Quantile Hedging

by

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Abstract. In a complete financial market every contingent claim can be hedged perfectly. In an incomplete market it is possible to stay on the safe side by superhedging. But such strategies may require a large amount of initial capital. Here we study the question what an investor can do who is unwilling to spend that much, and who is ready to use a hedging strategy which succeeds with high probability.

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1. Introduction

The problem of pricing and hedging of contingent claims is well understood in the context of arbitrage-free models which are complete. In such models every contingent claim is attainable, i.e., it can be replicated by a self-financing trading strategy. The cost of replication defines the price of the claim, and it can be computed as the expectation of the claim under the unique equivalent martingale measure.

In an incomplete market the equivalent martingale measure is no longer unique, and not every contingent claim is attainable. Such claims carry an intrinsic risk. There is an interval of arbitrage-free prices, given by the expected values under the different equivalent martingale measures. It is still possible to stay on the safe side by using a "superhedging" strategy, cf. El Karoui and Quenez (1995) and Karatzas (1997). The cost of carrying out such a strategy is given by the supremum of the expected values over all equivalent martingale measures. The corresponding value process is a supermartingale under any equivalent martingale measure, and the superhedging strategy is determined by the "optional decomposition" of such a universal supermartingale, cf. Kramkov (1996). But in many situations the cost of superhedging is too high from a practical point of view.

What if the investor is unwilling to put up the initial amount of capital required by a perfect hedging or superhedging strategy? What is the maximal probability of a successful hedge the investor can achieve with a given smaller amount? Equivalently one can ask how much initial capital an investor can save by accepting a certain shortfall probability, i.e., by being willing to take the risk of having to supply additional capital at maturity in, e.g., 1% of the cases. This question seems to be relevant from an applied point of view. Even in complete markets many investors do not want a perfect hedge because it takes away completely the opportunity to make a profit together with the risk of a loss. Also the total amount of capital available to an investor is often limited, and the investor will look for the most efficient allocation of capital to participate in as many business opportunities as possible while keeping the total business risk under control.

In this paper our aim is to construct a hedging strategy which maximizes the probability of a successful hedge under the objective measure P, given a constraint on the required cost. Alternatively, we can fix a bound ε for the shortfall probability and minimize the cost in the class of hedging strategies such that the probability of covering the claim is at least $1-\varepsilon$. This concept of *quantile hedging* can be considered as a dynamic version of the familiar *value at risk* concept (VaR). Just as in VaR a certain level of security (e.g. 99%) is chosen. However the amount of capital required to reach this level is less than in the static VaR approach because we are going to allow for dynamic strategies which react to the price movements of the underlying. In the context of the classical Black-Scholes model, the idea of quantile hedging was proposed by the first author in March 1995 at the Isaac Newton Institute, triggered by a talk of David Heath on the results in Kulldorff (1993); see also Karatzas (1997, p.58), Schwarz (1996), Cvitanic and Spivak (1998). A closely related idea appears in Browne (1997).

In section 2 we consider the general complete case where there is a unique equivalent martingale measure P^*. Here the problem of quantile hedging is solved in a straightforward manner. We simply translate to a general setting the method of Kulldorff (1993) for maximizing the probability of reaching a given level up to a given time by trading on a Brownian motion with drift. In a first step, we determine a set of maximal probability under the constraint that the cost of hedging the given claim on that set satisfies a given bound. Using the Neyman-Pearson lemma, this set is constructed as an optimal test where the alternative is given by the objective measure P, and where the hypothesis is defined in terms of the contingent claim and the equivalent martingale measure P^*. In a second step, we use the completeness of the model in order to replicate the knockout option obtained by restricting the claim to this maximal set. This strategy maximizes the probability of a successful hedge.

In section 4 we consider the general incomplete case. Here the representation theorem for contingent claims does no longer hold. Instead we use the technique of superhedging. This is combined with the Neyman-Pearson lemma for compound hypotheses defined in terms of the contingent claim and the class of equivalent martingale measures: For a given claim H, the optimal strategy consists in superhedging the modified claim $H\varphi$, where φ is the optimal randomized test provided by the Neyman-Pearson lemma.

In order to illustrate our approach we compute the strategy of quantile hedging for a call option in different models for the price fluctuation of the underlying asset. In section 3 we consider the standard case of geometric Brownian motion with known volatility. In section 5 we pass to an incomplete extension of the Black-Scholes model where volatility is subject to a random jump.

Just as the static VaR approach, the dynamical concept of quantile hedging invites critique since it does not take into account the *size* of the shortfall, only the probability of its occurence. In other words, we evaluate the shortfall in terms of a very simple binary loss function. In section 2.5 we point out how our method can be extended to the case where we minimize the expected size of the shortfall, given a constraint on the cost. A systematic discussion of the general case, where the shortfall is measured by a convex loss function, and some explicit case studies are given in Föllmer and Leukert (1998). In the present paper, our purpose is to explain the basic idea in its simplest form.

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2. Quantile hedging: The complete case

In section 2.1 the problem of quantile hedging is formulated in a general semimartingale context. In sections 2.2 - 2.4 we show how the problem is solved in the complete case where the equivalent martingale measure is unique. Section 2.5 explains how the method can be extended to other versions of the hedging problem which take into acount the size of the shortfall.

2.1 Formulation of the problem

We assume that the discounted price process of the underlying is given as a semimartingale $X = (X_t)_{t\in[0,T]}$ on a probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)_{t\in[0,T]}$. For simplicity we assume that $\mathcal{F}_0$ is trivial. Let $\mathcal{P}$ denote the set of all equivalent martingale measures. We assume absence of arbitrage in the sense that $\mathcal{P} \neq \emptyset$.

A *self-financing strategy* is defined by an initial capital $V_0 \geq 0$ and by a predictable process ξ which serves as an integrand for the semimartingale X. Such a strategy (V_0, ξ) will be called *admissible* if the resulting value process V defined by

$$V_t = V_0 + \int_0^t \xi_s dX_s \quad \forall t \in [0,T] \ , \quad P-\text{a.s.} \tag{2.1}$$

satisfies

$$V_t \geq 0 \quad \forall t \in [0,T] \ , \quad P-\text{a.s.} \tag{2.2}$$

In the *complete* case there is a unique equivalent martingale measure $P^* \approx P$. Consider a contingent claim given by a $\mathcal{F}_T$-measurable, nonnegative random variable H such that $H \in L^1(P^*)$. Completeness implies that there exists a perfect hedge, i.e., a predictable process ξ^H such that

$$E^*\left[H \mid \mathcal{F}_t\right] = H_0 + \int_0^t \xi_s^H dX_s \quad \forall t \in [0,T] \quad P-\text{a.s.}, \tag{2.3}$$

where E^* denotes expectation with respect to P^*; cf., e.g., Jacka (1992). Thus the claim can be duplicated by the self-financing trading strategy (H_0, ξ^H). This assumes, of course, that we are ready to allocate the required initial capital

$$H_0 = E^*\left[H\right] \ . \tag{2.4}$$

But what if the investor is unwilling or unable to put up the initial capital H_0? What is the best hedge the investor can achieve with a given smaller amount $\tilde{V}_0 < H_0$? As our optimality criterion we take the probability that the hedge is successful. Thus we are looking for an admissible strategy (V_0, ξ) such that

$$P\left[V_0 + \int_0^T \xi_s dX_s \geq H\right] = \max \tag{2.5}$$

under the constraint

$$V_0 \leq \tilde{V}_0 \ . \tag{2.6}$$

2.2 Maximizing the probability of success

Let us call the set

$$\{V_T \geq H\} \tag{2.7}$$

the *"success set"* corresponding to the admissible strategy (V_0, ξ), where V_T is given by (2.1). In a first step we reduce our problem to the construction of a success set of maximal probability:

(2.8) **Proposition.** *Let $\tilde{A} \in \mathcal{F}_T$ be a solution of the problem*

$$P[A] = \max \tag{2.9}$$

under the constraint

$$E^*[HI_A] \leq \tilde{V}_0 \ , \tag{2.10}$$

where P^* is the unique equivalent martingale measure. Let $\tilde{\xi}$ denote the perfect hedge for the knockout option $\tilde{H} = HI_{\tilde{A}} \in L^1(P^*)$, i.e.,

$$E^*\left[HI_{\tilde{A}} \mid \mathcal{F}_t\right] = E^*[HI_{\tilde{A}}] + \int_0^t \tilde{\xi}_s dX_s \quad \forall t \in [0,T] \quad P-\text{a.s.} \tag{2.11}$$

Then $(\tilde{V}_0, \tilde{\xi})$ solves the optimization problem defined by (2.5) and (2.6), and the corresponding success set coincides almost surely with $\tilde{A}$.

Proof. 1) Let (V_0, ξ) be any admissible strategy such that $V_0 \leq \tilde{V}_0$. The corresponding value process defined by (2.1) is a nonnegative local martingale under P^*, hence a supermartingale under P^*. If A denotes the success set corresponding to (V_0, ξ), then we have

$$V_T \geq HI_A \qquad P-\text{a.s.} \tag{2.12}$$

since $V_T \geq 0 \quad P-\text{a.s.}$ due to (2.2). Thus

$$\tilde{V}_0 \geq V_0 \geq E^*\left[V_T\right] \geq E^*\left[HI_A\right] \ , \tag{2.13}$$

i.e., A satisfies the constraint (2.10). This implies

$$P\left[A\right] \leq P\left[\tilde{A}\right] \tag{2.14}$$

due to (2.9).

2) Let us now show that any strategy $(V_0, \tilde{\xi})$ such that $E^*[HI_{\tilde{A}}] \leq V_0 \leq \tilde{V}_0$ is optimal. Note first that the strategy is admissible:

$$V_0 + \int_0^t \tilde{\xi}_s dX_s \geq E^*\left[HI_{\tilde{A}}\right] + \int_0^t \tilde{\xi}_s dX_s = E^*\left[HI_{\tilde{A}} \mid \mathcal{F}_t\right] \geq 0 \quad P-\text{a.s.} \tag{2.15}$$

due to (2.11), since $HI_{\tilde{A}} \geq 0$. The success set

$$A = \left\{ V_0 + \int_0^T \tilde{\xi}_s dX_s \geq H \right\} \tag{2.16}$$

corresponding to $(V_0, \tilde{\xi})$ satisfies

$$\tilde{A} \subseteq \{HI_{\tilde{A}} \geq H\} \subseteq A \qquad P-\text{a.s.} \tag{2.17}$$

by (2.11) since $V_0 \geq E^*[HI_{\tilde{A}}]$. On the other hand, part 1) shows that the success set A must satisfy (2.14), and this implies $A = \tilde{A} \quad P-$ a.s.. Thus we have identified $\tilde{A}$ as the success set corresponding to $(V_0, \tilde{\xi})$. In particular, the strategy $(\tilde{V}_0, \tilde{\xi})$ is optimal in the sense of (2.5) and (2.6).

The problem of constructing a maximal success set is now solved by applying the Neyman-Pearson lemma. To this end we introduce the measure Q^* given by

$$\frac{dQ^*}{dP^*} = \frac{H}{E^*[H]} = \frac{H}{H_0} \ . \tag{2.18}$$

The constraint (2.10) can be written as

$$Q^*[A] \leq \alpha := \frac{\tilde{V}_0}{H_0} \ . \tag{2.19}$$

Define the level

$$\tilde{a} = \inf\Big\{a : Q^*\big[\frac{dP}{dP^*} > a \cdot H\big] \leq \alpha\Big\} \ . \tag{2.20}$$

and the corresponding set

$$\tilde{A} := \Big\{\frac{dP}{dP^*} > \tilde{a} \cdot H\Big\} \ . \tag{2.21}$$

(2.22) **Theorem.** *Assume that the set $\tilde{A}$ satisfies*

$$Q^*[\tilde{A}] = \alpha \ . \tag{2.23}$$

Then the optimal strategy solving (2.5) and (2.6) is given by $(\tilde{V}_0, \tilde{\xi})$ where $\tilde{\xi}$ is the perfect hedge for the knockout option $HI_{\tilde{A}}$.

Proof. P and Q^* are both dominated by P^*, and the set $\tilde{A}$ is of the form

$$\tilde{A} = \Big\{\frac{dP}{dP^*} > \text{const}\,\frac{dQ^*}{dP^*}\Big\} \ . \tag{2.24}$$

The Neyman–Pearson lemma states that

$$P[A] \leq P[\tilde{A}] \tag{2.25}$$

for all sets $A \in \mathcal{F}_T$ such that

$$Q^*[A] \leq Q^*[\tilde{A}] \ ; \tag{2.26}$$

cf., e.g., Witting (1985). Under our condition (2.23), the theorem now follows from proposition (2.8).

Theorem (2.22) shows that the problem of quantile hedging is solved by hedging a suitable knockout option. From a practical point of view the hedging strategy for a knockout option has some inconvenient features, and we refer to Shreve and Wystup (1998) for a careful analysis of this issue.

2.3 Maximizing the expected success ratio

Condition (2.23) is clearly satisfied if

$$P\Big[\frac{dP}{dP^*} = \tilde{a} \cdot H\Big] = 0 \,. \tag{2.27}$$

But in general it may not be possible to find any set $A \in \mathcal{F}_T$ which assumes the bound in (2.19). In this case, the Neyman-Pearson theory suggests to replace the "critical region" $A \in \mathcal{F}_T$ by a "randomized test", i.e., by a $\mathcal{F}_T$-measurable function φ such that $0 \le \varphi \le 1$. Let $\mathcal{R}$ denote the class of all these functions, and consider the following optimization problem:

$$E[\tilde{\varphi}] = \max_{\varphi \in \mathcal{R}} E[\varphi] \tag{2.28}$$

under the constraint

$$\int \varphi \, dQ^* \le \alpha = \frac{\tilde{V}_0}{H_0} \,. \tag{2.29}$$

In its extended form, the Neyman–Pearson lemma states that the solution is given by

$$\tilde{\varphi} = I_{\{\frac{dP}{dP^*} > \tilde{a} \cdot H\}} + \gamma I_{\{\frac{dP}{dP^*} = \tilde{a} \cdot H\}} \tag{2.30}$$

where $\tilde{a}$ is given by (2.20), and where γ is defined by

$$\gamma = \frac{\alpha - Q^*[\frac{dP}{dP^*} > \tilde{a} \cdot H]}{Q^*[\frac{dP}{dP^*} = \tilde{a} \cdot H]} \tag{2.31}$$

in case that condition (2.23) does not hold. This provides the solution to the following extension of our hedging problem.

(2.32) **Definition.** For any admissible strategy (V_0, ξ) we define the corresponding "*success ratio*" as

$$\varphi = I_{\{H \le V_T\}} + \frac{V_T}{H} I_{\{V_T < H\}} \,. \tag{2.33}$$

Note that $\varphi \in \mathcal{R}$, and that the set $\{\varphi = 1\}$ coincides with the success set $\{V_T \geq H\}$ associated to the strategy (V_0, ξ). In the extended version of our original problem defined by (2.5) and (2.6), we are now looking for a strategy which maximizes the expected success ratio $E[\varphi]$ under the measure P:

(2.34) **Theorem.** *Let $\tilde{\xi}$ denote the perfect hedge for the contingent claim $\tilde{H} = H\tilde{\varphi}$ where $\tilde{\varphi}$ is defined by (2.30). Then*

i) $(\tilde{V}_0, \tilde{\xi})$ maximizes the expected success ratio $E[\varphi]$ under all admissible strategies (V_0, ξ) with $V_0 \leq \tilde{V}_0$,

ii) the success ratio of $(\tilde{V}_0, \tilde{\xi})$ is given by $\tilde{\varphi}$.

Note that condition (2.23) implies $\tilde{\varphi} = I_{\tilde{A}}$, and in this case the strategy $(\tilde{V}_0, \tilde{\xi})$ reduces to the strategy described in Theorem (2.22).

Proof. The argument is analogous to the proof of (2.8) and (2.22), and it is a special case of the proof of theorem (4.9) below.

2.4 Minimizing the cost for a given probability of success

Consider a given shortfall probability $\varepsilon \in (0, 1)$. We are looking for the least amount of initial capital which allows us to stay on the safe side with probability $1 - \varepsilon$, i.e., we want to determine the minimal value of V_0 such that there exists an admissible strategy (V_0, ξ) with

$$P\left[V_0 + \int_0^T \xi_s dX_s \geq H\right] \geq 1 - \varepsilon \, . \tag{2.35}$$

In analogy to the previous argument, this can be reduced to the problem of finding a set $A \in \mathcal{F}_T$ such that

$$E^*\left[HI_A\right] = \min \tag{2.36}$$

under the constraint

$$P\left[A\right] \geq 1 - \varepsilon \, . \tag{2.37}$$

Equivalently, we want to maximize $Q^*\left[A^c\right]$ under the constraint $P\left[A^c\right] \leq \varepsilon$, where Q^* is defined by (2.18). The solution is again provided by the Neyman-Pearson lemma: Choose

$$\tilde{b} = \inf\left\{b : P\left[\frac{dQ^*}{dP} > b\right] \leq \varepsilon\right\} \tag{2.38}$$

and define $\tilde{B}$ through its complement

$$\tilde{B}^c = \left\{\frac{dQ^*}{dP} > \tilde{b}\right\} = \left\{\frac{dP}{dP^*} < (\tilde{b}E^*[H])^{-1}H\right\} \, . \tag{2.39}$$

If $P[\tilde{B}] = 1-\varepsilon$ then $\tilde{B}^c$ maximizes $Q^*[A^c]$ under the constraint $P[A^c] \leq \varepsilon$. In other words, $\tilde{B}$ minimizes $E^*[HI_A]$ under the constraint (2.37). But this implies that the optimal strategy for the original problem is given by the duplicating strategy for the knockout option $HI_{\tilde{B}}$.

In the same way we can solve the extended problem where we require that the expected success ratio satisfies $E[\varphi] \geq 1-\varepsilon$. Define

(2.40) $$\tilde{\varphi} = I_{\{\frac{dQ^*}{dP} < \tilde{b}\}} + \gamma I_{\{\frac{dQ^*}{dP} = \tilde{b}\}}$$

where $\tilde{b}$ is given by (2.38) and γ is defined by

(2.41) $$\gamma = \frac{(1-\varepsilon) - P[\frac{dQ^*}{dP} < \tilde{b}]}{P[\frac{dQ^*}{dP} = \tilde{b}]} \ .$$

in case that $P[\frac{dQ^*}{dP} < \tilde{b}] < 1-\varepsilon$.

(2.42) **Theorem.** *Let $\tilde{\xi}$ denote the perfect hedge for the contingent claim $\tilde{H} = H\tilde{\varphi}$ and define $\tilde{V}_0 = E^*[\tilde{H}]$. Then*

i) $(\tilde{V}_0, \tilde{\xi})$ has minimal cost under all admissible strategies (V_0, ξ) with expected success ratio $E[\varphi] \geq 1-\varepsilon$,

ii) the success ratio of $(\tilde{V}_0, \tilde{\xi})$ is given by $\tilde{\varphi}$, and $E[\tilde{\varphi}] = 1-\varepsilon$.

2.5 Controlling the size of the shortfall

For a given strategy (V_0, ξ) the resulting shortfall is defined as the excess

(2.43) $$S = (H - V_T)^+$$

of the contingent claim over the final portfolio value. So far we have looked for a strategy which maximizes the probability that the shortfall S is 0. In other words, our aim was to minimize the expected loss

(2.44) $$E[L(S)]$$

for the special binary loss function $L(x) = I_{(0,\infty)}(x)$. But it is natural to take into account also the size of the shortfall S, not just the probability that it is strictly positive. This suggests to minimize (2.44) for a loss function of the form $L(x) = l(x)I_{(0,\infty)}(x)$ where l is some increasing convex function on $[0,\infty)$, for example $l(x) = x^p$ for some $p \geq 1$.

In the special case $l(x) = x$, our aim would be to minimize the expected shortfall

(2.45) $$E[S] = E[(H - V_T)^+]$$

under a constraint on the initial capital V_0. Without loss of generality we can assume that $0 \leq V_T \leq H$, i.e., the final portfolio value is of the form $V_T = H\varphi$ for some randomized test $\varphi \in \mathcal{R}$. Thus, the problem of minimizing the expected shortfall (2.45) is equivalent to the problem of maximizing the expected value $E[H\varphi]$ in the class $\mathcal{R}$ under a constraint of the form $E^*[H\varphi] \leq \tilde{V}_0$. A slight modification of the preceding discussion shows that the optimal strategy consists in replicating the modified claim $H\tilde{\varphi}$, where $\tilde{\varphi}$ is the optimal test of the simple hypothesis P^* against the alternative P.

A systematic discussion of the hedging problem in the case where l is a general convex loss function and some explicit case studies are given in Föllmer and Leukert (1998).

3. Quantile hedging in the Black-Scholes model

In the standard Black-Scholes model with constant volatility $\sigma > 0$, the underlying price process is given by a geometric Brownian Motion

$$dX_t = X_t(\sigma dW_t + m dt) \tag{3.1}$$

with initial value $X_0 = x_0$, where W is a Wiener process under P and m is a constant. For simplicity we set the interest rate equal to zero. The unique equivalent martingale measure is then given by

$$\frac{dP^*}{dP} = \exp\left(-\frac{m}{\sigma}W_T - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right) . \tag{3.2}$$

The process W^* defined by

$$W_t^* = W_t + \frac{m}{\sigma}t \tag{3.3}$$

is a Brownian motion under P^*. Since

$$\begin{aligned} X_T &= x_0 \exp(\sigma W_T + (m - \frac{1}{2}\sigma^2)T) \\ &= x_0 \exp(\sigma W_T^* - \frac{1}{2}\sigma^2 T) , \end{aligned} \tag{3.4}$$

we can also write

$$\frac{dP^*}{dP} = \text{const} \cdot X_T^{-m/\sigma^2} . \tag{3.5}$$

A European call $H = (X_T - K)^+$ can be hedged perfectly if we use the initial capital

$$H_0 = E^*[H] = x_0 \Phi(d_+) - K\Phi(d_-) , \tag{3.6}$$

where

$$d_{\pm} = -\frac{1}{\sigma\sqrt{T}} \ln\left(\frac{K}{x_0}\right) \pm \frac{1}{2}\sigma\sqrt{T} \tag{3.7}$$

and Φ denotes the distribution function of the standard normal distribution $N(0,1)$.

Suppose we want to use only an initial capital V_0 which is smaller than the Black-Scholes price H_0. By (2.22) the optimal strategy consists in duplicating the knockout option HI_A where the set A is of the form

$$A = \left\{ \frac{dP}{dP^*} > \text{const} \cdot H \right\} . \tag{3.8}$$

Due to (3.5) we can write

$$A = \{X_T^{m/\sigma^2} > \lambda (X_T - K)^+\} , \tag{3.9}$$

and the constant λ is chosen such that

$$E^*\left[HI_A\right] = V_0 . \tag{3.10}$$

We distinguish two cases.

i) $m \leq \sigma^2$:

In this case the success set takes the form

$$A = \{X_T < c\} = \{W_T^* < b\} \tag{3.11}$$

where

$$c = x_0 \exp\left(\sigma b - \frac{1}{2}\sigma^2 T\right) . \tag{3.12}$$

Thus, the modified option HI_A can be written as a combination

$$(X_T - K)^+ - (X_T - c)^+ - (c - K) I_{\{X_T > c\}} \tag{3.13}$$

of two call options and of a binary option. We get

$$P\left[A\right] = \Phi\left(\frac{b - \frac{m}{\sigma}T}{\sqrt{T}}\right) , \tag{3.14}$$

and the constant b can be determined from the condition

$$\begin{aligned} V_0 &= E^*[HI_A] \\ &= x_0 \Phi\left(d_+\right) - K\Phi\left(d_-\right) - x_0 \Phi\left(\frac{-b + \sigma T}{\sqrt{T}}\right) + K\Phi\left(\frac{-b}{\sqrt{T}}\right) . \end{aligned} \tag{3.15}$$

If instead of V_0 we prescribe a shortfall probability ε and require

$$1-\varepsilon = P[A] \ , \tag{3.16}$$

then b is defined by

$$b = \sqrt{T}\Phi^{-1}(1-\varepsilon) + \frac{m}{\sigma}T \ , \tag{3.17}$$

and the corresponding minimal cost V_0 can be computed via (3.15).

To illustrate the amount of initial capital that can be "saved" by accepting a certain shortfall probability consider the following numerical example: $T = 0.25$ (i.e., 3 months), $\sigma = 0.3, m = 0.08, , X_0 = 100, K = 110$. For the values $\varepsilon = 0.01, 0.05, 0.1$ the corresponding proportions V_0/H_0 are given, respectively, by $0.89, 0.59, 0.34$. Thus, we can reduce the initial capital by 41% if we are ready to accept a shortfall probability of 5%.

For later purposes we define $\alpha(y)$ as the maximal probability of a successful hedge for a given capital $y \geq 0$ and state the following properties of the function α.

(3.18) **Lemma.** *The function α belongs to $C[0,\infty) \cap C^1(0,\infty)$, increases strictly from $\alpha(0) = P[X_T \leq K]$ to 1 on $[0, H_0]$, is concave on $[0,\infty)$ and strictly concave on $[0, H_0]$, and satisfies $\frac{\partial}{\partial y}\alpha(0+) = \infty$.*

Proof. Let f resp. f^* denote the density functions of X_T under P and P^*. We have

$$\alpha(y) = \int_0^{c(y)} f(z)dz \tag{3.19}$$

where $c(y)$ is defined as the solution of the equation

$$y = \int_K^c (z-K)f^*(z)dz \tag{3.20}$$

(for $y \geq H_0$ we put $c(y) = +\infty$). Differentiation yields

$$1 = (c(y)-K)f^*(c(y))c'(y) \ , \tag{3.21}$$

hence

$$\alpha'(y) = f(c(y))c'(y) = \frac{f(c(y))}{f^*(c(y))}\frac{1}{c(y)-K} \ . \tag{3.22}$$

Since

$$\frac{f}{f^*}(z) = \text{const } z^{m/\sigma^2} \tag{3.23}$$

and since $m \leq \sigma^2$, the right hand side in (3.22) is a decreasing function of $c(y)$. But $c(y)$ increases from K to ∞ as y increases from 0 to H_0, and so we obtain the stated properties of α.

ii) $m > \sigma^2$:

In this case the function x^{m/σ^2} is convex. Since $P[A] < 1$, the success set A must have the form

$$\begin{aligned} A &= \{X_T < c_1\} \cup \{X_T > c_2\} \\ &= \{W_T^* < b_1\} \cup \{W_T^* > b_2\} \end{aligned} \tag{3.24}$$

where $c_1 < c_2$ are the two distinct solutions of the equation

$$x^{m/\sigma^2} = \lambda(x-K)^+ , \tag{3.25}$$

and where the constant λ is determined by the condition $E^*[HI_A] = V_0$. We have

$$P\left[A\right] = \Phi\left(\frac{b_1 - \frac{m}{\sigma}T}{\sqrt{T}}\right) + \Phi\left(-\frac{b_2 - \frac{m}{\sigma}T}{\sqrt{T}}\right) . \tag{3.26}$$

The modified option HI_A can again be written as a combination of call options and digital options, and the corresponding cost is given by

$$\begin{aligned} V_0 &= x_0 \Phi\left(d_+\right) - K\Phi\left(d_-\right) \\ &\quad - x_0 \Phi\left(\frac{-b_1 + \sigma T}{\sqrt{T}}\right) + K\Phi\left(\frac{-b_1}{\sqrt{T}}\right) + x_0\Phi\left(\frac{-b_2 + \sigma T}{\sqrt{T}}\right) - K\Phi\left(\frac{-b_2}{\sqrt{T}}\right) . \end{aligned} \tag{3.27}$$

The function α again has the properties stated in (3.18); we omit the proof.

4. Quantile hedging: The incomplete case

In this section we discuss the problem of quantile hedging in the general incomplete case where the equivalent martingale measure is no longer unique. The solution combines the Neyman-Pearson lemma for multiple hypotheses with the technique of superhedging.

4.1 Superhedging

In incomplete models not every contingent claim is attainable. Nonetheless it is possible to stay on the safe side by putting up a sufficiently high amount of initial capital and following a superhedging strategy. This approach was initiated by El Karoui and Quenez

(1995); see also Karatzas (1997) and the references given there. The least amount of initial capital required to be on the safe side is given by

$$\inf\{V_0 \mid \exists \xi \ : \ (V_0, \xi) \quad \text{admissible}, \quad V_0 + \int_0^T \xi_s dX_s \geq H \quad P-\text{a.s.}\} \ . \tag{4.1}$$

There is a basic duality which characterizes this least amount as the largest arbitrage-free price. More precisely, let us assume

$$U_0 := \sup_{P^* \in \mathcal{P}} E^*[H] < \infty \ , \tag{4.2}$$

and let us define (U_t) as a right-continuous version of the process defined by

$$U_t = \text{ess.sup}_{P^* \in \mathcal{P}} E^*[H \mid \mathcal{F}_t] \ . \tag{4.3}$$

The process (U_t) is a *$\mathcal{P}$-supermartingale*, i.e., a supermartingale simultaneously for all $P^* \in \mathcal{P}$. In fact it is the smallest non-negative $\mathcal{P}$-supermartingale with terminal value $\geq H$. As shown in full generality in Kramkov (1996) and in Föllmer and Kabanov (1998), such a $\mathcal{P}$-supermartingale admits an *optional decomposition* of the form

$$U_t = U_0 + \int_0^t \xi_s dX_s - C_t \tag{4.4}$$

where C is an increasing optional process, and where (U_0, ξ) is an admissible strategy. While the Doob-Meyer decomposition holds for a fixed probability measure and with a predictable increasing process, the optional decomposition is valid simultaneously for all measures $P^* \in \mathcal{P}$, and the increasing process is only optional.

The optional decomposition (4.4) of the process (U_t) can now be viewed as the following superhedging procedure: Put up the initial capital U_0, then follow the dynamic trading strategy ξ and withdraw the cumulative amount of capital C_t from the superhedging portfolio as one learns more and more about the development of the underlying price.

As a corollary of the optional decomposition, the value U_t can be characterized as the least amount of capital needed at time t to cover the claim H by following an admissible strategy ξ from time t up to time T, i.e.,

$$U_t = \text{ess.}\inf V_t \tag{4.5}$$

where V_t runs through the class of $\mathcal{F}_t$-measurable random variables ≥ 0 such that

$$V_t + \int_t^T \xi_s dX_s \geq H \quad P - a.s. \tag{4.6}$$

for some admissible strategy ξ. In other words, U_t is an upper bound for any arbitrage-free price of the claim computed at time t. If additional constraints are imposed on the strategies ξ in (4.6) then the dual description (4.3) of the process defined by (4.5) has a corresponding analogue in terms of a suitable extension of the class $\mathcal{P}$; see Karatzas (1997) and Föllmer and Kramkov (1997).

4.2 Quantile hedging and the extended Neyman-Pearson Lemma

Again we consider the question what an investor can do who is unwilling or unable to put up the high amount of initial capital U_0 required to stay on the safe side. In many incomplete models the cost of superhedging for a call option is given by $U_0 = X_0$, and so the superhedging strategy reduces to the trivial strategy of holding one unit of the underlying; see, e.g., Eberlein and Jacod (1997) and Frey (1997). Thus, our question is particularly relevant in such cases.

So let us fix a smaller amount $\tilde{V}_0 < U_0$. We can now ask for a strategy which maximizes the probability of a successful hedge under the constraint that the initial capital is not larger than $\tilde{V}_0$. In the extended version of the problem, we want to maximize the expected success ratio defined by (2.33). Thus, our aim is to find an admissible strategy $(\tilde{V}_0, \tilde{\xi})$ such that the corresponding success ratio $\tilde{\varphi}$ satisfies

$$E[\tilde{\varphi}] = \max\{E[\varphi] : (V_0, \xi) \text{ admissible, } V_0 \leq \tilde{V}_0\} \ . \tag{4.8}$$

This problem is easily reduced to the Neyman-Pearson Lemma for a composite hypothesis. As in section 2.3 we denote by $\mathcal{R}$ the class of all "*randomized tests*", i.e., all $\mathcal{F}_T$ - measurable functions φ such that $0 \leq \varphi \leq 1$ $P-$ a.s..

(4.9) **Theorem.** *There exists a function $\tilde{\varphi} \in \mathcal{R}$ such that*

$$E[\tilde{\varphi}] = \max_{\varphi \in \mathcal{R}} E[\varphi] \tag{4.10}$$

under the constraints

$$E^*[H\varphi] \leq \tilde{V}_0 \qquad \forall P^* \in \mathcal{P}. \tag{4.11}$$

The modified option $\tilde{H} := H\tilde{\varphi}$ may or may not be attainable. If it is attainable, then let $\tilde{\xi}$ denote the corresponding replicating strategy. If it is not attainable, let $\tilde{\xi}$ denote the superhedging strategy resulting from the optional decomposition of the $\mathcal{P}$-supermartingale

$$\tilde{U}_t = \text{ess.} \sup_{P^* \in \mathcal{P}} E^*[\tilde{H} \mid \mathcal{F}_t], \quad 0 \leq t \leq T. \tag{4.12}$$

In either case, $(\tilde{V}_0, \tilde{\xi})$ solves the optimization problem defined by (4.8).

Proof. 1) The existence of a solution $\tilde{\varphi} \in \mathcal{R}$ of the optimization problem defined by (4.10) and (4.11) follows from the Neyman-Pearson Lemma as explained below. Note

that we have $\tilde{\varphi} = 1$ on $\{H = 0\}$, P- almost surely (otherwise we could replace $\tilde{\varphi}$ by 1 on $\{H = 0\}$, thereby increasing the expectation in (4.10) without affecting the constraints in (4.11)).
2) Let (V_0, ξ) be any admissible strategy with $V_0 \leq \tilde{V}_0$. The resulting value process (V_t) is a $\mathcal{P}$-supermartingale. Since the success ratio φ satisfies $H\varphi = \min(H, V_T)$, we obtain

$$E^*[H\varphi] \leq E^*[V_T] \leq V_0 \qquad \forall P^* \in \mathcal{P} \ . \tag{4.13}$$

Thus φ satisfies the constraints in (4.11), and so we have

$$E[\varphi] \leq E[\tilde{\varphi}] \ . \tag{4.14}$$

3) Consider the admissible strategy $(\tilde{U}_0, \tilde{\xi})$ given by the optional decomposition (4.4) of the $\mathcal{P}$-supermartingale $(\tilde{U}_t)$ associated to the modified option $\tilde{H} = H\tilde{\varphi}$. Note that $\tilde{U}_0 = \tilde{V}_0$ since the optimal test $\tilde{\varphi}$ attains the bound $\tilde{V}_0$ in (4.11). The resulting value process $(\tilde{V}_t)$ defined by (2.1) satisfies

$$\tilde{V}_T \geq \tilde{H} = H\tilde{\varphi} \ . \tag{4.15}$$

Let $\tilde{\psi}$ denote the success ratio corresponding to the admissible strategy $(\tilde{V}_0, \tilde{\xi})$. We have $E[\tilde{\psi}] \leq E[\tilde{\varphi}]$ due to part 2). On the other hand, (4.15) implies $\tilde{\psi} \geq \tilde{\varphi}$ P-almost surely, and so we see that $\tilde{\varphi}$ is the success ratio associated to $(\tilde{V}_0, \tilde{\xi})$. Due to (4.14), we have shown that the strategy $(\tilde{V}_0, \tilde{\xi})$ solves the optimization problem (4.8).

We can express condition (4.11) in a more familiar form by introducing the family of probability measures $\{Q^* \mid P^* \in \mathcal{P}\}$ where Q^* is associated to P^* via the density

$$\frac{dQ^*}{dP^*} = \frac{H}{E^*[H]} \ . \tag{4.16}$$

Then the constraints take the form

$$\int \varphi dQ^* \leq \alpha(P^*) := \frac{\tilde{V}_0}{E^*[H]} \qquad \forall P^* \in \mathcal{P} \ . \tag{4.17}$$

Thus we are faced with the problem of testing the compound hypothesis $\{Q^* \mid P^* \in \mathcal{P}\}$ against the simple alternative P. The critical levels are given by α, viewed as a bounded measurable function on the parameter set $\mathcal{P}$ endowed with the $\sigma-$field generated by the integrals $\int f dP^*$ for bounded measurable functions f on $(\Omega, \mathcal{F}_T)$. The *existence* of an optimal test $\tilde{\varphi}$ now follows from the standard theory; cf., e.g., Witting (1985).

In addition to the basic existence result, the Neyman-Pearson theory also shows that optimal tests typically have a 0-1 structure. Consider a test $\varphi \in \mathcal{R}$ which satisfies the constraint (4.17). In our special context, the class $\{Q^* \mid P^* \in \mathcal{P}\}$ is *measure convex*, i.e., any mixture of measures in this class by some probability distribution on $\mathcal{P}$ belongs again to the class. Applying corollary 2.83 in Witting (1985), we see that the following form of $\tilde{\varphi}$ is sufficient (and often necessary) for optimality:

(4.18) *There exists a measure* $\tilde{P} \in \mathcal{P}$ *such that the following two conditions are satisfied:*

$$\tilde{\varphi} = \begin{cases} 1 & \text{if } \frac{dP}{d\tilde{P}} > \lambda H \\ 0 & \text{if } \frac{dP}{d\tilde{P}} < \lambda H \end{cases} \tag{4.19}$$

for some constant λ, *and*

$$\int \tilde{\varphi} H d\tilde{P} = \tilde{V}_0 \ . \tag{4.20}$$

In the context of section 5 below such a "worst case" martingale measure $\tilde{P}$ will be constructed explicitly.

4.3 Quantile Hedging for a given shortfall probability

Consider a given shortfall probability $\varepsilon \in (0,1)$. We are looking for the least amount

$$\inf \left\{ V_0 \mid \ \exists\, \xi \text{ admissible} : P[V_0 + \int_0^T \xi_s dX_s \geq H] \geq 1-\varepsilon \right\} \ . \tag{4.21}$$

of initial capital which allows us to be on the safe side with probability $1-\varepsilon$. As in the complete case, we pass from tests to randomized tests, and rephrase the problem in terms of success ratios rather than success sets. Thus we want to determine the least amount

$$\inf \{V_0 \mid \exists\, \xi \ : \ (V_0, \xi) \text{ admissible}, \ E[\varphi] \geq 1-\varepsilon\} \ , \tag{4.22}$$

where φ denotes the success ratio associated to the admissible strategy (V_0, ξ).

Again the problem can be reduced to finding a random variable $\varphi \in \mathcal{R}$ such that

$$\sup_{P^* \in \mathcal{P}} E^* [H\varphi] = \min \tag{4.23}$$

under the constraint

$$E[\varphi] \geq 1-\varepsilon. \tag{4.24}$$

As in the theory of optimal tests, weak compactness of $\mathcal{R}$ guarantees the *existence* of a random variable $\tilde{\varphi} \in \mathcal{R}$ which solves the optimization problem defined by (4.23) and (4.24)

(4.25) **Remark.** The set $\mathcal{P}$ of equivalent martingale measures is convex. For a given contingent claim the expectation $E^*[H]$ is linear in P^*. Thus the set $\{E^*[H] : P^* \in \mathcal{P}\}$ of arbitrage-free prices for H is an interval. For a given shortfall probability ε the above construction yields a strategy ξ^ε such that the investor has an expected sucess ratio of $(1-\varepsilon)100\%$. Typically - if φ^ε is not too small - we will have

$$\inf\{E^*[H] : P^* \in \mathcal{P}\} < \sup\{E^*[H\varphi^\varepsilon] : P^* \in \mathcal{P}\} < \sup\{E^*[H] : P^* \in \mathcal{P}\} \ , \tag{4.26}$$

i.e., the capital required to reach an expected sucess ratio $\geq 1-\varepsilon$ lies within the arbitrage-free interval. Thus there exists a particular $P^\varepsilon \in \mathcal{P}$ such that

$$\sup\{E^*[H\varphi^\varepsilon] : P^* \in \mathcal{P}\} = E_{P^\varepsilon}[H] \ . \tag{4.27}$$

Furthermore we know from the theory of tests that $\sup\{E^*[H\varphi^\varepsilon] : P^* \in \mathcal{P}\}$ is convex and continuous in $\varepsilon \in (0,1)$; cf. e.g. Ingster (1992, p.93). This suggests the following interpretation: For a given contingent claim H the seller is a priori faced with the task of choosing one equivalent martingale measure in order to price H in an arbitrage-free manner. Instead of choosing an element of $\mathcal{P}$ (which may have a rather complicated structure), the above approach suggests that the seller may simply choose a shortfall probability ε corresponding to the risk he or she is willing to bear.

5. Quantile hedging of a volatility jump

Consider a geometric Brownian motion with drift 0 where the volatility has a constant value $\sigma > 0$ up to time t_0 and then jumps to a new constant value η according to some distribution μ on $(0,\infty)$.

We use an explicit model $(\bar\Omega, \bar{\mathcal{F}}, \bar P)$ of the following form. Put $\bar\Omega = C[0,T] \times (0,\infty)$, and for $\bar\omega = (\omega,\eta)$ define $X_t(\omega) = \bar X_t(\bar\omega) = \omega(t)$. We fix a time $t_0 \in (0,T)$ and an initial value $x_0 > 0$. For each value $\eta > 0$ we define a time-dependent volatility by $\sigma_t(\eta) = \sigma$ for $t < t_0$ and $\sigma_t(\eta) = \eta$ for $t \geq t_0$. Let P^η denote the unique probability measure on $\Omega = C[0,T]$ such that the process (X_t) satisfies the stochastic differential equation

$$dX_t = X_t \sigma_t(\eta) dW_t^\eta \ , \ X_0 = x_0 \tag{5.1}$$

under P^η, where (W_t^η) is a Wiener process under P^η. The measure $\bar P$ on $\bar\Omega$ is defined by $\bar P(d\omega, d\eta) = \mu(d\eta) P^\eta(d\omega)$. We denote by $\bar{\mathcal{F}}$ the completion of the natural product σ-field on $\bar\Omega$ under $\bar P$, and by $(\bar{\mathcal{F}}_t)_{0\leq t\leq T}$ the right-continuous complete filtration on $\bar\Omega$ generated by the processes $(\bar X_t)$ and (σ_t). The projection of $\bar P$ on Ω is denoted by P, and $(\mathcal{F}_t)$ is the right-continuous complete filtration on Ω generated by (X_t).

Consider the European call option with strike price K and exercise time T, viewed both as a random variable $\bar H = h(\bar X_T)$ on $\bar\Omega$ and as a random variable $H = h(X_T)$ on Ω, with $h(x) = (x-K)^+$. At time t_0, the value $X_{t_0} = x$ is observed and the new volatility η is revealed. From this time on, the option can be replicated perfectly using the standard Black–Scholes hedging strategy in the complete model P^η. The required cost is given by

$$v^\eta(x) = E^\eta\left[h(X_T) | X_{t_0} = x\right] \leq x \ . \tag{5.2}$$

For $t < t_0$ the value η is still unknown. The cost of *superhedging* is given by

$$U_t = \operatorname*{ess\,sup}_{\bar{P}^* \in \bar{\mathcal{P}}} \bar{E}^* \left[h(\bar{X}_T) | \bar{\mathcal{F}}_t \right] , \tag{5.3}$$

where $\bar{\mathcal{P}}$ is the class of all equivalent martingale measures $\bar{P}^* \approx \bar{P}$. All measures $\bar{P}^* \in \bar{\mathcal{P}}$ have the same projection P on $(\Omega, \mathcal{F}_{t_0})$ and the same conditional expectation

$$\bar{E}^* \left[h(\bar{X}_T) | \bar{\mathcal{F}}_{t_0} \right] (\omega, \eta) = v^\eta \left(X_{t_0}(\omega) \right) \tag{5.4}$$

with respect to $\bar{\mathcal{F}}_{t_0}$. This implies that U_t, viewed as a random variable on Ω for $t < t_0$, is given by

$$U_t = \operatorname*{ess\,sup}_{\eta} E \left[v^\eta (X_{t_0}) | \mathcal{F}_t \right] \tag{5.5}$$

where the essential supremum is taken with respect to μ. If μ has unbounded support then we get $U_t = X_t$ for $t < t_0$, and in particular

$$U_0 = x_0 \ ; \tag{5.6}$$

cf., e.g., Frey and Sin (1997). In this case, the superhedging strategy is reduced to the following simple procedure: Buy one unit of the underlying asset at time 0 and hold it up to time t_0. At that time the value η is revealed. Pay out the refund $C_{t_0} = X_{t_0} - v^\eta(X_{t_0})$, and use the remaining capital $v^\eta(X_{t_0})$ to implement a perfect hedge of the option.

Let us now turn to the problem of *quantile hedging*. Thus, we want to maximize the probability of a successful hedge under the constraint that the initial cost is not larger than some fixed amount $\tilde{V}_0$ such that

$$0 < \tilde{V}_0 < U_0 \leq x_0 \ . \tag{5.7}$$

At time t_0, let $\alpha^\eta(x, y)$ denote the maximal probability of achieving a successful hedge, given the present state $x = X_{t_0}$ and some capital $y \geq 0$. We know from section 3 that

$$\alpha^\eta(x, y) = F_x^\eta \left(c^\eta(x, y) \right) \tag{5.8}$$

where F_x^η resp. f_x^η denote the distribution function and the density function of X_T, given $X_{t_0} = x$ and the volatility η, and where $c^\eta(x, y)$ is the solution of the equation

$$y = \int_K^c (z - K) f_x^\eta(z) dz \tag{5.9}$$

($:= +\infty$ for $y \geq v^\eta(x)$). Let us define

$$\alpha(x, y) := \int \alpha^\eta(x, y) \mu(d\eta) \tag{5.10}$$

and

$$u(x) := \operatorname{ess\,sup}_{\eta} v^\eta(x) \le x \; . \tag{5.11}$$

(5.12) **Lemma.** *The function $\alpha(x,\cdot)$ belongs to $C[0,\infty) \cap C^1(0,\infty)$, is strictly increasing to 1 on $[0,u(x)]$, concave on $[0,\infty)$ and strictly concave on $[0,u(x)]$, and satisfies $\frac{\partial}{\partial y}\alpha(x,0+) \equiv +\infty$.*

Proof. The properties of $\alpha^\eta(x,\cdot)$ established in section 3 imply $\alpha(x,\cdot) \in C[0,\infty) \cap C^1(0,\infty)$ and

$$\frac{\partial}{\partial y}\alpha(x,y) = \int \frac{1}{c^\eta(x,y) - K}\mu(d\eta) \; . \tag{5.13}$$

This decreases in y, and the decrease is strict on $\{y|\alpha(x,y) < 1\} = [0,u(x))$.

(5.14) **Proposition.** *Let ν denote the distribution of X_{t_0} under P. There exists a unique function $v \in C^1(0,\infty)$ such that*

$$\int \alpha\big(x,v(x)\big)\nu(dx) = \sup_f \int \alpha\big(x,f(x)\big)\nu(dx) \tag{5.15}$$

where the supremum is taken over all measurable functions $f \ge 0$ on $(0,\infty)$ with $\int f d\nu \le \tilde{V}_0$. We have

$$0 < v(x) < u(x) \; , \quad \int v d\nu = \tilde{V}_0 \; , \tag{5.16}$$

and v is the solution of the equation

$$\frac{\partial}{\partial y}\alpha\big(x,v(x)\big) = c \tag{5.17}$$

for some constant $c \in (0,\infty)$.

Proof. 1) It is easy to see that a function which maximizes the integral in (5.15) must belong to the class C of all measurable functions $f \ge 0$ such that $\int f d\nu = \tilde{V}_0$ and $0 \le f \le u$ ν-a.s. Since $\int u d\nu \le \int x d\nu < \infty$, the class C is convex and weakly compact in $\mathcal{L}^1(\nu)$. Existence and uniqueness of an optimal $v \in C$ now follow by standard arguments. As to existence, we can for instance argue as follows. Take a sequence (f_n) such that the integrals on the right hand side of (5.15) converge to the supremum. Using Lemma A.1.1 in Delbaen/Schachermayer (1994), we can choose functions $v_n \in \operatorname{conv}(f_n, f_{n+1}, \ldots) \subseteq C$ such that v_n converges a.s. to some $v \in C$. By Lebesgue's theorem, the function v must satisfy (5.15).

2) In order to clarify the structure of v, consider any $f \in C$ such that $f \leq c \cdot v$ for some $c > 1$. For any $\lambda \in \left[-\frac{1}{c-1}, 1\right]$, the function

$$f_\lambda := v(1-\lambda) + \lambda f$$

satisfies the constraints in our optimization problem (5.15), i.e., we have $\int f_\lambda d\nu = \tilde{V}_0$ and $f_\lambda \geq 0$. Thus, the concave function F defined by

$$F(\lambda) := \int \alpha\big(x, f_\lambda(x)\big)\nu(dx)$$

on the interval $I = \left[-\frac{1}{c-1}, 1\right]$ assumes its maximum in $\lambda = 0$. In particular we have

$$\frac{F(\lambda) - F(0)}{\lambda} = \int \frac{\alpha\big(x, f_\lambda(x)\big) - \alpha\big(x, v(x)\big)}{\lambda}\,\nu(dx) \;\leq 0 \tag{5.18}$$

for $\lambda \in (0, 1]$. For any $\lambda \in I - \{0\}$, the integrand vanishes on $\{f = v\}$, and on $\{f \neq v\} \subseteq \{v > 0\}$ it is bounded in absolute value by

$$\frac{\alpha\big(x, v(x)\big)}{v(x)}|f(x) - v(x)| \leq c\;; \tag{5.19}$$

here we use the relation $f_\lambda(x) - v(x) = \lambda(f(x) - v(x))$, the fact that $\alpha(x, \cdot)$ is concave on $[0, \infty)$ with values in $[0, 1]$, and the estimate $|f(x) - v(x)| \leq cv(x)$. Using Lebesgue's theorem we obtain differentiability of F in $\lambda = 0$ and the equation

$$0 = F'(0) = \int \frac{\partial}{\partial y}\alpha\big(x, v(x)\big)\big(f(x) - v(x)\big)\nu(dx). \tag{5.20}$$

Since (5.20) holds for all $f \geq 0$ such that $\int f d\nu = \tilde{V}_0$ and $f \leq c \cdot v$ for some $c > 1$, we can conclude that $\frac{\partial}{\partial y}\alpha\big(x, v(x)\big)$ must be constant ν-a.s. on $\{v > 0\}$, hence ν-a.s. due to part 3) below. This constant must be strictly positive, because otherwise we would get $v(x) = u(x)$ ν-a.s., hence $\int v\, d\nu = U_0 > \tilde{V}_0$. Thus, $v(x) < u(x)$ ν-a.s.

3) Let us now take any $f \in C$. In this case we have $f_\lambda \in C$ for any $\lambda \in [0, 1]$, and so the estimate (5.18) holds for any $\lambda \in (0, 1]$. For $\lambda \searrow 0$ we can apply monotone convergence separately on the sets $\{v < f\}$ and $\{v > f\}$. Using the bound in (5.18) we can conclude that the derivative $F'(0+)$ from the right exists and satisfies

$$0 \geq F'(0+) = \int \frac{\partial}{\partial y}\alpha\big(x, v(x)\big)\big(f(x) - v(x)\big)\nu(dx)\;, \tag{5.21}$$

and that the integrand belongs to $L^1(\nu)$. Taking $f > 0$ we see that the solution v of our optimization problem must satisfy $v(x) > 0$ ν-a.s. since $\frac{\partial}{\partial y}\alpha\big(x, 0+\big) = +\infty$.

(5.22) **Theorem.** *The probability of a successful hedge is maximized by the following strategy:*

i) Up to time t_0, use the strategy which replicates the contingent claim $v(X_{t_0})$, where v solves the optimization problem in (5.15)*.*

ii) From time t_0 on, use the strategy which maximizes the probability of a successful hedge under the new volatility η, given the initial capital $v(X_{t_0})$ (see section 3).

Proof. Consider any admissible strategy (V_0, ξ) with initial cost $V_0 \leq \tilde{V}_0$. The resulting value

$$V_t = V_0 + \int_0^t \xi_s dX_s \tag{5.23}$$

will be viewed as a random variable on $(\Omega, \mathcal{F}_{t_0})$ for any $t \leq t_0$. We have

$$E[V_{t_0}] \leq \tilde{V}_0 \leq U_0 \ , \tag{5.24}$$

and the strategy will achieve a successful hedge with conditional probability

$$\bar{P}[V_T \geq \bar{H} | \bar{\mathcal{F}}_{t_0}](\omega, \eta) \leq \alpha^{\eta}\big(X_{t_0}(\omega), V_{t_0}(\omega)\big) \ . \tag{5.25}$$

This implies

$$\bar{P}[V_T \geq \bar{H}] \leq E\big[\alpha(X_{t_0}, V_{t_0})\big] \ . \tag{5.26}$$

But $\alpha(x, \cdot)$ is concave, and so we get

$$\bar{P}[V_T \geq \bar{H}] \leq E\Big[\alpha\big(X_{t_0}, f(X_{t_0})\big)\Big] \tag{5.27}$$

by Jensen's inequality for conditional expectations, if f denotes a measurable function such that

$$f(X_{t_0}) = E[V_{t_0} | X_{t_0}] \quad P - a.s. \tag{5.28}$$

Since

$$E\big[f(X_{t_0})\big] = E[V_{t_0}] \leq \tilde{V}_0 \ , \tag{5.29}$$

we see that

$$E\Big[\alpha\big(X_{t_0}, f(X_{t_0})\big)\Big] \leq E\Big[\alpha\big(X_{t_0}, v(X_{t_0})\big)\Big] \tag{5.30}$$

due to proposition (5.14). Thus, the right hand side is an upper bound for the probability of a successful hedge under the constraint that the initial cost is bounded by $\tilde{V}_0$. But this upper bound is actually achieved if we use the strategy described in the theorem.

Let us now look at the structure of a "worst case" martingale measure $\tilde{P}$ as it appears in (4.18). The results in section 3 show that, in our case, the optimal $\tilde{\varphi}$ is of the form $\tilde{\varphi} = I_{\tilde{A}}$ where the success set $\tilde{A} \in \bar{\mathcal{F}}_T$ is given by

$$\tilde{A} = \left\{ X_T \le c^\eta\big(X_{t_0}, v(X_{t_0})\big) \right\} . \tag{5.31}$$

(5.32) **Theorem.** *There exists a measure $\tilde{P} \in \bar{\mathcal{P}}$ such that*

$$\tilde{E}[HI_{\tilde{A}}] = \tilde{V}_0 \tag{5.33}$$

and

$$\tilde{A} = \left\{ \frac{d\bar{P}}{d\tilde{P}} \ge \text{const}\ \bar{H} \right\} , \tag{5.34}$$

i.e., $\tilde{A}$ is the critical region for the optimal test of the hypothesis $\tilde{Q}$ defined by $d\tilde{Q}/d\tilde{P} =$ const $\cdot\, H$ against the alternative P.

Proof. Note first that, in our case, the constraint (5.33) is satisfied for *any* $\bar{P}^* \in \bar{\mathcal{P}}$ since

$$\bar{E}^*[\bar{H}I_{\tilde{A}}] = \bar{E}^*\big[E^\eta_{X_{t_0}}[HI_{\tilde{A}(\eta,\cdot)}]\big] = E\big[v(X_{t_0})\big] = \tilde{V}_0 \tag{5.35}$$

due to (5.4). Let us now define a strictly positive density $\tilde{\varphi}$ by

$$\tilde{\varphi}(\omega,\eta) = \frac{1}{c}\frac{\partial}{\partial y}\alpha^\eta\big(X_{t_0}(\omega), v\big(X_{t_0}(\omega)\big)\big) . \tag{5.36}$$

We have

$$\int \tilde{\varphi}(\omega,\eta)\mu(d\eta) \equiv 1 \tag{5.37}$$

due to (5.17). This implies that the measure $\tilde{P}$ defined by $\tilde{\varphi}$ has projection P on $(\Omega, \mathcal{F}_{t_0})$ and conditional expectation (5.4) with respect to $\bar{\mathcal{F}}_{t_0}$. Thus, we have $\tilde{P} \in \bar{\mathcal{P}}^*$. Now recall that

$$\frac{\partial}{\partial y}\alpha^\eta\big(X_{t_0}, v(X_{t_0})\big) = \frac{1}{c^\eta\big(X_{t_0}, v(X_{t_0})\big) - K} . \tag{5.38}$$

This implies (5.34) since

$$\left\{ \frac{d\bar{P}}{d\tilde{P}} \ge c \cdot \bar{H} \right\} = \left\{ c\big(c^\eta(X_{t_0}, v(X_{t_0})) - K\big) \ge c \cdot \bar{H} \right\} = \tilde{A} . \tag{5.39}$$

References

Browne, S.: Reaching Goals by a Deadline: Digital Options and Continuous-Time Active Portfolio Management. *Preprint Graduate School of Business, Columbia University* (1997)

Cvitanic, J., Spivak, G.: Maximizing the probability of a perfect hedge. *Preprint Department of Statistics, Columbia University* (1998)

Delbaen, F., Schachermayer, W.: A general version of the fundamental theorem of asset pricing. *Math. Annalen* **300**, 463 - 520 (1994)

Eberlein, E., Jacod, J.: On the range of option prices. *Finance and Stochastics* **1**, 131-140 (1997)

El Karoui, N., Quenez, M.C.: Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM J. Control and Optimization*, **33 (1)**, 29-66 (1995)

Föllmer, H., Kabanov, Yu,M..: Optional decomposition and Lagrange multipliers. *Finance and Stochastics* **2**, 69-81 (1998)

Föllmer, H., Kramkov, D.: Optional decompositions under constraints. *Prob. Theory Relat. Fields* **109**, 1-25 (1997)

Föllmer, H., Leukert, P.: Partial hedging with minimal shortfall risk. *Preprint, Humboldt-Universität Berlin* (1998)

Frey, R., Sin, C.A.: Bounds on European Option Prices under Stochastic Volatility. *Discussion paper SFB 303 Bonn* **405**, (1997)

Heath, D.: A continuous-time version of Kulldorff's result. *Working paper* (1995)

Ingster, Yu.: Asymptotically minimax hypothesis testing for nonparametric alternatives I. *Mathematical Methods of Statistic*, **2 (2)**, 85-114 (1993)

Jacka, S.D.: A martingale representation result and an application to incomplete financial markets. *Math. Finance* **2**, no. 4, 239-250 (1992)

Karatzas, I.: Lectures in Mathematical Finance. Providence: American Mathematical Society 1997

Karlin, S.: Mathematical Methods and Theory in Games, Programming and Economics, vol. 2. Reading, Massachusetts: Addison-Wesley 1959

Kramkov, D.O.: Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets. *Probability Theory and Related Fields*, **105**, 459-479 (1996)

Kulldorff, M.: Optimal control of favorable games with a time-limit. *SIAM J. Control and Optimization*, **31**, 52-69 (1993)

Schwarz, O.: Absicherungsstrategien mit maximaler Erfolgswahrscheinlichkeit für Optionen bei vorgegebenem Anfangskapital. *Diplomarbeit, Humboldt-Universität Berlin* (1996)

Shreve, S., Wystup, U.: Hedging in-the-money knock-out options. *Forthcoming* (1998)

Witting, H.: Mathematische Statistik I. Stuttgart: Teubner 1985