The minimal entropy martingale measure

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Abstract: Suppose discounted asset prices in a financial market are given by a P-semimartingale S. Among all probability measures Q that turn S into a local Q-martingale, the minimal entropy martingale measure is characterised by the property that it minimises the relative entropy with respect to P. Via convex duality, it is intimately linked to the problem of maximising expected exponential utility from terminal wealth. It also appears as a limit of p-optimal martingale measures as p decreases to 1. Like for most optimal martingale measures, finding its explicit form is easy if S is an exponential Lévy process, and quite difficult otherwise.

Key words: martingale measure, relative entropy, exponential utility maximisation, duality, exponential Lévy process, Esscher transform, utility indifference valuation, backward stochastic differential equations

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Consider a stochastic process $S = (S_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, P)$ and adapted to a filtration $I\!F = (\mathcal{F}_t)_{t\geq 0}$. Each S_t takes values in $I\!R^d$ and models the discounted prices at time t of d basic assets traded in a financial market. An *equivalent local martingale measure (ELMM)* for S, possibly on $[0, T]$ for a time horizon $T < \infty$, is a probability measure Q equivalent to the original (historical, real-world) measure P (on $\mathcal{F}_T$, if there is a T) such that S is a local Q-martingale (on $[0, T]$, respectively); see eqf04/007 [equivalent martingale measure and ramifications]. If S is a nonnegative P-semimartingale, the fundamental theorem of asset pricing says that the existence of an ELMM Q for S is equivalent to the absence-of-arbitrage condition (NFLVR) that S admits no free lunch with vanishing risk; see eqf04/002 [fundamental theorem of asset pricing].

Definition. Fix a time horizon $T < \infty$. An ELMM Q^E for S on $[0, T]$ is called *minimal entropy martingale measure (MEMM)* if Q^E minimises the relative entropy $H(Q|P)$ over all ELMMs Q for S on $[0, T]$.

Recall that the *relative entropy* is defined as

$$H(Q|P) := \begin{cases} E_P\left[\frac{dQ}{dP}\log\frac{dQ}{dP}\right] & \text{if } Q \ll P, \\ +\infty & \text{otherwise.} \end{cases}$$

This is one example of the general concept of an *f-divergence* of the form

$$D_f(Q|P) := \begin{cases} E_P\left[f\left(\frac{dQ}{dP}\right)\right] & \text{if } Q \ll P, \\ +\infty & \text{otherwise,} \end{cases}$$

where f is a convex function on $[0, \infty)$; see [49], [26], or [22] for a number of examples. The minimiser $Q^{*,f}$ of $D_f(\,\cdot\,|P)$ is then called *f-optimal ELMM*.

In many situations arising in mathematical finance, f-optimal ELMMs come up via duality from expected utility maximisation problems; see eqf04/009[expected utility maximization], eqf14/008 [expected utility maximization]. One starts with a utility function U (see eqf03/007 [utility function]) and obtains f (up to an affine function) as the convex conjugate of U, i.e.

$$f(y) - \alpha y - \beta = \sup_x \big(U(x) - xy\big).$$

Finding $Q^{*,f}$ is then the dual to the primal problem of maximising the expected utility

$$\vartheta \mapsto E\left[U\left(x_0 + \int_0^T \vartheta_r\, dS_r\right)\right]$$

from terminal wealth over allowed investment strategies ϑ. Moreover, under suitable conditions, the solutions $Q^{*,f}$ and $\vartheta^{*,U}$ are related by

$$\frac{dQ^{*,f}}{dP} = \text{const. } U'\left(x_0 + \int_0^T \vartheta_r^{*,U}\, dS_r\right). \tag{1}$$

More details can for instance be found in [41], [46], [67], [26], [68]. Relative entropy comes up with $f_E(y) = y \log y$ when one starts with the exponential utility functions $U_\alpha(x) = -e^{-\alpha x}$ with risk aversion $\alpha > 0$. The duality in this special case has been studied in detail in [8], [18], [40].

Since f_E is strictly convex, the minimal entropy martingale measure is always unique. If S is locally bounded, the MEMM (on $[0, T]$) exists if and only if there is at least one ELMM Q for S on $[0, T]$ with $H(Q|P) < \infty$; see [21]. For general unbounded S, the MEMM need not exist; [21] contains a counterexample, and [1] shows how the duality above will then fail. In [21] it is also shown that the MEMM is automatically equivalent to P, even if it is defined as the minimiser of $H(Q|P)$ over all P-absolutely continuous local martingale measures for S on $[0, T]$, provided only that there exists some ELMM Q for S on $[0, T]$ with $H(Q|P) < \infty$. Moreover, the density of Q^E with respect to P on $\mathcal{F}_T$ has a very specific form; it is given by

$$\left.\frac{dQ^E}{dP}\right|_{\mathcal{F}_T} = Z_T^E = Z_0 \exp\left(\int_0^T \vartheta_r^E\, dS_r\right) \tag{2}$$

for some constant $Z_0 > 0$ and some predictable S-integrable process ϑ^E. This has been proved in [21] for models in finite discrete time and in [28] and [26] in general; see also [23] for an application to finding optimal strategies in a Lévy process setting. Note, however, that the representation (2) only holds at the time horizon, T; the density process

$$Z_t^E = \left.\frac{dQ^E}{dP}\right|_{\mathcal{F}_t} = E_P\left[Z_T^E \,\middle|\, \mathcal{F}_t\right], \qquad 0 \le t \le T,$$

is usually quite difficult to find. We remark that the above results on the equivalence to P and the structure of the f_E-optimal Q^E both have versions for more general f-divergences; see [26]. (Essentially, (2) is the relation (1) in the case of exponential utility; but it can also be proved directly without using general duality.)

The history of the minimal entropy martingale measure Q^E is not straightforward to trace. A general definition and an authoritative exposition are given by Frittelli in [21]. But the idea of so-called minimax measures to link martingale measures via duality to utility maximisation already appears for instance in [30], [31] and [41]; see also [8]. Other early contributors include Miyahara [53], who used the term "canonical martingale measure", and Stutzer [70]; some more historical comments and references are contained in [71]. Even before, in [20], it was shown that the property defining the MEMM is satisfied by the so-called minimal martingale measure if S is continuous and the so-called mean-variance tradeoff of S has constant expectation over all ELMMs for S; see also eqf04/015 [minimal martingale measure]. The most prominent example for this occurs when S is a Markovian diffusion; see [53].

After the initial foundations, work on the MEMM has mainly concentrated on three major areas. The first aims to determine or describe the MEMM and in particular its density process Z^E more explicitly in specific models. This has been done, among others, for

- *stochastic volatility* models: see [9], [10], [35], [62], [63], and compare also eqf19/019 [modelling and measuring volatility], eqf08/017 [Barndorff-Nielsen/Shephard (BNS) models];
- jump-diffusions ([54]); and
- *Lévy processes* (compare eqf02/004 [Lévy processes]), both in general and in special settings: see [36] for an overview and [42], [43] for some examples. In particular, many studies have considered *exponential Lévy models* (see eqf08/031 [exponential Lévy models]) where $S = S_0\,\mathcal{E}(L)$ and L is a Lévy process under P. There, existence of the MEMM Q^E reduces to an analytical condition on the Lévy triplet of L. Moreover, Q^E is then given by an Esscher transform (see eqf21/024 [Esscher transform]) and L is again a Lévy process under Q^E; see for instance [13], [19], [24], [39].

For continuous semimartingales S, an alternative approach is to characterise Z^E via semimartingale backward equations or backward stochastic differential equations; see [50], [52]. The results in [56], [57] use a mixture of the above ideas in a specific class of models.

The second major area is concerned with convergence questions. Several authors have proved in several settings and with various techniques that the minimal entropy martingale measure Q^E is the limit, as $p \searrow 1$, of the so-called p-optimal martingale measures obtained by minimising the f-divergence associated to the function $f(y) = y^p$. This line of research was initiated in [27], [28], and later contributions include [39], [52], [65]. In [45], [60], this convergence is combined with the general duality (1) to utility maximisation in order to obtain convergence results for optimal wealths and strategies as well.

The third and by far most important area of research on the MEMM is centered on its link to the exponential utility maximisation problem; see [8], [18] for a detailed exposition of this issue. More specifically, the MEMM is very useful when one studies the valuation of contingent claims by (exponential) *utility indifference valuation*; see eqf04/011 [utility indifference valuation]. To explain this, we fix an initial capital x_0 and a random payoff H due at time T. The maximal expected utility one can obtain by trading in S via some strategy ϑ, if one starts with x_0 and has to pay out H in T, is

$$\sup_{\vartheta} E\left[U\left(x_0 + \int_0^T \vartheta_r\, dS_r - H\right)\right] =: u(x_0; -H),$$

and the *utility indifference value* x_H is then implicitly defined by

$$u(x_0 + x_H; -H) = u(x_0; 0).$$

Hence x_H represents the monetary compensation required for selling H if one wants to achieve utility indifference at the optimal investment behaviour. If $U = U_\alpha$ is exponential, its multiplicative structure makes the analysis of the utility indifference value x_H tractable, in remarkable contrast to all other classical utility functions. Moreover, $u(x_0; -H)$ as well as x_H and the optimal strategy ϑ^*_H can be described with the help of a minimal entropy martingale measure (defined here with respect to a new, H-dependent reference measure P_H instead of P). This topic has first been studied in [4], [58], [59], [64]; later work has examined intertemporally dynamic extensions ([5], [51]), descriptions via BSDEs in specific models ([6], [51]), extensions to more general payoff structures ([38], [47], [48], [61]), etc.; see also [29], [37], [69].

Apart from the above, there are a number of other areas where the minimal entropy martingale measure has come up; these include

– option price comparisons ([7], [11], [32], [33], [34], [55]);
– generalisations or connections to other optimal ELMMs ([2], [14], [15], [66]); see also eqf04/015 [minimal martingale measure] and [20];
– utility maximisation with a random time horizon ([12]);
– good deal bounds ([44]); see also eqf04/016 [good-deal bounds];
– a calibration game ([25]).

There are also many papers who simply choose the MEMM as pricing measure for option pricing applications; especially in papers from the actuarial literature, this approach is often motivated by the connections between the MEMM and the Esscher transformation. Finally, we mention that the idea of looking for a martingale measure subject to a constraint on relative entropy also naturally comes up in calibration problems; see for instance [3], [16], [17], and compare eqf05/009 [calibration], eqf08/002 [model calibration].

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