Some examples of risk measures via $g-$expectations

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Abstract

This paper shows how $g-$expectations and conditional $g-$expectations (introduced by Peng [21]) provide examples of some families of static and dynamic risk measures. Some sufficient conditions for a dynamic risk measure to come from a $g-$expectation are deduced as a straightforward corollary of a result of Coquet et al. [7] on nonlinear expectations.

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1 Introduction

Different families of risk measures have been proposed in literature. Coherent risk measures were introduced by Artzner et al. [1] and Delbaen [9] as axiomatic tools able to quantify riskiness of financial positions. Weakening coherence axioms, Frittelli [15] proposed sublinear risk measures. Finally, convex risk measures were studied firstly by Heath [18] and later, in general probability spaces, by Föllmer & Schied [12], [13], [14] and, indipendently, by Frittelli & Rosazza [16], [17].

We wish to emphasize that all the risk measures just cited are static ones. Furthermore, their definition depends only on the axioms imposed by the authors.

In this paper, we will show how "nonlinear expectations", introduced by Peng [21] via Backward Stochastic Differential Equation (shortly, BSDE), provide examples of risk measures. In Section 1.2, we recall from Peng [21] the definition of $g-$expectation and conditional $g-$expectation, i.e. particular static and dynamic "nonlinear expectations" depending on a functional g.

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In Section 2, some properties of $g-$expectations are recalled by Briand et al. [4], Coquet et al. [7] and Peng [21] and other ones are established. In particular, we will see that these properties depend on the axioms imposed on the functional g.

In Section 3, *we will show that, under suitable hypothesis on g, examples of coherent and convex risk measures can be obtained through $g-$expectations.* On the contrary, $g-$expectations don't provide examples of sublinear (but not coherent!) risk measures.

Moreover, conditional $g-$expectation, which in some sense can be seen as the dynamic version of $g-$expectation, "suggests" an axiomatic definition of a dynamic risk measure. A discussion about this definition and other dynamic risk measures present in literature is postponed to Section 4 and to [17].

As previously, under suitable hypothesis on g, conditional $g-$expectations provide *examples of dynamic risk measures* (see Section 4 for details). As a simple rewriting of a result of Coquet et al. [7], in Section 4.1 we will give *some sufficient conditions under which a dynamic coherent or convex risk measure comes from a conditional $g-$expectation.*

1.1 Notations and axioms

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(B_t)_{t\geq 0}$ a standard $d-$dimensional Brownian motion.

Let $\{\mathcal{F}_t^B\}_{t\geq 0}$ be the filtration generated by $(B_t)_{t\geq 0}$, i.e.

$$\mathcal{F}_t^B \triangleq \sigma\left\{B_s; 0 \leq s \leq t\right\}, \ \forall t \geq 0,$$

and $\{\mathcal{F}_t\}_{t\geq 0}$ the augmented filtration associated to $\{\mathcal{F}_t^B\}_{t\geq 0}$, i.e.

$$\mathcal{F}_t \triangleq \sigma\left\{\mathcal{F}_t^B \cup \mathcal{N}\right\}, \ \forall t \geq 0, \tag{1}$$

where $\mathcal{N}$ is the collection of all $P-$null sets. We remind (see, for instance, Karatzas & Shreve [19]) that the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ previously defined is continuous and that $(B_t)_{t\geq 0}$ is also a $\{\mathcal{F}_t\}_{t\geq 0}-$Brownian motion. In particular, $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the so called "usual conditions", i.e. right-continuity and $\mathcal{F}_0 \supseteq \mathcal{N}$.

We will now introduce a class of non linear backward stochastic differential equations. We will take as basic references on this subject El Karoui [10], Pardoux & Peng [20] and Peng [21], and we will essentially keep the same notations of the last author.

Let $T > 0$ be a fixed horizon of time. Let $L^2(\mathcal{F}_t) = L^2(\Omega, \mathcal{F}_t, P)$ (with $t \in [0,T]$) denote the space of all real-valued, $\mathcal{F}_t-$measurable and square integrable random variables endowed with the L^2-norm $\|\cdot\|_2$ topology and let $L^2_{\mathcal{F}}(T;\mathbb{R}^n)$ denote the space of all $\mathbb{R}^n-$valued, adapted processes $(V_t)_{t\in[0,T]}$ such that

$$E\left[\int_0^T \|V_t\|^2 \ dt\right] < +\infty,$$

where $\|\cdot\|$ stands for the Euclidean norm on $\mathbb{R}^n$. $\mathbf{1}$ will denote the random variable $P-a.s.$ equal to 1.

In the furthering, except otherwise stated, any equality/inequality involving stochastic processes has to be understood as: "for any fixed $t \in [0,T]$ such an equality/inequality holds true $P-$a.s.". For simplifying notations, we will often omit "$P-$a.s.".

Consider a functional $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfying at least the following usual assumptions (the same as in Coquet et al. [7]). We'll often write $g(t,y,z)$ instead of $g(\omega,t,y,z)$.

"Usual" assumptions on g

(A) g is Lipschitz in (y,z), i.e. there exists a constant $C>0$ such that, $P-a.s.$, for any $t \in [0,T]$ and any (y_0,z_0), $(y_1,z_1) \in \mathbb{R}\times\mathbb{R}^d$,

$$|g(t,y_0,z_0) - g(t,y_1,z_1)| \leq C\left(|y_0-y_1| + \|z_0-z_1\|\right).$$

(B) $g(\cdot,y,z) \in L^2_{\mathcal{F}}(T;\mathbb{R})$ for any $y \in \mathbb{R}$ and any $z \in \mathbb{R}^d$.

(C) $P-$a.s., $\forall t \in [0,T]$ and $\forall y \in \mathbb{R}$, $g(t,y,0)=0$.

Axioms for g

(D) $g(\cdot,y,z)$ is continuous in t for any $\omega \in \Omega$, any $y \in \mathbb{R}$ and any $z \in \mathbb{R}^d$.

(E) g is sublinear in (y,z), i.e.

(E1) positively homogeneous in (y,z):
$P-$a.s., $\forall t \in [0,T]$, $\forall \alpha \geq 0$, $\forall (y,z) \in \mathbb{R}\times\mathbb{R}^d$,
$g(t,\alpha y,\alpha z) = \alpha g(t,y,z)$;

(E2) subadditive in (y,z):
$P-$a.s., $\forall t \in [0,T]$, $\forall (y_0,z_0)$, $(y_1,z_1) \in \mathbb{R}\times\mathbb{R}^d$,
$g(t,y_0+y_1,z_0+z_1) \leq g(t,y_0,z_0) + g(t,y_1,z_1)$;

(F) g is convex in (y,z):
$P-$a.s., $\forall t \in [0,T]$, $\forall (y_0,z_0)$, $(y_1,z_1) \in \mathbb{R}\times\mathbb{R}^d$, $\forall \alpha \in (0,1)$,
$g(t,\alpha y_0 + (1-\alpha)y_1, \alpha z_0 + (1-\alpha)z_1) \leq \alpha g(t,y_0,z_0) + (1-\alpha)g(t,y_1,z_1)$;

(G) g does *not* depend on y.

It is well known (see Karatzas & Shreve [19]) that any g satisfying axiom (B) and (D) (continuity) is progressively measurable.

1.2 $g-$expectations

We recall (see Pardoux and Peng [20] -Theorem 4.1, Peng [22] and Coquet et al. [7]) that, under the usual assumptions on g, for every $X \in L^2(\mathcal{F}_T)$ the following backward stochastic differential equation (shortly, BSDE)

$$\begin{aligned} -dY_t &= g(t,Y_t,Z_t)dt - Z_t dB_t, \ \forall\ 0 \leq t \leq T; \\ Y_T &= X, \end{aligned} \qquad (2)$$

or, equivalently,

$$Y_t = X + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \ \forall\ 0 \le t \le T,$$

has a unique solution, i.e. there is a unique pair $(Y_t, Z_t)_{t \in [0,T]} \in L^2_{\mathcal{F}}(T; \mathbb{R}) \times L^2_{\mathcal{F}}(T; \mathbb{R}^d)$ which solves (2).

We recall from Peng [21] the following definitions, which extend the notions of expectations and conditional expectations to nonlinear ones. As we will recall later, the usual expectation corresponds to the case of $g \equiv 0$.

Definition 1 (see Peng; def. 36.1; [21]) .

The $g-$expectation $\mathcal{E}_g : L^2(\mathcal{F}_T) \to \mathbb{R}$ *is defined by*

$$\mathcal{E}_g[X] \triangleq Y_0, \tag{3}$$

where Y_0 *is (the first component of) the solution at* $t = 0$ *of the BSDE (2) with terminal condition* X*.*

Definition 2 (see Peng; def. 36.5; [21]) *For every* $X \in L^2(\mathcal{F}_T)$ *and for every* $t \in [0, T]$*, the* conditional $g-$expectation of X under $\mathcal{F}_t$ *(denoted by* $\mathcal{E}_g[X|\,\mathcal{F}_t]$ *) is defined by*

$$\mathcal{E}_g[X|\,\mathcal{F}_t] \triangleq Y_t, \tag{4}$$

where Y_t *is (the first component of) the solution at time* t *of the BSDE (2) with terminal condition* X*.*

$\mathcal{E}^{\mu}[\cdot|\,\mathcal{F}_t]$ *(respectively,* $\mathcal{E}^{-\mu}[\cdot|\,\mathcal{F}_t]$*) will denote the conditional* $g-$ *expectation for* $g(t, y, z) = \mu\|z\|$ *with* $\mu > 0$ *(respectively,* $g(t, y, z) = -\mu\|z\|$ *with* $\mu > 0$*).*

We recall from Peng (see Proposition 36.4 of [21]) that the conditional $g-$expectation defined as in (4) is the unique random variable in $L^2(\mathcal{F}_t)$ which satisfies the following condition:

$$\mathcal{E}_g[\mathbf{1}_A X] = \mathcal{E}_g[\mathbf{1}_A \mathcal{E}_g[X|\,\mathcal{F}_t]], \ \forall A \in \mathcal{F}_t. \tag{5}$$

2 Properties of $g-$expectations

We recall from Briand et al. [4], Coquet et al. [7] and Peng [21] some properties of $g-$expectation and of conditional $g-$expectation which will be useful later.

Proposition 3 (Peng; Lemma 36.3 - 36.6; [21]) *Under the usual assumptions on* $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$*, the* $g-$*expectation* $\mathcal{E}_g$ *and the conditional* $g-$*expectation* $\mathcal{E}_g[\cdot|\,\mathcal{F}_t]$ *satisfy the following properties:*

1) $\mathcal{E}_g[\mathrm{c}] = c$, $\forall c \in \mathbb{R}$;

2) *a.* *if* $X_1 \geq X_2$ $P-$*a.s.* $\Rightarrow \mathcal{E}_g[X_1] \geq \mathcal{E}_g[X_2]$;

b. *if* $X_1 \geq X_2$ $P-$*a.s.* *and if* $P(X_1 > X_2) > 0 \Rightarrow \mathcal{E}_g[X_1] > \mathcal{E}_g[X_2]$;

3) *there exists a constant* $C > 0$ *such that* $\forall X_1, X_2 \in L^2(\mathcal{F}_T)$

$$[\mathcal{E}_g(X_1) - \mathcal{E}_g(X_2)]^2 \leq C \|X_1 - X_2\|_2^2$$

4) *if* X *is* $\mathcal{F}_t-$*measurable* $\Rightarrow \mathcal{E}_g[X|\mathcal{F}_t] = X$;

5) $\mathcal{E}_g[\mathcal{E}_g[X|\mathcal{F}_t]|\mathcal{F}_r] = \mathcal{E}_g[X|\mathcal{F}_{t\wedge r}]$, $\forall X \in L^2(\mathcal{F}_T)$, $\forall r, t \in [0,T]$;

6) *if* $X_1 \geq X_2$ $P-$*a.s.* $\Rightarrow \mathcal{E}_g[X_1|\mathcal{F}_t] \geq \mathcal{E}_g[X_2|\mathcal{F}_t]$.

In particular, we remind that property 1) is due essentially to assumption (C) on g and that properties 2) and 6) follow from the Comparison Theorem stated by Peng [21] in Theorem 35.3 and extended later in Theorem 2.2 of El Karoui et al. [11].

Proposition 4 (Briand et al. [4]; Coquet et al. [7] and Peng [21]) . *Let* $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *satisfy the usual assumptions. Then*

7) *if* $g \equiv 0 \Rightarrow \mathcal{E}_g[X] = E[X]$ *and* $\mathcal{E}_g[X|\mathcal{F}_t] = E[X|\mathcal{F}_t]$, $\forall X \in L^2(\mathcal{F}_T)$;

8) *if* g *is convex in* (y,z), *then* $\mathcal{E}_g$ *and* $\mathcal{E}_g[\cdot|\mathcal{F}_t]$ *are convex functionals;*

9) *if* g *does not depend on the variable* y, *then*

$$\mathcal{E}_g[X + \beta|\mathcal{F}_t] = \mathcal{E}_g[X|\mathcal{F}_t] + \beta,\ \forall \beta \in L^2(\mathcal{F}_t),\ \forall X \in L^2(\mathcal{F}_T). \tag{6}$$

Moreover, if g *satisfies also axiom (D) (continuity), then (6) holds for any* $t \in [0,T]$ *if and only if* g *does not depend on* y.

10) *if* $g(t,y,z) = \mu\|z\|$ *for some* $\mu \in \mathbb{R}$, *then*

$$\begin{aligned}
\mathcal{E}^{\mu}[\alpha X|\mathcal{F}_t] &= \alpha\mathcal{E}^{\mu}[X|\mathcal{F}_t],\ \forall\alpha \geq 0,\ \forall X \in L^2(\mathcal{F}_T), \\
\mathcal{E}^{\mu}[\alpha X|\mathcal{F}_t] &= -\alpha\mathcal{E}^{\mu}[-X|\mathcal{F}_t],\ \forall\alpha < 0,\ \forall X \in L^2(\mathcal{F}_T), \\
\mathcal{E}^{\mu}[X|\mathcal{F}_t] &= -\mathcal{E}^{-\mu}[-X|\mathcal{F}_t],\ \forall X \in L^2(\mathcal{F}_T).
\end{aligned}$$

The following result shows that $g-$expectation and conditional $g-$expectation are positively homogeneous not only for $g(t,y,z) = \mu\|z\|$, but for any positively homogeneous g.

Proposition 5 *Let* g *satisfy the usual assumptions.*
If $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *is positively homogeneous in* (y,z), *then, for any* $t \in [0,T]$, $\mathcal{E}_g[\cdot|\mathcal{F}_t]$ *satisfies*

$$\mathcal{E}_g[\alpha X|\mathcal{F}_t] = \alpha\mathcal{E}_g[X|\mathcal{F}_t],\ \forall\alpha \geq 0,\ \forall X \in L^2(\mathcal{F}_T), \tag{7}$$

$$\mathcal{E}_g[\alpha X|\mathcal{F}_t] = -\alpha\mathcal{E}_g[-X|\mathcal{F}_t],\ \forall\alpha < 0,\ \forall X \in L^2(\mathcal{F}_T). \tag{8}$$

Proof. Note that (7) implies (8) and that for $\alpha = 0$ the thesis is trivial. Hence, we need only to study the case where $\alpha > 0$.

By definition, $\mathcal{E}_g[X|\mathcal{F}_t] = Y_t$, where Y_t is the first component of the solution (Y_t, Z_t) of the following BSDE

$$Y_t = X + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s \tag{9}$$

and $\mathcal{E}_g[\alpha X|\mathcal{F}_t] = Y_t^*$ solves

$$Y_t^* = \alpha X + \int_t^T g(s, Y_s^*, Z_s^*)ds - \int_t^T Z_s^* dB_s. \tag{10}$$

Since g is positively homogeneous in (y, z) and $\alpha > 0$, we can rewrite (10) as

$$\begin{aligned} Y_t^* &= \alpha X + \alpha \int_t^T g\left(s, \frac{Y_s^*}{\alpha}, \frac{Z_s^*}{\alpha}\right) ds - \alpha \int_t^T \frac{Z_s^*}{\alpha} dB_s \\ \frac{Y_t^*}{\alpha} &= X + \int_t^T g\left(s, \frac{Y_s^*}{\alpha}, \frac{Z_s^*}{\alpha}\right) ds - \int_t^T \frac{Z_s^*}{\alpha} dB_s, \end{aligned}$$

hence also the pair $\left(\frac{Y_t^*}{\alpha}, \frac{Z_t^*}{\alpha}\right)_{t\geq 0}$ solves (9). Since from Theorem 35.1 of Peng [21] the solution of (9) is unique, then from the definitions of Y_t and of Y_t^*, it follows that for any $t \in [0, T]$

$$\mathcal{E}_g[\alpha X|\mathcal{F}_t] = Y_t^* = \alpha Y_t = \alpha \mathcal{E}_g[X|\mathcal{F}_t].$$

■

Corollary 6 *Let $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfy the usual assumptions.*

If g is sublinear in (y, z), i.e. g is positively homogeneous and subadditive in (y, z), then $\mathcal{E}_g$ and $\mathcal{E}_g[\cdot|\mathcal{F}_t]$ are sublinear.

Proof. Since sublinearity is equivalent to convexity plus positive homogeneity, the thesis follows immediately from Proposition 4- item 8)- and Proposition 5. ■

Proposition 7 *Let g satisfy the usual assumptions and axiom (D) (continuity).*

i) g is convex in (y, z) if and only if, for any $t \in [0, T]$, $\mathcal{E}_g[\cdot|\mathcal{F}_t]$ is convex in $X \in L^2(\mathcal{F}_T)$.

ii) g is positively homogeneous in (y, z) if and only if, for any $t \in [0, T]$, $\mathcal{E}_g[\cdot|\mathcal{F}_t]$ is positively homogeneous in $X \in L^2(\mathcal{F}_T)$.

iii) g is sublinear in (y, z) if and only if, for any $t \in [0, T]$, $\mathcal{E}_g[\cdot|\mathcal{F}_t]$ is sublinear in $X \in L^2(\mathcal{F}_T)$.

Proof. From Propositions 4 and 5 and Corollary 6, we know that convexity (resp. positive homogeneity or sublinearity) of g implies convexity (resp. positive homogeneity or sublinearity) of the functional $\mathcal{E}_g[\cdot|\mathcal{F}_t]$. We will show now that, under the further assumption of continuity on g, also the converse implications are true. This proof is a simple modification of that one given by Briand et al. [4] for Theorem 4.1 (here recalled in Proposition 4- item 9)).

i) Let $(t,y,z) \in [0,T) \times \mathbb{R} \times \mathbb{R}^d$ be fixed and consider a real number $\varepsilon > 0$ small enough. Set

$$\xi_\varepsilon \triangleq y + z\left[B_{t+\varepsilon} - B_t\right].$$

Note that, by replacing vector z with p, ξ_ε defined above corresponds to the term $\left(y + p\left(X^{t,x}_{t+\varepsilon} - x\right)\right)$ which appears in ${}^{\varepsilon}Y^{t,x,y,p}_s$ of Proposition 2.3 of Briand et al. [4]. Indeed, $\left(X^{t,x}_{t+\varepsilon} - x\right)$ coincides with $\left(B_{t+\varepsilon} - B_t\right)$ when function b is identically equal to 0 and function σ identically equal to 1.

Since g is assumed to satisfy the usual axioms plus continuity, from Proposition 2.3 of Briand et al. [4] and the remark above it follows that

$$\frac{1}{\varepsilon}\left[\mathcal{E}_g\left[\xi_\varepsilon \middle| \mathcal{F}_t\right] - y\right] \underset{\varepsilon \to 0}{\overset{L^2}{\to}} g\left(t,y,z\right). \tag{11}$$

Consider now $\alpha \in (0,1)$ and set

$$\xi^{(i)}_\varepsilon \triangleq y^{(i)} + z^{(i)}\left[B_{t+\varepsilon} - B_t\right], \text{ for } i = 1,2.$$

Then

$$\begin{aligned} \Lambda_\varepsilon &\triangleq \frac{1}{\varepsilon}\left[\mathcal{E}_g\left[\alpha\xi^{(1)}_\varepsilon + (1-\alpha)\xi^{(2)}_\varepsilon \middle| \mathcal{F}_t\right] - \left(\alpha y^{(1)} + (1-\alpha)y^{(2)}\right)\right] \underset{\varepsilon \to 0}{\overset{L^2}{\to}} \\ &\underset{\varepsilon \to 0}{\overset{L^2}{\to}} g\left(t, \alpha y^{(1)} + (1-\alpha)y^{(2)}, \alpha z^{(1)} + (1-\alpha)z^{(2)}\right). \end{aligned} \tag{12}$$

On the other hand,

$$\begin{aligned} &\frac{1}{\varepsilon}\alpha\left[\mathcal{E}_g\left[\xi^{(1)}_\varepsilon \middle| \mathcal{F}_t\right] - y^{(1)}\right] \underset{\varepsilon \to 0}{\overset{L^2}{\to}} \alpha g\left(t, y^{(1)}, z^{(1)}\right); \\ &\frac{1}{\varepsilon}(1-\alpha)\left[\mathcal{E}_g\left[\xi^{(2)}_\varepsilon \middle| \mathcal{F}_t\right] - y^{(2)}\right] \underset{\varepsilon \to 0}{\overset{L^2}{\to}} (1-\alpha)\, g\left(t, y^{(2)}, z^{(2)}\right), \end{aligned}$$

hence

$$\begin{aligned} \Psi_\varepsilon &\triangleq \frac{1}{\varepsilon}\left\{\alpha\mathcal{E}_g\left[\xi^{(1)}_\varepsilon \middle| \mathcal{F}_t\right] + (1-\alpha)\,\mathcal{E}_g\left[\xi^{(2)}_\varepsilon \middle| \mathcal{F}_t\right] - \left[\alpha y^{(1)} + (1-\alpha)\, y^{(2)}\right]\right\} \underset{\varepsilon \to 0}{\overset{L^2}{\to}} \\ &\underset{\varepsilon \to 0}{\overset{L^2}{\to}} \alpha g\left(t, y^{(1)}, z^{(1)}\right) + (1-\alpha)\, g\left(t, y^{(2)}, z^{(2)}\right). \end{aligned} \tag{13}$$

Furthermore, from convexity of any $\mathcal{E}_g\left[\cdot \middle| \mathcal{F}_t\right]$, we get

$$\Lambda_\varepsilon \le \Psi_\varepsilon \quad P - \text{a.s.} \tag{14}$$

It is well known (see, for instance, Shiryaev [27]) that for any sequence $\{X_n\}_{n \ge 0}$ converging in L^2 to X, there exists a subsequence which converges to X almost surely. Thus, from (14) and by taking the *a.s.*$-$limit on the suitable subsequences in (12) and in (13), it follows that g is convex in (y,z) for any $t \in [0,T)$. Moreover, since g is continuous in $t \in [0,T]$, it is convex for any $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$.

Proof of item ii) is analogous to the previous one.

iii) Since sublinearity is equivalent to convexity plus positive homogeneity, the thesis follows from i) and ii). ■

3 Static risk measures

Let $\mathcal{X}$ be the space of all financial positions whose riskiness we would like to quantify. From now on, we will assume that $\mathcal{X} = L^2(\mathcal{F}_T)$, that is the space of all $\mathcal{F}_T-$measurable, at least square integrable random variables. Note that risk measures are usually defined on the whole $L^p(\Omega, \mathcal{F}, P)$, with $p \geq 1$; here only "enough integrable" $\mathcal{F}_T-$measurable risks are taken into account.

We recall that a *static risk measure* is a functional $\rho : \mathcal{X} \to \mathbb{R}$ satisfying some desirable properties.

Axioms for ρ

(a) convexity:
$\rho(\alpha X + (1-\alpha)Y) \leq \alpha \rho(X) + (1-\alpha)\rho(Y)$, $\forall \alpha \in (0,1)$, $\forall X, Y \in \mathcal{X}$;
(b) *positivity: $X \geq 0$ $P-a.s. \Rightarrow \rho(X) \leq \rho(\mathbf{0})$;
(c) *constancy: $\rho(\boldsymbol{\alpha}) = -\alpha$, $\forall \alpha \in \mathbb{R}$;
(d) *translability: $\rho(X + \boldsymbol{\beta}) = \rho(X) - \beta$, $\forall \beta \in \mathbb{R}$, $\forall X \in \mathcal{X}$;
(e) sublinearity:
  (e1) positive homogeneity: $\rho(\alpha X) = \alpha \rho(X)$, $\forall \alpha \geq 0$, $\forall X \in \mathcal{X}$;
  (e2) subadditivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$, $\forall X, Y \in \mathcal{X}$;
(f) *normalization: $\rho(\mathbf{1}) = -1 = -\rho(-\mathbf{1})$;
(g) lower semi-continuity: $\{X \in \mathcal{X} : \rho(X) \leq \gamma\}$ is closed in $\mathcal{X}$ for any $\gamma \in \mathbb{R}$.

Remark 8 (Frittelli & Rosazza; Remark 8; [17]) *Let $\rho : \mathcal{X} \to \mathbb{R}$ satisfy the *translability axiom (d).*
ρ is lower semi-continuous iff the set $\{X \in \mathcal{X} : \rho(X) \leq 0\}$ is closed in $\mathcal{X}$.

For the financial interpretation of the axioms above, see, among others, Artzner et al. [1], Delbaen [9], Föllmer & Schied [12], [13], [14], Frittelli [15], Frittelli & Rosazza [16], [17] and Heath [18].

Notice that we have emphasized with "*" the axioms where risk measures invert signs. The reason of this change is due to the well known interpretation of ρ: a financial position X having $\rho(X) \leq 0$ is *acceptable*, otherwise it is unacceptable.

We will see in a while that, under suitable assumptions on g, $g-$expectations provide examples of some families of risk measures. However, since, under the usual assumptions on g, any $g-$expectation satisfies properties of monotonicity and constancy (see Proposition 3), we *can't* expect to find examples of risk measures ρ which don't satisfy axioms (b) and (c).

3.1 Coherent risk measures

We recall from Artzner et al. [1] and Delbaen [9] the definition of coherent risk measures.

Definition 9 *A functional* $\rho : \mathcal{X} \to \mathbb{R}$ *is a* coherent risk measure *if it satisfies (b) (*positivity), (d) (*translability) and (e) (sublinearity).*

The following result shows that it is possible to find some examples of coherent risk measures via $g-$expectations.

Proposition 10 *If g satisfies the usual assumptions and axiom (E) (sublinearity), then the functional* $\rho_g : L^2(\mathcal{F}_T) \to \mathbb{R}$ *defined by*

$$\rho_g(X) \triangleq \mathcal{E}_g[-X], \ \forall X \in L^2(\mathcal{F}_T) \tag{15}$$

is a coherent risk measure satisfying lower semi-continuity (g).

Proof. Consider a $g-$expectation $\mathcal{E}_g$ with g satisfying the usual assumptions and axiom (E) and set $\rho_g(X) \triangleq \mathcal{E}_g[-X]$ for all $X \in L^2(\mathcal{F}_T)$.

Property (b) (*positivity) of ρ_g follows immediately from Proposition 3- items 1) and 2).

Furthermore, if g is sublinear in (y,z) (or, equivalently, convex and positively homogeneous in (y,z)) and satisfies the usual assumptions, then, from a remark of Briand et al. [4], g does not depend on y. Hence, from Proposition 4- item 9)- and Corollary 6, ρ_g satisfies also axioms (d) (*translability) and (e) (sublinearity).

From Remark 8, it remains to show that the acceptance set $\mathcal{A} \triangleq \{X \in L^2(\mathcal{F}_T) : \rho(X) \leq 0\}$ is closed in L^2 which follows immediately from Proposition 3- item 3). ■

From the representation of coherent risk measures satisfying lower semi-continuity established by Delbaen (see [9]) and the result above, we deduce the following corollary. A particular case (when $g(t,z) = \mu|z|$) of it was previously established by Chen & Peng [5] in both a static and dynamic setting. We will recall it in Section 4.1.

Corollary 11 *Let* $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *satisfy the usual assumptions and axiom (E) (sublinearity) and the risk measure* $\rho_g : L^2(\mathcal{F}_T) \to \mathbb{R}$ *be defined as in (15).*

Then there exists a non empty set $\mathcal{M}$ of $P-$absolutely continuous probability measures on $(\Omega, \mathcal{F}_T)$ such that

$$\rho_g(X) = \mathcal{E}_g[-X] = \sup_{Q \in \mathcal{M}} E_Q[-X], \ \forall X \in L^2(\mathcal{F}_T).$$

3.2 Convex risk measures

We recall now the definition of convex risk measures, firstly proposed by Heath [18] and later, in general probability spaces, by Föllmer & Schied [12], [13], [14] and, indipendently, by Frittelli & Rosazza [16], [17].

Definition 12 (see Frittelli & Rosazza [17]) *A functional* $\rho : \mathcal{X} \to \mathbb{R}$ *is a* convex risk measure *if it satisfies axiom (a) (convexity), (g) (lower semi-continuity) and* $\rho(\mathbf{0}) = 0$.

Proposition 13 *If* g *satisfies the usual assumptions and axiom (F) (convexity), then the functional* $\rho_g : L^2(\mathcal{F}_T) \to \mathbb{R}$ *defined by*

$$\rho_g(X) \triangleq \mathcal{E}_g[-X], \ \forall X \in L^2(\mathcal{F}_T)$$

*is a convex risk measure satisfying axioms (b) (*positivity), (c) (*constancy) and (d) (*translability).*

Proof. Consider a $g-$expectation $\mathcal{E}_g$ with g satisfying the usual hypothesis and axiom (F) and set $\rho_g(X) \triangleq \mathcal{E}_g[-X]$ for all $X \in L^2(\mathcal{F}_T)$.

Properties (c) (*constancy) and (b) (*positivity) of ρ_g follow immediately from Proposition 3- items 1) and 2).

Furthermore, since by hypothesis g is sublinear, hence convex, in (y, z) and it satisfies the usual assumptions, then, from a remark of Briand et al. [4], g does not depend on y. Hence, from Proposition 4- items 8) and 9)- ρ_g satisfies also axioms (a) (convexity) and (d) (*translability).

Lower semi-continuity of ρ can be shown exactly as in Proposition 10. ■

As for the coherent case, from the previous result and the representation of convex risk measures (see [12], [13], [14], [16] and [17]), we deduce the following corollary.

Corollary 14 *Let* $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *satisfy the usual assumptions and axiom (F) (convexity) and the risk measure* $\rho_g : L^2(\mathcal{F}_T) \to \mathbb{R}$ *be defined as in (15).*

Then there exist a convex set $\mathcal{M}$ *of* $P-$*absolutely continuous probability measures on* $(\Omega, \mathcal{F}_T)$ *and a convex functional* $F : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$ *such that* $\inf_{Q \in \mathcal{M}} F(Q) = 0$ *and*

$$\rho_g(X) = \mathcal{E}_g[-X] = \sup_{Q \in \mathcal{M}} \{E_Q[-X] - F(Q)\}, \ \forall X \in L^2(\mathcal{F}_T).$$

4 Dynamic risk measures

As shown previously, $g-$expectations provide examples of some families of static risk measures. Since properties of conditional $g-$expectations (see Briand et al. [4], Coquet et al. [7] and Peng [21]) are "similar" to those of $g-$expectations, the former ones seem to suggest a natural extension of these examples in a dynamic setting. We will come back to conditional $g-$expectations later.

For an exhaustive discussion on the existent literature see Frittelli & Rosazza [17]. Here, we wish only to mention that the first approaches to dynamic risk measures are due to Cvitanic & Karatzas [8] and Wang [28]. More recent studies on this subject can be found in Artzner et al. [2], [3], Pflug & Ruszczynski [23], Riedel [24] and Scandolo [26]. In particular, we will compare the approach of Artzner et al. [2] and [3] to ours.

In this section, we will consider a general filtration $\{\mathcal{F}_t\}_{t\geq 0}$, not necessarily the Brownian one.

Let $L^0(\mathcal{F}_t) = L^0(\Omega, \mathcal{F}_t, P)$ denote the space of all random variables defined on $(\Omega, \mathcal{F}_t, P)$.

For the first part of this section, we will take as reference Frittelli & Rosazza [16].

We recall the definition of a dynamic risk measure $(\rho_t)_{t\in[0,T]}$. Since we are interested in "monitoring" the riskiness of a risky position X at any intermediate time t between the initial date 0 and the final T, for any t a random variable ρ_t is introduced. Indeed, ρ_t represents the riskiness at time t and takes into account all the information available up to date t. Moreover, two "boundary conditions" at times 0 and T are imposed to $(\rho_t)_{t\in[0,T]}$: ρ_0 has to be a static risk measure and ρ_T has to reduce, reasonably, to the opposite of our risky position.

Definition 15 *We call* dynamic risk measure *any map such that:*
a) $\rho_t : \mathcal{X} \to L^0(\mathcal{F}_t)$, *for all* $t \in [0,T]$*;*
b) ρ_0 *is a static risk measure;*
c) $\rho_T(X) = -X \quad P-a.s.$, *for all* $X \in \mathcal{X}$.

We will list now some desirable properties for $(\rho_t)_{t\in[0,T]}$.

Axioms for $(\rho_t)_{t\in[0,T]}$

(aD) dynamic convexity: $\forall t \in [0,T]$, ρ_t is convex;
(bD) dynamic *positivity: $X \geq 0$ $P-$a.s. $\Rightarrow \forall t \in [0,T]$, $\rho_t(X) \leq \rho_t(\mathbf{0})$;
(cD) dynamic *constancy: $\forall t \in [0,T]$, $\forall c \in \mathbb{R}$, $\rho_t(\mathbf{c}) = -\mathbf{c}$;
(dD) dynamic *translability:
$\forall t \in [0,T]$, $\forall a$ $\mathcal{F}_t-$measurable in $\mathcal{X}$, $\forall X \in \mathcal{X}$, $\rho_t(X+a) = \rho_t(X) - a$;
(eD) dynamic sublinearity:
(e1D) dynamic positive homogeneity:
$\forall t \in [0,T]$, $\forall \alpha \geq 0$, $\forall X \in \mathcal{X}$, $\rho_t(\alpha X) = \alpha \rho_t(X)$;
(e2D) dynamic subadditivity:
$\forall t \in [0,T]$, $\forall X, Y \in \mathcal{X}$, $\rho_t(X+Y) \leq \rho_t(X) + \rho_t(Y)$;
(fD) dynamic *normalization: $\forall t \in [0,T]$, $\rho_t(\mathbf{1}) = -\mathbf{1} = -\rho_t(-\mathbf{1})$.

The financial motivation of axioms (aD), (bD), (cD), (eD) and (fD) is analogous to that one in the static case, since these axioms are the exact dynamic copy of the static (a), (b), (c), (e) and (f), respectively. On the contrary, axiom (dD) (dynamic *translability) is stronger. It requires, indeed, translation invariance not only with respect to constants, but also with respect to any $\mathcal{F}_t-$measurable random variable. A similar property is assumed by Riedel [24] under the name of "predictable translation invariance".

We recall now a possible extension of the definitions of coherent and convex risk measures to a dynamic setting.

Definition 16 .

1) A dynamic risk measure $(\rho_t)_{t\in[0,T]}$ is called coherent *if it satisfies axioms (bD) (dynamic *positivity), (dD) (dynamic *translability) and (eD) (dynamic sublinearity).*

2) A dynamic risk measure $(\rho_t)_{t\in[0,T]}$ is called convex *if it satisfies axiom (aD) (dynamic convexity) and $\rho_t(\mathbf{0}) = \mathbf{0}$.*

3) $(\rho_t)_{t\in[0,T]}$ is said to be time-consistent *if $\forall t \in [0,T]$, $\forall X \in \mathcal{X}$, $\forall A \in \mathcal{F}_t$*

$$\rho_0 [X\mathbf{1}_A] = \rho_0 [-\rho_t (X) \mathbf{1}_A] . \tag{16}$$

While the definitions of dynamic coherent and convex risk measures are dynamic versions of the static ones (except for having imposed no hypothesis on the continuity of $(\rho_t)_t$ in the dynamic convex case), time-consistency (as well as dynamic *translability which has been discussed previously) is a new requirement. Indeed, it is nothing but the rewriting for risk measures of the "filtration-consistency" property introduced by Coquet et al. [7].

We wish to emphasize some interesting aspects about time-consistency. First of all, the sign "-" in (16) is a consequence of the sign interpretation of $\rho_t(X)$.

Furthermore, the financial interpretation of time-consistency is the following. In order to quantify the riskiness of X at the initial time 0, the two approaches below are equivalent:

1) using directly the static risk measure ρ_0, i.e. computing $\rho_0(X)$;

2) calculating $\rho_0(X)$ in two steps, i.e. valuing firstly the riskiness $\rho_t(X)$ of X at an intermediate date t and then quantifying at time 0 the risk of $-\rho_t(X)$ through ρ_0.

Finally, as we will see later, time-consistency (already defined in the first version of this paper and in [25]) is linked to the "recursivity" property introduced by Artzner et al. [2].

We present now some examples of dynamic coherent and convex risk measures deduced as a quite natural "dynamic extension" of the characterization of static lower semi-continuous coherent and convex risk measures (see Delbaen [9], Föllmer & Schied [12], [13], [14] and Frittelli & Rosazza [16], [17]). Take note that example 17 is one among the two dynamic risk measures proposed and studied by Artzner et al. [2], [3].

Example 17 (dynamic coherent risk measure) ***(see also Artzner et al.*** *[2], [3]) Let $\mathcal{M}$ be a convex set of $P-$absolutely continuous probability measures defined on $(\Omega, \mathcal{F}_T)$ and set*

$$\rho_t(X) \triangleq ess.\sup_{Q\in\mathcal{M}} E_Q\left[-X|\,\mathcal{F}_t\right], \ \forall X \in \mathcal{X}, \ \forall t \in [0,T]. \tag{17}$$

From properties of conditional expectations and essential supremum[1]*, it follows that $(\rho_t)_{t\in[0,T]}$ is a dynamic coherent risk measure.*

Example 18 (dynamic convex risk measure) *Let $\mathcal{M}$ be a convex set of $P-$absolutely continuous probability measures defined on $(\Omega, \mathcal{F}_T)$ and for any $t \in [0,T]$ let $F_t : \mathcal{M} \to \mathbb{R}\cup\{+\infty\}$ be a convex functional such that* $\inf_{Q\in\mathcal{M}} F_t(Q) = 0$. *Set*

$$\rho_t(X) \triangleq ess.\sup_{Q\in\mathcal{M}} \left\{E_Q\left[-X|\,\mathcal{F}_t\right] - F_t(Q)\right\}, \ \forall X \in \mathcal{X}, \ \forall t \in [0,T]. \tag{18}$$

*Then $(\rho_t)_{t\in[0,T]}$ is a dynamic convex risk measure satisfying axioms (bD) (dynamic *positivity), (cD) (dynamic *constancy) and (dD) (dynamic *translability).*

4.1 Dynamic risk measures coming from conditional $g-$expectations

We have just seen some examples of dynamic convex risk measures provided by the natural extensions of representations of coherent and convex risk measures to a dynamic setting. We will present now examples of dynamic risk measures coming from conditional $g-$expectations.

To this aim, we come back to the augmented Brownian filtration $\{\mathcal{F}_t\}_{t\geq 0}$ taken in equation (1).

Proposition 19 *Let g satisfy the usual assumptions and set*

$$\rho_t^g(X) \triangleq \mathcal{E}_g\left[-X|\,\mathcal{F}_t\right], \ \forall X \in L^2(\mathcal{F}_T), \ \forall t \in [0,T]. \tag{19}$$

1) If g satisfies axiom (E) (sublinearity) (for instance, $g(z) = \mu\|z\|$ with $\mu > 0$), then $(\rho_t^g)_{t\in[0,T]}$ is a dynamic coherent and time-consistent risk measure.

2) If g satisfies axiom (F) (convexity), then $(\rho_t^g)_{t\in[0,T]}$ is a dynamic convex and time-consistent risk measure satisfying axioms (bD), (cD) and (dD).

Remark 20 *i) As previously pointed out, time-consistency so defined is linked with the "recursivity property" required by Artzner et al. [2], [3]. Moreover, it*

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[1] We recall from Föllmer & Schied (see Definition A.19 in [14]) that a random variable Y^* is the *essential supremum* of a (eventually uncountable) set of random variables $\mathcal{Y}$ if:

(•) $Y^* \geq Y$ $P-$a.s., for any $Y \in \mathcal{Y}$;

(••) if Y^{**} satisfies (•), then $Y^{**} \geq Y^*$ $P-$a.s.

is easy to check that any dynamic risk measure of the form (19) satisfies the following "continuous-time recursivity": for any $0 \leq s \leq t \leq T$

$$\rho_s^g(X) = \rho_s^g(-\rho_t^g(X)), \ \forall X \in L^2(\mathcal{F}_T). \tag{20}$$

By defining $\mathcal{E}_g[\cdot|\,\mathcal{F}_\tau]$ *for any stopping time* $\tau \leq T$ *as in Coquet et al. [7], equation (20) holds true also when times* s *and* t *are replaced by stopping times* σ *and* τ *such that* $\sigma \leq \tau \leq T$.

ii) It is interesting to notice that for the dynamic risk measure $(\rho_t^g)_{t\in[0,T]}$ *defined in (19) also the following property (whose stronger version with stopping times is called "time-consistency" by Artzner et al. [3]) holds true:*

if, for some $t \in (0,T]$, $\rho_t^g(X) \leq \rho_t^g(Y) \ \Rightarrow$ *for any* $s \in [0,t]$, $\rho_s^g(X) \leq \rho_s^g(Y)$.

Proof. i) follows immediately from (19) and Proposition 3- item 5).

ii) Suppose that, for $t \in (0,T]$, $\rho_t^g(X) \leq \rho_t^g(Y)$. By (19) we get

$$\rho_t^g(X) = \mathcal{E}_g[-X|\,\mathcal{F}_t] \leq \mathcal{E}_g[-Y|\,\mathcal{F}_t] = \rho_t^g(Y).$$

Therefore, from (5) and Proposition 3- item 5) and 6), we get for any $s \in [0,T]$

$$\begin{aligned} \rho_s^g(X) &= \mathcal{E}_g[-X|\,\mathcal{F}_s] = \mathcal{E}_g[\mathcal{E}_g[-X|\,\mathcal{F}_t]|\,\mathcal{F}_s] \\ &\leq \mathcal{E}_g[\mathcal{E}_g[-Y|\,\mathcal{F}_t]|\,\mathcal{F}_s] = \mathcal{E}_g[-Y|\,\mathcal{F}_s] = \rho_s^g(Y). \end{aligned}$$

■

Remark 21 (Chen & Peng; Lemma 3; [5]) *Suppose that the process* $(Z_t)_{t\in[0,T]}$ *is one dimensional. Recall that* $\mathcal{E}^\mu$ *and* $\mathcal{E}^\mu[\cdot|\,\mathcal{F}_t]$ *denote the* $g-$*expectation and the conditional* $g-$*expectation, respectively, when* $g(t,y,z) \equiv \mu|z|$ *and* $\mu > 0$.

Chen & Peng [5] proved that

$$\mathcal{E}^\mu[X|\,\mathcal{F}_t] = ess. \sup_{Q\in\mathcal{M}_G} E_Q[X|\,\mathcal{F}_t], \ \forall X \in L^2(\mathcal{F}_T), \ \forall t \in [0,T], \tag{21}$$

where $\mathcal{M}_G = \left\{Q_G : E\left[\left.\frac{dQ_G}{dP}\right|\mathcal{F}_T\right] = \exp\left\{-\frac{1}{2}\int_0^T \phi_t^2 dt - \int_0^T \phi_t dB_t\right\}; |\phi| \leq \mu\right\}$.

In other words, from (21) it follows that risk measures of the forms (17) and (19) can coincide for suitable sets $\mathcal{M}$ *and functionals* g.

In Proposition 19, we have seen that functionals of the form (19) are dynamic time-consistent risk measures. On the other hand, it seems reasonable to wonder under what conditions dynamic time-consistent risk measures come from conditional $g-$expectations. The answer is due to straightforward corollaries of a result of Coquet et al. [7] on nonlinear expectations.

From now on, we will consider only the case where the stochastic process $(Z_t)_{t\in[0,T]}$ is one dimensional, therefore $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$.

We recall the following definitions from Coquet et al. [7].

Definition 22 (see: Coquet et al.; deff. 3.1, 3.2, 4.1; [7]) .

- *A functional* $\mathcal{E} : L^2(\mathcal{F}_T) \to \mathbb{R}$ *is called a* nonlinear expectation *if it satisfies the following properties:*

  i) (strict monotonicity):

  * *if* $X \geq Y \quad P-a.s. \Rightarrow \mathcal{E}(X) \geq \mathcal{E}(Y)$;

  ** *if* $X \geq Y \quad P-a.s. : \mathcal{E}(X) = \mathcal{E}(Y) \Leftrightarrow X = Y \quad P-a.s.$;

  ii) (constancy) $\mathcal{E}(\mathbf{c}) = c$, $\forall c \in \mathbb{R}$.

- *A nonlinear expectation* $\mathcal{E}$ *is said to be an* $\{\mathcal{F}_t\}_{t\in[0,T]}-$consistent expectation *(or, simply, an* $\{\mathcal{F}_t\}_{t\in[0,T]}-$expectation*) if for any* $X \in L^2(\mathcal{F}_T)$ *and any* $t \in [0,T]$ *there exists* $\eta \in L^2(\mathcal{F}_t)$ *such that*

$$\mathcal{E}[\mathbf{1}_A X] = \mathcal{E}[\mathbf{1}_A \eta], \ \forall A \in \mathcal{F}_t.$$

Coquet et al. (see Lemma 3.1 in [7]) proved that, if such a random variable η *exists, then it is unique. Such* η *is called* conditional $\{\mathcal{F}_t\}_{t\in[0,T]}-$expectation *of* X *under* $\mathcal{F}_t$ *and denoted by* $\mathcal{E}[X|\mathcal{F}_t]$.

- *An* $\{\mathcal{F}_t\}_{t\in[0,T]}-$ *consistent expectation* $\mathcal{E}$ *is said to be* dominated by $\mathcal{E}^\mu$ *(with* $\mu > 0$*) if*

$$\mathcal{E}[X+Y] - \mathcal{E}[X] \leq \mathcal{E}^\mu[Y], \ \forall X, Y \in L^2(\mathcal{F}_T),$$

where $\mathcal{E}^\mu$ *stands for the* $g-$*expectation with* $g(t,y,z) = \mu|z|$.

- *An* $\{\mathcal{F}_t\}_{t\in[0,T]}-$ *consistent expectation* $\mathcal{E}$ *satisfies the* translability condition (T) *if*

$$\mathcal{E}[X+\beta|\mathcal{F}_t] = \mathcal{E}[X|\mathcal{F}_t] + \beta, \ \forall X \in L^2(\mathcal{F}_T), \ \forall \beta \in L^2(\mathcal{F}_t). \qquad \text{(T)}$$

Coquet et al. [7] proved the following result which provides a sufficient condition for nonlinear expectations to come from $g-$expectations. We recall from the above authors that $\{\mathcal{F}_t\}_{t\in[0,T]}-$consistency (see Lemma 3.2 in [7]) and $\mathcal{E}^\mu-$domination (see Remark 4.1 in [7]) are necessary conditions for $\mathcal{E}$ to come from a $g-$expectation with g satisfying the usual assumptions plus axiom (G).

Theorem 23 (Coquet et al.; Theorem 7.1; [7]) .

Let $\mathcal{E} : L^2(\mathcal{F}_T) \to \mathbb{R}$ *be a nonlinear* $\{\mathcal{F}_t\}_{t\in[0,T]}-$*consistent expectation.*

If $\mathcal{E}$ *is* $\mathcal{E}^\mu-$*dominated for some* $\mu > 0$ *and if it satisfies the translability condition (T), then there exists a unique* g *satisfying axioms (A), (B), (C) (usual assumptions), (G) (independence from* y*) and* $|g(t,z)| \leq \mu|z|$ *such that*

$$\mathcal{E}[X] = \mathcal{E}_g[X] \quad \textit{and} \quad \mathcal{E}[X|\mathcal{F}_t] = \mathcal{E}_g[X|\mathcal{F}_t], \ \forall X \in L^2(\mathcal{F}_T). \qquad (22)$$

Just for completeness, we will state the following result, which is a straightforward consequence of properties of $g-$expectations proved by Briand et al. [4], Coquet et al. [7] and Peng [21].

Corollary 24 *If the functional g is continuous in t for all $z \in \mathbb{R}$, then in Theorem 23 also the converse implication holds true.*

Proof. We are going to prove now the "converse implication" of Theorem 23, i.e. we need to show that any functional $\mathcal{E}$ as in (22) is a nonlinear expectation satisfying the $\{\mathcal{F}_t\}_{t\in[0,T]}$ −consistency, the $\mathcal{E}^\mu$−domination and the translability condition (T).

Nonlinearity: it follows from Proposition 3- items 1) and 2).

Consistency: if g satisfies axiom (A), (B), (C) and if (22) holds true for such g, from Lemma 3.2 of Coquet et al. [7] functional $\mathcal{E}$ is $\{\mathcal{F}_t\}_{t\in[0,T]}$ −consistent.

Domination: from Remark 4.1 (whose proof is omitted) of Coquet et al. [7] the property of domination follows immediately. For completeness, we will sketch a proof of it.

We know that g satisfies axioms (A), (B), (C) (usual assumptions), (D) (continuity) and (G) (independence from y). Hence, it is well known (see, for instance, Karatzas & Shreve [19]) that g is also progressively measurable. Moreover, $|g(t,z)| \le \mu |z|$ or, equivalently, $-\mu |z| \le g(t,z) \le \mu |z|$.

By definition of g−expectation, we have that

$$\begin{aligned} \mathcal{E}_g(X+Y) &= X+Y+\int_0^T g(t,Z_t)\,dt - \int_0^T Z_t dB_t \\ \mathcal{E}_g(X) &= X+\int_0^T g\left(t,\hat{Z}_t\right)dt - \int_0^T \hat{Z}_t dB_t \end{aligned}$$

hence for any fixed $X \in L^2(\mathcal{F}_T)$ we get

$$\begin{aligned} \mathcal{E}_{g,X}(Y) &\triangleq \mathcal{E}_g(X+Y) - \mathcal{E}_g(X) \\ &= Y+\int_0^T \left[g(t,Z_t) - g\left(t,\hat{Z}_t\right)\right]dt - \int_0^T \left[Z_t - \hat{Z}_t\right]dB_t. \qquad (23) \end{aligned}$$

Setting $Z_t^* \triangleq Z_t - \hat{Z}_t$ and $g_{\hat{Z}}(t,Z_t^*) \triangleq g\left(t,Z_t^*+\hat{Z}_t\right) - g\left(t,\hat{Z}_t\right)$, it follows that

$$\mathcal{E}_{g,X}(Y) = Y + \int_0^T \left[g_{\hat{Z}}(t,Z_t^*)\right]dt - \int_0^T Z_t^* dB_t. \qquad (24)$$

From proof of Theorem 23 (see [6]) it is clear that the Lipschitzian constant of g is μ, therefore it follows that $g_{\hat{Z}}(t,Z_t^*) = g(t,Z_t) - g\left(t,\hat{Z}_t\right) \le \mu\left|Z_t - \hat{Z}_t\right| = \mu |Z_t^*|$. Hence, from the Comparison Theorem (see, for instance, Theorem 2.2 of El Karoui et al. [11] or Theorem 2.1. of Coquet et al. [6]), we get from (23)

$$\mathcal{E}_{g,X}(Y) = \mathcal{E}_g(X+Y) - \mathcal{E}_g(X) \le \mathcal{E}^\mu(Y), \ \forall Y. \qquad (25)$$

Since X is arbitrary and the bound in (25) is independent from X, then $\mathcal{E}^\mu$−domination follows.

Translability: since g satisfies axiom (G), from Proposition 4- item 9)- $\mathcal{E}_g\left[\cdot\right|\mathcal{F}_t]$ satisfies the translability condition, hence also $\mathcal{E}\left[\cdot\right|\mathcal{F}_t]$. ■

The following two propositions are simple consequences of Theorem 23 and provide sufficient conditions for dynamic risk measures (defined on $L^2(\mathcal{F}_T)$) to come from $g-$expectations: the former for coherent, the latter for convex ones.

Proposition 25 .

Let $(\rho_t)_{t\in[0,T]}$ *be a dynamic coherent and time-consistent risk measure defined on* $L^2\left(\mathcal{F}_T\right)$ *and let* $(\pi_t)_{t\in[0,T]}$ *be defined by* $\pi_t\left(X\right) \triangleq \rho_t\left(-X\right)$.

If π_0 *is strictly monotone and* $\mathcal{E}^{\mu}-$*dominated, then there exists a unique* g *satisfying axioms (A), (B), (C) (usual assumptions), (G) (independence from* y*) and* $|g(t,z)| \leq \mu\,|z|$ *such that*

$$\rho_0\left(X\right) = \mathcal{E}_g\left[-X\right] \quad \text{and} \quad \rho_t\left(X\right) = \mathcal{E}_g\left[-X\right|\mathcal{F}_t],\ \forall X \in L^2\left(\mathcal{F}_T\right). \tag{26}$$

Moreover, if such a g *is continuous in* t *for all* $z \in \mathbb{R}$*, then it is also sublinear in* z.

Proof. Set $\mathcal{E}\left[X\right] \triangleq \pi_0\left(X\right) = \rho_0\left(-X\right)$ and $\mathcal{E}\left[X\right|\mathcal{F}_t] \triangleq \pi_t\left(X\right) = \rho_t\left(-X\right)$.

We will check that $\mathcal{E}$ satisfies all the hypothesis of Theorem 23 and hence deduce the thesis.

Since (ρ_t) is a dynamic coherent risk measure, $\mathcal{E} = \pi_0$ satisfies (bD) (dynamic *positivity), (dD) (dynamic *translability) and (eD) (dynamic sublinearity), therefore also (cD) (dynamic *constancy) holds true. From strict monotonicity of π_0, $\mathcal{E}$ is a nonlinear expectation.

Time-consistency (resp. $\mathcal{E}^{\mu}-$domination, resp. dynamic translability) of (π_t) implies $\{\mathcal{F}_t\}-$consistency (resp. $\mathcal{E}^{\mu}-$domination, resp. translability (T)) of $\mathcal{E}$.

Since $\mathcal{E} = \pi_0$ satisfies all the hypothesis of Theorem 23, the first part of the thesis follows immediately.

Assume now that function g is also continuous in t for all $z \in \mathbb{R}$. Since $\mathcal{E}_g\left[\cdot\right]$ and $\mathcal{E}_g\left[\cdot\right|\mathcal{F}_t]$ are sublinear (by (26) and dynamic coherence of (ρ_t)), from Proposition 7- item iii)- and by continuity of g, it follows that g is also sublinear. ■

Proposition 26 .

Let $(\rho_t)_{t\in[0,T]}$ *be a dynamic convex and time-consistent risk measure defined on* $L^2\left(\mathcal{F}_T\right)$ *satisfying axioms (cD) (dynamic *constancy) and (dD) (dynamic *translability). Let* $(\pi_t)_{t\in[0,T]}$ *be defined by* $\pi_t\left(X\right) \triangleq \rho_t\left(-X\right)$.

If π_0 *is strictly monotone and* $\mathcal{E}^{\mu}-$*dominated, then there exists a unique* g *satisfying axioms (A), (B), (C) (usual assumptions), (G) (independence from* y*) and* $|g(t,z)| \leq \mu\,|z|$ *such that*

$$\rho_0\left(X\right) = \mathcal{E}_g\left[-X\right] \quad \text{and} \quad \rho_t\left(X\right) = \mathcal{E}_g\left[-X\right|\mathcal{F}_t],\ \forall X \in L^2\left(\mathcal{F}_T\right). \tag{27}$$

Moreover, if such a g *is continuous in* t *for all* $z \in \mathbb{R}$*, then it is also convex in* z.

Proof. Set $\mathcal{E}[X] \triangleq \pi_0(X)$ and $\mathcal{E}[X|\mathcal{F}_t] \triangleq \pi_t(X)$.

As in the proof above, we are going to show that $\mathcal{E}$ satisfies all the hypothesis of Theorem 23 and hence deduce the thesis.

Since, by hypothesis, $\mathcal{E} = \pi_0$ satisfies strictly monotonicity and (cD) (dynamic constancy), $\mathcal{E}$ is a nonlinear expectation.

Time-consistency (resp. $\mathcal{E}^\mu$−domination) of (π_t) implies $\{\mathcal{F}_t\}_t$ −consistency (resp. $\mathcal{E}^\mu$−domination) of $\mathcal{E}$. Furthermore, translability (T) of $\mathcal{E}[\cdot|\mathcal{F}_t]$ follows from axiom (dD) (dynamic translability) on (π_t).

Since $\mathcal{E} = \pi_0$ satisfies all the hypothesis of Theorem 23, the first part of the thesis follows immediately.

Assume now that function g is also continuous in t for all $z \in \mathbb{R}$. Since $\mathcal{E}_g[\cdot]$ and $\mathcal{E}_g[\cdot|\mathcal{F}_t]$ are convex (by (27) and dynamic convexity of (π_t)), from Proposition 7- item i)- and by continuity of g, it follows that g is also convex. ■

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