with respect to w (H does not depend on w) :

$$\lambda \frac{\partial^2 J}{\partial \psi \partial w} + \frac{\partial^2 J}{\partial w^2} (v - H) = 0$$

We differentiate the previous equation with respect to w and ψ:

$$\lambda \frac{\partial^3 J}{\partial \psi \partial^2 w} + \frac{\partial^3 J}{\partial w^3} (v - H) = 0 \tag{A.9}$$

$$\lambda \frac{\partial^3 J}{\partial \psi^2 \partial w} + \frac{\partial^3 J}{\partial \psi \partial w^2} (v - H) - \frac{\partial^2 J}{\partial w^2} \frac{\partial H}{\partial \psi} = 0 \tag{A.10}$$

Thus, (A.8) becomes

$$\begin{aligned} 0 &= \frac{\partial^2 J}{\partial t \partial \psi} + \lambda \frac{\partial^2 J}{\partial \psi^2} \sigma \mu_t^\nu + \sigma \lambda \frac{\partial J}{\partial \psi} \frac{\mu_t^\nu}{\partial \psi} + \frac{3}{2} \sigma^2 (H - v) \frac{\partial^2 J}{\partial w^2} \frac{\partial H}{\partial \psi} \\ &\quad + \frac{1}{2} \lambda \sigma^2 \frac{\partial J}{\partial w} \frac{\partial^2 H}{\partial \psi^2} + \frac{1}{2\lambda} \sigma^2 (H - v)^3 \frac{\partial^3 J}{\partial w^3} \end{aligned} \tag{A.11}$$

Let us differentiate (I) with respect to t :

$$\lambda' \frac{\partial J}{\partial \psi} + \lambda \frac{\partial^2 J}{\partial \psi \partial t} + \frac{\partial^2 J}{\partial t \partial w} (v - H) - \frac{\partial J}{\partial w} \frac{\partial H}{\partial t} = 0 \tag{A.12}$$

Let us differentiate (II) with respect to w :

$$\frac{\partial^2 J}{\partial t \partial w} + \lambda \frac{\partial^2 J}{\partial \psi \partial w} \sigma \mu_t^\nu + \frac{1}{2} \lambda^2 \sigma^2 \frac{\partial^3 J}{\partial \psi^2 \partial w} = 0$$

Hence using (A.9) and (A.10), we get :

$$\frac{\partial^2 J}{\partial t \partial w} + \sigma \mu_t^\nu (H - v) \frac{\partial^2 J}{\partial w^2} + \frac{1}{2} \sigma^2 (H - v)^2 \frac{\partial^3 J}{\partial w^3} + \frac{1}{2} \lambda \sigma^2 \frac{\partial^2 J}{\partial w^2} \frac{\partial H}{\partial \psi} = 0 \tag{A.13}$$

Substituting $\frac{\partial^2 J}{\partial t \partial \psi}$ in (A.13) and (A.11), we get :

$$\begin{aligned} 0 &= \lambda' \frac{\partial J}{\partial \psi} - \lambda^2 \frac{\partial^2 J}{\partial \psi^2} \sigma \mu_t^\nu - \frac{3}{2} \lambda \sigma^2 (H - v) \frac{\partial^2 J}{\partial w^2} \frac{\partial H}{\partial \psi} - \frac{1}{2} \lambda^2 \sigma^2 \frac{\partial J}{\partial w} \frac{\partial^2 H}{\partial \psi^2} \\ &\quad + \sigma \mu_t^\nu (H - v)^2 \frac{\partial^2 J}{\partial w^2} + \frac{1}{2} \lambda \sigma^2 (H - v) \frac{\partial^2 J}{\partial w^2} \frac{\partial H}{\partial \psi} - \frac{\partial J}{\partial w} \frac{\partial H}{\partial t} \\ &\quad - \lambda^2 \sigma \frac{\partial J}{\partial \psi} \frac{\partial \mu_t^\nu}{\partial \psi} \end{aligned} \tag{A.14}$$

To finish, we need to have an expression of $\frac{\partial J}{\partial \psi}$ and $\frac{\partial^2 J}{\partial \psi^2}$. In differentiating (I) with respect to ψ, we get :

$$\lambda \frac{\partial^2 J}{\partial \psi^2} = \frac{\partial J}{\partial w}\frac{\partial H}{\partial \psi} - \frac{1}{\lambda}(v-H)^2 \frac{\partial^2 J}{\partial w^2}$$

We plug this expression and the one of $\frac{\partial J}{\partial \psi}$ which comes from (I) in (A.14), and we get :

$$\begin{aligned} \text{(A.15)}\ 0 \quad = \quad & (H-v)\left(\frac{\partial J}{\partial w}\left(\frac{\lambda'}{\lambda} - \lambda\sigma\frac{\partial \mu_t^\nu}{\partial \psi}\right) + \frac{\partial^2 J}{\partial w^2}\left(2\lambda\sigma^2\frac{\partial H}{\partial \psi} + 2(H-v)\sigma\mu_t^\nu\right)\right) \\ & -\frac{\partial J}{\partial w}\left(\frac{\partial H}{\partial t} + \lambda\sigma\mu_t^\nu\frac{\partial H}{\partial \psi} + \frac{1}{2}\lambda^2\sigma^2\frac{\partial^2 H}{\partial \psi^2}\right) \end{aligned}$$

We use the same kind of arguments as in Lasserre (2003). The prices are fixed by the market maker which does not observe V, hence H does not depend on v. Hence, this tells us that :

$$\begin{aligned} \frac{\partial J}{\partial w}\left(\frac{\partial H}{\partial t} + \lambda\sigma\mu_t^\nu\frac{\partial H}{\partial \psi} + \frac{1}{2}\lambda^2\sigma^2\frac{\partial^2 H}{\partial \psi^2}\right) &= 0 \\ \frac{\partial J}{\partial w}\left(\frac{\lambda'}{\lambda} - \lambda\sigma\frac{\partial \mu_t^\nu}{\partial \psi}\right) + \frac{\partial^2 J}{\partial w^2}\left(2\lambda\sigma^2\frac{\partial H}{\partial \psi} + 2(H-v)\sigma\mu_t^\nu\right) &= 0 \end{aligned}$$

If U is strictly increasing, it is easy to show that J is strictly increasing in w, hence we have :

$$\text{(A.16)} \qquad \frac{\partial H}{\partial t} + \lambda\sigma\mu_t^\nu\frac{\partial H}{\partial \psi} + \frac{1}{2}\lambda^2\sigma^2\frac{\partial^2 H}{\partial \psi^2} = 0$$

To finish the proof, it suffices to remains that $\mu_t^\nu = F(t,\Psi_t)$.□

Proof of Lemma 3.5: Knowing that $S_t = H(t,\Psi_t)$ and using Ito's formula, we have

$$\text{(A.17)} \qquad dS_t = \left(\lambda\alpha_t\frac{\partial H}{\partial \psi} + \frac{\partial H}{\partial t} + \frac{1}{2}\lambda^2\sigma^2\frac{\partial^2 H}{\partial \psi^2}\right)dt + \lambda(\beta+\sigma)\frac{\partial H}{\partial \psi}dB_t$$

where $(B_t)_t$ is a $(\mathbb{P},\mathbb{F})$-Brownian motion. Using Lemma 3.4, we get that (3.3) is satisfied, hence we have the following dynamics for S_t :

$$\text{(A.18)} \qquad dS_t = \lambda(\alpha_t - \sigma\mu_t^\nu)\frac{\partial H}{\partial \psi}dt + \lambda(\beta+\sigma)\frac{\partial H}{\partial \psi}dB_t$$

A necessary and sufficient condition for S_t to be a $(\mathbb{P},F)$-local martingale is :

$$\text{(A.19)} \qquad \lambda(\alpha_t - \sigma\mu_t^\nu)\frac{\partial H}{\partial \psi} = 0$$

Hence, using the fact that (H,λ) is a pricing rule so H is strictly increasing in ψ from Definition 2.3, a necessary and sufficient condition for S to be a $(\mathcal{F},\mathbb{P})$-local martingale is

$$\text{(A.20)} \qquad \alpha_t = \sigma\mu_t^\nu = \sigma F(t,\Psi_t)$$

□

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