On Kiyosi Itô's Work and its Impact

by Hans Föllmer (Berlin)

About a week before the start of the International Congress, an anonymous participant in a weblog discussion of potential candidates for the Fields medals voiced his concern that there might be a bias against applied mathematics and went on to write: "*I am hoping that the Gauss prize will correct this obvious problem and they will pick someone really wonderful like Kiyosi Itô of Itô Calculus fame*". Indeed this has happened: The

Gauss Prize 2006 for Applications of Mathematics

has been awarded to Kiyosi Itô *"for laying the foundations of the theory of Stochastic Differential Equations and Stochastic Analysis"*. However, in his message to the Congress Kiyosi Itô says that he considers himself a pure mathematician, and while he was delighted to receive this honor, he was also surprised to be awarded a prize for applications of mathematics. So why is the Gauss prize so appropriate in his case, and why was this anonymous discussant who obviously cares about applied mathematics so enthusiastic ?

The statutes of the Gauss prize say that it is *"to be awarded for*

- *outstanding mathematical contributions that have found significant applications outside of mathematics, or*
- *achievements that made the application of mathematical methods to areas outside of mathematics possible in an innovative way"*.

My aim is to show why, on both accounts, Kiyosi Itô is such a natural choice.

Kiyosi Itô was born in 1915. The following photo was taken in 1942 when he was working in the Statistical Bureau of the Japanese Government:

At this time he had just achieved a major breakthrough in the theory of Markov processes. The results first appeared in 1942 in a mimeographed paper *"Differential equations determining a Markov process"* written in Japanese (Zenkoku Sizyo Sugaku Danwakai-si).

English versions and further extensions of these initial results were published between 1944 and 1951 in Japan; see [24]. These papers laid the foundations of the field which later became known as Stochastic Analysis. A systematic account appeared in the *Memoirs of the American Mathematical Society* in 1951 under the title "*On stochastic differential equations*" [23], thanks to J.L. Doob who immediately recognized the importance of Itô's work.

What was the breakthrough all about ? A Markov process is usually described in terms of the transition probabilities $P_t(x, A)$ which specify, for each state x and any time $t \geq 0$, the probability of finding the process at time t in some subset A of the state space, given that x is the initial state at time 0. These transition probabilities should satisfy the Chapman-Kolmogorov equations

$$P_{t+s}(x, A) = \int P_t(x, dy) P_s(y, A).$$

For the purpose of this exposition we limit the discussion to the special case of a diffusion process with state space $\mathbb{R}^d$. A fundamental extension theorem of Kolmogorov guarantees, for each initial state x, the existence of a probability measure P_x on the space of continuous paths

$$\Omega = C([0, \infty), \mathbb{R}^d)$$

such that the conditional probabilities governing future positions are given by the transition probabilities, i.e.,

$$P_x[X_{t+s} \in A | \mathcal{F}_t] = P_s(X_t, A).$$

Here we use the notation $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$, and $\mathcal{F}_t$ denotes the σ-field generated by the path behavior up to time t. In analytical terms, the infinitesimal structure of the Markov process is described by the infinitesimal generator

$$\mathcal{L} := \lim_{t \downarrow 0} \frac{P_t - I}{t}. \tag{1}$$

In the diffusion case, this operator takes the form

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i} \tag{2}$$

with a state-dependent diffusion matrix $a = (a_{ij})$ and a state-dependent drift vector $b = (b_i)$, and for any smooth function f the function u defined by $u(x,t) := P_t f(x)$ satisfies Kolmogorov's backward equation

$$\partial_t u = \mathcal{L} u \quad \text{on} \quad \mathbb{R}^d \times (0, \infty). \tag{3}$$

Itô's aim was to reach a deeper understanding of the dynamics by describing the infinitesimal structure of the process in probabilistic terms. His basic idea was to

i) identify the "tangents" of the process, and to

ii) (re-) construct the process pathwise from its tangents.

At the level of stochastic processes, the role of “straight lines” is taken by processes whose increments are independent and identically distributed over time intervals of the same length. Such processes are named in honor of Paul Lévy. Kiyosi Itô had already investigated in depth the pathwise behavior of Lévy processes by proving what is now known as the Lévy-Itô decomposition [21]. In the continuous case and in dimension $d = 1$, the prototype of such a Lévy process is a Brownian motion with constant drift, whose increments have a Gaussian distribution with mean and variance proportional to the length of the time interval. This process had been introduced in 1900 by Louis Bachelier as a model for the price fluctuation on the Paris stock market, five years before Albert Einstein used the same model in connection with the heat equation. A standard Brownian motion, which starts in 0 and whose increments have zero mean and variance equal to the length of the time interval, is also named in honor of Norbert Wiener who in 1923 gave the first rigorous construction, and the corresponding measure on the space of continuous paths is usually called Wiener measure. An explicit construction of a Wiener process with time interval $[0, 1]$ can be obtained as follows: Take a sequence of independent Gaussian random variables $Y_1, Y_2, \dots$ with mean 0 and variance 1, defined on some probability space $(\Omega, \mathcal{F}, P)$, and some orthonormal basis $(\varphi_n)_{n=1,2,\dots}$ in $L^2[0, 1]$. Then the random series

$$W_t(\omega) = \sum_{n=1}^{\infty} Y_n(\omega) \int_0^t \varphi_n(s)\, ds$$

is uniformly convergent and thus defines a continuous curve, P-almost surely. Wiener had studied the special case of a trigonometric basis, and Lévy had simplified the computations by using the Haar functions. But the definitive proof that the construction works in full generality was given by Itô and Nisio [32] in 1968.

In the case of a diffusion it is therefore natural to say that a “tangent” of the Markov process in a state x should be an affine function of the Wiener process with coefficients depending on that state. Thus Itô was led to describe the infinitesimal behavior of the diffusion by a “stochastic differential equation” of the form

$$dX_t = \sigma(X_t)\, dW_t + b(X_t)\, dt. \tag{4}$$

In d dimensions, the Wiener process is of the form $W = (W^1, \cdots, W^d)$ with d independent standard Brownian motions, and $\sigma(x)$ is a matrix such that $\sigma(x)\sigma^T(x) = a(x)$. The second part of the program now consisted in solving the stochastic differential equation, i.e., constructing the trajectories of the Markov process in the form

$$X_t(\omega) = x + \int_0^t \sigma(X_s(\omega))\, dW_s(\omega) + \int_0^t b(X_s(\omega))\, ds. \tag{5}$$

At this point a major difficulty arose. Wiener et al. had shown that the typical path of a Wiener process is continuous but nowhere differentiable. In particular, a Brownian path is not of bounded variation and thus cannot be used as an integrator in the Lebesgue-Stieltjes sense. In order to make sense out of equation (5) it was thus necessary to introduce what is now known as the theory of "stochastic integration".

In their introduction to the *Selected Papers* [24] of Kiyosi Itô, D. Stroock and S.R.S. Varadhan write: "*Everyone who is likely to pick up this book has at least heard that there is a subject called the theory of stochastic integration and that K. Itô is the Lebesgue of this branch of integration theory (Paley and Wiener were its Riemann)*". Wiener and Paley had in fact made a first step, using integration by parts to define the integral

$$\int_0^t \xi_s \, dW_s := \xi_t W_t - \int_0^t W_s \, d\xi_s$$

for deterministic integrands of bounded variation, and then using isometry to pass to deterministic integrands in $L^2[0,t]$. But this "Wiener integral" is no help for the problem at hand, since the integrand $\xi_t = \sigma(X_t)$ is neither deterministic nor of bounded variation. In a decisive step, Itô succeeded in giving a construction of much wider scope. Roughly speaking, he showed that the stochastic integral

$$\int_0^t \xi_s \, dW_s \approx \sum_i \xi_{t_i}(W_{t_{i+1}} - W_{t_i}) \tag{6}$$

can be defined as a limit of non-anticipating Riemann sums for a wide class of stochastic integrands $\xi = (\xi_t)$. These sums are non-anticipating in two ways. First, the integrand is evaluated at the beginning of each time interval. Secondly, the values ξ_t only depend on the past observations of the Brownian path up to time t and not on its future behavior. To carry out the construction, Kiyosi Itô used the isometry

$$E[(\int_0^t \xi_s \, dW_s)^2] = E[\int_0^t \xi_s^2 \, ds].$$

This is clearly satisfied for simple non-anticipating integrands which are piecewise constant along a fixed partition of the time axis. The appropriate class of general integrands and the corresponding stochastic integrals are obtained by taking L^2-limits on both sides. In particular the Itô integral has zero expectation, since this property obviously holds for the non-anticipating Riemann sums in (6).

Once Kiyosi Itô had introduced the stochastic integral in this way, it was clear how to define a solution of the stochastic differential equation in rigorous terms. In order to prove the existence of the solution, Itô used a stochastic version of the method of successive approximation, having first clarified the dynamic properties of stochastic integrals viewed as stochastic processes with time parameter t.

In order to complete his program, Itô had to verify that his solution of the stochastic differential equation indeed yields a pathwise construction of the given Markov process. To do so, Itô invented a new calculus for smooth functions observed along the highly non-smooth paths of a diffusion. In particular he proved what is now known as Itô's formula. In fact there are nowadays many practioners who may not know or may not care about Lebesgue and Riemann, but who do know and do care about Itô's formula.

In 1987 Kiyosi Itô received the Wolf Prize in Mathematics. The laudatio states that "*he has given us a full understanding of the infinitesimal development of Markov sample paths. This may be viewed as Newton's law in the stochastic realm, providing a direct translation between the governing partial differential equation and the underlying probabilistic mechanism. Its main ingredient is the differential and integral calculus of functions of Brownian motion. The resulting theory is a cornerstone of modern probability, both pure and applied*". The reference to Newton stresses the fundamental character of Itô's contribution to the theory of Markov processes. Let us also mention Leibniz in order to emphasize the fundamental importance of Itô's work from another point of view. In fact Itô's approach can be seen as a natural extension of Leibniz's algorithmic formulation of the differential calculus. In a manuscript written in 1675 Leibniz argues that the whole differential calculus can be developed out of the basic product rule

$$d(XY) = XdY + YdX, \tag{7}$$

and he writes: "*Quod theorema sane memorabile omnibus curvis commune est*". In particular, this implies the rule $dX^2 = 2XdX$ and, more generally,

$$df(X) = f'(X)dX \tag{8}$$

for a smooth function f observed along the curve X. Since the 19th century we know, of course, that these rules are not "*common to all (continuous) curves*", since a continuous curve does not have to be differentiable. But it was Kiyosi Itô who discovered how these rules can be modified in such a way that they generate a highly efficient calculus for the non-differentiable trajectories of a diffusion process. In Itô's calculus, the classical rule $dX^2 = 2XdX$ is replaced by

$$dX^2 = 2XdX + d\langle X\rangle,$$

where

$$\langle X\rangle_t = \lim_n \sum_{\substack{t_i \in D_n \\ t_i < t}} (X_{t_{i+1}} - X_{t_i})^2 \tag{9}$$

denotes the quadratic variation (along dyadic partitions) of the path up to time t. Lévy had shown that a typical path of the Wiener process has quadratic variation $\langle W\rangle_t = t$. Itô proved that the solution of the stochastic differential equation (4) for $d = 1$ admits a quadratic variation of the form

$$\langle X\rangle_t = \int_0^t \sigma^2(X_s)\, ds. \tag{10}$$

He then went on to show that the behavior of a function $f \in C^2$ observed along the paths of the solution is described by the rule

$$df(X) = f'(X)dX + \frac{1}{2}f''(X)\, d\langle X\rangle, \tag{11}$$

which is now known as *Itô's formula*. Note that a continuous curve of bounded variation has quadratic variation 0, and so Itô's formula may indeed be viewed as an extension of the classical differentiation rule (8).

More generally, the classical product rule (7) becomes a special case of Itô's product rule

$$d(XY) = XdY + YdX + d\langle X, Y\rangle,$$

where $\langle X, Y\rangle$ denotes the quadratic covariation of X and Y, defined in analogy to (9) or, equivalently, by polarization:

$$\langle X, Y\rangle = \frac{1}{2}(\langle X + Y\rangle - \langle X\rangle - \langle Y\rangle).$$

For a smooth function f on $\mathbb{R}^d \times [0, \infty)$ and a continuous curve $X = (X^1, \cdots, X^d)$ such that the quadratic covariations $\langle X^i, X^j\rangle$ exist, the d-dimensional version of Itô's formula takes the form

$$df(X, t) = \nabla_x f(X, t)\, dX + f_t(X, t)\, dt + \frac{1}{2}\sum_{i,j=1}^{d} f_{x_i x_j}(X, t)\, d\langle X^i, X^j\rangle. \tag{12}$$

Let us now come back to the original task of identifying the solution of the stochastic differential equation (4) as a pathwise construction of the original Markov process. In a first step, Itô showed that the solution is indeed a Markov process. Moreover he proved that the solution has quadratic covariations of the form

$$\langle X^i, X^j\rangle_t = \int_0^t \sum_k \sigma_{i,k}(X_s)\sigma_{j,k}(X_s)\, ds.$$

Thus Itô's formula for a smooth function observed along the paths of the solution reduces to

$$df(X, t) = \nabla_x f(X, t)\sigma(X)\, dW + (\mathcal{L} + \frac{\partial}{\partial t})f(X, t)\, dt, \tag{13}$$

where $\mathcal{L}$ is given by (2). In order to show that $\mathcal{L}$ is indeed the infinitesimal generator of the Markovian solution process, it is now enough to take a smooth function on $\mathbb{R}^d$ and to use Itô's formula in order to write

$$E_x[f(X_t) - f(X_0)] = E_x[\int_0^t \nabla_x f(X_s)\sigma(X_s)\, dW_s + \int_0^t \mathcal{L}f(X_s)\, ds].$$

Recalling that the Itô integral appearing on the right-hand side has zero expectation, dividing by t and passing to the limit, we see that the infinitesimal generator associated to the transition probabilities of the Markovian solution process as in (1) coincides with the partial differential operator $\mathcal{L}$ defined by (2). With a similar application of Itô's formula, Kiyosi Itô also showed that the solution of the stochastic differential equation satisfies Kolmogorov's backward equation (3).

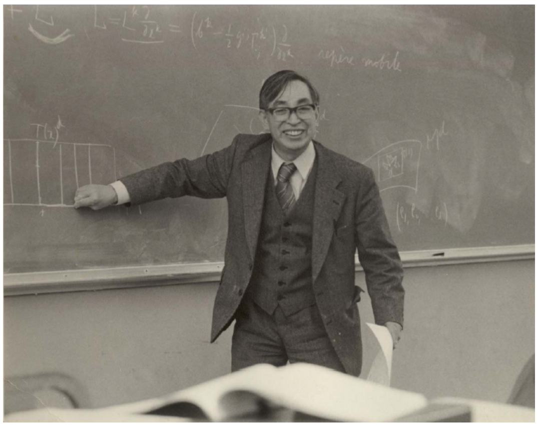
This concludes our sketch of Itô's construction of Markov processes as solutions of a corresponding stochastic differential equation. Let us emphasize, however, that we have outlined the argument only in the special case of a time-homogeneous diffusion process. In fact, Kiyosi Itô himself succeeded immediately in solving the problem in full generality, including time-inhomogeneous Markov processes with jumps and making full use of his previous analysis of general Lévy processes. For a comprehensive view of the general picture we refer to D. Stroock's book *Markov Processes from K. Itô's Perspective* [46] and, of course, to Kiyosi Itô's original publications [24].

At this point let us make a brief digression to mention a parallel approach to the construction of diffusion processes which was discovered by Wolfgang Doeblin. Born in Berlin in 1915, son of the prominent Jewish writer Alfred Döblin who took his family into exile in 1933, he studied mathematics in Paris and published results on Markov chains which became famous in the fifties. It was much less known, however, that he had also worked on the probabilistic foundation of Kolmogorov's equation. In February 1940, while serving in the French army and shortly before he took his life rather than surrender himself to the German troops, Wolfgang Doeblin sent a manuscript to the Academy of Sciences in Paris as a *pli cacheté*. This sealed envelope was finally opened in May 2000. The manuscript contains a representation of the paths of the diffusion process where the stochastic integral on the right hand side of equation (5) is replaced by a time change of Brownian motion. While Doeblin's approach does not involve the theory of stochastic integration which was developed by Kiyosi Itô and which is crucial for the applications described below, it does provide an alternative solution to the pathwise construction problem, and it anticipates important developments in martingale theory related to the idea of a random time change; see Bru and Yor [4] for a detailed account of the human and the scientific aspects of this startling discovery.

Over the last 50 years the impact of Itô's breakthrough has been immense, both within Mathematics and over a wide range of applications in other areas. Within Mathematics, this process took some time to gain momentum, at least in the West. On receiving Itô's manuscript *On stochastic differential equations*, J.L. Doob immediately recognized its importance and made sure that it was published in the *Memoirs of the AMS* in 1951. Moreover, in his book on *Stochastic processes* [9] which appeared in 1953, Doob devoted a whole chapter to Itô's construction of stochastic integrals and showed that it carries over without any major change from Brownian motion to general martingales. But when Kiyosi Itô came to Princeton in 1954, at that time a stronghold of probability theory with William Feller as the central figure, his new approach to diffusion theory did not attract much attention. Feller was mainly interested in the general structure of one-dimensional diffusions with local generator

$$\mathcal{L} = \frac{d}{dm}\frac{d}{ds},$$

motivated by his intuition that a "one-dimensional diffusion traveler makes a trip in accordance with the road map indicated by the scale function s and with the speed indicated by the measure m"; see [30]. Together with Henry McKean, at that time a graduate student of Feller, Kiyosi Itô started to work on a probabilistic construction of these general diffusions in terms of Lévy's local time. This program was carried out in complete generality in their joint book *Diffusion Processes and Their Sample Paths* [31], a major landmark in the development of probability theory in the sixties. At that time I was a graduate student at the University of Erlangen, and when a group of us organized an informal seminar on the book of Itô and McKean we found it very hard to read. But then we were delighted to discover that Itô's own *Lectures on Stochastic Processes* [25] given at the Tata Institute were much more accessible; see also [26] and [27]. This impression was fully confirmed when Professor Itô came to Erlangen in the summer of 1968: We thoroughly enjoyed the stimulating style of his lectures as illustrated by the following photo (even though it was taken ten years later at Cornell University), and also his gentle and encouraging way of talking to the graduate students.



Ironically, however, neither stochastic integrals nor stochastic differential equations were mentioned anywhere in the book, in the Tata lecture notes, or in his talks in Erlangen.

The situation began to change in the sixties, first in the East and then in the West. G. Maruyama [40] and I.V. Girsanov [18] used stochastic integrals in order to describe the transformation of Wiener measure induced by an additional drift. First systematic expositions of stochastic integration and of stochastic differential equations appeared in E.B. Dynkin's monograph [10] on Markov processes and, following earlier work of I.I. Gihman [16], [17] where some results of Itô had been found independently, in Gihman and Skorohod [19]. Kunita and Watanabe [34] clarified the geometry of spaces of martingales in terms of stochastic integrals. In the West, H.P. McKean published his book *Stochastic Integrals* [41] (dedicated to K. Itô) in 1969, and P.A. Meyer, C. Dellacherie, C. Doléans-Dade, J. Jacod and M. Yor started their systematic development of stochastic integration theory in the general framework of semimartingales; see, e.g. [8]. As a result, Stochastic Analysis emerged as one of the dominating themes of Probability Theory in the seventies. At the same time it began to interact increasingly with other mathematical fields. For example, J. Eells, K.D. Elworthy, P. Malliavin and others explored the idea of stochastic parallel transport presented by Kiyosi Itô at the ICM in Stockholm [28] and began to shape the new field of stochastic differential geometry; see, e.g., [12] and [13]. Connections to statistics, in particular to estimation and filtering problems for stochastic processes, were developed by R.S. Liptser and A.N. Shiryaev [35].

Infinite-dimensional extensions of stochastic analysis began to unfold in the eighties.

Measure-valued diffusions and "superprocesses" arising as scaling limits of large systems of branching particles became an important area of research where the techniques of Itô calculus were crucial; see, e.g., [6], and [14]. Stochastic differential equations were studied in various infinite-dimensional settings, see, e.g., [1] and [5]. With his lectures *Foundations of Stochastic Differential Equations in Infinite Dimensional Spaces* [29], given at ETH Zurich and at Lousiana State University, Kiyosi Itô himself made significant contributions to this development. In fact, in his foreword to [24] Kiyosi Itô says that "*it became my habit to observe even finite-dimensional facts from the infinite-dimensional viewpoint*". Paul Malliavin developed the stochastic analysis of an infinite-dimensional Ornstein-Uhlenbeck process and showed that this approach provides powerful new tools in order to obtain regularity results for the distributions of functionals of the solutions of stochastic differential equations [37]. His ideas led to what is now known as the Malliavin calculus, a highly sophisticated methodology with a growing range of applications which emerged in the eighties and nineties as one of the most important advances of stochastic analysis; see, e.g., [38] and [42].

While the impact of Itô's ideas within mathematics took some time to become really felt, their importance was recognized early on in several areas outside of mathematics. I will briefly mention some of them in anecdotical form before I describe one case study in more detail, namely the application of Itô's calculus in Finance. Already in the sixties engineers discovered that Itô's calculus provides the right concepts and tools for analyzing the stability of dynamical systems perturbed by noise and to deal with problems of filtering and control. When I was an instructor at MIT in 1969/70, stochastic analysis did not appear in any course offered in the Department of Mathematics. But I counted 4 courses in Electrical Engineering and 2 in Aeronautics and Astronautics in which stochastic differential equations played a role. The first systematic exposition in Germany was the book *Stochastische Differentialgleichungen* [2] by Ludwig Arnold, with the motion of satellites as a prime example. It was based on seminars and lectures at the Technical University Stuttgart which he was urged to give by his colleagues in Engineering. In the seventies the relevance of Itô's work was also recognized in physics and in particular in quantum field theory. When I came to ETH Zurich in 1977, Barry Simon gave a series of lectures for Swiss physicists on path integral techniques which included the construction of Itô's integral for Brownian motion, an introduction to stochastic calculus, and applications to Schrödinger operators with magnetic fields; see chapter V in [45]. When Kiyosi Itô was awarded a honorary degree by ETH Zurich in 1987, this was in fact due to a joint initiative of mathematicians and physicists. In another important development, the methods of Itô's calculus were crucial in analyzing scaling limits of models in population genetics in terms of measure-valued diffusions; see, e.g., [44] and the chapter on genetic models in [15], and [14].

I will now describe the application of Itô's calculus in Finance which began around 1970 and which has transformed the field in a spectacular manner, in parallel with the explosive growth of markets for financial derivatives. Consider the price fluctuation of some liquid financial asset, modeled as a stochastic process $S = (S_t)_{0 \le t \le T}$ on some probability space

$(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)_{0 \le t \le T}$. Usually S is assumed to be the solution of some stochastic differential equation (4), and then the volatility of the price fluctuation as measured by the quadratic variation process $\langle X \rangle$ is governed by the state-dependent diffusion coefficent $\sigma(x)$ as described in equation (10). The best-known case is geometric Brownian motion, where the coefficients are of the form $\sigma(x) = \sigma x$ and $b(x) = bx$. This is known as the Black-Scholes model, and we will return to this special case below. In general, the choice of a specific model involves statistical and econometric considerations. But it also has theoretical aspects which are related to the idea of *market efficiency.*

In its strong form, market efficiency requires that at each time t the available information and the market's expectations are immediately "priced in". Assuming a constant interest rate r, this means that the discounted price process $X = (X_t)_{0 \le t \le T}$ defined by $X_t = S_t \exp(-rt)$ satisfies the condition

$$E[X_{t+s}|\mathcal{F}_t] = X_t.$$

In other words, the discounted price process is assumed to be a *martingale* under the given probability measure P, and in this case P is called a *martingale measure* with respect to the given price process. In this strong form market efficiency has a drastic consequence: There is no way to generate a systematic gain by using a dynamic trading strategy. This follows from Itô's theory of the stochastic integral, applied to a general martingale instead of Brownian motion. Indeed, a trading strategy specifies the amount ξ_t of the underlying asset to be held at any time t. It is then natural to say that the resulting net gain at the final time T is given by Itô's stochastic integral

$$V_T = \int_0^T \xi_t \, dX_t \approx \sum_i \xi_{t_i} (X_{t_{i+1}} - X_{t_i}).$$

Note in fact that the non-anticipating construction of the Itô integral matches exactly the economic condition that each investment decision is based on the available information and is made before the future price increment is known. But if X is a martingale under the given probability measure P, as it is required by market efficiency in its strong form, then the stochastic integral inherits this property. Thus the expectation of the net gain under P is indeed given by

$$E[V_T] = 0.$$

There is a much more flexible notion of market efficiency, also known as the "absence of arbitrage opportunities". Here the existence of a trading strategy with positive expected net gain is no longer excluded. But it is assumed that there is no such profit opportunity without some downside risk, i.e.,

$$E[V_T] > 0 \Longrightarrow P[V_T < 0] \neq 0.$$

As shown by Harrison and Kreps [20], and then in much greater generality by Delbaen and Schachermayer [7], this relaxed notion of market efficiency is equivalent to the condition that the measure P, although it may not be a martingale measure itself, does admit an equivalent martingale measure $P^* \approx P$.

Equivalent martingale measures provide the key to the problem of pricing and hedging *financial derivatives*. Such derivatives, also known as *contingent claims*, are financial contracts based on the underlying price process. The resulting discounted outcome can be described as a nonnegative random variable H on the probability space $(\Omega, \mathcal{F}_T, P)$. The simplest example is a European call-option with maturity T, where $H = (X_T - c)^+$ only depends on the value of the stock price at the final time T. A more exotic example is the look-back option given by the maximal stock price observed up to time T.

For simple diffusion models such as the Black-Scholes model the equivalent martingale measure P^* is in fact unique, and in this case the financial market model is called *complete*. In such a complete situation any contingent claim H admits a unique arbitrage-free price, and this price is given by the expectation $E^*[H]$ under the martingale measure P^*. As shown by Jacod and Yor in the eighties, uniqueness of the equivalent martingale measure P^* is indeed equivalent to the fact that each contingent claim H admits a representation as a stochastic integral of the underlying price process:

$$H = E^*[H] + \int_0^T \xi_t \, dX_t. \tag{14}$$

This result may in fact be viewed an an extension of a fundamental theorem of K. Itô on the representation of functionals of Brownian motion as stochastic integrals. For a simple diffusion model it is actually a direct consequence of Itô's formula, as we will see below. In financial terms, the representation (14) means that the contingent claim H admits a perfect replication by means of a dynamic trading strategy, starting with the initial capital $E^*[H]$. But this implies that the correct price is given by the initial capital, since otherwise there would be an obvious arbitrage opportunity.

In the financial context, the crucial insight that arbitrage-free prices of derivatives should be computed as expectations under an equivalent martingale measure goes back to Black and Scholes [3]. They considered the problem of pricing a European call-option of geometric Brownian motion and realized that the key to the solution is provided by Itô's formula. More generally, suppose that the price fluctuation is modeled by a stochastic differential equation (4) and that the contingent claim is of the form $H = h(X_T)$ with some continuous function h. Note first that we can rewrite Itô's formula (13) as

$$df(X,t) = \nabla_x f(X,t)\, dX + (\mathcal{L}^* + \frac{\partial}{\partial t}) f(X,t)\, dt$$

in terms of the operator $\mathcal{L}^* = \mathcal{L} - b\nabla_x$. Thus the contingent claim can be written as

$$H = f(x,0) + \int_0^T \nabla_x f(X_t,t)\, dX_t \tag{15}$$

if the function f on $\mathbb{R}^d \times [0,T]$ is chosen to be a solution of the partial differential equation

$$(\mathcal{L}^* + \frac{\partial}{\partial t}) f = 0 \tag{16}$$

with terminal condition $f(\cdot, T) = h$. The representation (15) shows that the contingent claim admits a perfect replication, or a *perfect hedge*, by means of the strategy $\xi_t = \nabla_x f(X_t, t)$. Therefore its arbitrage-free price is given by $E^*[H] = f(x, 0)$. In the same way, the arbitrage-free price at any time t is given by the value $f(X_t, t)$. Thus Itô's formula provides an explicit method of computing the hedging strategy and the arbitrage-free price which involves the associated partial differential equation (16).

This approach can be extended to arbitrarily exotic derivatives. Indeed, applying the preceding argument stepwise to products of the form $H = \prod h_i(X_{t_i})$ and using an approximation of general derivatives by such finitely based functionals, one obtains the crucial representation (14) of a general contingent claim H as a stochastic integral of the underlying diffusion process. While this approach clarifies the picture from a conceptual point of view, the explicit computation of the price and the hedging strategy usually becomes a major challenge when moving beyond the simple case of a call option. At this stage additional methods of numerical analysis and of stochastic analysis may be needed. In particular, the Malliavin calculus and the analysis of "cubature on Wiener space" developed by T. Lyons have started to play an important role in this context; see, e.g., Malliavin and Thalmaier [39] and Lyons and Victoir [36].

New conceptual problems arise as soon as the financial market model becomes *incomplete*, i.e., if the martingale measure P^* is no longer unique. This happens if, for example, the driving Brownian motion in (4) is replaced by a general Lévy process as in Itô's original work, or if volatility becomes stochastic in the sense that the diffusion coefficient σ is replaced by a stochastic process. The issue of pricing and hedging financial derivatives in such an incomplete setting has led to new optimization problems and has opened new connections to convex analysis and to microeconomic theory. It has also become the source of new directions in martingale theory. In particular it has led to new variants of some fundamental decomposition theorems such as the Kunita-Watanabe decomposition and the Doob-Meyer decomposition, and it has motivated the systematic development of the theory of backward stochastic differential equations; see, e.g., [33] and [11]. In all these ramifications, however, Itô's stochastic analysis continues to provide the crucial concepts and tools.

In the beginning we recalled the statutes of the Gauss prize. We can now see more clearly why each and every one of their requirements is so well met by Kiyosi Itô's contributions. In the first place, these contributions are outstanding and in fact of fundamental importance from a strictly mathematical point of view. Secondly, they have found significant applications outside of mathematics as illustrated by the preceding case study: There is no doubt that the field of quantitative finance has been thoroughly transformed by the basic insights provided by Itô's calculus, both on a conceptual and on a computational level. Finally, this transformation of the field has paved the way to the innovative application of a wide range of mathematical methods, not only from stochastic analysis but also, following in their wake, methods from PDE's, convex analysis, statistics, and numerical analysis.

In their introduction to [24] quoted above, Stroock and Varadhan say that Kiyosi Itô "*has molded the way in which we all think about stochastic processes*". When this was written, "*we all*" referred to a rather small group of specialists. Over the last three decades this group has increased dramatically, both within and beyond the boundaries of mathematics. And I am sure that there is overwhelming agreement with the anonymous weblog discussant that the Gauss prize has been awarded to "*someone really wonderful*".

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