Modeling continuous-time financial markets with capital gains taxes *

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Abstract

We formulate a model of continuous-time financial market with risky asset subject to capital gains taxes. We study the problem of maximizing expected utility of future consumption within this model both in the finite and infinite horizon. Our main result is that the maximal utility does not depend on the taxation rule. This is shown by exhibiting maximizing strategies which tracks the classical Merton optimal strategy in tax-free financial markets. Hence, optimal investors can avoid the payment of taxes by suitable strategies, and there is no way to benefit from tax credits.

Key Words and phrases: Optimal consumption and investment in continuous-time, capital gains taxes.

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1 Introduction

Since the seminal papers of Merton [10, 11], there has been a continuous interest in the theory of optimal consumption and investment decision in financial markets. A large literature has focused particularly on the effect of market imperfections on the optimal consumption and investment decision, see e.g. Cox and Huang [2] and Karatzas, Lehoczky and Shreve [8] for incomplete markets, Cvitanić and Karatzas [3] for markets with portfolio constraints, Davis and Norman [4] for markets with proportional transaction costs.

However, there is a very limited literature on the capital gains taxes which apply to financial securities and represent a much higher percentage than transaction costs. Compared to ordinary income, capital gains are taxed only when the investor sells the security, allowing for a deferral option. One may think that the taxes on capital gains have an appreciable impact on individuals consumption and investment decisions. Indeed, under taxation of capital gains, an investor supports supplementary charges when he rebalances his portfolio, which alters the available wealth for future consumption, possibly depreciating consumption opportunities compared to a tax-free market. On the other hand, since taxes are paid only when embedded capital gains are actually realized, the investor may choose to defer the realization of capital gains and liquidate his position in case of capital losses, particularly when the tax code allows for tax credits. Previous works attempted to characterize intertemporal consumption and investment decisions of investors who have permanently to choose between two conflicting issues : realize the transfers needs for an optimally diversified portfolio, or use the ability to defer capital gains taxes.

The taxation code specifies the basis to which the price of a security has to be compared in order to evaluate the capital gains (or losses). The tax basis is either defined as (i) the specific share purchase price, or (ii) the weighted average of past purchase prices. In some countries, investors can chose either one of the above definitions of the tax basis. A deterministic model with the above definition (i) of the tax basis, together with the *first in first out* rule of priority for the stock to be sold, has been introduced and studied by Jouini, Koehl and Touzi [6, 7].

The case where the tax basis is defined as the weighted average of past purchase prices is easier to analyze, as the tax basis can be described by a controlled Markov dynamics. Therefore, it can be treated as an additional state variable in a classical stochastic control problem. A discrete-time formulation of this model with short sales constraints and linear taxation rule has been studied by Dammon, Spatt and Zhang [5]. They considered the problem of maximizing the expected discounted utility of future consumption, and provided a numerical analysis of this model based on the dynamic programming principle. In particular, they showed that investors may optimally sell assets with embedded capital gains, and that the Merton tax-free optimal strategy is *approximately* optimal for "young investors". We refer to Gallmeyer, Kaniel and Tompaidis [14] for an extension of this analysis to the multi-asset framework.

The first contribution of this paper is to provide a continuous-time formulation of the utility maximization problem under capital gains taxes, see Section 2. The financial market consists of a tax exempt riskless asset and a risky one. Transfers are not subject to any transaction costs. The holdings in risky assets are subject to the no-short sales constraints, and the total wealth is restricted by the no-bankruptcy condition. The risky asset is subject to taxes on capital gains. The tax basis is defined as the weighted average of past purchase prices. We also introduce a possible fixed delay in the tax basis. In contrast with [5], we consider a general nonlinear taxation rule. Our results hold both for finite and infinite horizon models. Section 4 shows that the reduction of our model to the tax-free case produces the same indirect utility than the classical Merton model.

The main result of our paper states that the value function of the continuous-time utility maximization problem with capital gains taxes coincides with the Merton tax-free value function. In other words, investors can optimally avoid taxes and realize the same indirect utility as in the tax-free market. We also provide a maximizing strategy which shows how taxes can be avoided. The particular tractability of the linear taxation rule case allows to prove that it is optimal to take advantage of the tax credits by realizing immediately capital losses.

From an economic viewpoint, our result shows that capital gains taxes do not induce any tax payment by optimal investors. This suggests that the incorporation of capital gains taxes in financial market models should be accompanied by another market imperfection, as transaction costs, which prevents optimal investors from implementing the maximizing strategies exhibited in this paper. This aspect is left for future research.

2 Consumption-investment models with capital gain taxes

2.1 The financial Market

We consider a financial market consisting of one bank account with constant interest rate $r > 0$, and one risky asset with price process evolving according to the Black and Scholes model:

$$dS_t \;=\; \mu S_t dt + \sigma S_t dW_t, \tag{2.1}$$

where μ is a constant instantaneous mean rate of return, $\sigma > 0$ is a constant volatility parameter, and the process $W = \{W_t, 0 \leq t\}$ is a standard Brownian motion defined on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $\mathbb{F}$ be the $\mathbb{P}$-completion of the natural filtration of the Brownian motion. In order for positive investment in the risky asset to be interesting, we shall assume throughout that

$$\mu \;>\; r\,. \tag{2.2}$$

We also assume that the financial market is not subject to any transaction costs, and the shares of the stock are infinitely divisible.

2.2 Relative tax basis

The sales of the stock are subject to *taxes on capital gains*. The amount of tax to be paid for each sale of risky asset is computed by comparison of the current price to the weighted average of price of the assets in the investor portfolio. We therefore introduce the *relative* *tax basis process* B_t which records the ratio of the weighted average price of the assets in the investor portfolio to the current price. When B_t is less than 1, the current price of the risky asset is greater than the weighted-average purchase price of the investor so if she sells the risky asset, she would *realize a capital gain.* Similarly, when B_t is larger then 1, the sale of the risky asset corresponds to the *realization of a capital loss.*

Example 2.1 Let $0 \leq t_0 < t_1 < t_2 < t_3$ be some given trading dates, and consider the following discrete portfolio strategy. :
- buy 5 units of risky asset at time t_0,
- sell 1 unit of risky asset at time t_1,
- buy 2 units of risky assets at time t_2,
- sell 4 units of risky asset at time t_3,
- buy 2 units of risky assets at time t_3.

The relative tax basis is not defined strictly before the first purchase date t_0, and is equal to one exactly at t_0. We set by convention

$$B_t = 1 \quad \text{for} \quad t \leq t_0.$$

Sales do not alter the basis. Therefore, we only care about purchases in order to determine the basis at each time. At times t_2 and t_4, the relative tax basis is given by

$$B_{t_2} = \frac{5S_{t_0} + 2S_{t_2}}{7S_{t_2}} \quad \text{and} \quad B_{t_3} = \frac{5S_{t_0} + 2S_{t_2} + 2S_{t_3}}{9S_{t_3}} .$$

Although no purchases occur in the time intervals (t_0, t_2), (t_2, t_3), the relative tax basis moves because of the change of the current price :

$$(BS)_t = \begin{cases} (BS)_{t_0} \text{ for } t_0 \leq t < t_2 , \\ (BS)_{t_2} \text{ for } t_2 \leq t < t_3 \\ (BS)_{t_3} \text{ for } t \geq t_3 . \end{cases}$$

2.3 Taxation rule

Each monetary unit of stock sold at some time t is subject to the payment of an amount of tax computed according to the relative tax basis observed at the prior time

$$t_\delta := (t - \delta)^+ = \max\{0, t - \delta\} . \tag{2.3}$$

Here the delay $\delta \geq 0$ is a fixed characteristic of the taxation rule. Another characteristic of the taxation rule is the amount of tax to be paid per unit of sale. This is defined by

$$f\left(B_{t_\delta -}\right) , \tag{2.4}$$

where f is a map from $\mathbb{R}_+$ into $\mathbb{R}$ satisfying

$$f \text{ non-increasing} \quad \text{and} \quad f(1) = 0 . \tag{2.5}$$

Example 2.2 *(Proportional tax on non-negative gains)* Let

$$f(b) \; := \; \alpha(1-b)^+ \quad \text{for some constant} \quad 0 < \alpha < 1 \, .$$

When the relative tax basis is less than unity, the investor realizes a capital gain, and pays the amount of tax $\alpha(1 - B_{t_\delta})$ per unit amount of sales.

Example 2.3 *(Proportional tax with tax credits)* Let

$$f(b) \; := \; \alpha(1-b) \quad \text{for some constant} \quad 0 < \alpha < 1 \, .$$

When the relative tax basis is less than unity, the investor realizes a capital gain, and pays the amount of tax $\alpha(1 - B_{t_\delta})$ per unit amount of sales. When the relative tax basis is larger then unity, the investor receives the tax credit $\alpha(B_{t_\delta} - 1)$ per unit amount of sales.

Remark 2.1 When there are no tax credits, i.e. $f \geq 0$, it is clear that the total tax paid by the investor is non-negative, and the investor can not do better than in a tax-free market. However, when f is not non-negative, it is not obvious that the investor can not take advantage of the tax credits, and do better than in a tax-free model. Of course, this would not be acceptable from the economic viewpoint. Our analysis of this situation in Section 7 shows that the presence of tax credits does not produce such a non-desirable effect.

2.4 Consumption-investment strategies

An investor start trading with an initial capital x in cash and y monetary units in the risky asset. At each time, trading occurs by means of transfers between the two investment opportunities.

We denote by $\tilde{L} := (\tilde{L}_t, t \geq 0)$ the process of cumulative transfers form the bank account to the risky assets one, and $\tilde{M} := (\tilde{M}_t, t \geq 0)$ the process of cumulative transfers from the risky assets account to the bank. Here, $\tilde{L}$ and $\tilde{M}$ are two $\mathbb{F}-$adapted, right-continuous, non-decreasing processes with $\tilde{L}_{0^-} = \tilde{M}_{0^-} = 0$.

In addition to the trading activity, the investor consumes in continuous-time at the rate $C = \{C_t, t \geq 0\}$. The process C is $\mathbb{F}-$adapted and nonnegative.

Given a consumption-investment strategy $(C, \tilde{L}, \tilde{M})$, we denote by X_t the position on the bank, Y_t the position on the risky assets account, and B_t the relative tax basis at time t.

2.5 Portfolio constraints

We first restrict the strategies to satisfy the solvency condition

$$Z_t \; := \; X_t + [1 - f(B_{t_\delta -})] \, Y_t \; \geq \; 0 \; \mathbb{P} - \text{a.s.} \quad \text{for all} \quad t \geq 0 \, , \tag{2.6}$$

i.e. the total wealth of the investor after liquidation is non-negative at any time. We also impose the no-short sales constraint

$$Y_t \; \geq \; 0 \; \mathbb{P} - \text{a.s.} \quad \text{for all} \quad t \geq 0 \, , \tag{2.7}$$

together with the absorption condition

$$Y_{t_0}(\omega) = 0 \text{ for some } t_0 \implies Y_t(\omega) = 0 \text{ for a.e. } \omega \in \Omega. \tag{2.8}$$

The latter is a technical condition which is needed for a rigorous continuous-time formulation of our problem.

The consumption-investment strategy $(C, \tilde{L}, \tilde{M})$ is said to be *admissible* if the resulting state variables (X, Y, B) satisfy the above conditions (2.6)-(2.7)-(2.8). In particular, the process $(Z_t, Y_t, B_{t_\delta})_{t\geq 0}$ is valued in the closure $\bar{\mathcal{S}}$ of the subset of $\mathbb{R}^3$:

$$\mathcal{S} \;:=\; (0,\infty)^3 . \tag{2.9}$$

We shall denote the boundaries $y = 0$ and $z = 0$ of $\mathcal{S}$ by

$$\partial^y\mathcal{S} \;:=\; \left\{(y,z,b) \in \bar{\mathcal{S}} \;:\; y = 0\,\right\} \quad \text{and} \quad \partial^z\mathcal{S} \;:=\; \left\{(y,z,b) \in \bar{\mathcal{S}} \;:\; z = 0\right\}.$$

Finally, given an admissible strategy $(C, \tilde{L}, \tilde{M})$, we introduce the stopping time :

$$\tau \;:=\; \inf\{t \geq 0 \;:\; Y_t \notin (0,\infty)\} \;=\; \inf\{t \geq 0 \;:\; Y_t = 0\}\;,$$

where the last equality follows from (2.7). In view of (2.8), it is clear that the trading strategy can be described by means of the non-decreasing right-continuous processes $(L_t, M_t)_{t\geq 0}$ which are related to $(\tilde{L}_t, \tilde{M}_t)_{t\geq 0}$ by

$$L_t \;:=\; \int_0^t Y_t^{-1} d\tilde{L}_t \quad \text{and} \quad M_t \;:=\; \int_0^t Y_t^{-1} d\tilde{M}_t, \quad t < \tau\;.$$

Here, dL_t and dM_t represent the proportion of transfers of risky assets.

2.6 Controlled dynamics

Let $(C, \tilde{L}, \tilde{M})$ be an admissible strategy, and define (L, M) as in the previous paragraph. We shall denote $\nu := (C, L, M)$, and $(X^\nu, Y^\nu, B^\nu) := (X, Y, B)$ the corresponding state variables.

Given an initial capital x on the bank account, the evolution of the wealth on this account is described by the dynamics :

$$dX_t^\nu \;=\; (rX_t^\nu - C_t)dt - Y_{t-}^\nu dL_t + Y_{t-}^\nu\left(1 - f(B_{t_\delta -}^\nu)\right) dM_t \quad \text{and} \quad X_{0-} \;=\; x\;, \tag{2.10}$$

recall from (2.3) that $t_\delta := (t-\delta)^+$. Given an initial endowment y on the risky assets account, the evolution of the wealth on this account is also clearly given by

$$dY_t^\nu \;=\; Y_{t-}^\nu\left(\frac{dS_t}{S_t} + dL_t - dM_t\right) \quad \text{and} \quad Y_{0-} \;=\; y\;. \tag{2.11}$$

In order to specify the dynamics of the relative tax-basis, we introduce the auxiliary process $K^\nu := B^\nu Y^\nu$. By definition of B^ν, we have :

$$dK_t^\nu \;=\; Y_{t-}^\nu dL_t - K_{t-}^\nu dM_t \quad \text{and} \quad K_{0-}^\nu \;=\; yb\;, \tag{2.12}$$

since $B^\nu_{0-} = b$. Observe that the contribution of the sales in the dynamics of K_t is evaluated at the basis price. We then define the relative basis process B^ν by

$$B^\nu_t = \mathbf{1}_{\{Y^\nu_t=0\}} + \frac{K^\nu_t}{Y^\nu_t} \mathbf{1}_{\{Y^\nu_t \neq 0\}} \,. \tag{2.13}$$

When the stopping time $\tau := \inf\{t \geq 0 : Y_t = 0\}$ is positive, the dynamics of the relative tax basis is described on $[0, \tau]$ by

$$dB^\nu_t = -B^\nu_t \left[(\mu - \sigma^2)dt + \sigma W_t\right] + (1 - B^\nu_{t-}) \left(dL_t - \Delta L_t \frac{\Delta L_t - \Delta M_t}{1 + \Delta L_t - \Delta M_t} \right) . \tag{2.14}$$

Hence, the position of the investor resulting from the strategy ν is described by the triple (Y^ν, Z^ν, B^ν), where the dynamics of Y^ν and B^ν are defined respectively by (2.11), (2.14), and Z^ν is defined by (2.6). When the tax function f is smooth, one can deduce the dynamics of Z^ν out of the dynamics of (X^ν, Y^ν, B^ν) by direct application of Itô's lemma; see subsection 2.8. We call (Y^ν, Z^ν, B^ν) the state process associated to the control ν.

Proposition 2.1 *Let $\nu = (C, L, M)$ be a triple of adapted process such that*
A1 *L, M are right-continuous, non-decreasing and $L_{0^-} = M_{0^-} = 0$*
A2 *the jumps of M satisfy $\Delta M \leq 1$*
A3 *$C \geq 0$ and $\int_0^t C_s ds < \infty$ a.s. for all $t \geq 0$*
Then, there exists a unique solution (Y^ν, Z^ν, B^ν) to (2.10)-(2.11)-(2.12)-(2.14).
Moreover, (Y^ν, Z^ν, B^ν) satisfies conditions (2.7)-(2.8).

Proof. Equation (2.11) clearly defines a unique solution Y^ν. Given Y^ν, it is also clear that (2.12) has a unique solution $Y^\nu B^\nu$, and (2.14) defines B^ν uniquely. Finally, given (Y^ν, B^ν), it is an obvious fact that equation (2.10) has a unique solution X^ν. □

Remark 2.2 The statement of the above proposition is still valid when A2 is replaced by the following weaker condition
A2' the jumps of the pair process (L, M) satisfy $\Delta L - \Delta M \geq -1$.
However in the case where tax credits are allowed by the taxation rule, see Section 7, it is easy to construct consumption-investment strategies, satisfying A1-A2'-A3, which increase without bound the value function of the problem (2.16) defined below, starting from some fixed positive initial holding in stock. Hence such a model allows for a weak notion of arbitrage opportunities. Indeed, for each $\varepsilon > 0$ and $\lambda > 0$, let $\Delta L_t = \Delta M_t := 0$ for $t \neq \tau$, $\Delta L_\tau = \Delta M_\tau := \Lambda$ where $\tau := \inf\{t : B_t > 1 + \varepsilon\}$. By sending Λ to infinity, the value function of the problem (2.16) converges to $+\infty$.

Definition 2.1 *Let $\nu = (C, L, M)$ be a triple of $\mathbb{F}-$adapted processes, and $(y, z, b) \in \bar{\mathcal{S}}$. We say that ν is a $(y, z, b)-$admissible consumption-investment strategy if it satisfies Conditions* A1-A2-A3 *together with the solvency condition* (2.6). *We shall denote by $\mathcal{A}^f(y, z, b)$ the collection of all $(y, z, b)-$admissible consumption-investment strategies.*

Remark 2.3 We used the absorption at zero condition in order to express an investment strategy by means of the proportions $dL_t = \frac{d\tilde{L}_t}{Y_{t-}}$ and $dM_t = \frac{d\tilde{M}_t}{Y_{t-}}$, instead of the volume of transfers, $d\tilde{L}_t$ and $d\tilde{M}_t$. This modification was needed for the specification of the tax basis by means of the process K defined above. Indeed, in terms of $(\tilde{L}, \tilde{M})$, the dynamics of the state variables X and Y are given by :

$$dY_t = Y_{t-}\frac{dS_t}{S_t} + d\tilde{L}_t - d\tilde{M}_t \quad \text{and} \quad dX_t = r(X_t - c_t)dt - d\tilde{L}_t + (1 - f(B_{t_\delta -}))\, d\tilde{M}_t\ ,$$

but the relative basis is defined by means of the process K whose dynamics are given by

$$dK_t \;=\; d\tilde{L}_t - \frac{K_{t-}}{Y_{t-}} d\tilde{M}_t\ .$$

Since the event $\{Y = 0\}$ has positive probability, this may cause trouble for the definition of the model.

2.7 The consumption-investment problem

Throughout this paper, we consider a power utility function :

$$U(c) \;:=\; \frac{c^p}{p} \quad \text{for all} \quad c \geq 0,$$

where $0 < p < 1$ is a given parameter. We next consider the investment-consumption criterion

$$J_t^f(y, z, b; \nu) \;:=\; \mathbb{E}\left[\int_0^t e^{-\beta u} U(C_u)dt + e^{-\beta t} U(Z_t^\nu)\mathbf{1}_{\{t<\infty\}}\right] \tag{2.15}$$

for $t \in \mathbb{R}_+ \cup \{+\infty\}$, $(y, z, b) \in \bar{\mathcal{S}}$ and $\nu \in \mathcal{A}^f(y, z, b)$. Let

$$T \;\in\; \mathbb{R}_+ \cup \{+\infty\}$$

be a given time horizon, so that our analysis holds for both finite and infinite horizon. The consumption investment problem is defined by

$$V_T^f(y, z, b) \;:=\; \sup_{\nu \in \mathcal{A}^f(y,z,b)} J_T^f(y, z, b; \nu)\ , \qquad (y, z, b) \in \bar{\mathcal{S}}\ . \tag{2.16}$$

In the context of financial markets without taxes, i.e. $f \equiv 0$, a slight modification of this problem has been solved by Merton [11, 10] by means of a verification argument. In the finite horizon case ($T < \infty$), the tax-free problem can be solved directly by passing to a dual formulation, [12, 2, 8]. In the infinite horizon tax-free problem, Merton [10] singled out the condition

$$\gamma \;:=\; \frac{1}{1-p}\left[\beta - rp - \frac{1}{2}\frac{p}{1-p}\left(\frac{\mu - r}{\sigma}\right)^2\right] \;>\; 0\,, \tag{2.17}$$

in order to ensure that the value function is finite. The (explicit) solution in this context is simply obtained by sending the time horizon to infinity in the solution of the finite horizon problem.

We conclude this section by the following easy result which states that, the value function V^f is non-increasing in f.

Proposition 2.2 *Let (y,z,b) be some initial data in $\mathcal{S}$, and let $f \geq g$ be two maps from $\mathbb{R}_+$ into $\mathbb{R}$. Then $V_T^f(y,z,b) \leq V_T^g(y, z+(f-g)(b), b)$.*

Proof. Consider some admissible consumption-investment strategy $\nu = (C,L,M) \in \mathcal{A}^f(y,z,b)$. In order to prove the required result, we verify that

$$\nu \;\in\; \mathcal{A}^g(y, z+(f-g)(b)y, b)\,.$$

We denote by (Y^f, Z^f, B^f) the state process implied by the strategy ν and the initial data (y,z,b) under the taxation rule f. Similarly, we denote by (Y^g, Z^g, B^g) the state process implied by the strategy ν and the initial data $(y, z+(f-g)(b)y, b)$ under the taxation rule g. Set $X^f := Z^f - [1-f(B^f)]Y^f$ and $X^g = Z^g - [1-g(B^g)]Y^g$.

We first observe that $(Y^f, B^f) = (Y^g, B^g) =: (Y,B)$. Then, since $f \geq g$, it follows that

$$Z^g - Z^f \;=\; X^g - X^f + (f-b)(B)Y \;\geq\; X^g - X^f\,.$$

Since $X^g_{0-} = z + (f-g)(b)y - [1-g(b)]y = X^f_{0-}$, we directly compute by (2.10) that :

$$X^g_t - X^f_t \;=\; \int_0^t r(X^g_s - X^f_s)ds + \int_0^t (f-g)(B_{s_\delta -})Y_{s-}dM_s \;\geq\; \int_0^t r(X^g_s - X^f_s)ds\,,$$

since $f \geq g$. This shows that $X^g \geq X^f$, so that $Z^g \geq Z^f \geq 0$ by the fact that $\nu \in \mathcal{A}^f(y,z,b)$. Hence $\nu \in \mathcal{A}^g(y, z+(f-g)(b)y, b)$. □

2.8 On the dynamics of the process Z

For later use, we provide the dynamics of the process Z defined in (2.6) in the case where the delay $\delta = 0$, and the tax function f is smooth, i.e.

$$f \;\in\; C^2(\mathbb{R}_+)\,.$$

This follows directly by applying Itô's lemma for processes with jumps. The result is

$$\begin{aligned} dZ \;=\;& \left\{ rZ_t - C_t + Y_t\left[(\mu - r)(1-f(B_t)) + \mu B_t f'(B_t) - \frac{1}{2}\sigma^2 B_t^2 f''(B_t) \right] \right\} dt \\ &+\sigma Y_t\left[(1-f(B_t)) + B_t f'(B_t) \right] dW_t \\ &-Y_{t-}\left[f(B_{t-}) + (1-B_{t-})f'(B_{t-}) \right](dL_t - \Delta L_t) \\ &-Y_{t-}\left[f(B_{t-})\Delta L_t + \Delta f(B_t)(1+\Delta L_t - \Delta M_t) \right]\,. \end{aligned} \tag{2.18}$$

An important observation is that the above dynamics are considerably simplified in the linear taxation rule case. Indeed

$$f(b) = \alpha(1-b) \implies 0 \;=\; f(B_{t-})\Delta L_t + \Delta f(B_t)(1+\Delta L_t - \Delta M_t) \tag{2.19}$$

$$\;=\; f(B_{t-}) + (1-B_{t-})f'(B_{t-})\,, \tag{2.20}$$

so that the coefficient of dL as well as the jump term are zero.

3 Main results

It is clear that in the tax-free market, i.e. when $f \equiv 0$, the set of admissible consumption-investment strategies $\mathcal{A}^f$, and the value function V_T^f, do not depend on the relative tax basis. We will then simply write

$$\mathcal{A}^0(y,z) \;=\; \mathcal{A}^0(y,z,b) \text{ and } V_T^0(y,z) \;=\; V_T^0(y,z,b) \quad \text{for all} \quad (y,z,b) \in \bar{\mathcal{S}}\,.$$

The first main result of this paper provides a lower bound for the value function V^f, which readily implies that the equality $V^f = V^0$ holds when there are no tax credits and the initial relative tax basis is $B_{0-} = 1$.

Theorem 3.1 *Let $T \in \mathbb{R}_+ \cup \{+\infty\}$ be some given maturity, and $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ a tax function satisfying* (2.5)*. Assume further that f^+ is locally Lipschitz at $b = 1$, Condition* (2.17) *holds, and*

$$\frac{\mu - r}{\sigma^2(1-p)} \;<\; 1 \quad \textit{and} \quad \min\{\mathbf{1}_{T<\infty}, \gamma\} \;>\; r(1-p)\,. \tag{3.1}$$

(i) *For any delay parameter $\delta \geq 0$, we have*

$$V_T^f(y,z,1) \;\geq\; V_T^0(y,z) \quad \textit{for all} \quad (y,z) \in \mathbb{R}_+^2\,. \tag{3.2}$$

In particular, when $f \geq 0$, the value function of the consumption-investment problem under taxes coincides with the value function of the tax-free problem for initial relative tax basis $B_{0-} = 1$, i.e. $V_T^f(y,z,1) = V_T^0(y,z)$ for all $(y,z) \in \mathbb{R}_+^2$.
(ii) *Set $\delta = 0$. Then*

$$V_T^f(y,z,b) \;\geq\; V_T^0(y,z) \quad \textit{for all} \quad (y,z,b) \in \bar{\mathcal{S}}\,. \tag{3.3}$$

The proof of the lower bounds (3.2)-(3.3) for the value function V^f will be provided in Section 5. Given this result, the last statement follows from the trivial inequality $V^f \leq V^0$ which is a consequence of Proposition 2.2.

Notice that the proof of the above result also provides a precise description of optimal consumption-investment behavior. Indeed, we shall construct a sequence of maximizing strategies by forcing the relative tax basis to be as close as possible to unity, and tracking Merton's optimal strategy.

By using a verification technique, we also obtain the following upper bound for the value function $V^f(y,z,b)$ in the case of a linear taxation rule without delay.

Theorem 3.2 *Let $T \in \mathbb{R}_+ \cup \{+\infty\}$ be some given maturity, and assume that* (2.17) *holds. Consider a taxation rule without delay, $\delta = 0$, defined by the linear tax function*

$$\ell(b) \;:=\; \alpha(1-b) \quad \textit{for all} \quad b \geq 0\,,$$

where α is some given positive constant. Then

$$V_T^\ell(y,z,b) \;\leq\; V_T^0(y,z) \quad \textit{for all} \quad (y,z,b) \in \bar{\mathcal{S}}\,. \tag{3.4}$$

The proof of this result is reported in Section 6. Combining Theorems 3.1 and 3.2, we easily obtain the following result.

Corollary 3.1 *Let $T \in \mathbb{R}_+ \cup \{+\infty\}$ be some given maturity, and set $\delta = 0$. Let Conditions* (2.17) *and* (5.1) *hold. Consider the linear tax function $\ell(b) := \alpha(1-b)$, and let f be another tax functions satisfying* (2.5)*, f^+ locally Lipschitz at $b = 1$. Assume further that $f \geq \ell$, then :*

$$V_T^0(y,z) \;\leq\; V_T^f(y,z,b) \;\leq\; V_T^0\left(y, z+(f-\ell)(b)\right) \quad \text{for all} \quad (y,z,b) \in \bar{\mathcal{S}} .$$

In particular, equality holds at any point $(y,z,b) \in \bar{\mathcal{S}}$ satisfying $f(b) = \ell(b)$.

This result may seem counter-intuitive, since one may expect that the investor can take profit from tax credits and obtain a value function V^f larger then V^{f+}. The explanation lies in the fact that the problem of maximizing $J_T^f(y,z,b;\nu)$ can be restricted to those admissible control processes ν inducing a accumulated amount of tax credits bounded by ε, for all constant $\varepsilon > 0$.

Proposition 3.1 *Assume that Conditions* (2.17) *and* (5.1) *hold. Consider the linear taxation rule of* (7.1)*, and let $\delta = 0$. Then, for all $T \in \mathbb{R}+ \cup \{+\infty\}$ and $(y,z,b) \in \bar{\mathcal{S}}$, there exists a sequence of maximizing strategies $\nu_n = (C^n, L^n, M^n)$ such that for all $n \geq 1$,*

$$\int_0^T \left(B_{u-}^{\nu^n} - 1\right) Y_{u-}^{\nu^n} dM_u^n \;\leq\; \frac{1}{n} \quad a.s.$$

The proof of this result is provided in Section 7.

4 Financial market without taxes

In this section, we review the solution of the consumption-investment problem in a financial market without capital gain taxes, i.e. when $f \equiv 0$. Since the reduction of our problem to this case is slightly different from the classical Merton model, we shall study both problems. We will show that they have essentially the same value functions, and discuss the issue of optimal strategies.

4.1 The classical Merton model

In the classical formulation of the tax-free consumption-investment problem, the investment control variable is described by means of unique process π which represents the proportion of wealth invested in risky assets at each time. Given a consumption plan C, the total wealth process is then defined by the dynamics :

$$d\bar{Z}_t^{(C,\pi)} \;=\; \left(r\bar{Z}_t^{(C,\pi)} - C_t\right) dt + \pi_t \bar{Z}_t^{(C,\pi)} \left[(\mu - r)dt + \sigma dW_t\right] \;, \quad \text{and} \quad \bar{Z}_0^{(C,\pi)} \;=\; z. \quad (4.1)$$

In this context, a consumption-investment strategy is a pair of $\mathbb{F}-$adapted processes (C,π), where C is non-negative and

$$\int_0^T C_s ds + \int_0^T |\pi_s|^2 ds \quad < \quad \infty \quad \mathbb{P}-\text{a.s.}$$

We shall denote by $\bar{\mathcal{A}}(z)$ the collection of all such consumption-investment strategies which satisfy the additional no-bankruptcy condition

$$\bar{Z}_t^{(C,\pi)} \;\geq\; 0 \;\; \mathbb{P}-\text{a.s.} \quad \text{for all} \quad 0 \leq t \leq T\,.$$

The *relaxed* tax-free consumption-investment problem is then defined by :

$$\bar{V}_T(z) \;\;:=\;\; \sup_{(C,\pi)\in\bar{\mathcal{A}}(z)} \;\mathbb{E}\left[\int_0^T e^{-\beta t}U(C_t)dt + e^{-\beta T}U\left(\bar{Z}_T^{(\pi,C)}\right)\mathbf{1}_{\{T<\infty\}}\right]. \tag{4.2}$$

Let (y,z) be an arbitrary initial data in $(0,\infty)\times\mathbb{R}_+$. Clearly, for any admissible consumption-investment strategy $\nu = (C,L,M) \in \mathcal{A}^0(y,z)$, one can define a pair $(C,\pi) \in \bar{\mathcal{A}}(z)$ such that $Z^\nu = \bar{Z}^{(C,\pi)}$. This shows that

$$V_T^0(y,z) \;\leq\; \bar{V}_T(z) \quad \text{for all} \quad (y,z) \in (0,\infty)\times\mathbb{R}_+\;, \tag{4.3}$$

and justifies the name of the problem $\bar{V}_T$. The value function V_T^0 on the boundary $y=0$ will be studied separately.

We shall prove later on (Proposition 4.2) that equality holds in (4.3) by exhibiting a maximizing strategy for the problem V_T^0. In preparation to this, let us first recall the explicit solution of the relaxed tax-free consumption-investment problem.

Theorem 4.1 *Let Condition* (2.17) *hold. Then, for all $z>0$:*

$$\bar{V}_T(z) \;\;=\;\; \frac{z^p}{p}\left[\frac{1}{\gamma}+\left(1-\frac{1}{\gamma}\right)e^{-\gamma T}\right]^{1-p}\,.$$

Moreover, existence holds for the problem $\bar{V}_T(z)$ with optimal consumption-investment strategy given by :

$$\bar{\pi}_t \;=\; \bar{\pi} \;:=\; \frac{\mu-r}{(1-p)\sigma^2}\,, \qquad \bar{C}_t \;:=\; \bar{c}(t)\bar{Z}_t\,,$$

where $\bar{c}(.)$ is the deterministic function

$$\bar{c}(t) \;\;:=\;\; \left[\frac{1}{\gamma}+\left(1-\frac{1}{\gamma}\right)e^{-\gamma(T-t)}\right]^{-1},$$

and $\bar{Z} := \bar{Z}^{(\bar{C},\bar{\pi})}$ is the wealth process defined by the strategy $(\bar{C},\bar{\pi})$:

$$\bar{Z}_0 = z, \qquad d\bar{Z}_t \;=\; \bar{Z}_t\left[(r-\bar{c}(t))\,dt - \frac{\mu-r}{(1-p)\sigma}\left(\frac{\mu-r}{\sigma}dt + dW_t\right)\right]\,.$$

Observe that

- the optimal investment strategy $\bar{\pi}$ is constant both in the finite and infinite horizon cases,

- the optimal consumption process is a linear deterministic function of the wealth process, with slope defined by the function $\bar{c}(t)$; in the infinite horizon case, the function $\bar{c}$ reduces to the constant γ,

Remark 4.1 Consider the infinite horizon case $T = +\infty$. Then $\bar{c}(t) = \gamma$. By direct computation, we see that

$$e^{-\beta t}\mathbb{E}\left[U(\bar{Z}_t)\right] = z^p e^{-\gamma t} \quad \text{for all} \quad t \geq 0 \,.$$

Hence Condition (2.17) guarantees that $e^{-\beta t}\mathbb{E}\left[U(\bar{Z}_t)\right] \longrightarrow 0$ as $t \to \infty$, and therefore :

$$\bar{V}_\infty(z) = \lim_{t\to\infty} \mathbb{E}\left[\int_0^t e^{-\beta s}U(\gamma \bar{Z}_s)ds + e^{-\beta t}U(\bar{Z}_t)\right] .$$

4.2 Connection with our tax-free model

We now focus on the reduction of the model of Section 2 to the tax-free case, i.e. $f \equiv 0$. In this context, the state variable B is not relevant any more. Given an initial data $(y, z) \in \mathbb{R}^2_+$ and an admissible control $\nu = (C, L, M) \in \mathcal{A}^0(y, z)$, the controlled state process reduces to the pair (Y^ν, Z^ν) which evolves according to the dynamics :

$$\begin{aligned} dY_t^\nu &= Y_{t^-}^\nu \left[(\mu - r)dt + \sigma dW_t + dL_t - dM_t\right] \\ dZ_t^\nu &= \left(rZ_t^\nu - C_t\right)dt + Y_{t^-}^\nu \left[(\mu - r)dt + \sigma dW_t\right] \end{aligned}$$

together with the initial condition $(Y, Z)_{0-} = (y, z)$.

This model presents some minor differences with the classical Merton model of Section 4.1. First, the investment strategies are constrained to have bounded variation. We shall see that this induces a non-existence of an optimal control for the problem $V^0(y, z)$, but does not entail any difference between V^0 and $\bar{V}$. Second, the above dynamics imply that zero is an absorbing boundary for the Y variable which describes the holdings in stock. From the solution of the classical Merton model reported in Theorem 4.1, notice that the investment in stock is always positive, except the case of zero initial capital $z = 0$. We therefore expect that the value functions $\bar{V}$ and V^0 do coincide except on the boundary $y = 0$.

Observe that the analysis of both problems $\bar{V}$ and V^0 is trivial on the boundary $z = 0$ since there is no possibility neither for consumption nor for investment. The following result characterizes the value function V^0 on the boundary of $\mathbb{R}^2_+$.

Proposition 4.1 *The solution of the problem V^0 on the boundary $\partial\mathbb{R}^2_+$ is given by :*
(i) *For all $y > 0$,*

$$V^0(y, 0) = \bar{V}(0) = 0 \quad \textit{with optimal controls} \quad (\hat{C}, \hat{L}, \hat{M})_t = (0, 0, 1), \; t \geq 0 \,,$$

and optimal state process $(\hat{Z}, \hat{Y})_t = 0$ for $t \geq 0$.
(ii) *Set $\gamma_0 := \frac{\beta - rp}{1-p}$ and assume $\beta > r$ whenever $T = +\infty$. Then, for all $z \geq 0$,*

$$V_T^0(0, z) = \frac{z^p}{p}\left[\frac{1}{\gamma_0} + \left(1 - \frac{1}{\gamma_0}\right)e^{-\gamma_0 T}\right]^{1-p}, \tag{4.4}$$

with optimal controls

$$(\hat{C}, \hat{L}, \hat{M})_t = (\hat{c}_0(t)\hat{Z}_t, 0, 0), \qquad \hat{c}_0(t) := \left[\frac{1}{\gamma_0} + \left(1 - \frac{1}{\gamma_0}\right)e^{-\gamma_0(T-t)}\right]^{-1} \tag{4.5}$$

for $0 \le t \le T$; the optimal state processes are $\hat{Y} = 0$ and

$$\hat{Z}_t \ := \ z \exp\left[rt - \int_0^t \hat{c}_0(s) ds \right] , \qquad 0 \le t \le T .$$

Proof. For item (i), it is sufficient to observe that the investment strategy $\{(\hat{C}, \hat{L}, \hat{M})_t = (0,0,1),\ t \ge 0\}$ is the only admissible strategy. We now concentrate on item (ii). Since the boundary $y = 0$ is an absorbing boundary, we are reduced to the (deterministic) control problem :

$$V_T^0(t,z,0) \ = \ \sup \int_t^T e^{-\beta t} U\left(C_t\right) dt + e^{-\beta(T-t)} U\left(Z_T\right) \mathbf{1}_{\{T<\infty\}} , \qquad (4.6)$$

where the state dynamics are given by

$$dZ_t \ = \ (rZ_t - C_t) dt .$$

1. We first solve the finite horizon problem $T < \infty$. We shall use a verification argument by guessing a solution to the Hamilton-Jacobi equation of this problem :

$$\begin{aligned} 0 \ &= \ -\beta V_T^0(t,z) + \frac{\partial V_T^0}{\partial t}(t,z) + rz \frac{\partial V_T^0}{\partial z}(t,z) + \sup_{\xi \ge 0} \left\{ U(\xi) - \xi \frac{\partial V_T^0}{\partial z}(t,z) \right\} \\ &= \ -\beta V_T^0(t,z) + \frac{\partial V_T^0}{\partial t}(t,z) + rz \frac{\partial V_T^0}{\partial z}(t,z) + \frac{1-p}{p} \left(\frac{\partial V_T^0}{\partial z}(t,z) \right)^{p/p-1} ; \end{aligned}$$

the argument $y = 0$ has been omitted for notational simplicity. We guess a solution to the above first order partial differential equation in the separable form $z^p h(t)$, and determine h so that the terminal condition

$$V_T^0(T,z,0) \ = \ U(z) \quad \text{or, equivalently,} \quad h(T) \ = \ \frac{1}{p}$$

is satisfied. This leads to the candidate solution defined in (4.4). By usual verification arguments, this candidate is then shown to be the solution of the problem, and the optimal controls are identified.

2. We now concentrate on the infinite horizon case $T = +\infty$. It is clear that the optimal state process Z should be set to zero at infinity. In terms of optimal control, this is a natural transversality condition for the problem. In order to take advantage of this information, we solve the problem by the calculus of variation approach. Direct calculation from the local Euler equation of the problem leads to the following characterization of the optimal state :

$$(p-1)\left(r\dot{Z} - \ddot{Z} \right) \ = \ (\beta - r)(rZ - \dot{Z}) ,$$

where $\dot{Z} = dZ/dt$ denotes the time derivative of the state $Z_.$. this ordinary differential equation can be solved explicitly by the technique of variation of the constant. In view of the boundary conditions $Z_0 = z$ and $Z_\infty = 0$, this provides the unique solution to the local Euler equation :

$$Z_t \ = \ z \exp - \left(\frac{\beta - r}{1-p} \right) t \quad \text{with optimal consumption} \quad C_t \ = \ rZ_t - \dot{Z}_t \ = \ \gamma_0 Z_t .$$

Notice that the condition $\beta > r$ is here necessary in order to ensure that $Z_\infty = 0$. $\square$

To conclude our analysis of the tax-free model, we now focus on the value function in the interior of the domain $(0, \infty)^2$. in contrast with the situation on the boundary, trading in the stock is now possible. The following result shows that the value function V^0 coincides with $\bar{V}$, the maximal utility in the classical Merton model. The price for the control restriction to the class of bounded variation processes is that existence does not hold any more for the problem V^0.

Proposition 4.2 *For all $(y, z) \in (0, \infty)^2$, we have $V_T^0(y, z) = \bar{V}_T(z)$.*

In view of (4.3), the only non-trivial inequality in the above result is that $V_T^0(y, z) \geq \bar{V}_T(z)$. This follows directly from our main Theorem 3.1, by considering the case $f \equiv 0$.

5 Lower bound for the value function $V_T^f(y, z, b)$

Consider a general tax function f satisfying (2.5), and f^+ is Lipschitz continuous at $b = 1$.
The chief goal of this section is to prove Theorem 3.1 which states that, starting from an initial data (y, z, b) in $\bar{\mathcal{S}}$, and under the condition

$$\bar{\pi} \;=\; \frac{\mu - r}{\sigma^2(1-p)} \;<\; 1 \quad \text{and} \quad \inf_{0 \leq s \leq T} \bar{c}(s) \;=\; \min\{\mathbf{1}_{T<\infty}, \bar{c}(0)\} \;>\; r(1-p)\,. \tag{5.1}$$

the maximal utility in the financial market is bounded from below by the maximal utility achieved in a tax-free market, i.e :

$$V_T^f(y, z, b) \;\geq\; V_T^0(y, z) \quad \text{for} \quad \delta \;=\; 0 \text{ and} (y, z, b) \;\in\; \bar{\mathcal{S}}.$$

When the delay parameter δ is positive, the above inequality holds for an initial relative tax basis $B_{0-} = 1$.
Notice that it is sufficient to prove this result when $f = f^+$, this follows from the inequality $V^f \leq V^{f^+}$ which we deduce from (2.2). For the rest of this section we suppose that $f \;=\; f^+$. This result is of course trivial when the initial holding in stock is zero, $Y_{0-} = 0$, since $\partial^y \mathcal{S}$ is an absorbing boundary. For a non-zero initial holding y in stock, we shall prove this result by constructing a sequence of admissible strategies $\hat{\nu}^k$ such that

$$V_T^0(y, z) \quad \leq \quad \limsup_{k \to \infty} \; J_T^f(y, z, b; \hat{\nu}^k)\;.$$

The strategies $\hat{\nu}^k$ are obtained by forcing the relative tax basis B_t to be as close as desired to unity, and by *tracking* Merton's optimal strategy, i.e. keeping the proportion of wealth invested in the risky asset

$$\pi_t \quad := \quad \frac{Y_t}{X_t + Y_t}\,\mathbf{1}_{\{X_t + Y_t \neq 0\}}, \;\; 0 \leq t \leq T,$$

and the proportion of wealth dedicated for consumption

$$c_t \quad := \quad \frac{C_t}{X_t + Y_t}\,\mathbf{1}_{\{X_t + Y_t \neq 0\}}, \;\; 0 \leq t \leq T,$$

close to the pair $(\bar\pi, \bar c(t))$ defined in Theorem 4.1.

To do this, we first fix some $t > 0$, and define a convenient sequence $(\nu^{t,n})_{n\ge 1} := (C^{t,n}, L^{t,n}, M^{t,n})_{n\ge 1}$ for all $(y,z,b) \in \mathcal{S} \cup \partial^z\mathcal{S}$. We shall denote by $\left(Y^{t,n}, Z^{t,n}, B^{t,n}\right) = \left(Y^{\nu^{t,n}}, Z^{\nu^{t,n}}, B^{\nu^{t,n}}\right)$ the corresponding state processes. For each integer $n \ge 1$, the consumption-investment strategy $\nu^{t,n}$ is defined as follows.

1. At time 0 choose the transfer $(\Delta L_0^{t,n}, \Delta M_0^{t,n})$ so as to set the relative tax basis to unity and the proportion of wealth invested in the risky asset to $\bar\pi$:

$$\Delta L_0^{t,n} := \bar\pi \frac{z}{y} \quad \text{and} \quad \Delta M_0^{t,n} := 1 .$$

Since the jumps of the processes X, Y and B are given by

$$\Delta X_s^{t,n} = -Y_{s-}^{t,n}\Delta L_{s-}^{t,n} + \left[1 - f(B_{s_\delta -}^{t,n})\right] Y_{s-}^{t,n} \Delta M_s^{t,n} , \tag{5.2}$$

$$\Delta Y_s^{t,n} = Y_{s-}^{t,n}(\Delta L_s^{t,n} - \Delta M_s^{t,n}) , \tag{5.3}$$

$$\Delta B_s^{t,n} = (1 - B_{s-}^{t,n})\left(1 + \Delta L_s^{t,n} - \Delta M_s^{t,n}\right)^{-1} \Delta L_s^{t,n} , \tag{5.4}$$

we verify that

$$B_0^{t,n} = 1 , \quad \pi_0^{t,n} = \bar\pi , \quad \text{and} \quad X_0^{t,n} + Y_0^{t,n} = z .$$

2. At the final time t, fix the jumps $\left(\Delta L^{t,n}, \Delta M^{t,n}\right)_t$ so that all the wealth is transferred to the bank :

$$\Delta L_t^{t,n} := 0 \quad \text{and} \quad \Delta M_t^{t,n} = 1 .$$

This implies that

$$Y_t^{t,n} = 0 \quad \text{and} \quad X_t^{t,n} = Z_t^{t,n} .$$

3. In Step 3 below, we shall construct a sequence of stopping times $(\tau_k^{t,n})_{k\ge 1}$. Our consumption strategy is defined by

$$C_s^{t,n} := \bar c(s)(X_s^{t,n} + Y_s^{t,n}) \quad \text{for} \quad 0 \le s \le T ,$$

where $\bar c$ is given as in Theorem 4.1 by

$$\bar c(s) = \left[\frac{1}{\gamma} + \left(1 - \frac{1}{\gamma}\right) e^{-\gamma(t-s)}\right]^{-1} .$$

The investment strategy is piece-wise constant :

$$dL_s^{t,n} = dM_s^{t,n} = 0 \quad \text{for all} \quad s \in [0,T] \setminus \{\tau_k^{t,n},\ k \ge 1\} .$$

4. We now introduce the sequence of stopping times $\tau_k^{t,n}$ as the hitting times of the pair process (π, B) of some barrier close to $(\bar\pi, 1)$. Set $\tau_0^{t,n} := 0$, and for $k \ge 1$:

$$\tau_k^{t,n} := \tau_k^{\pi} \wedge \tau_k^{B} ,$$

where

$$\begin{aligned}
\tau_k^B &:= \inf\left\{s \geq \tau_{k-1}^{t,n} \ : \ B_{s_\delta}^{t,n} \ < \ 1-n^{-1}\right\} \ , \\
\tau_k^\pi &:= \inf\left\{s \geq \tau_{k-1}^{t,n} \ : \ |\pi_s^{t,n} - \bar\pi| \ > \ h_n\right\} , \\
(h_n)_{n\geq 1} & \ \text{ is } \ \text{a sequence of positive numbers with } \ h_n \ \longrightarrow \ 0 \, .
\end{aligned}$$

5. To conclude the definition of $\nu^{t,n}$, it remains to specify the jumps $\left(\Delta L^{t,n}, \Delta M^{t,n}\right)$ at each time $\tau_k^{t,n}$. The idea here is to re-set the proportion $\pi^{t,n}$ to the constant $\bar\pi$, and to push-back the relative tax basis B to unity. To do this, we define for all $s \in \{\tau_k^{t,n}, \ k \geq 1\}$:

$$\Delta L_s^{t,n} \ := \ \frac{\bar\pi}{\pi_{s-}^{t,n}} \, \frac{1-\pi_{s-} f\left(B_{s_\delta -}^{t,n}\right)}{1+\sqrt{n}\left[1-\bar\pi f\left(B_{s_\delta -}^{t,n}\right)\right]} \quad \text{and} \quad \Delta M_s^{t,n} \ := \ 1 - \sqrt{n}\,\Delta L_s^{t,n} \, .$$

Since

$$\pi_s^{t,n} \ = \ \frac{Y_s^{t,n}}{Z_s^{t,n}} \ = \ \frac{\pi_{s-}^{t,n}(1+\Delta L_s^{t,n} - \Delta M_s^{t,n})}{1-\pi_{s-}^{t,n}\Delta M_s^{t,n} f\left(B_{s_\delta -}^{t,n}\right)} \, ,$$

it follows from (5.2)-(5.3)-(5.4) that :

$$\pi_s^{t,n} \ = \ \bar\pi \text{ and } B_s^{t,n} \ = \ \frac{1+\sqrt{n}B_{s-}^{t,n}}{1+\sqrt{n}} \quad \text{for} \quad s \in \{\tau_k^{t,n}, \ k \geq 0\} \, .$$

In the following remarks, we verify that $\nu^{t,n}$ is an $(y,z,b)-$admissible consumption-investment strategy for all maturity t. At this point, we need the restriction

$$B_{0-} \ = \ 1 \quad \text{whenever} \quad \delta > 0 \, ,$$

which ensures that

$$\tau_1^B \ > \ 0 \ \ P-\text{a.s.} \tag{5.5}$$

Remark 5.1 It follows from (5.5) that the sequence $\left(\tau_k^{t,n}\right)_{k\geq 0}$ is strictly increasing, and converges to $+\infty$. To see this, we first make the trivial observation that $\tau_k^{t,n} < \tau_{k+1}^\pi$ $\mathbb{P}-$a.s. On the other hand, since L and M are constant in the stochastic interval $[\tau_{k-1}^{t,n}, \tau_k^{t,n})$, we have $\ln B_{\tau_k^{t,n}-}^{t,n} \geq n^{-1}$. Since B_s is a weighted average of 1 and B_{s-}, it follows immediately that Then :

$$B_{\tau_k^{t,n}}^{t,n} \ > \ 1-n^{-1} \, .$$

This guarantees that $\tau_k^{t,n} < \tau_{k+1}^B$ $\mathbb{P}-$a.s. In particular $\tau_k^{t,n} \longrightarrow \tau_\infty^{t,n}$ as $k \longrightarrow \infty$. The proof of our claim is completed by observing that the limit $\tau_\infty^{t,n}$ is necessarily equal to $+\infty$; this is a classical result for hitting times of constant barriers □

As a consequence of the last remark, we deduce that

$$B_{s_\delta -}^{t,n} \ \geq \ 1-n^{-1} \ \text{ and } \ |\pi_s^{t,n} - \bar\pi| \ \leq \ h_n \quad \text{for all} \quad s > 0 \, . \tag{5.6}$$

Remark 5.2 Since $f(1) = 0$, f^+ is continuous at $b = 1$, $f(b) \leq 0$ for $b \geq 1$, and (5.6) holds, it is immediately checked from the above definitions that there is a constant C such that, for sufficiently large n,

$$0 \; < \; \Delta L_s^{t,n} \; < \; n^{-1/2} \text{ and } 0 \; < \; \Delta M_s^{t,n} \; \leq \; C n^{-1/2} \quad \text{for} \quad s \in \{\tau_k^{t,n}, \; k \geq 0\} \,.$$

This guarantees that the process of holdings in risky assets $Y^{t,n}$ is positive $\mathbb{P}-$a.s. for sufficiently large n. □

Remark 5.3 In view of the previous remark, it remains to show that $Z_s^{t,n} \geq 0$ for all $s > 0$, in order to conclude that $\nu^{t,n}$ is an $(y, z, b)-$admissible consumption-investment strategy. To see this, notice that the process $\hat{Z}^{t,n} := X^{t,n} + Y^{t,n}$ satisfies the linear stochastic differential equation :

$$d\hat{Z}_s^{t,n} \quad = \quad \hat{Z}_{s-}^{t,n} \left[\left(r - \bar{c}(s) + \pi_s^{t,n}(\mu - r) \right) ds + \sigma \pi_s^{t,n} dW_s - \pi_{s-}^{t,n} f(B_{s_\delta -}^{t,n}) dM_s^{t,n} \right] \,.$$

Using (5.6), we see that the jumps of this process satisfy

$$\Delta \hat{Z}_s^{t,n} \;=\; -\pi_{s-}^{t,n} f(B_{s_\delta -}^{t,n}) \Delta M_s^{t,n} \;\; \geq \;\; -(\bar{\pi} + 1) f(1 - n^{-1}) \; > \; -1 \,,$$

for sufficiently large n. Since $\hat{Z}_0^{t,n} = z > 0$, this guarantees that the process $\hat{Z}$ is non-negative, and using again (5.6), it follows that for sufficiently large n :

$$Z_s^{t,n} \quad = \quad \hat{Z}_s^{t,n} \left[1 - f(B_{s_\delta -}^{t,n}) \; \pi_s^{t,n} \right] \; \geq \; \hat{Z}_s^{t,n} \left(1 - f(1 - n^{-1}) \; (1 + \bar{\pi}) \right) \; \geq \; 0 \,.$$

□

The main property of the sequence $\left(\nu^{t,n}\right)_n$ is the following.

Lemma 5.1 *Let Conditions* (**??**) *and* (5.1) *hold, and let* $t > 0$ *be some fixed time horizon. Consider a taxation rule defined by a delay* $\delta \geq 0$ *and a function* f *satisfying* (2.5) *with* f *locally Lipschitz at* $z = 1$.

Let (y, z, b) *be some initial data in* $\mathcal{S}$ *with* $B_{0-} = b = 1$ *whenever* $\delta > 0$. *Then the consumption-investment strategy* $\nu^{t,n}$ *defined in this section is admissible, i.e.* $\nu^{t,n} \in \mathcal{A}^f(y, z, b)$.

Assume further that $nh_n^2 \longrightarrow \infty$. *Then, the process* $\hat{Z}^{t,n} := X^{t,n} + Y^{t,n} = X^{\nu^{t,n}} + Y^{\nu^{t,n}}$ *satisfies :*

$$\mathbb{E}\left[\sup_{0 \leq s \leq t} \left\{ \left(\bar{Z}_s - \hat{Z}_s^{t,n} \right)^+ \right\}^2 \right] \;\; \leq \;\; \alpha \left(h_n^2 + n^{-1} h_n^{-2} \right) \,,$$

for some constant α *depending on* t.

Proof. 1. We first observe that we may reduce the problem to the case where no tax credits are offered, i.e.

$$f \;\; \geq \;\; 0 \,.$$

This follows from the non-decrease of the process $\hat{Z} := X + Y$ in terms of the tax function f.
2. By definition of the sequence of stopping times $\left(\tau_k^{t,n}\right)$, we have

$$\sup_{0\le s\le t} \left|\pi_s^{t,n} - \bar{\pi}\right| \;\le\; h_n \quad \text{and} \quad \inf_{0\le s\le t} B_{s-}^{t,n} - 1 \;\ge\; n^{-1} \text{ for all } k \ge 1 \,. \tag{5.7}$$

Set $D := \hat{Z}^{t,n} - \bar{Z}$. Since $D_0 = 0$, we decompose D into :

$$D_s \;=\; F_s + G_s + H_s \,,$$

where

$$\begin{aligned}
F_s &:= \int_0^s D_u \left[(r - \bar{c}(t))du + \pi_u^{t,n}\left((\mu - r)du + \sigma dW_u\right)\right] \,, \\
G_s &:= \int_0^s \bar{Z}_u \left(\pi_s^{t,n} - \bar{\pi}\right)\left((\mu - r)du + \sigma dW_u\right) \\
H_s &:= -\int_0^s \pi_{u-}^{t,n} f\left(B_{u_\delta -}^{t,n}\right)\left(D_{u-} + \bar{Z}_{u-}\right) dM_u^{t,n} \\
&= -\sum_{u\le s} \pi_{u-}^{t,n} f\left(B_{u_\delta -}^{t,n}\right)\left(D_{u-} + \bar{Z}_{u-}\right) \Delta M_u^{t,n} \,.
\end{aligned}$$

In the subsequent calculation, A will denote a generic ($t-$dependent) constant whose value may change from line to line. We shall also denote by $V_s^* := \sup_{0\le u\le s} |V_u|$ for all process $(V_s)_s$.

We first start by estimating the first component F. Observe that $\bar{c}(.)$ is bounded and the process $\pi^{t,n}$ is bounded by $2\bar{\pi}$ for large n. Then

$$\begin{aligned}
|F_s|^2 &\le 2\left(\int_0^s D_u(r - \bar{c}(t) + \pi_u^{t,n}(\mu - r))du\right)^2 + 2\left(\int_0^s D_u \pi_u^{t,n} \sigma dW_u\right)^2 \\
&\le A\int_0^s |D_u^*|^2 du + 2\left(\int_0^s D_u \pi_u^{t,n} \sigma dW_u\right)^2 \,.
\end{aligned}$$

By the Buckholder-Davis-Gundy inequality, this provides

$$\begin{aligned}
\mathbb{E}|F_s^*|^2 &\le A\left(\int_0^s \mathbb{E}|D_u^*|^2 du + \mathbb{E}\int_0^s |D_u|^2 |\pi_u^{t,n}|^2 \sigma^2 du\right) \\
&\le A\int_0^s \mathbb{E}|D_u^*|^2 du \,.
\end{aligned} \tag{5.8}$$

Similarly, :

$$|G_s|^2 \;\le\; 2h_n^2 \left[(\mu - r)^2 \left(\int_{\tau_0}^s \bar{Z}_u du\right)^2 + \sigma^2 \left(\int_{\tau_0}^s \bar{Z}_u dW_u\right)^2\right] .$$

Using again the Buckholder-Davis-Gundy inequality, this provides

$$E|G_s^*|^2 \;\le\; A h_n^2 \,. \tag{5.9}$$

Finally, since the jumps of $M^{t,n}$ are bounded by $n^{-1/2}$ and $D_{0-} + \bar{Z}_{0-} = 0$, we estimate the component H by :

$$\begin{aligned} e^{-rt}n^{1/2}|H_s| &\leq \sum_{k\geq 1} e^{-rt}|\pi^{t,n}_{\tau^{t,n}_{k}\,{}_\delta -}|f\left(B^{t,n}_{\tau^{t,n}_{k_\delta}-}\right)\left(|D_{\tau^{t,n}_{k_\delta}-}| + \bar{Z}_{\tau^{t,n}_{k_\delta}-}\right)\mathbf{1}_{\tau^{t,n}_k\leq s} \\ &\leq \sum_{k\geq 1} e^{-r\tau^{t,n}_{k_\delta}}|\pi^{t,n}_{\tau^{t,n}_{k_\delta}-}|f\left(B^{t,n}_{\tau^{t,n}_{k_\delta}-}\right)\left(|D^*_s| + \bar{Z}^*_s\right)\mathbf{1}_{\tau^{t,n}_k\leq t} \\ &\leq A\left(|D^*_s| + |\bar{Z}^*_t|\right)\sum_{k\geq 1} e^{-r\tau^{t,n}_{k_\delta}} f\left(B^{t,n}_{\tau^{t,n}_{k_\delta}-}\right)\mathbf{1}_{\tau^{t,n}_k\leq t} \end{aligned} \tag{5.10}$$

where the last inequality follows from the boundedness of the investment process $\pi^{t,n}$. Since $b = 1$ whenever $\delta > 0$, $1 - B^{t,n}$ is bounded from below by n^{-1}, $f \geq 0$ satisfies (2.5) and is locally Lipschitz at $b = 1$, this provides :

$$|H_s| \leq An^{-3/2}\left(|D^*_s| + |\bar{Z}^*_t|\right)\sum_{k\geq 1} e^{-r\tau^{t,n}_{k_\delta}}\mathbf{1}_{\tau^{t,n}_k\leq t}$$

In the next step, we shall prove that

$$\limsup_{n\to\infty} n^{-2}h_n^2\,\mathbb{E}\left[\sum_{k\geq 1} e^{-r\tau^{t,n}_{k_\delta}}\mathbf{1}_{\tau^{t,n}_k\leq t}\right]^2 < \infty . \tag{5.11}$$

Then, it follows from the Cauchy-Schwartz inequality that

$$\mathbb{E}|H_s|^2 \leq An^{-1}h_n^{-2}\left(1 + \mathbb{E}|D^\star_s|^2\right) . \tag{5.12}$$

We now collect the estimates from (5.8), (5.9) and (5.12) to see that :

$$\left(1 - An^{-1}h_n^{-2}\right)E|D^*_s|^2 \leq A\left(h_n^2 + n^{-1}h_n^{-2} + \int_0^s \mathbb{E}|D^*_u|^2du\right) \quad \text{for all} \quad s \leq t .$$

Assuming that the sequence h_n satisfies $nh_n^2 \longrightarrow 0$, it follows from the previous inequality that

$$E|D^*_s|^2 \leq A\left(h_n^2 + n^{-1}h_n^{-2} + \int_0^s \mathbb{E}|D^*_u|^2du\right) \quad \text{for all} \quad s \leq t ,$$

and the required result follows from the Gronwall inequality.

3. In this step, we complete the proof of the lemma by verifying that the claim (5.11) holds.

3.1. Introduce the process

$$\eta^{t,n}_s := \varphi\left(\pi^{t,n}_s\right) := \ln\left(\frac{\pi^{t,n}_s}{1 - \pi^{t,n}_s}\right).$$

By an immediate application of Itô's lemma, we see that the processes $\eta^{t,n}$ is governed by the dynamics :

$$\begin{aligned} d\eta^{t,n}_s &= \lambda\left(s, \pi^{t,n}_s\right)ds + \sigma dW_s \text{ for } \tau_k \leq s < \tau_{k+1} , \\ \text{and } \eta^{t,n}_s &= \varphi(\bar{\pi}) \text{ for } s \in \left\{\tau^{t,n}_k,\ k \geq 0\right\} , \end{aligned}$$

where $\lambda(.,.)$ is given by

$$\lambda(s,p) \quad := \quad \mu - r - \frac{\sigma^2}{2} + \frac{\bar{c}(s)}{1-p},$$

recall that $\bar{\pi} \in (0,1)$ by (5.1), and therefore $\pi_s^{t,n} \in (0,1)$ for large n. Also, since $B^{t,n}$ is pulled up towards unity at each stopping time, we have

$$B^{t,n} \quad \geq \quad \underline{B}^{t,n}, \tag{5.13}$$

where $\underline{B}^{t,n}$ is defined by :

$$d\underline{B}_s^{t,n} \;=\; -\underline{B}_s^{t,n}\left[(\mu-\sigma^2)ds + \sigma dW_s\right] \text{ for } \tau_k^{t,n} \leq s < \tau_{k+1}^{t,n},$$
$$\text{and } \underline{B}_s^{t,n} \;=\; \bar{b} \;:=\; \tfrac{1+\sqrt{n}(1-n^{-1})}{1+\sqrt{n}} \text{ for } s \in \left\{\tau_k^{t,n},\; k \geq 0\right\}.$$

Under Condition (5.1), we have

$$\mu - \frac{\sigma^2}{2} \;<\; \lambda(s,\bar{\pi}) \quad \text{for al} \quad s \leq t.$$

By the comparison result for stochastic differential equations this implies that

$$-\ln B_s^{t,n} + \ln \bar{b} \;\leq\; \eta_s - \varphi(\bar{\pi}) \;,\; s \geq 0 \quad \text{for sufficiently large} \quad n. \tag{5.14}$$

We also need to introduce the martingale $P^{t,n}$ defined by :

$$P_0^{t,n} = 0 \quad \text{and} \quad dP_s^{t,n} = -\lambda(s,\pi_s^{t,n})P_s^{t,n}dW_s.$$

We shall denote by $Q^{t,n}$ the probability measure on $\mathcal{F}_t$ defined by the density

$$\left.\frac{dQ^{t,n}}{dP}\right|_{\mathcal{F}_t} \quad = \quad P_t^{t,n}.$$

3.2. Observe that the random variables

$$\theta_k^{t,n} \;:=\; \tau_k^{t,n} - \tau_{k-1}^{t,n} \quad \text{for all} \quad k \geq 1,$$

are independent. Then :

$$\begin{aligned}
R \;:=\; \left\{\mathbb{E}\left[\sum_{k\geq 1} e^{-r\tau_k^{t,n}} \mathbf{1}_{\tau_k^{t,n}\leq t}\right]^2\right\}^2 \quad &= \quad \left\{\mathbb{E}^{Q^{t,n}}\left[\left(P_t^{t,n}\right)^{-1} \sum_{k\geq 1} e^{-r\tau_k^{t,n}} \mathbf{1}_{\tau_k^{t,n}\leq t}\right]^2\right\}^2 \\
&\leq \quad \mathbb{E}^{Q^{t,n}}\left[\left(P_t^{t,n}\right)^{-2}\right] \mathbb{E}^{Q^{t,n}}\left[\sum_{k\geq 1} e^{-r\tau_k^{t,n}} \mathbf{1}_{\tau_k^{t,n}\leq t}\right]^4 \\
&\leq \quad C\,\mathbb{E}^{Q^{t,n}}\left[\sum_{k\geq 1} e^{-r\tau_k^{t,n}} \mathbf{1}_{\tau_k^{t,n}\leq t}\right]^4,
\end{aligned}$$

for some $t-$dependent constant C, where $\mathbb{E}^{Q^{t,n}}\left[\left(P_t^{t,n}\right)^{-2}\right]$ inherits the bound of the function λ. This provides

$$R \;\le\; C\sum_{j_1\ge 0}\left(\prod_{1\le k\le j_1}\mathbb{E}^{Q^{t,n}}\left[e^{-4r\theta_k^{t,n}}\right]\right)\sum_{j_2\ge j_1}\left(\prod_{j_1\le k\le j_2}\mathbb{E}^{Q^{t,n}}\left[e^{-3r\theta_k^{t,n}}\right]\right) \qquad (5.15)$$
$$\sum_{j_3\ge j_2}\left(\prod_{j_2\le k\le j_3}\mathbb{E}^{Q^{t,n}}\left[e^{-2r\theta_k^{t,n}}\right]\right)\sum_{j_4\ge j_3}\left(\prod_{j_3\le k\le j_4}\mathbb{E}^{Q^{t,n}}\left[e^{-r\theta_k^{t,n}}\right]\right).$$

3.3. We now define a convenient lower bound for the $\theta_k^{t,n}$'s. Set

$$\bar{k}_n \;=\; \varphi(\bar\pi+h_n)-\varphi(\bar\pi)\,,\quad \underline{k}_n \;=\; \varphi(\bar\pi)-\varphi(\bar\pi-h_n)\,,\quad \text{and}\;\; \ell_n \;:=\; -\ln\left(1-n^{-1}\right)+\ln\bar b\,.$$

Then, it follows (5.14) that

$$\theta_k^{t,n} \;\ge\; \hat\theta_k^n \qquad (5.16)$$

where

$$\hat\theta_k^n+\tau_{k-1}^{t,n} \;:=\; \inf\left\{s>\tau_{k-1}^{t,n}\;:\eta_s^{t,n}\ge\varphi(\bar\pi)+\overline{k}_n\wedge\ell_n \text{ or } \eta_s^{t,n}\le\varphi(\bar\pi)-\underline{k}_n\right\}$$

It is immediately checked that

$$\ell_n\approx n^{-1}\,,\quad \overline{k}_n\approx h_n\,,\quad\text{and}\quad \underline{k}_n\approx h_n\,. \qquad (5.17)$$

Then, since $nh_n^2\longrightarrow\infty$, we have $\overline{k}_n\wedge\ell_n=\ell_n$ for large n, and therefore

$$\begin{aligned}\hat\theta_k^n &= \inf\left\{s>\tau_k^{t,n}\;:\eta_s^{t,n}\ge\varphi(\bar\pi)+\ell_n \text{ or } \eta_s^{t,n}\le\varphi(\bar\pi)-\underline{k}_n\right\}\\ &= \inf\left\{s>\tau_k^{t,n}\;:\sigma W_s^{t,n}\ge\ell_n \text{ or } \sigma W_s^{t,n}\le-\underline{k}_n\right\}\end{aligned}$$

where, by the Girsanov theorem, $W_s^{t,n}:=W_s+\int_0^s\lambda(u,\pi_u^{t,n})du$ is a Brownian motion under the probability measure $Q^{t,n}$.

3.4. Observe that the stopping times $\hat\theta_k^n=\hat\theta_1^n$ in distribution. We then deduce from (5.15) end (5.16) that :

$$\begin{aligned}R &\le C\sum_{j_1\ge 0}\left(\mathbb{E}^{Q^{t,n}}\left[e^{-4r\hat\theta_1^n}\right]\right)^{j_1}\sum_{j_2\ge j_1}\left(\mathbb{E}^{Q^{t,n}}\left[e^{-3r\hat\theta_1^n}\right]\right)^{j_2-j_1}\\ &\qquad\sum_{j_3\ge j_2}\left(\mathbb{E}^{Q^{t,n}}\left[e^{-2r\hat\theta_1^n}\right]\right)^{j_3-j_2}\sum_{j_4\ge j_3}\left(\mathbb{E}^{Q^{t,n}}\left[e^{-r\hat\theta_1^n}\right]\right)^{j_4-j_3}\\ &= C\prod_{j=1}^{4}\left\{1-\mathbb{E}^{Q^{t,n}}\left[e^{-jr\hat\theta_1^n}\right]\right\}^{-1}.\end{aligned}$$

The stopping time $\hat\theta_1^n$ is the minimum between two first hitting times of the Brownian motion of constant barriers. The distribution of this random variable is well-known. In particular

$$\mathbb{E}^{Q^{t,n}}\left[e^{-jr\hat\theta_1^n}\right] \;=\; \frac{\cosh\left(\sqrt{2jr}\left(\frac{k_n-\ell_n}{2\sigma}\right)\right)}{\cosh\left(\sqrt{2jr}\left(\frac{k_n+\ell_n}{2\sigma}\right)\right)}$$

see e.g. Karatzas and Shreve [9] p100. By (5.17), we then directly compute that

$$1-\mathbb{E}^{Q^{t,n}}\left[e^{-jr\hat\theta_1^n}\right] \;\approx\; \frac{\sqrt{2jr}}{2\sigma}\,\frac{h_n}{n}\,,$$

so that

$$\sqrt{R} \;=\; \mathbb{E}\left[\sum_{k\geq 1} e^{-r\tau_k^{t,n}}\mathbf{1}_{\tau_k^{t,n}\leq t}\right]^2 \;\leq\; C\,\frac{n^2}{h_n^2} \quad \text{for large} \quad n\,,$$

and (5.11) follows. □

We are now ready for the

Proof of Theorem 3.1 We proceed in three steps.
1. We first show that

$$\liminf_{n\to\infty} J_t^f(y,z,b;\nu^{t,n}) \;\geq\; \bar J_t(z) \;\;:=\;\; \mathbb{E}\int_0^t e^{-\beta s}U(\bar c(s)\bar Z_s)ds + e^{-\beta T}U(\bar Z_t)$$

for all (y,z) be in $\mathcal{S}$ and $t\in\mathbb{R}_+$. Indeed, since the utility function U is non-decreasing, we have

$$J_t^f(y,z,b,\nu^{t,n}) \;\;\geq\;\; \hat J \;:=\; \mathbb{E}\int_0^t e^{-\beta s}U\left(\bar c(s)(\hat Z_s^{t,n}\wedge\bar Z_s)\right)ds + e^{-\beta t}U(\hat Z_t^{t,n}\wedge\bar Z_t)\,.$$

Next, since U is p-holder continuous

$$|U\left(\bar c(s)(\hat Z_s^{t,n}\wedge\bar Z_s)\right) - U(\bar c(s)\bar Z_s)| \;\leq\; \bar c(s)|\left(\bar Z_s\hat Z_s^{t,n}\right)^-|^p$$

Then using the Jensen inequality with the concave function $x:\mapsto x^{\frac{p}{2}}$:

$$\mathbb{E}|\left(\bar Z_s\hat Z_s^{t,n}\right)^-|^p \;=\; \mathbb{E}\left[|\left(\bar Z_s\hat Z_s^{t,n}\right)^-|^2\right]^{\frac{p}{2}} \;\leq\; \left[\mathbb{E}|\left(\bar Z_s\hat Z_s^{t,n}\right)^-|^2\right]^{\frac{p}{2}}$$

Now, using the estimate provided by lemma (5.1)

$$\left(\mathbb{E}|\left(\bar Z_s\hat Z_s^{t,n}\right)^-|^2\right)^{\frac{p}{2}} \;\leq\; (n^{-2}\,\alpha e^{\alpha t})^{\frac{p}{2}} \;=\; n^{-p}\,\alpha e^{\frac{p}{2}\alpha t}$$

It follows that :

$$\left|\bar J_t(z)-\hat J\right| \;\;\leq\;\; n^{-p}\left[\left(\frac{2}{p}c(0)^p+\alpha\right)e^{\frac{p}{2}\alpha t}-\frac{2}{p}\right]\int_0^t e^{\frac{p}{2}\alpha s}ds\,.$$

2. In the finite horizon case, the proof is completed by taking $t = T$ in Step 1. We next concentrate on the infinite horizon case $T = +\infty$. Fix some positive integer k. By Remark 4.1, we have :

$$\bar{V}_T(z) \quad = \quad \lim_{t\to\infty} \bar{J}_t(z) \; .$$

Then

$$\bar{J}_{t_k}(z) \quad \geq \quad \bar{V}_T(z) \; - \; \frac{1}{k} \; ,$$

for some $t_k > 0$. By the first step of this proof :

$$\lim_{n\to\infty} J^f_{t_k}(y, z, b; \nu^{t_k,n}) \quad \geq \quad \bar{J}_{t_k}(z) \; .$$

Then, there exists some integer n_k

$$J^f_{t_k}(y, z, b; \nu^{t_k,n_k}) \;\geq\; \bar{J}_{t_k}(z) \; - \; \frac{1}{2k} \quad \geq \quad \bar{V}_T(z) - \frac{1}{k} \; .$$

3. Finally, we define the consumption-investment strategies $\hat{\nu}^k$ consisting in following ν^{t_k,n_k} up to t_k, then liquidating at t_k the risky asset position and making a null consumption on the time interval (t_k, T). Then:

$$J^f_T(y, z, b; \hat{\nu}^k) \quad \geq \quad J^f_{t_k}(y, z, b; \nu^{t_k,n_k}) \;\geq\; \bar{V}_T(z) - \frac{1}{k} \text{ for all } k \geq 0 \; .$$

This proves that $V^f_T(y, z, b) \geq \limsup_{k\to\infty} J^f_T(y, z, b; \hat{\nu}^k) \geq \bar{V}_T(z)$. □

6 Upper bound for the value function $V^\ell_T(y, z, b)$

We now consider a linear taxation rule

$$\ell(b) \;=\; \alpha\,(1 - b) \quad \text{for all} \quad b \;\geq\; 0 \, ,$$

where α is some positive constant. We also assume that capital gains are taxed without delay, i.e. $\delta = 0$.

The purpose of this section is to prove Theorem 3.2 which states that, despite the presence of tax credits, it is not possible to do better than in the tax-free financial markets; in other words, there is no way to take advantages of the tax credits.

Recall from Section 2.8 that, for all admissible consumption-investment process ν, the process Z is pathwise continuous in the linear taxation rule case, and its dynamics simplify to :

$$dZ^\nu_t \quad = \quad [rZ^\nu_t + \lambda Y^\nu_t - r\alpha K^\nu_t - C_t]\, dt + \kappa Y^\nu_t dW_t \; , \tag{6.1}$$

where $K^\nu = Y^\nu B^\nu$,

$$\lambda \;:=\; (1-\alpha)(\mu - r) \quad \text{and} \quad \kappa \;:=\; (1-\alpha)\sigma.$$

In preparation of the proof of Theorem 3.2, we need the following observation.

Lemma 6.1 *Let $(y,z,b) \in \mathcal{S}^f$, and $\nu \in \mathcal{A}^\ell(y,z,b)$ be given. Consider the stopping time*

$$\theta \;:=\; T \wedge \inf\{t \geq 0 \;:\; Z^\nu_t = 0\}\,.$$

Then, on the event set $\{\theta < T\}$:

$$Y^\nu \;=\; C \;=\; 0 \text{ and } Z^\nu = Z^\nu_\theta = 0 \quad \text{on the stochastic interval} \quad (\theta, T)\,.$$

Proof. By easy calculation, we deduce from (6.1) that

$$Z^\nu_t \;=\; e^{r(t-t_0)} Z^\nu_{t_0} + \int_{t_0}^t e^{rs}\left(\lambda Y^\nu_s - r\alpha K^\nu_s\right) ds - \int_{t_0}^t e^{rs} C_s ds + \kappa \int_{t_0}^t e^{rs} Y^\nu_s dW_s ds\,.$$

Since ν is an admissible consumption-investment process, the solvency condition $Z^\nu_t \geq 0$ must hold for any $t \in [0,T)$. The required result follows from the law of iterated logarithm.
□

Proof of Theorem 3.2 Throughout this proof, the maturity date T is fixed in $\mathbb{R}_+ \cup \{\infty\}$. When the initial position (y,z,b) is on the boundary of the solvency region $\partial\mathcal{S}^\ell$ the result is trivial. Indeed, on the absorbing boundary $\partial^y\mathcal{S}^\ell$, the problem reduces to the (deterministic) control problem (4.6), hence $V^\ell_T(0,z,b) = V^0_T(0,z)$. On $\partial^z\mathcal{S}^\ell$ it is easy to verify that all admissible strategies have a null consumption and investment processes. We then concentrate on the case where (y,z,b) is in $\mathcal{S}^\ell$.

1. Define the function $v(t,z)$ by

$$v(t,z) \;=\; \bar{V}_{T-t}(z) \quad \text{for all} \quad (t,z) \in [0,T) \times \mathbb{R}_+\,,$$

where $T - t = \infty$ whenever $T = \infty$, and $\bar{V}$ is the Merton value function. Since the function v is smooth, it is easily seen that it is a classical solution to the associated Hamilton-Jacobi-Bellman equation. In particular, the supersolution part reads :

$$\left(-\beta v + \frac{\partial v}{\partial t}\right)(t,z) \;\leq\; -U(c) - [rz - c + \zeta(\mu - r)]Dv(t,z) - \frac{1}{2}\sigma^2\zeta^2 D^2 v(t,z)$$
$$\text{for all} \quad (t,z) \in [0,T) \times \mathbb{R}_+ \;\text{ and } (c,\zeta) \in \mathbb{R}_+ \times \mathbb{R}, \tag{6.2}$$

where D and D^2 denotes the gradient and the Hessian with respect to the z variable.

3. Fix some initial data $(y,z,b) \in \mathcal{S}$, and let ν be an arbitrary admissible consumption-investment strategy in $\mathcal{A}^\ell(y,z,b)$. For all integer $n \geq 1$, define the stopping time

$$\tau_n \;:=\; T \wedge \inf\left\{\, s \;\geq\; 0 \;:\; \min\left(Z^\nu_s, |Z^\nu_s|^{-1}, Y^\nu_s\right) \;>\; n \,\right\}\,,$$

so that the stopped processes $\left(Z^\nu_{t\wedge\tau_n}\right)_{0\leq t\leq T}$ and $\left(Y^\nu_{t\wedge\tau_n}\right)_{0\leq t\leq T}$ are respectively valued in the bounded intervals $[n^{-1}, n]$ and $[0,n]$.

We next apply Itô's lemma to the function v to get :

$$e^{-\beta\tau_n} v(\tau^n, Z^\nu_{\tau_n}) - v(0,z) \;=\; \int_0^{\tau_n} e^{-\beta s}\left[\left(-\beta v + \frac{\partial v_t}{\partial t}\right)(s, Z^\nu_s)\, ds\right.$$
$$\left. + Dv\,(s, Z^\nu_s)\, dZ^\nu_s + \frac{1}{2}\kappa^2 |Y^\nu_s|^2 D^2 v\,(s, Z^\nu_s)\, ds\right]$$

Since the process $Y_s^\nu Dv(s, Z_s^\nu)$ is bounded up to the stopping time τ_n, we obtain by substitution of (6.1) in the above equality :

$$\mathbb{E}\left[e^{-\beta\tau_n} v(\tau^n, Z_{\tau_n}^\nu)\right] - v(0,z) \;=\; \mathbb{E}\int_0^{\tau_n} e^{-\beta s}\left[\left(-\beta v + \frac{\partial v_t}{\partial t}\right)(s, Z_s^\nu) + \left(rZ_t^\nu - C_t + \lambda Y_t^\nu - r\alpha K_t^\nu\right) Dv\,(s, Z_s^\nu) + \frac{1}{2}\kappa^2 |Y_s^\nu|^2 D^2 v\,(s, Z_s^\nu)\right] ds. \tag{6.3}$$

Now, writing (6.2) with parameters $c = C_t(\omega)$ and $\zeta = (1-\alpha)Y_s^\nu(\omega)$, we see that $P-$a.s.

$$\left(-\beta v + \frac{\partial v}{\partial t}\right)(s, Z_s^\nu) \;\le\; -U(C_s) - [rZ_s^\nu - C_s + \lambda Y_s^\nu]Dv(s, Z_s^\nu) - \frac{1}{2}\kappa^2 |Y_s^\nu|^2 D^2(s, Z_s^\nu)\,,$$

and we deduce from (6.3) that

$$\mathbb{E}\left[e^{-\beta\tau_n} v(\tau^n, Z_{\tau_n}^\nu)\right] - v(0,z) \;\le\; -r\alpha\,\mathbb{E}\int_0^{\tau_n} e^{-\beta s}\left[K_s^\nu Dv\,(s, Z_s^\nu) - U(C_s)\right] ds\,.$$

Since $Dv(t,z) > 0$ for all $z > 0$, this provides :

$$\begin{aligned} v(0,z) \;&\ge\; \mathbb{E}\left[\int_0^{\tau_n} e^{-\beta s} U(C_s) ds + e^{-\beta\tau_n} v(\tau^n, Z_{\tau_n}^\nu)\right] \\ &\ge\; \mathbb{E}\left[\int_0^{\tau_n} e^{-\beta s} U(C_s) ds + e^{-\beta\tau_n} v(\tau^n, Z_{\tau_n}^\nu)\mathbf{1}_{\{T<\infty\}}\right]. \end{aligned} \tag{6.4}$$

3. Observe that

$$\tau_n \longrightarrow \theta\,, \qquad P - \text{a.s.}$$

where θ is defined in Lemma 6.1. We now send n to infinity in (6.4). It follows from Fatou's lemma together with Lemma 6.1 that

$$\begin{aligned} v(0,z) \;&\ge\; \mathbb{E}\left[\int_0^{\theta} e^{-\beta s} U(C_s) ds + e^{-\beta\theta} v(\theta, Z_\theta^\nu)\mathbf{1}_{\{T<\infty\}}\right] \\ &=\; \mathbb{E}\left[\int_0^{T} e^{-\beta s} U(C_s) ds + e^{-\beta T} U(Z_T^\nu)\mathbf{1}_{\{T<\infty\}}\right]. \end{aligned}$$

By arbitrariness of $\nu \in \mathcal{A}^f(y,z,b)$, this show that $V_T^f(y,z,b) \ge \bar{V}_T(z)$. □

7 Immediate realization of capital losses

In this section we consider a linear taxation rule without delay, $\delta = 0$, defined by the function

$$\ell(b) \;=\; \alpha(1-b)\,. \tag{7.1}$$

In this context, we have already proved that :

$$V^f(y,z,1) \;=\; V^{f+}(y,z,1) \;=\; V_T^0(y,z)\,, \quad \text{for all} \quad (y,z) \in [0,\infty) \times [0,\infty)\,;$$

in particular, the investor does not take profit from tax credits. For a better understanding of this fact, we now construct explicitly a maximizing strategy in the linear taxation rule, which exhibits tax credits.

We first intend to prove that it is always worth realizing capital losses whenever the tax basis exceeds unity. In other words, for each (y, z, b) in $\bar{\mathcal{S}}$, any admissible consumption-investment strategy for which the relative tax basis exceeds 1 at some stopping time τ can be improved strictly by increasing the sales of the risky asset at τ. We shall refer to this property as the *optimality of the immediate realization of capital losses.* In a discrete-time framework, this property was stated without proof by [5].

Proposition 7.1 *Let ℓ be the linear taxation rule of* (7.1), $\delta = 0$, *and consider some initial data (y, z, b) in $\bar{\mathcal{S}}$. Consider some consumption-investment strategy $\nu := (C, L, M)$ $\in \mathcal{A}^\ell(y, z, b)$, and suppose that there is a finite stopping time $\tau \le T$ with $P[\tau < T] > 0$ and $B^\nu_\tau > 1$ a.s. on $\{\tau < T\}$.*

Then there exists an admissible strategy $\tilde{\nu} = \mathcal{V}(\nu, \tau)$ such that

$$Y^{\tilde{\nu}} = Y^\nu, \quad Z^{\tilde{\nu}} \ge Z^\nu, \quad B^{\tilde{\nu}} \le B^\nu, \ \tilde{C} \ge C,$$

and

$$J^\ell_T(y, z, b; \tilde{\nu}) \quad > \quad J^\ell_T(y, z, b; \nu) \; .$$

We start by proving the following lemma which shows how to take advantage of the tax credit at time τ.

Lemma 7.1 *Let ℓ be as in* (7.1), $\delta = 0$, *and consider some initial holdings (y, z, b) in $\bar{\mathcal{S}}$. Consider some consumption-investment strategy $\nu := (C, L, M) \in \mathcal{A}^\ell(y, z, b)$, and suppose that there is a finite stopping time $\tau \le T$ with $P[\tau < T] > 0$ and $B^\nu_\tau > 1$ a.s. on $\{\tau < T\}$. Define $\bar{\nu} = (\bar{C}, \bar{L}, \bar{M})$ by*

$$\bar{C} \ := \ C \quad \text{and} \quad (\bar{L}, \bar{M}) \ := \ (L, M) + (1,1)(1 - \Delta M_\tau) 1_{t \ge \tau} \; . \tag{7.2}$$

Then $\bar{\nu} \in \mathcal{A}^\ell_T(y, z, b)$ and the resulting state processes are such that

$$Y^{\bar{\nu}} = Y^\nu, \quad Z^{\bar{\nu}} \ge Z^\nu, \quad B^{\bar{\nu}} \le B^\nu,$$

and, almost surely,

$$B^{\bar{\nu}}_\tau = 1, \quad K^{\bar{\nu}}_\tau < K^\nu_\tau \quad \text{on} \quad \{\tau < T\} \; .$$

Proof. 1. Since ν and $\bar{\nu}$ differ only by the jump at the stopping time τ, and $\Delta\bar{L}_\tau = \Delta\bar{M}_\tau$, we have

$$Y^{\bar{\nu}} \quad = \quad Y^\nu \; ,$$

and

$$\left(Z^{\bar{\nu}}_t, B^{\bar{\nu}}_t\right) \ = \ \left(Z^\nu_t, B^\nu_t\right) \quad \text{for all} \quad t < \tau \; .$$

Notice that the jump at time τ consists in selling out the whole portfolio as $\Delta\bar{M}_\tau = \Delta M_\tau + (1 - \Delta M_\tau) = 1$. Then, we clearly have :

$$B^{\bar\nu}_\tau \;=\; 1$$

and

$$K^{\bar\nu}_\tau - K^\nu_\tau \;=\; Y^\nu_{\tau-}\,(1 - B^\nu_{\tau-})\,(1 - \Delta M_\tau)\;. \tag{7.3}$$

Since

$$0 \;>\; (1 - B^\nu_\tau) \;=\; (1 - B^\nu_{\tau-})\frac{1 - \Delta M_\tau}{1 + \Delta L_\tau - \Delta M_\tau} \quad \text{on} \quad \{\tau < T\}\;, \tag{7.4}$$

it follows that $B_{\tau^-} > 1$ and $1 - \Delta M_\tau > 0$ a.s. on $\{\tau < T\}$. We therefore deduce from (7.3) that

$$K^{\bar\nu}_\tau - K^\nu_\tau \;<\; 0 \quad \text{on} \quad \{\tau < T\}\;.$$

2. We next examine the state variable $B^{\bar\nu}_t$ for $t > \tau$. Since $Y^{\bar\nu} = Y^\nu$, we have $K^{\bar\nu} - K^\nu = Y^\nu\,(B^{\bar\nu} - B^\nu)$, and

$$d(K^{\bar\nu}_t - K^\nu_t) \;=\; -(K^{\bar\nu}_{t-} - K^\nu_{t-})\,dM_t, \quad \text{with } K^{\bar\nu}_\tau - K^\nu_\tau \;=\; Y^\nu_\tau\,(1 - B^\nu_\tau)\,. \tag{7.5}$$

This linear stochastic differential equation can be solve explicitly :

$$K^{\bar\nu}_t - K^\nu_t \;=\; (K^{\bar\nu}_\tau - K^\nu_\tau)\,e^{-M^c_t + M^c_\tau} \prod_{\tau < u \le t} (1 - \Delta M_u), \quad \text{for all} \quad t \ge \tau\;,$$

where M^c denotes the continuous part of M. Since $K^{\bar\nu}_\tau - K^\nu_\tau = Y^\nu_\tau(1 - B^\nu_\tau) < 0$ on $\{\tau < T\}$, this shows that

$$K^{\bar\nu}_t \;\le\; K^\nu_t \quad \text{and therefore} \quad B^{\bar\nu}_t \;\le\; B^\nu_t \quad \text{for } t \ge \tau\;. \tag{7.6}$$

3. In this step, we intend to prove that $Z^{\bar\nu}_t \ge Z^\nu_t$ for $t \ge \tau$. Recall from (6.1) the dynamics of the processes $Z^{\bar\nu}$ and Z^ν. Using the fact that $Y^{\bar\nu} = Y^\nu$, we have for $t > \tau$:

$$e^{-r(t-\tau)}\left(Z^{\bar\nu}_t - Z^\nu_t\right) \;=\; Z^{\bar\nu}_\tau - Z^\nu_\tau \;-\; \alpha r \int_\tau^t e^{-r(u-\tau)}\left(K^{\bar\nu}_u - K^\nu\right) du$$

We next use (7.6) to see that, for $t \ge \tau$:

$$e^{-r(t-\tau)}\left(Z^{\bar\nu}_t - Z^\nu_t\right) \;\ge\; Z^{\bar\nu}_\tau - Z^\nu_\tau \;\ge\; 0\;.$$

4. Clearly, $\bar\nu$ satisfies Conditions A1-A2-A3, and $Z^{\bar\nu} \ge Z^\nu \;\ge\; 0$ by the previous steps of this proof. Hence $\bar\nu \in \mathcal{A}^\ell(y, z, b)$. □

Proof of Proposition 7.1. Consider the slight modification $\tilde\nu = (\tilde C, \tilde L, \tilde M)$ of the consumption-investment strategy introduced in Lemma 7.1 :

$$\tilde C_t \;:=\; \bar C_t \;-\; r\xi\left(K^{\tilde\nu}_t - K^\nu_t\right) 1_{t \ge \tau} \quad \text{and} \quad \left(\tilde L, \tilde M\right) \;:=\; \left(\bar L, \bar M\right)\;, \tag{7.7}$$

where ξ is a positive constant such that $\xi < \alpha$. Observe that $(Y^{\tilde{\nu}}, B^{\tilde{\nu}}) = (Y^{\bar{\nu}}, B^{\bar{\nu}})$, and $Z_t^{\tilde{\nu}} = Z_t^{\bar{\nu}}$ for $t \le \tau$. In order to check the admissibility of the triple $(\tilde{C}, \tilde{L}, \tilde{M})$, we repeat Step 3 of the proof of Lemma 7.1 and use $K^{\tilde{\nu}} = K^{\bar{\nu}} \le K^{\nu}$:

$$\begin{aligned} e^{-r(t-\tau)}\left(Z_t^{\tilde{\nu}} - Z_t^{\nu}\right) &= Z_\tau^{\tilde{\nu}} - Z_\tau^{\nu} - r(\alpha-\xi)\int_\tau^t e^{-r(u-\tau)}\left(K_{u-}^{\tilde{\nu}} - K_{u-}^{\nu}\right) du \\ &\ge Z_\tau^{\tilde{\nu}} - Z_\tau^{\nu} . \end{aligned}$$

Since $Z_\tau^{\tilde{\nu}} - Z_\tau^{\nu} = Z_\tau^{\bar{\nu}} - Z_\tau^{\nu} \ge 0$ by Lemma 7.1, it follows that $Z_t^{\tilde{\nu}} \ge Z_t^{\nu}$ a.s., and $\tilde{\nu} \in \mathcal{A}_T^\ell(y,z,b)$.
Recall from Lemma 7.1 that $K_\tau^{\tilde{\nu}} < K_\tau^{\nu}$ on $\{\tau < T\}$. Since the process $(K^{\bar{\nu}} - K^{\nu})$ is right-continuous, the strict inequality holds on some nontrivial time interval almost surely on $\{\tau < T\}$. Hence $\bar{C} > C$ with positive Lebesgue$\otimes P$ measure, and

$$J_T^\ell(y,z,b;\tilde{\nu}) \quad > \quad J_T^\ell(y,z,b;\nu) .$$

□

In view of the optimality of the immediate realization of capital losses stated in Proposition 7.1, we will now prove that, given $(y,z) \in \bar{\mathcal{S}}$, and for all constant $\varepsilon > 0$, the problem of maximizing $J_T^\ell(y,z;\nu)$ can be restricted to those admissible control processes ν inducing an accumulated amount of tax credit bounded by ε.

Lemma 7.2 *Let ℓ be the linear taxation rule defined in* (7.1), $\delta = 0$, *and let $t > 0$ be some finite maturity, $\varepsilon > 0$, and ν in $\mathcal{A}^\ell(y,z)$. Then, there exists $\nu^\varepsilon = (C^\varepsilon, L^\varepsilon, M^\varepsilon)$ in $\mathcal{A}^\ell(y,z)$ such that*

$$J_t^\ell(z,y;\nu^\varepsilon) \;\ge\; J_t^\ell(z,y;\nu) \quad \text{and} \quad \int_0^t \left(B_{u-}^{\nu^\varepsilon} - 1\right) Y_{u-}^{\nu^\varepsilon} dM_u^{\nu^\varepsilon} \;\le\; \varepsilon \quad a.s.$$

Proof. Let $\theta^0 := 0$, $\nu^0 := \nu$,

$$\theta^{n+1} \;:=\; t \wedge \inf\{s > \theta^n \;:\; (B_s^{\nu^n} - 1)\left(1 \vee H_s^{\nu^n}\right) > \varepsilon\} ,$$

with

$$H_s{}^{\nu^n} \;:=\; Y_{s-}^{\nu^n} e^{M_s^{\nu^n c}} \prod_{u \le s} (1 - \Delta M_u^{\nu^n})^{-1} ,$$

and

$$\nu^{n+1} \;:=\; \nu^n 1_{\{\theta^{n+1} = t\}} + \mathcal{V}(\nu^n, \theta^{n+1}) 1_{\{\theta^{n+1} < t\}} .$$

where $M^{\nu^n c}$ denotes the continuous part of M^{ν^n}, and $\mathcal{V}$ is defined in Proposition 7.1 so as to take advantage of the tax credit while decreasing the relative tax basis. We shall simply denote $(Y^n, Z^n, B^n) := (Y^{\nu^n}, Z^{\nu^n}, B^{\nu^n})$. Then :

$$C^{n+1} \;\ge\; C^n, \; Y^n \;=\; Y^0, \; Z^{n+1} \;\ge\; Z^n, \qquad B^{n+1} \;\le\; B^n, \tag{7.8}$$

and

$$\left(B_s^n - 1\right)\left(1 \vee H_s^{\nu^n}\right) \le \varepsilon \quad \text{for} \quad t \le \theta^{n+1}. \tag{7.9}$$

Clearly, for a.e. $\omega \in \Omega$, $\theta^n(\omega) = t$ for $n \ge N(\omega)$, where $N(\omega)$ is some sufficiently large integer. Therefore the sequence $\nu^n(\omega)$ is constant for $n \ge N(\omega)$ and

$$\nu^n \quad \longrightarrow \quad \nu^\varepsilon \ \text{ a.s.}$$

for some $\nu^\varepsilon \in \mathcal{A}^\ell(y, z, b)$. Also, by construction of the sequence ν^n, we have :

$$\nu_s^\varepsilon \ = \ \nu_s^n \quad \text{for} \quad s \le \theta^{n+1} \ . \tag{7.10}$$

By (7.8), it is immediately checked that $J_t^\ell(y, z, b; \nu^\varepsilon) \ge J_t^\ell(y, z, b; \nu)$. We finally use (7.9) and (7.10), together with Itô's lemma, to compute that

$$\begin{aligned}
\int_0^t \left(B_{u-}^{\nu^\varepsilon} - 1\right) Y_{u-}^{\nu^\varepsilon} dM_u^{\nu^\varepsilon} \quad &\le \quad \varepsilon \int_0^t Y_{u-}^{\nu^\varepsilon} (H_s^{\nu^\varepsilon})^{-1} dM_u^{\nu^\varepsilon} \\
&= \quad \varepsilon \int_0^t e^{-M_s^{\nu^\varepsilon c}} \prod_{u \le s} (1 - \Delta M_u^{\nu^\varepsilon}) dM_u^{\nu^\varepsilon} \\
&= \quad \varepsilon \left[1 - e^{-M_t^{\nu^\varepsilon}} \prod_{u \le t} (1 - \Delta M_u^{\nu^\varepsilon}) \right] \\
&\le \quad \varepsilon \ ,
\end{aligned}$$

by the fact that $1 - \Delta M^{\nu^\varepsilon} \le 1$. □

We are now ready for the

Proof of Proposition 3.1 Let (y, z, b) some initial date in $\bar{\mathcal{S}}$, and let $(\hat{\nu}^n)_{n \ge 1}$ be the sequence of consumption-investment strategies defined in Step 3 of Section 5, which satisfies

$$J_T^\ell(y, z, b; \hat{\nu}^n) \quad \ge \quad \bar{V}_T(z) - \frac{1}{n} \, .$$

Recall that $(\hat{\nu}^n)_{n \ge 1}$ is defined from $(\nu^{t,n})_{n \ge 1}$ by means of a diagonal extraction argument, and $\hat{\nu}^n = \nu^{T,n}$ for $T < \infty$. In view of (3.2), we deduce that $(\hat{\nu}^n)_{n \ge 1}$ is a maximizing strategy for the problem $V_T^\ell(y, z, b)$.

1. When the time horizon T is finite, this result is an immediate consequence of Lemma (7.2). Indeed, it suffices to define for each $n \ge 1$ the strategy $\tilde{\nu}^n := \mathcal{V}^{1/n}(\hat{\nu}^n, T)$ as in Lemma (7.2), then $(\tilde{\nu}^n)_{n \ge 1}$ is a sequence of maximizing strategies satisfying the requirement $\int_0^t \left(B_{u-}^{\tilde{\nu}^n} - 1\right) Y_{u-}^{\tilde{\nu}^n} dM_u^{\tilde{\nu}^n} \le n^{-1}$ $P-$a.s.

2. We now concentrate on the case $T = \infty$. Observe that for all $n \ge 1$

$$J_\infty^\ell(y, z, b; \hat{\nu}^n) \quad = \quad \lim_{t \to \infty} J_t^\ell(y, z, b; \hat{\nu}^n) \ .$$

This follows by the the monotone convergence theorem and the fact that the utility function is nonnegative. Then, there is a sequence $t_n \longrightarrow \infty$ such that

$$J_\infty^\ell(y, z, b; \hat{\nu}^n) \quad \le \quad J_{t_n}^\ell(y, z, b; \hat{\nu}^n) + \frac{1}{n} \, .$$

Clearly the sequence $\tilde{\nu}^n := \mathcal{V}^{1/n}(\hat{\nu}^n, t_n)$ is a maximizing sequence for the problem $V_\infty^\ell(y, z, b)$ which satisfies the requirement of the proposition. □

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