Finite-dimensional Models of the Yield Curve*

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July 14, 2003

Abstract

Models of the HJM family generally represent the yield curve as the solution of a stochastic differential equation evolving in an infinite-dimensional space. It is an important question to determine under what conditions the solution is actually finite-dimensional, i.e. the yield curve at any time can be expressed as a point function of some finite-dimensional state vector process. Several authors have discovered special volatility structures under which this is the case, while others have investigated general geometric conditions.

This paper takes a different point of view. In reality, the yield curve is intrinsically finite-dimensional in that it is constructed from a finite set of market data: short-term interest rates, interest rate futures and swap rates. We ask whether it is possible, given a stochastic differential equation model for the market rates, to construct consistent and arbitrage-free prices for zero-coupon bonds of arbitrary maturities. We demonstrate that this can be done in some simple cases, but that naive algorithms such as those used in conventional 'yield curve generators' can easily create arbitrage opportunities.

1 Introduction

Give the idea

2 Background

Summarize HJM and finite-dimensional versions (Ritchken-Sub, Chiarella, Björk *et al.*, Filipović & Teichmann). Outline paper.

2.1 Factor Models

The easiest way to build a finite-factor model of the yield curve is as follows. Let $(\Omega, \mathcal{F}, P)$ be a probability space and (x_t) be an R^n-valued Markov process on $(\Omega, \mathcal{F}, P)$ with differential generator $\mathcal{A}$. We define the short rate as

$$r_t = c(x_t)$$

*Preliminary version. Paper presented at the conference *Quantitative Methods in Finance*, Sydney, Australia, December 2002.

where $c : R^n \to R^+$ is a given function. Zero coupon bond values are now *defined* by

$$p(t,T) = F(t,T,x_t) \tag{1}$$

where

$$F(t,T,x) = E_{t,x}\left[e^{-\int_t^T c(x_s)ds}\right]. \tag{2}$$

Then the process $t \to e^{-\int_0^T r(s)ds} p(t,T)$ is a martingale for $t \in [0,T]$, so (**??**), (**??**) provides an arbitrage-free specification of the zero-coupon bond prices, in which P is the risk-neutral measure (i.e. (P,N) is a numéraire pair, where $N_t = \exp(\int_0^t r_s ds)$.)

All the conventional single-factor short rate models – the Hull-White model, the CIR model, etc. – fit into this framework. The multi-factor case has been studied in detail by Duffie and Kan [**?**]. In their paper (x_t) is a multi-dimensional diffusion process and the authors study "affine factor models" in which ZC bonds take the form

$$p(t,T) = \exp(A(t,T) + B(t,T)x_t)$$

where A and B are deterministic functions (A scalar, B a row vector). This turns out to be the case if and only if the function c and all the coefficients of the generator $\mathcal{A}$ are affine functions of the state vector x. Some quite complex geometric conditions are required to ensure that $c(x_t) \geq 0$ in an affine factor model.

A further question is whether the state vector can be chosen to have an economic interpretation, for example the k'th component x_t^k is the T_k-maturity ZC yield, i.e.

$$p(t,T_k) = \exp(-x_t^k(T_k - t)), \quad t < T_k. \tag{3}$$

This is possible, but some compatability conditions are needed. For example, if $c(x) = x^1$, i.e. the first factor is the short rate, then we have two ways to compute $p(t,T_k)$, namely (**??**) and

$$p(t,T_k) = E_{t,x}\left[e^{-\int_t^{T_k} x_s^1 ds}\right]\Big|_{x=x_t} \tag{4}$$

Obviously, (**??**) and (**??**) must give the same answer, and this leads to quite complicated conditions on the coefficients of $\mathcal{A}$.

2.2 HJM Models

Recall that the (instantaneous) forward rate at time T as seen at time t is

$$f(t,T) = -\frac{\partial}{\partial T}\log p(t,T) \tag{5}$$

In particular the short rate is $r(t) = f(t,t)$. The starting point in HJM is an Ito process representation for $f(t,T)$ in the form

$$df(t,T) = \mu^0(t,T)dt + \sigma^0(t,T)dw_t, \quad 0 \leq t \leq T \tag{6}$$

where w_t is an n-vector Brownian motion on $(\Omega, \mathcal{F}, P)$ and the coefficients are random. We can invert (**??**) to recover the bond price as

$$p(t,T) = \exp\left(-\int_t^T f(t,u)du\right).$$

In (**??**), (w_t) is a Brownian motion under measure P, the risk-neutral measure (corresponding to $B(t) = \exp(\int_0^t r(s)ds)$ as numéraire .) The famous HJM drift condition states that (**??**) is arbitrage-free if and only if

$$\mu^0(t,T) = \sigma^0(t,T)\int_t^T (\sigma^0(t,u))' du.$$

See Musiela and Rutkowski [**?**]. Thus the whole model is specified by the volatility function σ and the initial condition $f(0,T), T \geq 0$, today's forward rate curve. In the so-called *Musiela parameterization* we introduce the forward rate at fixed maturity x:

$$r(t,x) = f(t, t+x) \tag{7}$$

This satisfies

$$dr(t,r) = \mu(t,x)dt + \sigma(t,x)dw_t \tag{8}$$

where

$$\sigma(t,x) = \sigma^0(t,t+x), \quad \mu(t,t+x) = \mu^0(t,t+x) + \frac{\partial}{\partial x} r(t,x).$$

Generally, (**??**) or (**??**) is an infinite-dimensional system. Several authors have investigated special forms of the volatility function under which the solution can be expressed in finite-dimensional form. Recent papers in this direction - which include references to earlier work - are Inui and Kijima [**?**] and Chiarella and Kwon [**?**]. In [**?**] it is shown that if

$$\frac{\partial}{\partial T}\sigma_i^0(t,T) = \kappa_i(T)\sigma_i^0(t,T) \tag{9}$$

for some non-random function κ_i, and the spot volatility $\sigma_i^0(t,t)$ takes the form

$$\sigma_i(t,t) = \nu(t, f(t,t))$$

then (**??**) can be transformed into a $2n$-dimensional Markov system. In [**?**] this result is generalized to volatilities of the form

$$\sigma_i^0(t,T) = \nu(t,T,f(t,t+\varsigma_1),\ldots,f(t,t+\varsigma_n)) \tag{10}$$

where the functions σ_i^0 satisfy, as functions of T, higher order linear differential equations generalizing (**??**).

A major new approach to these questions was initiated by Björk et al. [**?**],[**?**]. Using the Musiela parameterization, the function $x \to r(t,x)$ is considered as an element of a Hilbert Space $\mathcal{H}$ of functions with a weighted Sobolev-type norm such that ∂/∂_k is a bounded operator. The volatility is assumed to be a function $\sigma^1 : \mathcal{H} \to R^n$ and then (**??**) can be written as an $\mathcal{H}$-valued SDE

$$dr_t = \mu^1(r_t)dt + \sum_i \sigma_i^1(r_t) \circ dw_t^i \tag{11}$$

in Stratonovich form ('$\circ$' denotes the Stratonovich integral). We say that equation (**??**) has a finite-dimensional realization when there exists an R^d valued process (Z_t) satisfying an SDE

$$dZ_t = a(t, Z_t)dt + \sum_i b_i(t, Z_t) \circ dw_t^i, \tag{12}$$

and a map $G : R^d \to \mathcal{H}$ such that $r_t = G(z_t)$ satisfies (**??**). Assuming that μ^1 and σ_i^1 in (**??**) are smooth vector fields, a necessary and sufficient condition for the existence of a finite-dimensional realization in a neighbourhood of the initial point r_0 is

$$\dim\{\mu^1, \sigma_1^1, \ldots, \sigma_n^1\}_{\mathrm{LA}} < \infty \tag{13}$$

where $\{\cdots\}_{\mathrm{LA}}$ denotes the Lie algebra spanned by the vector fields in the brackets.

Various special cases of this result are studied in [**?**], [**?**]. In particular, if $\sigma_i^1(r, x)$ are separable functions, i.e.

$$\sigma_i^1(r, x) = \sum_{j=1}^{N} \varphi_{ij}(r)\lambda_j(x) \quad i = 1, \ldots, n \tag{14}$$

for scalar functions φ_{ij} and smooth vector fields λ_j, then condition (9) is satisfied if all the functions λ_j are 'quasi-exponential', i.e. take the form $c'e^{Ax}b$ for some square matrix A and vectors b, c. It turns out that this covers the case of Chiarella and Kwon [**?**].

2.3 Libor Market Models

3 Market Data and Yield Curve Generators

Table **??** shows the discount factor curve for sterling (GBP) obtained from a London bank on 29 November 2002. The data used in constructing the yield curve consists of

- Short-term interest rates
- Short sterling futures
- Swap rates

Table **??** identifies which data is used for which points. The longer end of the curve is determined entirely by swap rates. Table **??** shows the discount factors up to 10 years. The curve stretches out to 52 years, but there are only 5 data points beyond 10 years (swap rates for 12, 15, 20, 40 and 52 years.) Constructing the yield curve is a black art, covered briefly in section 4.4 of Hull [**?**] but not described in detail in any textbook. Roughly, one employs a 'bootstrapping' procedure to determine points on the curve inductively, using interpolation where necessary. For example, the GBP 3-year swap rate on 29 November 2002 was 4.673%. Let $t_1, \ldots, t_6$ denote the semi-annual coupon dates associated with this swap, and suppose we have determined $d_1, \ldots, d_4$ the discount factors at times $t_1, \ldots, t_4$. Then we know that

$$0.04673 = \frac{(1 - d_6)}{\sum_i \theta_i d_i + \theta_5 d_5 + \theta_6 d_6} \tag{15}$$

where θ_i are the accrual factors in the market day-count convention.

We have no direct information about d_5, so we assume that the corresponding rate interpolates linearly between d_4 and d_6, i.e. if we define r_i by

$$d_i = \exp(-r_i t_i) \tag{16}$$

then

$$r_5 = r_4 + \frac{(t_6 - t_4)}{(t_6 - t_4)} r_4 \tag{17}$$

If we know $d_1, \ldots, d_4$ then (**??**),(**??**),(**??**) determine d_5 and d_6.

The short sterling future, traded on LIFFE, settles into the 3-month BBA Libor rate at maturity (third Wednesday of the month in March, June, September and December). The futures price is the expected price in the risk-neutral measure (corresponding to a continuously-compounding money market numéraire), whereas the forward rate is the expected rate in the forward measure, so a convexity adjustment is needed to obtain the implied forward rate from the futures prices. The latter[1] are shown in table **??** together with the convexity correction $\sigma^2 t_1 t_2 / 2$ derived from the Ho-Lee model with short rate volatility $\sigma = .012$ (see [**?**],section 21.16). The correction is subtracted from the futures rate.

The problem with the bootstrapping procedure from a theoretical standpoint is that it immediately introduces arbitrage into the model if we are prepared to trade zero-coupon bonds at interpolated prices. To demonstrate this, suppose we have an arbitrage-free model for two zero-coupon bonds maturing at times $T_1 < T_2$. Let $y_1(t), y_2(t)$ be the yields, so that

$$p(t, T_1) = e^{y_1(t)(T_1 - t)}, \quad p(t_1, T_2) = e^{y_2(t)(T_2 - t)}.$$

Take $T \in (T_1, T_2)$ and define the interpolated price as

$$p(t, T) = e^{y(t)(T-t)}$$

where

$$y(t) = \frac{T_2 - T}{T_2 - T_1} y_1(t) + \frac{T - T_1}{T_2 - T_1} y_2(t) = (1 - \alpha) y_1(t) + \alpha y_2(t).$$

Taking the T_2-bond as numéraire , absence of arbitrage demands that $p(t, T)/p(t, T_2)$ be a martingale in the T_2-forward measure write

$$\frac{p(t, T)}{p(t, T_2)} = \phi(t, y_1 Y_2(t)) = \exp((\beta_1 + t) y_1 + (\beta_2 + \beta_3 t) y_2),$$

[1] recall that futures prices are quoted in the form 100(1-r) where r is the rate.

where $\beta_1 = -(1-\alpha)T, \beta_2 = (1-\alpha), \beta_3 = T + T_2, \beta_4 = -(1+\alpha)$ If $y_t = (y_1(t), y_2(t))$ is a continuous semimartingale with decomposition $y_i(t) = M_i(t) + A_i(t)$, then by the Ito formula the

$$\begin{aligned} d\phi/\phi &= (\beta_2 y_1 + \beta_4 y_2)dt + (\beta_1 + \beta_2 t)dA_1 + (\beta_3 + \beta_4 t)dA_2 + (\beta_1 + \beta_2)^2 d < y_1 > \\ &\quad + \beta_3 + \beta_4 t)^2 d < y_2 > + 2(\beta_1 + \beta_2 t)(\beta_3 + \beta_4 t)d < y_1, y_2 > + dM_t \\ &\equiv dA_t + dM_t \end{aligned}$$

where M_t is a local martingale. For absence of arbitrage, A_t must vanish. However, the coefficients β_i depend on T, and it is not generically the case that $A_t \equiv 0$ for *all* T, given a fixed moel for y_t. Thus there will be arbitrage opportunities in the model. Presumably market friction in the form of bid-ask spreads is too great to allow these opportunities to be realized in practice.

Date	Discount Factor	
29-Nov-02	1.000000	
2-Dec-02	0.999666	
6-Dec-02	0.999221	
18-Dec-02	0.997905	f
31-Dec-02	0.996479	
2-Jan-03	0.996246	
19-Mar-03	0.988027	f
18-Jun-03	0.978104	f
17-Sep-03	0.967855	f
17-Dec-03	0.957271	f
31-Dec-03	0.955608	
2-Jan-04	0.955370	
17-Mar-04	0.946341	f
16-Jun-04	0.935175	f
15-Sep-04	0.923899	f
15-Dec-04	0.912655	f
29-Nov-05	0.870227	s
29-Nov-06	0.827609	s
29-Nov-07	0.786713	s
28-Nov-08	0.747604	s
30-Nov-09	0.709881	s
29-Nov-10	0.674063	s
29-Nov-12	0.606671	s

Table 1: Sterling discount factors, 29 November 2002, 0–10 years. Points labelled 'f','s' are derived from futures and swap rates respectively

Maturity Date	Future (close)	Convexity Adjustment	Implied Forward
Dec '02	95.990	0.01	4.010
Mar '03	95.990	0.12	4.009
Jun '03	95.850	0.32	4.147
Sep '03	95.650	0.60	4.344
Dec '03	95.430	0.98	4.560
Mar '04	95.250	1.45	4.735
Jun '04	95.140	2.00	4.840
Sep '04	95.070	2.64	4.903

Table 2: Short Sterling Futures, 29 November 2002

4 A One-Dimensional Model

In this section we construct an arbitrage free term structure from the yield associated with a zero coupon bond maturing at time T_2. We are given a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\in[0,T_2]}, \mathcal{P})$ satisfying the usual conditions, let $\{W_t\}_{t\in[0,T_2]}$ denote a one-dimensional standard $\mathcal{P}$-Brownian Motion. The yield follows a Markov process under $\mathcal{P}$

$$dy_t = \mu(y_t)dt + \sigma(y_t)dW_t \tag{18}$$

and the price of this bond is given by

$$P(t,T_2) = e^{-y_t(T_2-t)} \tag{19}$$

This bond obviously satisfies the terminal condition $P(T_2,T_2) = 1$. Let other zero coupon bonds have prices defined as a function $F \in C^{1,2}([0,\infty)\times D; \Re^+)$, to be determined, of the yield in **??** (D is some subset of $\Re$):

$$P(t,T) = F^T(t,y_t) \tag{20}$$

We supress the upper T index and we let $\tau_2 = T_2 - t$.

Proposition 1 *Fix $T \in [0,T_2]$. Let $F \in C^{1,2}([0,T_2]\times D)$ be a function of the yield y_t in* **??** *and solving the following PDE*

$$\tilde{\mathcal{A}}_t F(t,y) - F(t,y)h(t,y_t) = 0, \quad (t,y) \in [0,T]\times D \tag{21}$$

$$F(T,y) = 1, \quad y \in D \tag{22}$$

where

$$\begin{aligned} \tilde{\mathcal{A}}_t &= \frac{\partial}{dt} + (\mu(y_t) + \tau_2\sigma^2)\frac{\partial}{\partial y} + 1/2\sigma\frac{\partial^2}{\partial y^2} \\ h(t,y_t) &= y_t - \tau_2\mu(y_t) - 1/2\tau_2^2\sigma^2. \end{aligned}$$

Then if we define $P(t,T)$ and $P(t,T_2)$ by **??** *and* **??** *the ratio $P(t,T)/P(t,T_2)$ follows a martingale. Consequently, this model for $P(t,T)$ is arbitrage free.*

Furthermore the term structure can be represented as

$$P(t,T) = F(t,y_t) = \tilde{E}[\exp - \int_t^T h(t,y_s)ds] \tag{23}$$

where the expectation is taken under the forward measure given by $\frac{d\tilde{P}}{dP} = \mathcal{E}(\tau_2\sigma^2(y_t)dW_t)$

PROOF: Applying Ito's product rule to the ratio of bond prices we obtain

$$\begin{aligned} d(\frac{P(t,T)}{P(t,T_2)}) &= d(e^{y_t(T_2-t)}F) = e^{y_t\tau_2}dF + Fd(e^{y_t\tau_2}) + d < F, e^{y_t\tau_2} > \\ &= \frac{1}{P(t,T_2)}[\frac{\partial F}{d_t} + (\mu + \tau_2\sigma^2)\frac{\partial F}{\partial y} + 1/2\frac{\partial^2 F}{\partial y^2}\sigma^2]dt + \frac{1}{P(t,T_2)}\frac{\partial F}{\partial y}\sigma dW_t \\ &+F\frac{1}{P(t,T_2)}[-y_t + \tau_2\mu + 1/2\tau_2^2\sigma^2]dt + F\frac{1}{P(t,T_2)}\tau_2\sigma dW_t \end{aligned}$$

In order to obtain a martingale we need to cancel the drift term in the above ratio which gives us the above PDE

$$\frac{\partial F}{\partial t} + (\mu + \tau_2\sigma^2)\frac{\partial F}{\partial y} + 1/2\sigma^2\frac{\partial^2 F}{\partial^2 y} - F\{y_t - \tau_2\mu - 1/2\tau_2^2\sigma^2\} = 0 \tag{24}$$

The final result follows by applying the Feynman-Kac formula under the terminal condition in **??** $\diamondsuit$.

The above defines an arbitrage free term structure which depends on the drift and volatility of the yield and notice that we don't need to assume any function for the short rate. The formula also has an intuitive interpretation: it is the expected compounded percentage growth rate of the discont factor, ie. of $e^{y\tau_2}$. Since we have to rule out arbitrage in a continuum of assets we expect some infinitesimal object to do the job for us. It used to be the short rate which also happened to be the expected growth rate of the discount factor, but now we have discovered that we can not use the yield directly (it is not a limit) but the growth rate of the inverse of the numeraire instead. We hope to exploit this fact in order to extend the result to multifactor term structures where the convenience of the short rate becomes more obvious. This will allow us to answer the particular question of whether arbitrage opportunities are being created when traders construct term structures by interpolating between the market rates.

4.1 Example

We are going to illustrate the previous result with a particular example where an explicit formula for the term structure is derived. We will assume that the process for the yield of the T_2 bond follows a mean reverting Ornstein-Uhlenbeck diffusion process under the forward martingale measure

$$dy_t = (\lambda - ay_t)dt + \sigma dW_t$$

where λ, a and σ are constants. Therefore the yield follows a Gaussian mean reverting process as in section 3. The PDE that we need to solve is the following

$$F_t + F_y(\lambda - ay + \tau_2\sigma^2) + 1/2\sigma^2 F_{yy} - Fh(y_t, t) = 0$$

where $h(y_t, t) = (1 + a\tau_2)y_t - \lambda\tau_2 - 1/2\sigma^2\tau_2^2$. As it is an affine function of the yield we can use the result of Duffie and Kan [?] which tells us that the term structure is exponential affine, ie. of the form $P(t, T) = \exp[m(t, T) - n(t, T)y_t]$.

We use the method of the Riccati equations. Assuming the term structure is in fact exponential affine of the form above we can rewrite the PDE in terms of the m and n functions

$$\dot{m} - \dot{n}y - n(\lambda - ay + \tau_2\sigma^2) + 1/2\sigma^2 n^2 - h(y) = 0$$

we can then split the above equation into two ordinary differential equations (ODE's), one with y dependence the other one without it.

$$\dot{n} = an - (1 + a\tau_2) \tag{25}$$

$$\dot{m} = (\lambda + \tau_2\sigma^2)n - 1/2\sigma^2 n^2 - \tau_2\lambda - 1/2\tau_2^2\sigma^2 \tag{26}$$

with terminal condition $n(T, T) = m(T, T) = 0$. At this point the main advantage of this model can be appreciated; in traditional models the above ODEs **??**, **??** would contain the parameters defining the short rate which are not available in market data and which are not provided endogenously by the model either. The ODEs above are determined by the drift and volatility of the market yield alone which can be obtained directly from time series data. Denote $\tau = T - t$. Equation **??** has solution

$$n(t, T) = \tau_2 - (T_2 - T)e^{-a\tau} \tag{27}$$

We can now substitute this equation into **??** and integrate it to obtain $m(t, T)$.

$$m(t, T) = -\int_t^T (\lambda + \tau\sigma^2 n^2)ds + 1/2\sigma^2 \int_t^T n^2 ds + \int_t^T \tau_2\lambda + 1/2\tau_2^2\sigma^2 ds$$

The first integral is given by

$$-\frac{\lambda}{a}(T_2 - T)(1 - e^{-a\tau}) + \lambda T_2\tau - \frac{\lambda}{2}(T^2 - t^2) - \frac{\sigma^2}{a}(T_2 - T)\{(T_2 - T) - \tau_2 e^{-a\tau} + \frac{1}{a^2}(1 - e^{-a\tau})\}$$
$$+\sigma^2\{T_2^2\tau + 1/3(T^3 - t^3) - T_2(T^2 - t^2)\}$$

The second one

$$= \frac{\sigma^2}{4a}(T_2 - T)^2(1 - e^{-2a\tau})) + \frac{\sigma^2}{2}[T_2^2\tau + 1/3(T^3 - t^3)]$$
$$-\frac{\sigma^2}{a}T_2(T^2 - t^2) - \frac{\sigma^2}{a}(T_2 - T)[(T_2 - T) - \tau_2 e^{-a\tau} + 1/a(1 - e^{-a\tau})]$$

and the last one

$$-\lambda T_2\tau - \frac{\lambda}{2}(T^2 - t^2) + \frac{\sigma^2}{2}[T_2^2\tau + 1/3(T^3 - t^3) - T_2(T^2 - t^2)]$$

Putting all the terms together we have the following expression for m and hence we have solved for the term structure

$$P(t,T) = e^{m(t,T)-n(t,T)y_t} \tag{28}$$

where n is given by **??** and m is given by

$$m(t,T) = \frac{\sigma^2}{4a}(T_2 - T)^2(1 - e^{-2a\tau})] + \frac{\lambda}{a}(T_2 - T)(1 - e^{-a\tau}), \tag{29}$$

which has the additional property that if we set $T = T_2$ we recover the original bond price

$$E[\exp(-\int_t^{T_2} h(y_s,s)ds)|h(y_t,t)] = e^{-y(T_2-t)}$$

We can use this example to find the price of a call option on a zero coupon bond following the method as outlined in Jamshidian [?]. From **??** and **??** we have that

$$\begin{aligned} \ln P(t,T_2) &= -y_t(T_2 - t) \\ \ln P(t,T) &= m(t,T) - n(t,T)y_t. \end{aligned}$$

Since the state variable is assumed to be Gaussian, the bond prices follow log-normal processes and so we can apply Black-Scholes like formulae. We have that

$$\sigma_p^2 \equiv Var[\ln P(t,T)] = [(T_2 - T)e^{-a\tau} + \tau_2]^2\sigma(y_t)^2 \tag{30}$$

Hence the price of a call with strike K to be settled at date T on a zero coupon bond maturing at time U is

$$C(y,t,T,U,K) = P(y,t,U)N(h) - KP(y,t,T)N(h - \sigma_p) \tag{31}$$

where $h = \log(KP(y,T,U)/P(y,t,T))/\sigma_p + \sigma_p/2$.

5 Multi-Dimensional Models

The objective now is to extend the previous result to a multifactor case. It is clear from empirical evidence that it takes three factors to be able to generate a rich enough class of curves to describe actual term structure movements. With these considerations in mind we set out to model an n-dimensional term structure assuming that we live in the risk neutral world so that the market prices of risk are zero. To be more specific the next problem in order of difficulty is the following: given an n-dimensional process for the yields of bonds maturing at times $0 < T_1 < ... < T_n$

$$dy_t = \mu(y_t)dt + \sigma(y_t)dW_t \tag{32}$$

where $y_t \in R^n$, $W_t \in R^n$, $\mu : D \to R^n$, $\sigma : D \to R^{n\times n}$ where D, the state variables' domain, is a subset of R^n and (μ, σ) is regular enough for a strong solution to **??** to exist. We also assume a mild non-degeneracy condition $a(y) := \sigma(y)\sigma(y)^T > 0$. The bond prices are given as before by

$$P(t,T_i) = e^{-y_i(T_i-t)} \tag{33}$$

where $y_i \equiv y_i(t)$ for $i = 1,..,n$.

Proposition 2 *A necessary condition on (μ, σ) to rule out arbitrage opportunities among the bond prices in* **??** *using the bond with the longest maturity, T_n, as our numéraire is*

$$-< v(t) \cdot \mu(t, y) >= y_i - y_n + 1/2 v^T(t) a(y) v(t) \tag{34}$$

where $v(t) \equiv (-(T_i - t), T_n - t)^T$.

Proof: We require the following ratios to follow martingales:

$$\frac{P(t, T_i)}{P(t, T_n)} = \exp(y_2(T_n - t) - y_i(T_i - t)). \tag{35}$$

Let

$$f(t, y_i, y_n) = \exp((T_n - t)y_n - (T_i - t)y_i).$$

In order to apply Ito's Lemma we need the following

$$\begin{aligned}
\frac{\partial f}{\partial t} &= f \cdot (y_i - y_n) &(36)\\
\frac{\partial f}{\partial y_i} &= -f \cdot (T_i - t) &(37)\\
\frac{\partial f}{\partial y_n} &= f \cdot (T_n - t) &(38)\\
\frac{\partial^2 f}{\partial y_i^2} &= f \cdot (T_i - t)^2 &(39)\\
\frac{\partial^2 f}{\partial y_i \partial y_n} &= -(T_i - t)(T_n - t) \cdot f &(40)
\end{aligned}$$

$$\frac{df(t, y_t)}{f(t, y_t)} = ((y_i - y_n) - \mu_i(T_i - t) + \mu_n(T_n - t) + 1/2(T_i - t)^2 a_{ii} + 1/2(T_n - t)^2 a_{nn}$$

$$-(T_i - t)(T_n - t)a_{in})dt + \frac{\nabla f(t, y_t)^T}{f(t, y_t)} \sigma dW \tag{41}$$

and therefore under **?? ??** will follow a martingale $\diamondsuit$.

Condition **??** imposes a partial restriction on the drift vector in that it determines μ_i in terms of μ_n. Note also that μ_i are time varying. Having found the No Arbitrage condition on the state variables we can now proceed to "fill in the gaps", that is to construct the term structure from the state variables given in **??**. Assume the term structure is at least a twice differentiable function $F \in C^{2,1}(\Re^n \times [0, \infty); \Re^+)$ of the state variables, ie. $F(t, y_t) = P(t, T)$. Still using the T_n forward measure we have the next proposition

Proposition 3 *The price of a ZC bond $P(t, T)$ maturing at time $T < T_1$ is the solution to the following PDE*

$$\begin{aligned}
\mathcal{G}_t F(t, y_t) - F(t, y_t) h(y_n, t, T_n) &= 0 &(42)\\
F(y_T, T) &= 1 &(43)
\end{aligned}$$

where

$$
\begin{aligned}
\mathcal{G}_t &= \frac{\partial}{\partial t} + \sum_{i=1}^{n-1} \frac{\partial}{\partial y_i}(\mu_i + 1/2(T_n - t)a_{in}) + \frac{\partial}{\partial y_n}(\mu_n + (T_n - t)a_{nn}) + 1/2 \sum_{i,k=1} a_{ik} \frac{\partial}{\partial y_i \partial y_k} \\
h(t, y_n, T_n) &= y_n - \tau_n \mu - 1/2 \tau_n^2 a_{nn}
\end{aligned}
$$

Furthermore the solution has probabilistic representation as

$$
P(t,T) = F(t,y) = \tilde{E}_1[e^{-\int_t^T h(y_n, s, T_n) ds} | y_t] \tag{44}
$$

where the expectation is taken under the forward measure given by

$$
\frac{\partial \tilde{\mathcal{P}}_1}{\partial \mathcal{P}} = \mathcal{E}(\sum_{i=1}^{n-1} \mu_i + 1/2(T_n - t)a_{in} dW_t^i + (\mu_n + (T_n - t)a_{nn}) dW_t^n)
$$

PROOF: By the fundamental theorem of asset pricing we need the disconted bond price process to follow a martingale under the T_n forward measure

$$
d(\frac{F(t, y_t)}{P(t, T_n)}) = d(e^{y_n(T_n - t)} F(t, y_t)) := d(f^T(t, y_1, y_n))
$$

As before we need the following to apply Ito

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= e^{y_n(T_n-t)}(-y_n)F(y,t) + e^{y_n(T_n-t)} \frac{\partial F}{\partial t} \\
\frac{\partial f}{\partial y_i} &= e^{y_n(T_n-t)} \frac{\partial F}{\partial y_i} \\
\frac{\partial f}{\partial y_n} &= e^{y_n(T_n-t)}(T_n - t)F(y,t) + e^{y_n(T_n-t)} \frac{\partial F}{\partial y_n} \\
\frac{\partial^2 f}{\partial y_i^2} &= e^{y_n(T_n-t)} \frac{\partial^2 F}{\partial y_i^2} \\
\frac{\partial^2 f}{\partial y_n^2} &= e^{y_n(T_n-t)}(T_n - t)^2 F + 2(T_n - t)\frac{\partial F}{\partial y_n} + e^{y_n(T_n-t)} \frac{\partial^2 F}{\partial y_n^2} \\
\frac{\partial^2 f}{\partial y_n \partial y_i} &= e^{y_n(T_n-t)}(T_n - t)\frac{\partial F}{\partial y_i} + e^{y_n(T_n-t)} \frac{\partial F}{\partial y_n \partial y_1} \\
\frac{\partial^2 f}{\partial y_i \partial y_j} &= e^{y_n(T_n-t)} \frac{\partial^2 F}{\partial y_i \partial y_j}
\end{aligned}
$$

The differential then looks like

$$
\begin{aligned}
d(e^{y_n(T_n-t)}) = \frac{1}{P(t,T_n)}[-y_n F + \frac{\partial F}{\partial t} + \sum_{i=1}^{n} \frac{\partial F}{\partial y_i}\mu_i + (T_n - t)F\mu_n + 1/2 \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik} \frac{\partial^2 F}{\partial y_i \partial y_k} \\
+1/2(T_n - t)^2 F a_{22} + 1/2 \sum_{i=1}^{n-1} (T_n - t)\frac{\partial F}{\partial y_i} a_{in} + (T_n - t)\frac{\partial F}{\partial y_n} a_{22}]dt + \nabla f \sigma dW_t
\end{aligned} \tag{45}
$$

So the price of a zero coupon bond maturing at time $T < T_1$ will be given by the solution to **??**. The representation **??** is granted by the Feynman-Kac formula. $\diamondsuit$

Notice again that the exponential inside the expectations formula only depends on the parameters entering the diffusion equation for state variable of the numeraire. If we want all the state varables to enter the dyanamics of the term structure, we need to make the drift of the numeraire depend on the other state variables.

What we have been considering so far is a relatively easy task since our bond was maturing before the earliest factor-maturity. If the bond we are trying to price has maturity $T_{i-1} < T < T_i$ then we will be losing state variables as the bonds start maturing and we need the following

Proposition 4 *Let $P(t,T)$ be a zero coupon where $t < T_1 < T_{i-1} < T_i$. Then its arbitrage free price is given by*

$$P(t,T) = E_1[e^{-\int_t^{T_1} h^1 ds} E_2(e^{-\int_{T_1}^{T_2} h^2 ds} E_3\{e^{-\int_{T_2}^{T_3} h^3 ds} \ldots\ldots E_i[e^{-\int_{T_{i-1}}^{T} h^i ds} |\mathcal{F}_{T_{i-1}}] \ldots |\mathcal{F}_{T_2}\}|\mathcal{F}_{T_1})|\mathcal{F}_t] \tag{46}$$

where E_i denotes the expectation taken under the i-th forward measure given by

$$\frac{d\tilde{\mathcal{P}}_i}{d\mathcal{P}} = \mathcal{E}(\sum_{k=i}^{n-1} \mu_k + 1/2(T_n - t)a_{in} dW_t^k + (\mu_n + (T_n - t)a_{nn})dW_t^n) \tag{47}$$

Proof: We have a string of PDE's that we have to solve backwards where the first one solved becomes a boundary condition for the next one, and so on. The subindex i means we have to use the state variable from i onwards since the others have matured already. i.e. we need to solve

$$P(T_{i-1}, T) = E_i[e^{-\int_{T_{i-1}}^{T} h^i ds} |\mathcal{F}_{T_{i-1}}]$$

and use the solution as a boundary condition for the next PDE, ie.

$$P(T_{i-2}, T) = E_{i-1}[e^{-\int_{i-2}^{T_{i-1}} h^{i-1} ds} P(T_{i-1}, T)|\mathcal{F}_{T_{i-2}}]$$

and so on until

$$P(t,T) = E_1[e^{-\int_t^{T_1} h^1 ds} P(T_1, T)|\mathcal{F}_t]$$

$\diamondsuit$

5.1 Example

Let us have two state variables (y_1, y_2) where the bond maturing at time T_2 will act as numéraire . We want to price a bond with maturity T where $T < T_1 < T_2$. From () we need only specify the diffusion matrix (which we will assume constant with coefficients denoted by $a_{ij} = \sigma_i \sigma_j^*$ and one of the drifts. We take the drift for y_2 to be $\mu_2 = \lambda - a_1 y_1 - a_2 y_2$ where λ, a_1, a_2 are constants. The no arbitrage condition between the state variables forces us to specify drift for y_1 to be

$$\mu_1 = \tau_1^{-1}\{y_1 - y_2 + \tau_2(\lambda - a_1 y_1 - a_2 y_2) + 1/2(a_{11}\tau_1^2 + a_{22}\tau_2^2 + a_{12}\tau_1\tau_2)\}$$

Rearranging we have

$$\mu_1 = \alpha + < \beta, y > \tag{48}$$

where

$$\begin{aligned}
\alpha &= \tau_2\lambda + 1/2(a_{11}\tau_1^2 + a_{22}\tau_2^2 + a_{12}\tau_1 tau_2) \\
\beta_1 &= 1/\tau_1 - \tau_2 a_1 \\
\beta_2 &= -1/\tau_1 - \tau_2 a_2
\end{aligned}$$

Our PDE now reads

$$\dot{m} - \dot{n}_1 y_1 - \dot{n}_2 y_2 - n_1(\alpha - 1/2\tau_1 a_{12}) - n_1\beta_2 y_2 - \lambda n_2 + n_2 a_1 y_1 + n_2 a_2 y_2 - y_2 + \tau_2\lambda + \tau_2 < a, y >$$

$$+1/2\tau^2 a_{22} + diffussionterms = 0$$

Solving the next system of ODEs will give us the solution to the above PDE

$$\dot{m} - n_1(\alpha - 1/2\tau_1 a_{12}) - \lambda n_2 + \tau_2\lambda + 1/2\tau^2 a_{22} + diff = 0 \quad (49)$$

$$-\dot{n}_1 - n_1\beta_1 + n_2 a_1 + \tau_1 a_1 = 0 \quad (50)$$

$$-\dot{n}_2 - n_1\beta_2 + n_2 a_2 - 1 + \tau_2 a_2 = 0 \quad (51)$$

We can write the last two equations more succintly in matrix form, however notice that we cannot compute an explicit solution since the matrix does not commute in the following sense $A(t_1)A(t_2) = A(t_2)A(T_1)\forall t_1, t_2$. We need to use some numerical methods.

6 Models Based on Fixed-Maturity Data

As we saw in Section **??**, the market data comes in two forms: data such as futures rates associated with fixed maturity dates, and data such as Libor or swap rates where the quotes refer to a fixed horizon from today (for example, the 5-year swap rate). Our approach in Sections **??** and **??** deals with the former class, while here we give a very brief treatment of the latter in a one-dimensional case. The problem is more tricky in that there is no obvious numéraire , since successive daily quotes actually refer to different traded assets. We deal with this as follows.

Let θ be a fixed time (say 3 months) and z_t be the θ-maturity zero coupon yield, so that z_t is related to the ZC bond price and Libor rate at t by

$$p(t, t+\theta) = \frac{1}{1+\theta L_t} = e^{-\theta z_t}.$$

We consider the sequence of times $T_n = n\theta, n = 0, 1, \ldots$ and suppose that z_t satisfies the stochastic differential equation

$$dz_t = \mu(t, z_t)dt + \sigma(t, z_t)dw_t \quad (52)$$

where conditions for a unique strong solution are assumed. In (**??**), the scalar process (z_t) is a Brownian motion in the T_n-forward measure P_n on the interval $(T_{n-1}, T_n]$. We denote by $\mathcal{A}$ the time-varying differential generator for z_t, i.e. for $f \in C^{1,2}$

$$\mathcal{A}f(t, z) = \frac{\partial f}{\partial t} + \mu(t, z)\frac{\partial f}{\partial z} + \frac{1}{2}\sigma^2(t, z)\frac{\partial^2 f}{\partial z^2}.$$

The problem here is that $p(t, t+\theta)$ is a different traded asset for each t, whereas to get martingale conditions we need the price process for a *single* traded asset. For this purpose it is convenient to use the 'rolling reinvestment' numéraire[2] corresponding to always investing in the T_n-ZC bond of current shortest maturity. ZC bonds of arbitrary maturity are then expressed as follows.

Proposition 5 *For $j < k$ let $t \in [0, T_j]$ and $T \in [T_{k-1}, T_k]$, and denote $q_n = p(T_{n-1}, T_n)$. Then*

$$p(t,T) = p(t,T_j)E_j\left(q_{j+1}E_{j+1}\left(q_{j+2}\cdots E_{k-1}\left(q_k\left(\left.\frac{1}{p(T,T_k)}\right|\mathcal{F}_{T_{k-1}}\right)\right|\mathcal{F}_{T_{k-2}}\right)\cdots\left|\mathcal{F}_{T_j}\right)\right|\mathcal{F}_t\right) \tag{53}$$

PROOF: $p(s,T)/p(s,T_k)$ is a P_k-martingale for $s \in [T_k - 1, T_k]$, so

$$p(T_{k-1},T) = q_k E\left(\left.\frac{1}{p(T,T_k)}\right|\mathcal{F}_{T_{k-1}}\right).$$

The result follows by backward iteration. $\diamondsuit$

Proposition **??** shows that in order to specify the whole term structure it is enough to specify $p(t, T_k)$ for $t \in [T_{k-1}, T_k]$ and $k = 1, 2, \ldots$. Let us assume that there are functions F, G_T such that for $0 \le t \le T_1$

$$p(t,T_1) = \frac{1}{F(t,z_t)}, \tag{54}$$

where F is a $C^{1,2}$ function such that $0 < F(t,z) \le 1$, while for $0 \le t \le T \le T_1$,

$$p(t,T) = G_T(t,z_t) \tag{55}$$

where G_T satisfies the same conditions as F. We shall assume that the relationship between ZC bond values and the state variable z_t is the same in each interval T_{k-1}, T_k, so that for $T_{k-1} \le t \le T \le T_k$,

$$p(t,T_k) = \frac{1}{F(t-T_{k-1},z_t)}, \quad p(t,T) = G_{T-T_{k-1}}(t-T_{k-1},z_t).$$

It follows from Proposition **??** that if we have specified the functions F and G_T the we have determined $p(t,T)$ for arbitrary t, T.

Proposition 6 *Fix $T \in [0,\theta]$. Let $F \in C^{1,2}([0,\theta] \times R$ be a function such that $F(0,z) = e^{\theta z}$, $0 < F \le 1$, $F(\theta, z) = 1$ and $\mathcal{A}F(t,z) \le 0$ for all $(t,z) \in [0,\theta] \times R)$. Let $G \in C^{1,2}([0,T] \times R$ satisfy the equation*

$$\begin{aligned} \frac{\partial G}{\partial t} + \mathcal{J}G + \left(\frac{1}{F}\mathcal{A}F\right)G &= 0, \quad (t,z) \in [0,T] \times R \\ G(T,z) &= 1, \quad z \in R \end{aligned}$$

[2] cf. Hull [?], Section 22.3

where $\mathcal{J}$ is the operator

$$\mathcal{J} = \left(\mu + \sigma^2 \frac{1}{F}\frac{\partial F}{\partial z}\right)\frac{\partial}{\partial z} + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial z^2}.$$

*Then $0 < G(t,z) \leq 1$, and if we define $p(t,T), p(t,\theta)$ by (**??**),(**??**), where z_t satisfies (**??**), then $t \mapsto p(t,T)/p(t,T_1)$ is a martingale. Consequently, this model for $p(t,T)$ is arbitrage-free.*

PROOF: Assuming that a function F exists satisfying the conditions stated above (this is the subject of the next paper "A PDE problem") the result follows by applying Itô to the product FG. $\diamondsuit$

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