ASYMMETRIC CUBATURE FORMULAE WITH FEW POINTS IN HIGH DIMENSION FOR SYMMETRIC MEASURES

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Abstract. Let μ be a positive measure on $\mathbb{R}^d$ invariant under the group of reflections and permutations, and m a natural number. We describe a method to construct cubature formulae of degree m with respect to μ, with n positive weights and n points in the support of μ, and such that n grows at most like d^m with the dimension d. We apply this method to classical measures to explicitly construct cubature formulae of degree 5 with the number of points growing at most like d^3.

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1. Introduction. We will denote by $\mathbb{R}[X_1,\ldots,X_d]$ the space of polynomials in d variables with real coefficients, and by $\mathbb{R}_m[X_1,\ldots,X_d]$ its subspace made of polynomials of total degree less than or equal to m. Note that $\dim \mathbb{R}_m[X_1,\ldots,X_d] = \binom{m+d}{d}$. We will write $\delta_{\mathbf{x}}$ for the Dirac probability at the point $\mathbf{x}$.

Definition 1.1. *Let μ be a positive measure on $\mathbb{R}^d$, $d \geq 1$, and m a positive integer. We say that the points $\mathbf{x}_1,\ldots,\mathbf{x}_n \in \mathbb{R}^d$ and the weights $\lambda_1,\ldots,\lambda_n \in \mathbb{R}$ define a cubature formula of degree m, with respect to the measure μ, if*

$$\int_{\mathbb{R}^d} P(z)\mu(dz) = \sum_{k=1}^{n} \lambda_k P(\mathbf{x}_k) \tag{1.1}$$

holds for all polynomials P in $\mathbb{R}_m[X_1,\ldots,X_d]$.
When $d=1$ people use the name quadrature in place of cubature.

Remark 1.2. *The existence of a cubature formula of degree m is equivalent to the existence of a finite measure ξ (a measure of the form $\xi = \sum_{i=1}^n \lambda_i \delta_{\mathbf{x}_i}$), such that for all polynomials P of degree less than or equal to m, the integral of P with respect to ξ is equal to the integral of P with respect to μ.*
Once we have a cubature formula, using the notation of the previous definition, we can approximate the integral of a smooth function f with respect to μ by

$$\sum_{i=1}^{n} \lambda_i f(\mathbf{x}_i).$$

The following theorem was first published by Tchakaloff (in the special case of a compactly supported measure). See [25],[28],[29] for its proof.

Theorem 1.3. *Let d and m be positive integers and let μ be a positive measure on $\mathbb{R}^d$ with the property that $\int |P(z)|\mu(dz) < \infty$ for all $P \in \mathbb{R}_m[X_1,\ldots,X_d]$. Then, we can find n points $\mathbf{x}_1,\ldots,\mathbf{x}_n$ in the support of μ and n positive real numbers $\lambda_1,\ldots,\lambda_n$, with*

$$n \leq \dim \mathbb{R}_m[X_1,\ldots,X_d],$$

such that the cubature relation (1.1) holds for all $P \in \mathbb{R}_m[X_1,\ldots,X_d]$.

Unfortunately, this is only an existence theorem, and does not provide any efficient method to obtain such cubature formulae. Hundreds of papers are devoted to the construction of cubature formulae. See the books [9], [18],[23],[28], the papers [5],[6], [7],[8] and the references therein. In [1], some Gaussian cubature formulae (i.e. cubature formulae with a minimum number of points) were constructed in high dimension. However, the measures considered there are rather artificial and exotic. It seems that no one has constructed explicit cubature formulae for classical measures of degree $m \geq 4$ in high dimension (i.e. in any given dimension) with no more than $\dim \mathbb{R}_m [X_1, \ldots, X_d] = \binom{m+d}{m}$ points, despite Tchakaloff's theorem. Stroud, in the introduction of his celebrated book "Approximate calculation of multiple integrals" [28], explains the need to construct such formulae. There exist some cubature formulae of degree 3 with few points, but the degree is not high enough to make the approximation accurate. Few points means, in this paper, that the number of points grows polynomially with the dimension.

Paraphrasing [18], an ideal cubature formula with respect to a positive measure should have points within the domain of integration, as few points as possible (in particular fewer points than the dimension of the polynomial space $\mathbb{R}_m [X_1, \ldots, X_d]$), and positive weights. Also, cubature formulae of higher degree will provide more accurate approximations of integrals. Nonetheless, the "space of cubature formulae is not totally ordered". By this, we mean that a cubature formula of degree 7 with some negative weights and another one of degree 5 with positive weights and with the same number of points cannot really be compared. Indeed, it is easy to find some functions whose integral will be better approximated by the first method and some other functions whose integral will be better approximated by the second one.

In this paper, we describe a method to construct cubature formulae with few points in any given dimension with respect to measures which are invariant under reflection and permutation of the axes. This method works particularly well for cubature formulae of degree 3 (where we get formulae with $O(d)$ points) and 5 (where we get formulae with $O(d^3)$, and even sometimes $O(d^2)$, that is well under the Tchakaloff bound), but have not yet applied it for higher degrees.

In the next section, we will explain how combinatorics and coding theory help to construct cubature formulae with respect to the finite measure

$$\frac{1}{2^d} \sum_{\mathbf{g} \in \{-1,1\}^d} \delta_{\mathbf{g}}.$$

Formulae of degree $2m+1$ in d dimensions will only require $O(d^m)$ points. Some other combinatorial objects will allow us to construct cubature formulae with few points for measures of the form

$$\frac{1}{|\mathcal{S}_d.\mathbf{x}|} \sum_{\mathbf{y} \in \mathcal{S}_d.\mathbf{x}} \delta_{\mathbf{y}}$$

where we let $\mathcal{S}_d$, the group of permutation of order d, act naturally on $\mathbb{R}^d$. We also let $\mathcal{G}_d \simeq \left(\{-1,+1\}^d, *\right)$, the group of reflections of the axes, and $\mathcal{GS}_d$, the group of permutations and reflections of the axes, act naturally on $\mathbb{R}^d$. We will show that for a $\mathcal{GS}_d$-invariant measure μ, there exists a $\mathcal{GS}_d$-invariant cubature formula of degree m, i.e. a formula of the form

$$\int P(z)\mu(dz) = \sum_{i=1}^{k} \lambda_i \frac{1}{|\mathcal{GS}_d.\mathbf{x}_i|} \sum_{\mathbf{y} \in \mathcal{GS}_d.\mathbf{x}_i} P(\mathbf{y}), \text{ for all } P \text{ in } \mathbb{R}_m [X_1, \ldots, X_d].$$

In this formula, we can find the points $\mathbf{x}_1, \ldots, \mathbf{x}_k$ with the number of points k bounded by a term which depends only on m, the degree of the cubature formula. For example, for a cubature formula of degree 5, this upper bound is equal to 4, and it is possible to find two points that give the formula. Then, using the techniques described in the next two sections, we will construct cubature formulae with respect to $\frac{1}{|\mathcal{GS}_d.\mathbf{x}|}\sum_{\mathbf{y}\in\mathcal{GS}_d.\mathbf{x}}\delta_{\mathbf{y}}$. Those new approximations will give us a cubature formula of degree m with respect to μ and with few points. This explains why we first constructed some cubature formulae with respect to finite measures, that is, why we constructed finite measures approximating some other finite measures! The last section will deal with concrete examples of construction of such cubature formulae of degree 5 with respect to classical measures, such as the Gaussian measure, the Lebesgue measure on the unit hypercube, on the surface of the unit sphere, and on the whole unit sphere. We also include a table of formulae similar to the ones found in [7],[8].

2. Codes and Orthogonal Arrays.

2.1. Definitions and Link with Cubature Formulae. In this section, we describe how we can find a cubature formula of degree $2m+1$ with respect to the measure of total mass one which puts equal mass at all of the vertices of a d-dimensional hypercube, i.e. with respect to

$$\delta_{\mathcal{G}_d} = \frac{1}{2^d}\sum_{\mathbf{g}\in\{-1,1\}^d}\delta_{\mathbf{g}}.$$

We will need to define orthogonal arrays. See [14] for a full description of these combinatorial objects.

Definition 2.1. *An $N\times k$ array A with entries from a set S is said to be an orthogonal array with $|S|$ levels, strength t and index λ (for some t in the range $0\le t\le k$) if every $N\times t$ sub-array of A contains each t-uple based on S exactly λ times as a row.*

Such an array will be denoted by $OA(N,k,|S|,t)$. We do not put λ explicitly in this notation, as it is quite easy to see that $\lambda = N/|S|^t$.

Theorem 2.2. *The points defined to be the rows of an orthogonal array with parameters $OA(N,d,2,2m+1)$ and with entries in $\{-1,1\}$, associated to the constant weight $1/N$, define a cubature formula of degree $2m+1$ with respect to $\delta_{\mathcal{G}_d}$.*

Proof. Let $\mathbf{x}_1,\ldots,\mathbf{x}_N$ be the rows of an $OA(N,d,2,2m+1)$ with entries in $\{-1,1\}$. Let P_1 be the square of a monomial with leading coefficient 1. Then, using the particular form of P_1,

$$\sum_{k=1}^{N}\frac{1}{N}P_1(\mathbf{x}_k) = \sum_{k=1}^{N}\frac{1}{N}1 = \int P_1(z)\delta_{\mathcal{G}_d}(dz).$$

Now let $P_2 = Q^2R$, where $R = X_1^{\alpha_1}\ldots X_d^{\alpha_d}$, $\alpha_i\in\{0,1\}$ with $1\le\sum_{i=1}^d\alpha_i\le 2m+1$ and Q is any monomial. Then,

$$\sum_{k=1}^{N}\frac{1}{N}P_2(\mathbf{x}_k) = \sum_{k=1}^{N}\frac{1}{N}R(\mathbf{x}_k) = \sum_{k=1}^{N}\frac{1}{N}\prod_{i:\ \alpha_i=1}x_k^i.$$

If λ is the index of the orthogonal array $OA(N,d,2,2m+1)$, then for any functional

Υ, $l \leq 2m+1$ and different indices $i_1, ..., i_l$,

$$\sum_{k=1}^{N} \Upsilon(x_k^{i_1}, \ldots, x_k^{i_l}) = \lambda \sum_{(y^1,\ldots,y^l)\in\{-1,1\}^l} \Upsilon(y^1, \ldots, y^l).$$

Hence, if $l = \sum_{i=1}^{d} \alpha_i$,

$$\begin{aligned}\sum_{k=1}^{N} \frac{1}{N} \prod_{i:\ \alpha_i=1} x_k^i &= \frac{\lambda}{N} \sum_{(y^1,\ldots,y^l)\in\{-1,1\}^l} \prod_{j=1}^{l} y^j \\ &= 0 = \int P(z)\delta_{\mathcal{G}_d}(dz).\end{aligned}$$

We have proved that for all monomials of degree less than or equal to $2m+1$,

$$\sum_{k=1}^{N} \frac{1}{N} P(\mathbf{x}_k) = \int P(z)\delta_{\mathcal{G}_d}(dz).$$

□

We are now going to link orthogonal arrays and codes. Let us first give the definition of a code.

Definition 2.3. *Let S be a set of symbols of size s. A code is a collection C of vectors in S^k. These vectors are called codewords. The distance d_C of the code is defined by*

$$d_C = \min_{u,v\in C, u\neq v} card\left\{j \in \{1, \ldots, k\}, u^j \neq v^j\right\}.$$

Such a code will be denoted by $(k, card\ C, d_C)_s$.

Definition 2.4. *Let C be a $(k, N, d_C)_s$ code on S, and assume that S is a finite field. Define the dual code of C by*

$$C^\perp = \left\{v \in S^k, \forall u \in C, uv^\perp = \sum_{i=1}^{k} u_i v_i = 0\right\}.$$

Its distance $d_{C^\perp}$ will be called the dual distance of the code C.
The following is due to Delsarte [10].

Theorem 2.5. *Let A be the $N \times k$ array such that its rows are the codewords of a $(k, N, d)_s$ code over a finite field S, with dual distance $d^\perp$. Then A is an $OA(N, k, s, d^\perp - 1)$.*
Thus we see that the codewords of a code with parameters $(k, N, d_C)_2$ over $\mathbb{Z}/2\mathbb{Z}$, with dual distance $2m+2$, give us a cubature formula of degree $2m+1$ with respect to $\delta_{\mathcal{G}_d}$. We will now give some examples of such orthogonal arrays.

2.2. Degree 3. We are going to describe the orthogonal arrays with parameters $OA(N, d, 2, 3)$, preferably with N as low as possible. This is closely related to Hadamard matrices. Write I_n for the $n \times n$ identity matrix, and A^T for the transpose of a matrix A.

Definition 2.6. *A Hadamard matrix of degree n is a matrix $H \in M_n(\{-1, 1\})$ such that $HH^T = nI_n$.*

Proposition 2.7. *If there exists a Hadamard matrix of degree n, then $n \in \{1,2\}$ or 4 divides n.*

We can construct an $OA(N, d, 2, 3)$ in the following way. Consider the least n greater than or equal to d, for which we have a Hadamard matrix A of degree n. Then consider the $2n \times n$ matrix B, such that the rows of B are made of the rows of A and of $-A$. Then delete $n-d$ columns of B. That gives us an $OA(2n, d, 2, 3)$. The Hadamard matrices have been heavily studied. Hadamard conjectured that if $n \in \{1,2\}$ or if 4 divides n, then a Hadamard matrix exists. It still has not been proved or disproved. Such matrices have been constructed for all $n \leq 1000$, except for $n = 428, 668, 716, 764, 892$. Moreover, there exists an easy way to construct them when n is a power of 2.

Definition 2.8. *Let $A = (a_{ij})_{1\leq i,j\leq n} \in M_n(\mathbb{R})$ and $B \in M_m(\mathbb{R})$. Then we define the tensor product of A and B, $A \otimes B$ by the $nm \times nm$ matrix*

$$\begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{pmatrix}$$

Proposition 2.9. *Let H_a, H_b be Hadamard matrices of degree a and b. Then $H_a \otimes H_b$ is a Hadamard matrix of degree ab.*

Corollary 2.10. *Let $H_1 = (1)$ and $H_2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Then define $H_{2^m} = H_1 \otimes (\bigotimes_{i=1}^m H_2)$. Then $H_{2^m} \in M_{2^m}(\{-1,1\})$ is a Hadamard matrix.*

These matrices H_{2^m} are called the Hadamard matrices of Sylvester type. There exists an extensive literature (e.g. [12][14]) on techniques which deal with the construction of Hadamard matrices.

2.3. Degree 5. The arrays we are interested in are $OA(N, d, 2, 5)$. We saw that we can construct such an array by using the codewords of a code $(d, N, d_C)_2$ with dual distance 6. Lower dimension constructions are easier.

2.3.1. $d = 5$, $N = 2^d = 32$. The rows of an $OA(N, d, 2, 5)$ correspond to the $N = 2^d$ points in $\{-1,1\}^d$.

2.3.2. $d = 6, 7, 8$, $N = 2^{d-1}$. Let A be the $OA(2^d, d, 2, 5)$ constructed above. Then define $A_{i,d} = \prod_{j=1}^{d-1} A_{i,j}$. That gives us an $OA(2^{d-1}, d, 2, 5)$.

2.3.3. $d = 9$, $N = 128 = 2^{d-2}$. This is the first example of an orthogonal array constructed from a code, a cyclic code. We will describe in this particular case how to construct this array (or those codewords). Let G be the 7×9 matrix in $GF(2)$ (the Galois field with 2 elements, i.e. $\mathbb{Z}/2\mathbb{Z}$)

$$G = \begin{pmatrix} 1&1&1&0&0&0&0&0&0 \\ 0&1&1&1&0&0&0&0&0 \\ 0&0&1&1&1&0&0&0&0 \\ 0&0&0&1&1&1&0&0&0 \\ 0&0&0&0&1&1&1&0&0 \\ 0&0&0&0&0&1&1&1&0 \\ 0&0&0&0&0&0&1&1&1 \end{pmatrix}.$$

G is called a cyclic generator matrix. Define the set of points C in $GF(2)^9$

$$C = \{xG, x \in GF(2)^7\}$$

the calculation being done in $GF(2)$. Let

$$\begin{aligned}\Phi_2 : GF(2) &\longrightarrow \{-1,1\}\\ 0 &\mapsto -1\\ 1 &\mapsto 1\end{aligned}$$

Now we can construct a matrix A with rows that are the points in $\Phi_2(C) \subset \{-1,1\}^9$. A is an $OA(128,9,2,5)$.

2.3.4. $d = 10,\ldots,16$, $N = 256$. The $OA(256,16,2,5)$ is constructed using a Nordstrom-Robinson code. Define the generator matrix G in $M_{4,8}(\mathbb{Z}/4\mathbb{Z})$ by

$$G = \begin{pmatrix} 1 & 3 & 1 & 2 & 1 & 0 & 0 & 0\\ 1 & 0 & 3 & 1 & 2 & 1 & 0 & 0\\ 1 & 0 & 0 & 3 & 1 & 2 & 1 & 0\\ 1 & 0 & 0 & 0 & 3 & 1 & 2 & 1 \end{pmatrix}.$$

Define the set of points

$$C = \{xG, x \in (\mathbb{Z}/4\mathbb{Z})^4\}$$

the calculation being done in $\mathbb{Z}/4\mathbb{Z}$. Let us define the Gray map

$$\begin{aligned}\Phi_4 : \mathbb{Z}/4\mathbb{Z} &\longrightarrow \{-1,1\}^2\\ 0 &\mapsto (-1,-1)\\ 1 &\mapsto (-1,1)\\ 2 &\mapsto (1,1)\\ 3 &\mapsto (1,-1).\end{aligned}$$

Now, we can construct a matrix A with rows that are the elements of $\Phi_4(C) \subset \{-1,1\}^{16}$. A is an $OA(256,16,2,5)$. To get an $OA(256,k,2,5)$ for $k = 10,\ldots,15$, we just delete $16-k$ columns of A.

2.3.5. Higher dimension. $OA(512,20,2,5)$ and $OA(1024,24,2,5)$ are described in [14]. Many more $OA(N,d,2,5)$ for higher d are known, in particular with $d = 2^m+1$ and $N = 2^{2m+1}$, when $m \geq 5$. They come from BCH codes [14][20]. The X4 construction [14] allows the construction of some $OA(2^{4m+1}, 2^{2m}+2m, 2, 5)$, for $m \geq 2$. Finally, Kerdock codes ([20] for their original construction, or [13] for a simpler one) gives us orthogonal arrays of the form $OA(4^{2m}, 4^m, 2, 5)$, for $m \geq 2$. Those codes allow us to write $N_5(d) = O(d^2)$.

2.4. A Notation. We will denote by $\mathcal{G}_d^m$ the set of points of a cubature formula of degree m with respect to $\delta_{\mathcal{G}_d}$ (with points in $\{-1,+1\}^d$), such that we do not know another cubature formula with points in $\{-1,+1\}^d$, of the same degree with respect to the same measure but with fewer points. When $d \geq m$, $\mathcal{G}_d^m$ is described in term of an $OA(|\mathcal{G}_d^m|, d, 2, m)$. $\mathcal{G}_d^m$ can be also seen as a subset of the group $\mathcal{G}_d \simeq \left(\{-1,+1\}^d, *\right)$. When $d \leq m$, necessarily, we have $\mathcal{G}_d^m = \mathcal{G}_d$.

Hadamard matrices allow us to write $\left|\mathcal{G}_d^3\right| = 2d$ anytime that there exists a Hadamard matrix, and $\left|\mathcal{G}_d^3\right| = O(d)$. Kerdock and BCH codes allow us to write $\left|\mathcal{G}_d^5\right| = O(d^2)$. For a general m, coding theory tells us that $\left|\mathcal{G}_d^{2m+1}\right| = O(d^m)$.

3. Permutation Sets. Let us consider a point $\mathbf{x} = (x^1, \ldots, x^d) \in \mathbb{R}^d$ and the measure of total mass one which puts equal mass at all the points of the form $\sigma.\mathbf{x} = (x^{\sigma(1)}, \ldots, x^{\sigma(d)})$, for $\sigma \in \mathcal{S}_d$, i.e.

$$\delta_{\mathcal{S}_d,\mathbf{x}} = \frac{1}{|\mathcal{S}_d.\mathbf{x}|} \sum_{\mathbf{y} \in \mathcal{S}_d.\mathbf{x}} \delta_{\mathbf{y}}$$

where $\mathcal{S}_d$ denotes the symmetric group. We want to describe a cubature formula of degree m with respect to $\delta_{\mathcal{S}_d,\mathbf{x}}$.

3.1. Block Designs. We assume in this subsection that $x^1 = \ldots = x^k = \alpha$ and $x^{k+1} = \ldots = x^d = \beta$, where $\alpha \neq \beta$. Hence, we are looking for a cubature formula of degree t with respect to the measure on $\mathbb{R}^d$

$$\delta_{\mathcal{S}_d,\mathbf{x}} = \frac{1}{\binom{d}{k}} \sum_{\mathbf{y} \in \mathcal{S}_d.\mathbf{x}} \delta_{\mathbf{y}}.$$

Knowledge of some t-designs will allow us to construct some cubature formulae of degree t.

DEFINITION 3.1. *Let V be a finite set of size v and let B be a collection of k-subsets of V, called blocks. Then (V, B) is a t-design with parameters t-(v, k, λ) if every t-subset of V is in exactly λ blocks.*

The incidence matrix A of a design $(V, B) = (\{1, \ldots, v\}, \{B_1, \ldots, B_b\})$ is defined, for $i = 1, \ldots, b$ and $j = 1, \ldots, v$, by

$$A_{i,j} = \mathbf{1}_{\{j \in B_i\}}.$$

It is easily seen that a t-design with parameters (v, k, λ) is a $(t-1)$-design with parameters $\left(v, k, \lambda \frac{v-(t-1)}{k-(t-1)}\right)$ [2]. Also, every 0-subset (i.e. the empty set) is included in all the blocks, so a t-design with parameters (v, k, λ) is a 0-design with parameters (v, k, b), where $b = |B|$ is the number of blocks of the design. Hence, if a t-design has parameters (v, k, λ) and has b blocks, then

$$\lambda = b \frac{k(k-1)..(k-(t-1))}{v(v-1)..(v-(t-1))} = b \frac{\binom{k}{t}}{\binom{v}{t}}$$

A 2-design is usually called a balanced incomplete block design, or just a block design. For a block design, if the number of blocks b is equal to v, we say that the design is symmetric (or square for some authors).

EXAMPLE 3.2. *Let q be a prime power, $n \geq 2$ be an integer, and $V(n, q)$ be a n dimensional vector space over $GF(q)$ (the Galois Field of order q). $V(n, q)$ contains q^n vectors. A 1-dimensional space is made of $q-1$ non zero vectors (it is made of the vectors bx, where b ranges over $GF(q) - \{0\}$ and where x is a non zero vector), hence there are $\frac{q^n-1}{q-1}$ 1-dimensional spaces. If $x = (x_1, \ldots, x_n)$ is a non zero vector, then the set of vectors $y = (y_1, \ldots, y_n)$ such that*

$$x_1 y_1 + \ldots + x_n y_n = 0$$

define a subspace of $V(n, q)$ of dimension $n-1$ (a hyperplane), and conversely, for any hyperplane, we can find a vector $x = (x_1, \ldots, x_n)$ such that for each vector $y = (y_1, \ldots, y_n)$ of the hyperplane,

$$x_1 y_1 + \ldots + x_n y_n = 0.$$

So we have $\frac{q^n-1}{q-1}$ hyperplanes. Each hyperplane contains $\frac{q^{n-1}-1}{q-1}$ 1-dimensional spaces (itself being a vector space of dimension $n-1$), and the intersection of two hyperplanes is a subspace of $V(n,q)$ of dimension $n-2$, which contains $\frac{q^{n-2}-1}{q-1}$ 1-dimensional spaces. Hence, the hyperplanes of $V(n,q)$ as blocks and the 1-dimensional spaces of $V(n,q)$ as objects form a symmetric block design $\left(\frac{q^n-1}{q-1}, \frac{q^{n-1}-1}{q-1}, \frac{q^{n-2}-1}{q-1}\right)$.

We refer to [2],[3],[12] for more details on designs, and for tables of known block designs.

Let $J_{b,d}$ be the $b \times d$ matrix with each entry set to be 1, and we remind the reader that $\mathbf{x}$ is defined in this subsection by $x^1 = \ldots = x^k = \alpha$ and $x^{k+1} = \ldots = x^d = \beta$, with $\alpha \neq \beta$.

THEOREM 3.3. *Let A be the incidence matrix of a t-design with parameters $t-(d,k,\lambda)$ and with b blocks. Let $\mathbf{x}_1, \ldots, \mathbf{x}_b \in \mathbb{R}^d$ be the rows of $(\alpha-\beta)A + \beta J_{b,d}$. Then those points associated to the constant weight $1/b$ (b is the number of blocks of the t-design) define a cubature formula of degree t with respect to $\delta_{\mathcal{S}_d,\mathbf{x}}$.*

Proof. First of all, note that, by considering the polynomials which are products of some $\frac{(X_j-\beta)^k}{(\alpha-\beta)^k}$ in place of the monomials, we see that we may take $\alpha = 1$ and $\beta = 0$. Since for all $k_1, \ldots, k_d \geq 0$, $\int P(z_1^{k_1}, \ldots, z_d^{k_d})\delta_{\mathcal{S}_d,\mathbf{x}}(dz) = \int P(z_1, \ldots, z_d)\delta_{\mathcal{S}_d,\mathbf{x}}(dz)$, and since permuting the rows of the incidence matrix of a block design gives another block design with the same parameters, we only have to check that

$$\int P(z)\delta_{\mathcal{S}_d,\mathbf{x}}(dz) = \frac{1}{b}\sum_{i=1}^{b} P(\mathbf{x}_i)$$

for the polynomials $X_1, X_1X_2, \ldots, X_1\ldots X_t$. Define $\lambda_t = \lambda$, and for $i < t$, $\lambda_i = \lambda_{i+1}(d-i)/(k-i)$, so that A is the incidence matrix of a $i-(d,k,\lambda_i)$ design, $i = 1, \ldots, t$. Let $P = X_1 \ldots X_i$. Then

$$\frac{1}{b}\sum_{j=1}^{b} P(\mathbf{x}_j) = \frac{\lambda_i}{b} = \frac{\binom{k}{i}}{\binom{d}{i}}.$$

The proof is then finished by noticing that

$$\int P(z)\delta_{\mathcal{S}_d,\mathbf{x}}(dz) = \binom{d-i}{k-i}/\binom{d}{k} = \frac{\binom{k}{i}}{\binom{d}{i}}.$$

□

For cubature formulae of degree 2, symmetric block designs will be very interesting, as they provide formulae with the minimum number of points.

3.2. k-Homogeneous Permutation Sets. We know come back to the more general case, i.e. we do not assume anything about the structure of the point $\mathbf{x} = (x^1, \ldots, x^d)$, and we recall that we are looking for a cubature formula of degree t with respect to the d-dimensional measure

$$\delta_{\mathcal{S}_d,\mathbf{x}} = \frac{1}{|\mathcal{S}_d.\mathbf{x}|}\sum_{\mathbf{y}\in\mathcal{S}_d.\mathbf{x}} \delta_{\mathbf{y}}.$$

The symmetric group $\mathcal{S}_d$ acts on the set $T = \{1, \ldots, d\}$, and acts naturally on $T^{\{k\}}$, the set of k-element subsets of T. For A, B in $T^{\{k\}}$, $[A;B]$ denotes the subset of $\mathcal{S}_d$

which consists of all permutations that move A to B. A non empty subset Z_d of $\mathcal{S}_d$ is said to be k-homogeneous on T if the cardinality of $Z_d \cap [A;B]$ is independent of A and B in $T^{\{k\}}$. The cardinality of $Z_d \cap [A;B]$ is called the k-multiplicity of Z_d. When this cardinality is one, we say that Z_d is sharply k-homogeneous on T. If Z_d is k-homogeneous on T, then Z_d is $k-1$-homogeneous on T (when $2 \le k \le d/2$, [24]). See [4] for the construction of such sets. Particular cases of k-homogeneous sets are k-homogeneous groups ([17]).

Example 3.4. *Let $K = GF(q)$ be the Galois field of order q, where q is a prime power. Let Γ be the group of mappings from K to $K : x \to ax + b$, where $b \in K$ and a is in the group of non zero square of K. Then Γ is a sharply 2-homogeneous group on K. $|\Gamma| = q(q-1)/2$.*

The link with our problem is explained in the following theorem:

Theorem 3.5. *Let Z_d be a t-homogeneous set on $T = \{1, \ldots, d\}$. Then the set of points $\sigma.\mathbf{x}$, for $\sigma \in Z_d$, with the constant weight $1/|Z_d.\mathbf{x}|$ defines a cubature formula of degree t with respect to $\delta_{\mathcal{S}_d,\mathbf{x}}$.*

Proof. It is straightforward to check the cubature relation for all monomials of degree less than or equal to t. □

3.3. A Notation. We will denote by $\mathcal{S}_d^{m,\mathbf{x}}$ a subset of $\mathcal{S}_d$ such that the probability measure $\frac{1}{|\mathcal{S}_d^{m,x}.\mathbf{x}|}\sum_{\mathbf{y}\in\mathcal{S}_d^{m,x}.\mathbf{x}} \delta_{\mathbf{y}}$ defines a cubature formula of degree m with respect to $\delta_{\mathcal{S}_d,\mathbf{x}}$ (with points inside the support of $\delta_{\mathcal{S}_d,\mathbf{x}}$ and with positive weights), and such that we do not know another formula with fewer points (with points inside the support of $\delta_{\mathcal{S}_d,\mathbf{x}}$ and with positive weights).

4. Invariant Cubature Formulae. Now that we have constructed some cubature formulae with few points with respect to some discrete measures, we are going to see how these can be useful for the construction of cubature formulae with respect to a "symmetric" measure. First of all, we should remind the reader of the application to cubature of invariant theory. See [18] for a more detailed presentation.

4.1. Invariant theory. Let $\mathcal{G}$ be a group of bijective linear transformations on $\mathbb{R}^d$. A set Ω of $\mathbb{R}^d$ is said to be $\mathcal{G}$-invariant if for all $\mathbf{g} \in \mathcal{G}$, $\mathbf{g}.\Omega = \Omega$. A function f on Ω is said to be $\mathcal{G}$-invariant if for all $\mathbf{g} \in \mathcal{G}$, $f \circ \mathbf{g} = f$. Finally, a measure μ is said to be $\mathcal{G}$-invariant if its support is, and if for all measurable sets A and for all $\mathbf{g} \in \mathcal{G}$, $\mathbf{g}.A$ is measurable and $\mu(\mathbf{g}.A) = \mu(A)$. We will denote by $\mathbb{R}_m [X_1, \ldots, X_d] (\mathcal{G})$ the space of all $\mathcal{G}$-invariant polynomials of maximum degree m.

Definition 4.1. *A cubature formula with points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ and weights $\lambda_1, \ldots, \lambda_n$ is said to be $\mathcal{G}$-invariant if $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is $\mathcal{G}$-invariant and $\mathbf{g}.\mathbf{x}_i = \mathbf{x}_j$ implies $\lambda_i = \lambda_j$. Equivalently, the cubature formula is $\mathcal{G}$-invariant if $\sum_{i=1}^n \lambda_i \delta_{\mathbf{x}_i}$ is a $\mathcal{G}$-invariant measure.*
$\mathcal{G}.\mathbf{x}_i = \{\mathbf{g}.\mathbf{x}_i, \mathbf{g} \in \mathcal{G}\}$ is called the orbit of the point $\mathbf{x}_i$. All the weights associated to the points inside the same orbit are equal.
A subset $\{\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_k}\}$ is called a generator set of the above cubature formula if $\mathcal{G}.\mathbf{x}_{i_1}, \ldots, \mathcal{G}.\mathbf{x}_{i_k}$ forms a partition of $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$. We will also say that the collection of orbits $\mathcal{G}.\mathbf{x}_{i_j}$ and weights $\lambda_{i_j} \left|\mathcal{G}.\mathbf{x}_{i_j}\right|$, $j = 1, \ldots, k$ generate the above $\mathcal{G}$-invariant cubature formula.

The following theorem is due to Sobolev [26].

Theorem 4.2. *Let μ be a $\mathcal{G}$-invariant measure. Then the orbits $\mathcal{G}.\mathbf{x}_1, \ldots, \mathcal{G}.\mathbf{x}_k$ and their weights $\lambda_1, \ldots, \lambda_k$ are the generators of a $\mathcal{G}$-invariant cubature formula of degree m with respect to μ if and only if for all $\mathcal{G}$-invariant polynomials P of degree* *less than or equal to m,*

$$\int P(z)\mu(dz) = \sum_{i=1}^{k} \lambda_i P(\mathbf{x}_i).$$

From Tchakaloff's theorem and Sobolev's theorem, we obtain the following corollary.

COROLLARY 4.3. *Let d and m be positive integers and let μ be a positive $\mathcal{G}$-invariant measure on $\mathbb{R}^d$ with the property that $\int |P(z)|\mu(dz) < \infty$ for all $P \in \mathbb{R}_m[X_1, \ldots, X_d]$. Then, we can find k orbits $\mathcal{G}.\mathbf{x}_1, \ldots, \mathcal{G}.\mathbf{x}_k$ (in the support of μ) and weights $\lambda_1, \ldots, \lambda_k$ that generate a $\mathcal{G}$-invariant cubature formula of degree m with respect to μ with*

$$k \leq \dim \mathbb{R}_m[X_1, \ldots, X_d](\mathcal{G}).$$

Proof. By Tchakaloff's theorem, we can find n points $\widetilde{\mathbf{x}_1}, \ldots, \widetilde{\mathbf{x}_n}$ in the support of μ together with their weights $\widetilde{\lambda_1}, \ldots, \widetilde{\lambda_n}$ that defines a cubature formula of degree m with respect to μ. Then the points $\widetilde{\mathbf{x}_{\mathbf{g},i}} = \mathbf{g}.\widetilde{\mathbf{x}}_i$, $\mathbf{g} \in \mathcal{G}$, $i = 1, \ldots, n$ and the weights $\widetilde{\lambda_{\mathbf{g},i}} = \frac{\widetilde{\lambda}_i}{|\mathcal{G}|}$ define a $\mathcal{G}$-invariant cubature formula of degree m with respect to μ. Let $\mathcal{G}.\widehat{\mathbf{x}_1}, \ldots, \mathcal{G}.\widehat{\mathbf{x}_{k'}}$ be some orbits and $\widehat{\lambda_1}, \ldots, \widehat{\lambda_{k'}}$ some weights that generate this $\mathcal{G}$-invariant cubature formula. $\mathcal{G}.\widehat{\mathbf{x}_1}, \ldots, \mathcal{G}.\widehat{\mathbf{x}_{k'}}$ and $\widehat{\lambda_1}, \ldots, \widehat{\lambda_{k'}}$ generate a $\mathcal{G}$-invariant cubature formula of degree m with respect to μ. If $k^{'} > \dim \mathbb{R}_m[X_1, \ldots, X_d](\mathcal{G})$, using the same convex analysis argument as in Tchakaloff's theorem, it is possible to find $k \leq \dim \mathbb{R}_m[X_1, \ldots, X_d](\mathcal{G})$ orbits $\mathcal{G}.\mathbf{x}_1, \ldots, \mathcal{G}.\mathbf{x}_k \in \{\mathcal{G}.\widehat{\mathbf{x}_1}, \ldots, \mathcal{G}.\widehat{\mathbf{x}_{k'}}\}$ and some new weights $\lambda_1, \ldots, \lambda_k$ that generate a $\mathcal{G}$-invariant cubature formula of degree m with respect to μ. □

Assume that the orbits $\mathcal{G}.\mathbf{x}_1, \ldots, \mathcal{G}.\mathbf{x}_k$ together with their weights $\lambda_1, \ldots, \lambda_k$ generate a $\mathcal{G}$-invariant cubature formula of degree m with respect to μ. In other words, $\xi = \sum_{i=1}^{k} \lambda_i \xi_{\mathcal{G},\mathbf{x}_i}$, where $\xi_{\mathcal{G},\mathbf{x}_i} = \frac{1}{|\mathcal{G}.\mathbf{x}_i|} \sum_{\mathbf{y} \in \mathcal{G}.\mathbf{x}_i} \delta_{\mathbf{y}}$, is the measure associated to a cubature formula of degree m with respect to μ. Now assume that for all i, we can find a cubature formula with n_i points and positive weights with respect to the (discrete) measure $\xi_{\mathcal{G},\mathbf{x}_i}$. Let $\widetilde{\xi_{\mathcal{G},\mathbf{x}_i}}$ denote the measures associated to these cubature formulae. Then, $\widetilde{\xi} = \sum_{i=1}^{k} \lambda_i \widetilde{\xi_{\mathcal{G},\mathbf{x}_i}}$ is still a measure associated to a cubature formula of degree m with respect to μ. Note that this cubature formula is not anymore $\mathcal{G}$-invariant. Its number of points is $\sum_{i=1}^{k} n_i$. We are going to show that, in the case where $\mathcal{G}$ is the group of permutations and reflections of the axes, $\dim \mathbb{R}_m[X_1, \ldots, X_d](\mathcal{G})$ does not depend on the dimension d, and that we can find (and construct) cubature formulae with respect to $\xi_{\mathcal{G},\mathbf{x}_i}$ with the number of points bounded by the Tchakaloff bound. Hence, if we have some orbits and weights generating a $\mathcal{G}$-invariant cubature formula with respect to μ, it is possible to find a cubature formula with respect to μ which has a number of points which grows polynomially with the dimension (at least when $\mathcal{G}$ is the group of permutations and reflections of the axes).

4.2. Invariance under Reflection. Let us consider a positive measure μ on $\mathbb{R}^d$ invariant with respect to the group of reflections of the axes $\mathcal{G}_d \simeq \left(\{-1, +1\}^d, *\right)$. The Lebesgue measure on the hypercube, on the unit sphere, the Gaussian measure on $\mathbb{R}^d$ are examples of such a measure.

$\mathcal{G}_d$ acts on $\mathbb{R}^d$ in a natural way : for a point $\mathbf{x} = (x^1, \ldots, x^d)$ and an element of the group $\mathbf{g} = \left(g^1, \ldots, g^d\right)$, we define $\mathbf{g}.\mathbf{x} = \left(g^1 x^1, \ldots, g^d x^d\right)$.

For $\mathbf{x} \in \mathbb{R}^d$, $\mathcal{G}_d.\mathbf{x}$ is of cardinality $2^{e(\mathbf{x})}$, where $e(\mathbf{x})$ is the number of non-zero coordinates of $\mathbf{x}$. That allows us to define the action of $\mathcal{G}_{e(\mathbf{x})}$ on $\mathbf{x}$: assume that $i_1, \ldots, i_{e(\mathbf{x})}$ are the $e(\mathbf{x})$ non-zero coordinates of $\mathbf{x}$, then for $\mathbf{g} \in \mathcal{G}_{e(\mathbf{x})}$, we define the i_l coordinate of $\mathbf{g}.\mathbf{x}$ to be $g^l x^{i_l}$, while the coordinate which are zero remain zero under the action of $\mathbf{g}$. This just means that we only consider the reflection with respect to the i-th axis when $x_i \neq 0$. Obviously, $\mathcal{G}_d.\mathbf{x} = \mathcal{G}_{e(\mathbf{x})}.\mathbf{x}$.

Corollary 4.3 tells us that there exists k (with $k \leq \dim \mathbb{R}_m \left[X_1, \ldots, X_d\right] (\mathcal{G}_d)$) orbits $\mathcal{G}_d.\mathbf{x}_1, \ldots, \mathcal{G}_d.\mathbf{x}_k$ in the support of the measure μ and k weights $\lambda_1, \ldots, \lambda_k$ that generate a $\mathcal{G}_d$-invariant cubature formula of degree m with respect to μ. The measure $\sum_{i=1}^k \lambda_i \xi_{\mathcal{G}_d, \mathbf{x}_i}$, where $\xi_{\mathcal{G}_d, \mathbf{x}_i} = \frac{1}{|\mathcal{G}_d.\mathbf{x}_i|} \sum_{\mathbf{y} \in \mathcal{G}_d.\mathbf{x}_i} \delta_{\mathbf{y}}$, is then associated to a cubature formula of degree m with respect to μ. But the measure

$$\widetilde{\xi_{\mathcal{G}_d, \mathbf{x}_i}} = \frac{1}{2^{e(\mathbf{x}_i)}} \sum_{\mathbf{g} \in \mathcal{G}^m_{e(\mathbf{x}_i)}} \delta_{\mathbf{g}.\mathbf{x}_i}$$

defines a cubature formula of degree m with respect to $\xi_{\mathcal{G}_d, \mathbf{x}_i}$ ($\mathcal{G}^m_{e(\mathbf{x}_i)}$ has been defined in term of orthogonal arrays in section 2). Hence, the finite measure $\sum_{i=1}^k \lambda_i \widetilde{\xi_{\mathcal{G}_d, \mathbf{x}_i}}$ defines a cubature formula of degree m with respect to μ.

The number of points in this cubature formula is $\sum_{i=1}^k \left|\mathcal{G}^m_{e(\mathbf{x}_i)}\right| \leq k \left|\mathcal{G}^m_d\right|$. We saw that $|\mathcal{G}^m_d| = O(d^{[m/2]})$. We therefore need to find a way to find k generators, with k growing polynomially with the dimension. First, let us write the problem in equivalent terms.

For $\mathbf{x} \in \mathbb{R}^d$, let us denote by

$$\mathbf{x}^2 = \left(\left(x^1\right)^2, \ldots, \left(x^d\right)^2\right)$$

and

$$\sqrt{\mathbf{x}} = \left(\sqrt{x^1}, \ldots, \sqrt{x^d}\right).$$

To our measure μ, we associate the measure $\nu = \mu \circ \sqrt{.}$ with support included in $\mathbb{R}^d_+$, such that for all integrable functions f,

$$\int f(z^2)\mu(dz) = \int f(z)\nu(dz).$$

Assume that the orbits $\mathcal{G}_d.\mathbf{x}_1, \ldots, \mathcal{G}_d.\mathbf{x}_k$ in the support of the measure μ and the weights $\lambda_1, \ldots, \lambda_k$ generate a $\mathcal{G}_d$-invariant cubature formula of degree m with respect to μ. Then, the points $\mathbf{x}_1^2, \ldots, \mathbf{x}_k^2$ and the weights $\lambda_1, \ldots, \lambda_k$ define a $\mathcal{G}_d$-invariant cubature formula of degree $[m/2]$ with respect to ν. Indeed, let P be a polynomial of degree less than or equal to $[m/2]$, and $Q = P(X^2)$. Then

$$\begin{aligned}\sum_{i=1}^k \lambda_i P(\mathbf{x}_i^2) = \sum_{i=1}^k \lambda_i Q(\mathbf{x}_i) = \sum_{i=1}^k \lambda_i \frac{1}{2^{e(\mathbf{x}_i)}} \sum_{y \in \mathcal{G}_d.\mathbf{x}_i} Q(y) \\ = \int Q(z)\mu(dz) = \int P(z)\nu(dz).\end{aligned}$$

It is straightforward to check that, reciprocally, if the points $\mathbf{x}_1, \ldots, \mathbf{x}_k$ in the support of the measure ν and the weights $\lambda_1, \ldots, \lambda_k$ define a cubature formula of degree $[m/2]$ with respect to ν, then the orbits $\mathcal{G}_d.\sqrt{\mathbf{x}_1}, \ldots, \mathcal{G}_d.\sqrt{\mathbf{x}_k}$ in the support of the measure μ and the weights $\lambda_1, \ldots, \lambda_k$ generate a $\mathcal{G}_d$-invariant cubature formula of degree m with respect to μ.

So to find these k generators $\mathcal{G}_d.\mathbf{x}_1, \ldots, \mathcal{G}_d.\mathbf{x}_k$ is equivalent to finding a cubature formula of degree $[m/2]$ with respect to $\nu = \mu \circ \sqrt{.}$. We are now going to show that this problem is simpler when ν is invariant with respect to the group of permutation of the axes $\mathcal{S}_d$ (which is equivalent to the fact that μ is invariant with respect to the group of permutation of the axes $\mathcal{S}_d$).

4.3. Invariance under Permutation. Let us consider a positive measure ρ on $\mathbb{R}^d$ invariant with respect to the group of permutation of the axes $\mathcal{S}_d$. The Lebesgue measure on the hypercube, on the unit sphere, the Gaussian measure on $\mathbb{R}^d$ are again examples of such a measure.

Corollary 4.3 tells us that there exists k (with $k \leq \dim \mathbb{R}_m [X_1, \ldots, X_d] (\mathcal{S}_d)$) orbits $\mathcal{S}_d.\mathbf{x}_1, \ldots, \mathcal{S}_d.\mathbf{x}_k$ in the support of the measure ρ and k weights $\lambda_1, \ldots, \lambda_k$ that generate a $\mathcal{S}_d$-invariant cubature formula of degree m with respect to ρ. Then, by definition, $\sum_{i=1}^k \lambda_i \delta_{\mathcal{S}_d, \mathbf{x}}$, (where $\delta_{\mathcal{S}_d,\mathbf{x}} = \frac{1}{|\mathcal{S}_d.\mathbf{x}|} \sum_{\mathbf{y} \in \mathcal{S}_d.\mathbf{x}} \delta_{\mathbf{y}}$) is the measure associated to a cubature formula of degree m with respect to ρ. But the measure

$$\widetilde{\delta_{\mathcal{S}_d,\mathbf{x}}} = \frac{1}{|\mathcal{S}_d^{m,\mathbf{x}}.\mathbf{x}_i|} \sum_{\sigma \in \mathcal{S}_d^{m,\mathbf{x}_i}} \sigma.\mathbf{x}_i$$

defines a cubature formula of degree m with respect to $\delta_{\mathcal{S}_d,\mathbf{x}}$ ($\mathcal{S}_d^{m,\mathbf{x}}$ has been defined in term of designs and permutation sets in section 3). Hence, the finite measure $\sum_{i=1}^k \lambda_i \widetilde{\delta_{\mathcal{S}_d,\mathbf{x}}}$ defines a cubature formula of degree m with respect to ρ. This formula has $\sum_{i=1}^k |\mathcal{S}_d^{m,\mathbf{x}_i}|$ points.

$\mathbb{R}_m [X_1, \ldots, X_d] (\mathcal{GS}_d)$ is generated by $\sum_{\sigma \in \mathcal{S}_d} X_{\sigma(1)}^{p_1} \ldots X_{\sigma(d)}^{p_d}$ for $(p_1, \ldots, p_d)$ describing $\mathcal{P}_{m,d}$ where

$$\mathcal{P}_{m,d} = \{p = (p_1, \ldots, p_d), \quad |p| \leq m, \quad m \geq p_1 \geq p_2 \geq \ldots \geq p_d \geq 0\}.$$

For all $d \geq m$, $|\mathcal{P}_{m,d}| = |\mathcal{P}_{m,m}|$. This cardinality can be expressed in term of number of Young tableaux [19]. The very important point here is that k, the number of generators, is bounded by a term ($|\mathcal{P}_{m,m}|$) which *does not depend on the dimension*. So to get a cubature formula with respect to a symmetric measure with few points, one has to find k generators (and we know that we can do it with k bounded by $|\mathcal{P}_{m,m}|$) and "good" cubature formulae with respect to $\delta_{\mathcal{S}_d,\mathbf{x}}$ (this is a combinatorial problem; we gave some indications on how to do so in section 3).

4.4. Invariance under Permutation and Reflection . We now put the two previous sections together. Recall that $\mathcal{GS}_d$ is the group generated by all reflections and permutations of the axes. μ will now denote a positive measure on $\mathbb{R}^d$ which is $\mathcal{GS}_d$-invariant. The Lebesgue measure on the hypercube, on the unit sphere, the Gaussian measure on $\mathbb{R}^d$ are, once again, examples of such a measure.

We summarize our technique to find cubature formulae of degree m with respect to our $\mathcal{GS}_d$-invariant measure μ.

1 Find k orbits $\mathcal{GS}_d.\mathbf{x}_1, \ldots, \mathcal{GS}_d.\mathbf{x}_k$ (with their elements in the support of μ) and k positive weights $\lambda_1, \ldots, \lambda_k$ that generate a $\mathcal{GS}_d$-invariant cubature formula of degree m with respect to μ. This is equivalent to the fact that orbits

$\mathcal{S}_d.\mathbf{x}_1^2, \ldots, \mathcal{S}_d.\mathbf{x}_k^2$ and the weights $\lambda_1, \ldots, \lambda_k$ that generate a $\mathcal{S}_d$-invariant cubature formula of degree $[m/2]$ with respect to $\nu = \mu \circ \sqrt{\cdot}$. One should be able to find these generators with $k \leq \left|\mathcal{P}_{[m/2],[m/2]}\right|$.

2 For all $i = 1, \ldots, k$, construct, using the methods of section (3) a cubature formula of degree $[m/2]$ with respect to $\delta_{\mathcal{S}_d,\mathbf{x}}$. Those points are the points in $\mathcal{S}_d^{[m/2],x_i}.\mathbf{x}_i^2 = \{\mathbf{x}_{i,1}^2, \ldots, \mathbf{x}_{i,n_i}^2\}$. The points $\mathbf{x}_{i,j}^2$, $i = 1, \ldots, k$, $j = 1, \ldots, n_i$ with the weights $\frac{\lambda_i}{n_i}$ now define a cubature formula of degree $[m/2]$ with respect to $\mu \circ \sqrt{\cdot}$. Equivalently, the orbits $\mathcal{G}_d.\mathbf{x}_{i,j}$ and the weights $\frac{\lambda_i}{n_i}$ generate a $\mathcal{G}_d$-invariant cubature formula of degree m with respect to μ.

3 The points in $\mathbf{g}.\mathbf{x}_{i,j}$, $i = 1, \ldots, k$, $j = 1, \ldots, n_i$, $\mathbf{g} \in \mathcal{G}_{e(\mathbf{x}_i)}^m$ with the weights $\frac{\lambda_i}{n_i \left|\mathcal{G}_{e(\mathbf{x}_i)}^m\right|}$ define a cubature formula of degree m with respect to μ.

Example 4.4. *Consider a $\mathcal{GS}_d$ invariant measure μ_d on $\mathbb{R}^d$ (for example, the Gaussian measure, the Lebesgue measure on the unit cube, or on the unit sphere). Let V be the volume of μ_d, and assume that $\mathbf{x} = \sqrt{\frac{1}{V}\int z^2 \mu_d(dz)}$ belongs to the support of μ_d. Then the generator $\mathcal{G}_d.\mathbf{x}$ and its weight V generate a $\mathcal{G}_d$-invariant cubature formula of degree 3 with respect to μ_d (hence it provides a cubature formula with 2^d points). Using our construction, we see that the points $\mathbf{g}.\mathbf{x}$, $\mathbf{g} \in \mathcal{G}_d^3$, with their weight $V/\left|\mathcal{G}_d^3\right|$, define a cubature formula of degree 3 with respect to μ_d. It has $\left|\mathcal{G}_d^3\right| = O(d)$ points. Recall that $\mathcal{G}_d^3$ was defined in terms of Hadamard matrices, and that whenever there exists a Hadamard matrix of degree d, $\left|\mathcal{G}_d^3\right| = 2d$ (which is the Möller lower bound [22]).*

Let us now construct cubature formula of degree 5 with few points for some classical measures.

5. Application to Some Classical Regions.

In this section, we will describe cubature formula of degree 5 for the region C_d, $E_d^{r^2}$, U_d, S_d.

5.1. $E_d^{r^2}$, the Gaussian Measure on $\mathbb{R}^d$.

Our definition of $E_d^{r^2}$ differs very slightly from the one given by Stroud. We consider the measure on $\mathbb{R}^d$

$$\mu_d(d\mathbf{x}) = \frac{1}{(2\pi)^{d/2}} \exp(-\frac{x_1^2 + \ldots + x_d^2}{2}) dx_1 \ldots dx_d.$$

We choose this definition of $E_d^{r^2}$ so that $\int f(z)\mu_d(z) = \mathbb{E}(f(N))$ where N is a d-dimensional normal random variable. First of all, we write $\nu_d = \mu_d \circ \sqrt{\cdot}$. One can easily see that $\nu_d = \nu_1 \otimes \ldots \otimes \nu_1$, and that

$$\int 1\nu_1(dx) = 1, \quad \int x\nu_1(dx) = 1, \quad \int x^2\nu_1(dx) = 3.$$

We will provide two constructions.

5.1.1. First Solution.

When the dimension d is of the form $d = 3k - 2$, the orbits $\mathcal{GS}_d.x_0$ and $\mathcal{GS}_d.\mathbf{x}_1$, where

$$\mathbf{x}_0 = (0, \ldots 0), \quad \mathbf{x}_1 = (\sqrt{3}, \ldots, \sqrt{3}, 0, \ldots, 0).$$

(k coordinates of $\mathbf{x}_1$ are equal to $\sqrt{3}$, $2k - 2$ to 0) , with the weights

$$\omega_0 = \tfrac{2}{d+2}, \quad \omega_1 = \tfrac{d}{d+2}$$

generate a $\mathcal{GS}_d$-invariant cubature formula of degree 5 with respect to μ_d. So we need to find a cubature formula of degree 2 with respect to

$$\delta_{\mathcal{S}_d,\mathbf{x}} = \frac{1}{|\mathcal{S}_d.\mathbf{x}_1|} \sum_{\mathbf{y}\in\mathcal{S}_d.\mathbf{x}_1^2} \delta_{\mathbf{y}}.$$

According to section 3, this can be done by using a 2-design with parameters

$$2-(3k-2,k,\lambda),$$

for a given λ. A few symmetric block designs will give us good answers. There exist symmetric block designs with parameters $(7,3,1)$, $(16,6,2)$, $(25,9,3)$ [3], $(70,24,8)$ [16] and more generally with parameters $(3^{k+2}-2,3^{k+1},3^k)$ [15],[21] and $(2.9^{k+1}-2,6.9^k,2.9^k)$ [27]. Symmetric design of the form $(9\lambda-2,3\lambda,\lambda)$ do not exist when $\lambda=4,5,6,7$. We give an example of the simplest of these block designs, described by its incidence matrix,

$$A_{7,3,1} = \begin{pmatrix} 0&0&0&1&0&1&1\\ 1&0&0&0&1&0&1\\ 1&1&0&0&0&1&0\\ 0&1&1&0&0&0&1\\ 1&0&1&1&0&0&0\\ 0&1&0&1&1&0&0\\ 0&0&1&0&1&1&0 \end{pmatrix}. \tag{5.1}$$

So assume that there exists a symmetric block design A with parameters $2-(d,(d+2)/3,(d+2)/9)$. Now define $\mathbf{x}_{1,i}$, $i=1,\dots,d$ to be $\sqrt{3}$ times the i-th row of A. Then $\mathbf{x}_{1,1}^2,\dots,\mathbf{x}_{1,d}^2$ together with equal weights $1/d$ define a cubature formula of degree 2 with respect to $\delta_{\mathcal{S}_d,\mathbf{x}}$. Then the point $\mathbf{x}_0$ with its weight $\frac{2}{d+2}$ and the points $\mathbf{g}.\sqrt{\mathbf{x}_{1,i}}$ for $\mathbf{g}\in\mathcal{G}^5_{(d+2)/3}$, $i=1,\dots,d$, with their weights equal to $\frac{1}{\left|\mathcal{G}^5_{(d+2)/3}\right|(d+2)}$, define a cubature formula of degree 5 with respect to the Gaussian measure μ_d. So every time that there exists a design $(d=9\lambda-2,3\lambda,\lambda)$, we can construct a cubature formula of degree 5 with respect to the d-dimensional Gaussian measure, with the number of points being equal to $d\left|\mathcal{G}^5_{(d+2)/3}\right|+1=O(d^3)$. If we want to find a cubature formula for a given dimension for which the above method does not work (as we need the existence of some specific design), we choose the least d' greater than d such that the method works, then project our points on a d-dimensional subspace of $\mathbb{R}^{d'}$ to get a cubature formula of degree 5 with respect to μ_d. It is interesting to note that, for $\lambda=1$ (hence $d=7$) that leads to a formula with $7\left|\mathcal{G}^5_3\right|+1=57$ points, which is exactly the Möller lower bound [22]. Considering that known cubature formulae of degree greater than 4 in dimension greater than 3 attaining the Möller lower bound are quite rare, it is surprising that this formula is actually the second cubature formula (with positive weights) for $E_d^{r^2}$ attaining the Möller lower bound (see [28] for a description of the first one).

5.1.2. Second Solution. Let

$$\mathbf{x}_0=(r,0,\dots,0),\quad \mathbf{x}_1=(s,\dots,s),$$

where

$$r^2=\tfrac{d+2}{2},\quad s^2=\tfrac{d+2}{d-2}.$$

Then the orbits $\mathcal{GS}_d.\mathbf{x}_0$ and $\mathcal{GS}_d.\mathbf{x}_1$, with the weights

$$w_0 = \tfrac{8d}{(d+2)^2}, \quad w_1 = \left(\tfrac{d-2}{d+2}\right)^2$$

generate a $\mathcal{GS}_d$-invariant cubature formula of degree 5 with respect to μ_d. We denote, for $a, b \in \{1, \dots, d\}$, by (a, b) the permutation which swaps a and b and leaves invariant all the other elements.
Then the points $\pm(1, i).\mathbf{x}_0$, $i = 1, \dots, d$, with their weights $\frac{w_0}{2d}$ and $\mathbf{g}.\mathbf{x}_1$, $\mathbf{g} \in \mathcal{G}_d^5$ with their weights $\frac{w_1}{|\mathcal{G}_d^5|}$ define a cubature formula of degree 5 with respect to the Gaussian measure μ_d. The formula has $\left|\mathcal{G}_d^5\right| + 2d = O(d^2)$ points. The orbits and the weights were obtained from the cubature formula $E_n^{r^2} : 5-3$ page 317 in [28].

5.2. C_d, the d-Dimensional Cube. We put on $[-1, 1]^d$ the measure

$$\mu_d(d\mathbf{x}) = \frac{1}{2^d} dx_1 \dots dx_d.$$

So, once again, we want to find a symmetric cubature formula of degree 2 with respect to the measure associated to μ_d. This measure ν_d is actually equal to

$$\nu_d(d\mathbf{x}) = \frac{1}{2^d} \frac{1}{\sqrt{x_1 \dots x_d}} dx_1 \dots dx_d.$$

Assume that the dimension d is odd and let

$$\alpha = \sqrt{\tfrac{1}{3} + \tfrac{2}{3\sqrt{5}}}, \quad \beta = \sqrt{\tfrac{1}{3} - \tfrac{2}{3\sqrt{5}}}.$$

Define $\mathbf{x}_0 = (\alpha, \dots, \alpha)$ and $\mathbf{x}_1 \in \mathbb{R}^d$ such that $\mathbf{x}_1^i = \alpha$ if $i = 1, \dots, (d-1)/2$, and $\mathbf{x}_1^i = \beta$ if $i = (d+1)/2, \dots, d$. Then the orbits $\mathcal{GS}_d.x_0$ and $\mathcal{GS}_d.\mathbf{x}_1$, with the weights

$$w_0 = \tfrac{1}{d+1}, \quad w_1 = \tfrac{d}{d+1}$$

generate a $\mathcal{GS}_d$-invariant cubature formula of degree 5 with respect to μ_d. To find a cubature formula with respect to $\delta_{\mathcal{S}_d,\mathbf{x}}$, we need to find a $2-(d, (d-1)/2, \lambda)$ design.

Symmetric block design with parameter $(4k-1, 2k-1, k)$ are called Hadamard designs, and they exist whenever a Hadamard matrix of degree $4k$ exists ([2], chapter I.9). Indeed, let A be a Hadamard matrix of degree $4k$; we can always take its first column and first row to be full of 1's. Then replace the -1 by 0 and delete the first row and the first column. This gives the incidence matrix B of a 2-$(4k-1, 2k-1, k)$ design. Denote by $\mathbf{x}_{1,1}, \dots, \mathbf{x}_{1,d}$ the d rows of $(\alpha - \beta)B + \beta J_{d,d}$ $(x_{i,j} \in \mathbb{R}^d)$. Then $\mathbf{x}_{1,1}^2, \dots, \mathbf{x}_{1,d}^2$ and the equal weights $\frac{1}{d}$ define a cubature formula of degree 2 with respect to $\delta_{\mathcal{S}_d,\mathbf{x}}$.

This implies that $\mathbf{x}_0^2, \mathbf{x}_{1,1}^2, \dots, \mathbf{x}_{1,d}^2$ with equal weights $1/(d+1)$ defines a cubature formula of degree 2 with respect to ν_d. To simplify the notations, let $\mathbf{x}_{1,0} = \mathbf{x}_0$. The points $\mathbf{g}.\mathbf{x}_{1,i}$, $i = 0, \dots, d$, $\mathbf{g} \in \mathcal{G}_d^5$ with equal weights define a cubature formula of degree 5 with respect to μ_d. This formula has $(d+1)\left|\mathcal{G}_d^5\right|$ points when there exists a Hadamard matrix of degree $d+1$.

If we want to find a cubature formula for a given dimension which is not the degree of a Hadamard matrix minus one, we, once again, choose the lower d' greater than d such that we know a Hadamard matrix of degree $d+1$, then project our points on a d-dimensional subspace of $\mathbb{R}^{d'}$ to get a cubature formula of degree 5 with respect to μ_d. We get, once again, a formula with $O(d^3)$ points.

We could have described a formula of the same form as the one described in section 5.1.2, but that would lead to points outside the hypercube.

5.3. U_d**, the Surface of the Unit Sphere.** We consider the Lebesgue measure μ_d on the surface of the d -dimensional unit sphere, i.e. the set of points $\mathbf{x} = (x_1, \ldots, x_d)$ such that $x_1^2 + \ldots + x_d^2 = 1$. We denote by $V = \int 1\mu_d(dx)$ the surface of the unit sphere. We propose here a solution comparable to the one proposed in 5.1.2. Indeed, let

$$\mathbf{x}_0 = (1, 0, \ldots, 0), \quad \mathbf{x}_1 = \left(\sqrt{\tfrac{1}{d}}, \ldots, \sqrt{\tfrac{1}{d}}\right),$$

and

$$w_0 = \tfrac{2V}{d+2}, \quad w_1 = \tfrac{dV}{d+2}.$$

The orbits $\mathcal{GS}_d.x_0$ and $\mathcal{GS}_d.\mathbf{x}_1$, with the weights w_0 and w_1 generate a $\mathcal{GS}_d$-invariant cubature formula of degree 5 with respect to μ_d. This leads to a cubature formula of degree 5 with respect to the Lebesgue measure on the unit sphere, with $\left|\mathcal{G}_d^5\right| + 2d = O(d^2)$ points. The orbits and the weights were obtained from the cubature formula $U_n : 5 - 2$ page 294 in [28].

5.4. S_d**, the Unit Sphere.** We consider the Lebesgue measure μ_d on the unit sphere in $\mathbb{R}^d$, i.e. the set of points $\mathbf{x} = (x_1, \ldots, x_d)$ such that $x_1^2 + \ldots + x_d^2 \leq 1$. We denote by $V = \int 1\mu_d(dx)$ the volume of the unit sphere. Let

$$\mathbf{x}_0 = (r, 0, \ldots, 0), \quad \mathbf{x}_1 = (s, \ldots, s).$$

and

$$w_0 = \tfrac{2dV}{(d+2)(d+4)r^4}, \quad w_1 = \tfrac{V}{(d+2)(d+4)s^4}$$

where

$$r^2 = 1 - \sqrt{\tfrac{2}{d+4}}, \quad s^2 = \tfrac{d(d+4)+2\sqrt{2(d+4)}}{(d^2+2d-4)(d+4)}.$$

The orbits $\mathcal{GS}_d.x_0$ and $\mathcal{GS}_d.\mathbf{x}_1$, with the weights w_0 and w_1 generate a $\mathcal{GS}_d$-invariant cubature formula of degree 5 with respect to μ_d. This leads, once again, to a cubature formula of degree 5 with respect to the Lebesgue measure on the unit sphere, with $\left|\mathcal{G}_d^5\right| + 2d = O(d^2)$ points. The orbits and the weights were obtained from the cubature formula $S_n : 5 - 3$ page 270 in [28].

5.5. Tables. In this section, we present some tables of the same type as the ones found in [7],[8]. PI means that the weights are positive and the points are inside the support of the measure. EI means that the weights are all equal (and positive) with the points inside the support of the measure. We saw that the number of points depends on the existence on some combinatorial objects, and so cannot easily be expressed as a function of the dimension. We make precise for which d our cubature formulae are very close (or equal) to the Möller lower bound.

Region	Degree	Number of points	Quality	Ref.
$E_d^{r^2}$	3	$\left\|\mathcal{G}_d^3\right\| = O(d)$ $2d$ if $\exists$ Hadamard matrix of degree d	EI	example 4.4
C_d	3	$\left\|\mathcal{G}_d^3\right\| = O(d)$ $2d$ if $\exists$ Hadamard matrix of degree d	EI	example 4.4
S_d	3	$\left\|\mathcal{G}_d^3\right\| = O(d)$ $2d$ if $\exists$ Hadamard matrix of degree d	EI	example 4.4
$E_d^{r^2}$	5	$\left\|\mathcal{G}_d^5\right\| + 2d = O(d^2)$ $d^2 + 2d$ if d is a power of 4	PI	section 5.1.2
$E_d^{r^2}$	5	$O(d^3)$ Möller lower bound for $d = 7$	PI	section 5.1.1
C_d	5	$O(d^3)$	EI	section 5.2
U_d	5	$\left\|\mathcal{G}_d^5\right\| + 2d = O(d^2)$ $d^2 + 2d$ if d is a power of 4	PI	section 5.3
S_d	5	$\left\|\mathcal{G}_d^5\right\| + 2d = O(d^2)$ $d^2 + 2d$ if d is a power of 4	PI	section 5.4

For $d \in \{3, 24\}$, we determine the exact number of points in these formulae, so that one can compare them with the ones in [7][8]. No other formulae (of degree 5) with quality PI or EI have fewer points when $d \geq 8$.

dim	$E_d^{r^2}$ 5.1.1 degree 5	C_d degree 5	$U_d, S_d, E_d^{r^2}$ 5.1.2 degree 5	$E_d^{r^2}, C_d, S_d$ degree 3
3	19	32	14	8
4	25	128	24	8
5	35	256	42	16
6	41	256	44	16
7	57	512	78	16
8	149	1536	144	16
9	189	1536	146	24
10	225	3072	276	24
11	289	3072	278	24
12	321	4096	280	24
13	417	4096	282	32
14	481	4096	284	32
15	513	4096	286	32
16	513	5120	288	32
17	1027	10240	546	36
18	1185	10240	548	36
19	1473	10240	550	36
20	1761	12288	552	36
21	2049	24576	1066	40
22	2625	24576	1068	40
23	3009	24576	1070	40
24	3201	28672	1072	40

6. Conclusion and acknowledgment. The main but basic idea of this paper is to find some measures $\xi_1, \ldots, \xi_k$ such that their sum provides a cubature formula with respect to a given measure μ, and such that it is relatively easy to find cubature formulae with respect to the measures $\xi_1, \ldots, \xi_k$. Here, we have found these measures $\xi_1, \ldots, \xi_k$ using the invariance of μ with respect to some group of symmetries, and the cubature formulae with respect to $\xi_1, \ldots, \xi_k$ using some (well known) combinatorial objects. We believe that many new cubature formulae (with positive weights, points inside the support of μ, and with few points) can be found using this method.

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