On weak Brownian motions of arbitrary order[1]

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Abstract

We show the existence, for any $k \in \mathbb{N}$, of processes which have the same k-marginals as Brownian motion, although they are not Brownian motions. For $k = 4$, this proves a conjecture of Stoyanov. The law $\tilde{\mathbb{P}}$ of such a "weak Brownian motion of order k" can be constructed to be equivalent to Wiener measure $\mathbb{P}$ on $C[0, 1]$. On the other hand, there are weak Brownian motions of arbitrary order whose law is singular to Wiener measure. We also show that, for any $\varepsilon > 0$, there are weak Brownian motions whose law coincides with Wiener measure outside of any interval of length ε.

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1 Introduction

Our main aim in this paper is to construct families, indexed by $k \in \mathbb{N}$, of stochastic processes $(X_t)_{t\geq 0}$ which look more and more like Brownian motions, as k increases, although they are not Brownian motions.

Theorem 1.1. *Let $k \in \mathbb{N}$. There exists a process $(X_t)_{0\leq t\leq 1}$ which is not Brownian motion such that the k-dimensional marginals of X are identical to those of Brownian motion.*

In particular, we solve Stoyanov's conjecture which corresponds to the case $k = 4$:
There exists a process X with $X_0 = 0$ satisfying the following two conditions:
(i) $X_t - X_s \sim N(0, t-s)$, *for all $s < t$.*
(ii) *Any two increments $X_{t_2} - X_{t_1}$ and $X_{t_4} - X_{t_3}$ are independent, for $0 \leq t_1 < t_2 \leq t_3 < t_4$.*
But X is not a Brownian motion; see, Stoyanov ([19], p.292); also ([20], p.316).

A process $(X_t)_{0\leq t\leq 1}$ whose k-dimensional marginals coincide with those of Brownian motion will be called a *weak Brownian motion of order k*. For any $k \geq 1$, we are going to show that there is a weak Brownian motion of order k whose law is equivalent to, but differs from, Wiener measure. This amounts to the existence of a random variable $\Phi > 0$ on $C[0,1]$, such that for every $t_1, \cdots, t_k \leq 1$,

$$E_{\mathbb{P}}[\Phi \,|X_{t_1}, \cdots, X_{t_k}] = 1, \qquad \Phi \not\equiv 1 \tag{1}$$

with respect to Wiener measure $\mathbb{P}$. In order to obtain such Φ, it suffices to construct Ψ, uniformly bounded (by $1/2$, say), such that

$$E_{\mathbb{P}}[\Psi \,|X_{t_1}, \cdots, X_{t_k}] = 0, \qquad \Psi \not\equiv 0 \tag{2}$$

for every $t_1, \cdots, t_k \leq 1$. That (2) implies (1) is obvious: take $\Phi = 1 + \Psi$. As a consequence, densities Φ generating a weak Brownian motion of order k can be chosen arbitrarily uniformly close to 1.

In Section 3 we characterize functionals $\Psi \in L^2(\mathbb{P})$ with vanishing projections of order k (i.e., satisfying (2)) in terms of the integrands appearing in the representation of Ψ as a stochastic integral of Brownian motion. In Section 4, this characterization will be used in order to construct such Ψ's in $L^\infty(\mathbb{P})$, hence weak Brownian motions of any order k whose law is equivalent to Wiener measure. But we will also show that there are weak Brownian motions of arbitrary order whose law is singular to Wiener measure.

In Section 5 we construct weak Brownian motions with an even stronger property. We show that, for any $\varepsilon \in (0,1)$, there exists a probability measure $\tilde{\mathbb{P}}$ on $C[0,1]$ which is different from Wiener measure $\mathbb{P}$ but coincides with $\mathbb{P}$ outside of any interval of length ε. In particular, $\tilde{\mathbb{P}}$ defines a weak Brownian motion of order k for any $k \leq \varepsilon^{-1} - 1$, but the resemblance to Brownian motion goes much further. Actually, this construction is valid in a very general context, where the reference measure $\mathbb{P}$ is the law of a non-degenerate Markov process.

If X is a weak Brownian motion of order $k \geq 4$, then X admits a continuous version whose paths have quadratic variation

$$\langle X \rangle_t = t; \tag{3}$$

see Proposition 2.1. This property allows us to apply Itô calculus in a strictly pathwise manner (Föllmer [4]) even though X may not be a semimartingale. In particular, the Itô integral

$$\int_0^t f(X_s) dX_s$$

exists as a pathwise limit of non-anticipating Riemann sums along dyadic partitions for any bounded $f \in C^1$ and satisfies Itô's formula. Given the existence of the quadratic variation in (3), a weak Brownian motion of any order $k \geq 1$ satisfies

$$E\left[\int_0^t f(X_s) dX_s\right] = 0 \tag{4}$$

for any bounded $f \in C^1$, since Itô's formula allows us to compute the left hand side of (4) from the 1-dimensional marginals of X. Property (4) may be viewed as a weak form of the martingale property. In Section 7 we introduce the corresponding notion of a weak martingale. We show that, in the class of continuous semimartingales which satisfy condition (3), weak martingales can be characterized as weak Brownian motions of order 1.

In Section 6 we consider Gaussian semimartingales of the form

$$X_t = W_t - \int_0^t \int_0^u l(u,v) dW_v du,$$

where (W_t) is a Brownian motion and l is a continuous Volterra kernel. We formulate criteria in terms of l for X to be either a Brownian motion or a weak Brownian motion of order 1. In particular, we show that there are Gaussian semimartingales other than Brownian motion which have quadratic variation $\langle X\rangle_t = t$ and are weak Brownian motions of order 1.

2 General properties of weak Brownian motions

We first give a definition of weak Brownian motions of order k.

Definition 2.1. Let $k \in \mathbb{N}$, and let $X = (X_t)_{0\le t\le 1}$ be a real-valued stochastic process. We shall say that $(X_t)_{0\le t\le 1}$ is a *weak Brownian motion of order* k if for every k-tuple $(t_1, t_2, \cdots, t_k)$,

$$(X_{t_1}, X_{t_2}, \cdots, X_{t_k}) \stackrel{(law)}{=} (B_{t_1}, B_{t_2}, \cdots, B_{t_k}), \tag{5}$$

where $(B_t)_{0\le t\le 1}$ is a Brownian motion.

Using this terminology, we can write Stoyanov's conjecture as: *There exist weak Brownian motions of order* 4 *which are not Brownian motions.* Theorem 1.1 shows that this conjecture is true, and that it is also valid for any order k.

Proposition 2.1. *Let X be a weak Brownian motion of order k.*
1) *If $k \ge 2$, then X admits a continuous version.*
2) *If $k \ge 4$, then X has quadratic variation*

$$\langle X\rangle_t = t. \tag{6}$$

Proof. 1) The existence of a continuous version follows from Kolmogorov's criterion.
2) Under our hypothesis, we have

$$E\left[\left(\sum_{s_i\in\tau, s_i\le t} \left(X_{s_{i+1}} - X_{s_i}\right)^2 - t\right)^2\right] = E\left[\left(\sum_{s_i\in\tau, s_i\le t} \left(B_{s_{i+1}} - B_{s_i}\right)^2 - t\right)^2\right], \tag{7}$$

for any finite partition τ of $[0,1]$. These expectations converge to 0 if we consider any sequence of partitions (τ_n) such that $\sup_{s_i\in\tau_n}(s_{i+1} - s_i)$ goes to 0. Therefore, the quadratic variation $\langle X\rangle_t$ exists as a limit in $\mathcal{L}^2$ and satisfies $\langle X\rangle_t = t$. Moreover, if we choose the sequence (τ_n) of dyadic partitions, then the series (indexed by n) of the expectations in (7) converges, and so we get

$$\lim_n \sum_{i:s_i\in\tau_n, s_i\le t} (X_{s_{i+1}} - X_{s_i})^2 = t$$

a.s.. □

Remark 2.1. Suppose that $(X_t)_{0\le t\le 1}$ is a continuous weak Brownian motion of order $k\ge 1$, and denote by $\tilde{\mathbb{P}}$ its law on $C[0,1]$. Let us assume that $\tilde{\mathbb{P}}$ is concentrated on the set of continuous paths which have quadratic variation $\langle X\rangle_t = t$ along the sequence of dyadic partitions; the preceding proposition shows that this assumption is satisfied if $k\ge 4$. Under this assumption we can apply Itô calculus in a strictly pathwise manner; see Föllmer [4]. Denoting by (X_t) the coordinate process on $C[0,1]$, we obtain

$$\int_0^t f(X_s)dX_s = F(X_t) - F(X_0) - \frac{1}{2}\int_0^t f'(X_s)ds,$$

for any bounded function $f\in C^1(\mathbb{R}^1)$, where F satisfies $F'=f$. By Fubini's theorem we see that

$$\tilde{E}\left[\int_0^t f(X_s)dX_s\right] = \tilde{E}\left[F(X_t)\right] - \tilde{E}\left[F(X_0)\right] - \frac{1}{2}\int_0^t \tilde{E}\left[f'(X_s)\right]ds$$

only depends on the one-dimensional marginals of X. Thus, any weak Brownian motion of order $k\ge 1$ satisfies

$$\tilde{E}\left[\int_0^t f(X_s)dX_s\right] = 0 \tag{8}$$

under the additional assumption that (6) is satisfied. This implies that X is a weak martingale in the sense of Definition 7.1 below; see also Carmona-Petit-Yor [1] and Petit-Yor [15] for analogous "weak" notions.

Remark 2.2. A continuous weak Brownian motion may have a non-zero quadratic variation without being a semimartingale. Here is an example in the case $k=1$. Let $(B_t)_{0\le t\le 1}$ be a Brownian motion. The process X defined by

$$X_t = \begin{cases} B_t, & t\le \frac{1}{2}, \\ B_{\frac{1}{2}} + (\sqrt{2}-1)B_{t-\frac{1}{2}}, & t>\frac{1}{2}, \end{cases}$$

is a continuous weak Brownian motion of order 1 and satisfies

$$\begin{aligned} d\langle X\rangle_t &= dt, & t\le \tfrac{1}{2}, \\ d\langle X\rangle_t &= (\sqrt{2}-1)^2 dt, & t>\tfrac{1}{2}. \end{aligned}$$

But X is not a semimartingale.

Remark 2.3. For $k = 1$, condition (6) is not automatically deduced from (5). Indeed, the process with bounded variation

$$X_t = \sqrt{t}N,$$

where $N \sim N(0,1)$, satisfies (5) but not (6). We may even find Gaussian continuous semimartingales (X_t), with a non-trivial martingale part, which satisfy (5) for $k = 1$, but not (6). For example, the process $X = (X_t)_{0 \le t \le \pi/4}$ given by

$$X_t = M_t + \int_0^t M_u du,$$

where $(M_t)_{0 \le t \le \pi/4}$ is a Gaussian martingale with quadratic variation $e^{-t}\sin t$, i.e., $M_t = B_{e^{-t}\sin t}$ with a Brownian motion B. Clearly, $X_t \sim N(0,t)$ for all $t \in [0, \pi/4]$, but

$$\langle X \rangle_t = \langle M \rangle_t = e^{-t}\sin t.$$

Let us now assume that X is a continuous semimartingale with quadratic variation $\langle X \rangle_t = t$ and with absolutely continuous drift term. Thus, X takes the form

$$X_t = B_t + \int_0^t v_s \, ds, \tag{9}$$

where B is a Brownian motion and (v_t) is a previsible process satisfying

$$\int_0^t E\left[|v_u|\right] du < \infty$$

for all $t > 0$. The next proposition provides a characterization of weak Brownian motions of such type.

Proposition 2.2. *A process X of the form* (9) *is a weak Brownian motion of order k if and only if for every $t_1 \le t_2 \le \cdots \le t_{k-1}$, then dt-almost surely, for $t \ge t_{k-1}$,*

$$E[v_t | X_{t_1}, X_{t_2}, \cdots, X_{t_{k-1}}, X_t] = 0. \tag{10}$$

Proof. 1) Suppose that X given by (9) is a weak Brownian motion of order k. Then, for every bounded Borel function $\varphi : \mathbb{R}^{k-1} \to \mathbb{R}$, and every bounded function $f \in C^1(\mathbb{R})$, X satisfies

$$E\left[\varphi(X_{t_1}, X_{t_2}, \cdots, X_{t_{k-1}}) \int_{t_{k-1}}^{t_k} f(X_s) dX_s\right] = 0, \tag{11}$$

since the above stochastic integral is identical to

$$F(X_{t_k}) - F(X_{t_{k-1}}) - \frac{1}{2}\int_{t_{k-1}}^{t_k} f'(X_s)ds,$$

where $F(x) = \int_0^x f(y)dy$. Hence, from our hypothesis, the left-hand side of (11) is equal to the same quantity for Brownian motion, hence is equal to 0. (10) now follows easily.

2) Conversely, assuming that (10) holds, we proceed by iteration with respect to $j \leq k$, in order to show finally that the joint characteristic function

$$\phi_k(\lambda_1, \cdots, \lambda_k; t_1, \cdots, t_k) := E\left[\exp\left(i\sum_{j=1}^{k} \lambda_j (X_{t_j} - X_{t_{j-1}})\right)\right]$$

($\lambda_j \in \mathbb{R}$, $t_1 \leq t_2 \leq \cdots \leq t_{k-1} \leq t_k$) equals

$$\exp\left(-\frac{1}{2}\sum_{j=1}^{k} \lambda_j^2 (t_j - t_{j-1})\right).$$

Hence, let us assume, for instance, that with obvious notation

$$\phi_{k-1}(\lambda_1, \lambda_2, \cdots, \lambda_{k-1}; t_1, t_2, \cdots, t_{k-1}) = \exp\left(-\frac{1}{2}\sum_{j=1}^{k-1} \lambda_j^2 (t_j - t_{j-1})\right),$$

and applying Itô's formula to obtain, for $t \geq t_{k-1}$,

$$\begin{aligned} &\phi_k(\lambda_1, \lambda_2, \cdots, \lambda_k; t_1, t_2, \cdots, t_{k-1}, t) \\ = \ &\exp\left(-\frac{1}{2}\sum_{j=1}^{k-1} \lambda_j^2 (t_j - t_{j-1})\right) \\ &+ i\lambda_k \int_{t_{k-1}}^{t} E\left[v_u \exp\left(i\sum_{j=1}^{k-1} \lambda_j (X_{t_j} - X_{t_{j-1}})\right) \exp\left(i\lambda_k (X_u - X_{t_{k-1}})\right)\right] du \\ &- \frac{\lambda_k^2}{2}\int_{t_{k-1}}^{t} \phi_k(\lambda_1, \lambda_2, \cdots, \lambda_k; t_1, t_2, \cdots, t_{k-1}, u)du. \end{aligned} \tag{12}$$

It now follows from (10) that for $t > t_{k-1}$,

$$i\lambda_k \int_{t_{k-1}}^{t} E\left[v_u \exp\left(i\sum_{j=1}^{k-1} \lambda_j (X_{t_j} - X_{t_{j-1}})\right) \exp\left(i\lambda_k (X_u - X_{t_{k-1}})\right)\right] du = 0.$$

Hence, the identity (12) simplifies into a linear integral equation for $\phi_k(\lambda_1, \lambda_2, \cdots, \lambda_k; t_1, t_2, \cdots, t_{k-1}, t)$ which, when solved, yields the desired equality. □

Remark 2.4. Using a similar argument we may extend this proposition to a pair to processes X, Y given by

$$X_t = B_t + \int_0^t u_s ds, \qquad Y_t = B'_t + \int_0^t v_s ds,$$

where B and B' are two Brownian motions. Then (X_t) and (Y_t) have the same distributions of order k if and only if for all $t_1 \leq \cdots \leq t_k$, $j \in \{0, 1, \cdots, k-1\}$, $t \in [t_j, t_{j+1}]$, dt-a.s.,

$$E[u_t | X_{t_1} = x_1, \cdots, X_{t_j} = x_j, X_t = x] = E[v_t | Y_{t_1} = x_1, \cdots, Y_{t_j} = x_j, Y_t = x]. \quad (13)$$

This result also extends Petit-Yor [15].

Proof. By analogy with the proof of Proposition 2.2, we use ϕ_k^X for X and ϕ_k^Y for Y. Assume $\phi_{k-1}^X = \phi_{k-1}^Y$. If $\phi_k^X = \phi_k^Y$, from (12) we get

$$\begin{aligned} & E\left[u_t \exp\left(i \sum_{j=1}^{k-1} \lambda_j (X_{t_j} - X_{t_{j-1}})\right) \exp\left(i\lambda_k (X_t - X_{t_k})\right)\right] \\ = \; & E\left[v_t \exp\left(i \sum_{j=1}^{k-1} \lambda_j (Y_{t_j} - Y_{t_{j-1}})\right) \exp\left(i\lambda_k (Y_t - Y_{t_k})\right)\right], \end{aligned} \quad (14)$$

which implies (13). Conversely, if (14) holds, then we consider $\phi_k(t_1, \cdots, t_{k-1}, t)$ as the solution of a linear equation and this results in $\phi_k^X = \phi_k^Y$. □

Remark 2.5. In fact, it may be interesting to exploit (12) more completely by considering (12) as a linear equation for

$$t \longrightarrow \phi_k(\lambda_1, \cdots, \lambda_k; t_1, \cdots, t_{k-1}, t),$$

which can be expressed in terms of ϕ_{k-1} and

$$u \longrightarrow E\left[v_u \exp\left(i \sum_{j=1}^{k-1} \lambda_i (X_{t_j} - X_{t_{j-1}})\right) \exp\left(i\lambda_k (X_u - X_{t_{k-1}})\right)\right].$$

Ultimately, this method seems to relate $E[v_u | X_{t_1}, \cdots, X_{t_{k-1}}, X_u]$ to the k-marginals of X.

Of course, the only Gaussian process with two-dimensional Brownian marginals is Brownian motion. But, we may also ask, for $k \leq 3$, about the existence of semimartingales (X_t) which would satisfy (5) but not (6). In particular, does there exist an absolutely continuous process $X_t = \int_0^t v_s ds$ which satisfies (5) for $k \leq 3$? The preceding remark shows that the answer is positive for $k = 1$, and part (i) of the following proposition gives a characterization of such examples. Parts (ii) and (iii) provide partial negative answers in the case $k = 2$.

Proposition 2.3. (i) *If the process* (X_t) *given by*

$$X_t = \int_0^t v_u du, \tag{15}$$

is a weak Brownian motion of order 1*, then du-a.s*

$$E[v_u | X_u] = \frac{X_u}{2u}. \tag{16}$$

(ii) *If* X *satisfying* (15) *is a weak Brownian motion of order* 2*, then for any* s*, du-a.s.,* $u > s$*,*

$$E[v_u | X_s, X_u] = \frac{X_u - X_s}{2(u - s)}. \tag{17}$$

Furthermore, $u \longrightarrow v_u$ *cannot be right-continuous in* L^1*.*
(iii) *There is no continuous process with bounded variation* $(X_s; s \leq 1)$ *such that*

$$(X_s, X_t) \stackrel{(law)}{=} (B_s, B_t). \tag{18}$$

and

$$\int_0^1 |dX_s| \in L^2(\mathbb{P}). \tag{19}$$

Proof. 1) The first assertion is a general consequence of the "weak scaling" property

$$X_t \stackrel{(law)}{=} \sqrt{t} X_1, \tag{20}$$

for any given t; see Appendix of Pitman-Yor [16]. However, we sketch the proof. For $f \in C^1$ with compact support, we have

$$E[f(X_t)] = f(0) + E\left[\int_0^t f'(X_s) v_s ds\right].$$

Then we take derivative with respect to t, which yields

$$\frac{d}{dt}E[f(X_t)] = \frac{d}{dt}\left(E\left[f\left(\sqrt{t}X_1\right)\right]\right) = \frac{1}{2\sqrt{t}}E\left[f'\left(\sqrt{t}X_1\right)X_1\right] = \frac{1}{2t}E\left[f'(X_t)X_t\right].$$

On the other hand, we have

$$E[v_t f'(X_t)] = E[E[v_t|X_t]f'(X_t)],$$

which yields the result. Or we may use an argument similar to the proof of the second assertion.

2) Let $s < t$. From our assumption, we have

$$E\left[f(X_s)g(X_t)\right] = E[f(B_s)g(B_t)],$$

for all bounded functions f and g in $C^2(\mathbb{R})$. The left-hand side is equal to

$$E\left[f(X_s)\left(g(X_s) + \int_s^t g'(X_u)v_u du\right)\right],$$

while the right-hand side is given by

$$E\left[f(B_s)\left(g(B_s) + \frac{1}{2}\int_s^t g''(B_u)du\right)\right].$$

In particular, for fixed s and $u > s$, we have, for du-a.s.

$$E\left[f(X_s)g'(X_u)v_u\right] = \frac{1}{2}E\left[f(B_s)g''(B_u)\right]. \tag{21}$$

Let us introduce

$$\phi_{s,u}(x,y) := E[v_u|X_s = x, X_u = y].$$

Then, taking for g'' a bounded function φ with compact support, we may rewrite (21) as

$$E\left[f(B_s)\int_{-\infty}^{B_u}\varphi(y)dy\phi_{s,u}(B_s,B_u)\right] = \frac{1}{2}E\left[f(B_s)\varphi(B_u)\right]. \tag{22}$$

It follows from Fubini's Theorem, and the conditional distribution of B_u given B_s, that

$$\phi_{s,u}(B_s,y) = \frac{1}{2}\left(\frac{y-B_s}{u-s}\right), \quad dy - \text{a.s.},$$

which is equivalent to the above statement.

3) Suppose that X satisfying (15) is a weak Brownian motion of order 2 and that v is right-continuous in L^1. Then from (17), we get

$$E[v_s|X_s] = \frac{1}{2} \lim_{u \downarrow s} E\left[\frac{X_u - X_s}{u - s} \middle| X_s \right] = \frac{1}{2} E[v_s|X_s].$$

Hence, $E[v_s|X_s] = 0$. But (V_s) also satisfies (16); hence $X_s = 0$, which is not a weak Brownian motion of order 2.

4) We now prove the assertion (iii). If (18) and (19) are satisfied, then

$$E\left[\sum_{i: s_i \le t} \left(X_{s_{i+1}} - X_{s_i} \right)^2 \right] = t, \tag{23}$$

for $0 \le s_1 < s_2 < \cdots < s_n \le 1$. But on the other hand,

$$\sum_{i: s_i \le t} \left(X_{s_{i+1}} - X_{s_i} \right)^2 \le \left(\int_0^1 |dX_s| \right) \left(\sup_{i: s_i \le t} \left| X_{s_{i+1}} - X_{s_i} \right| \right) \le \left(\int_0^1 |dX_s| \right)^2.$$

Hence, by dominated convergence, the left-hand side of (23) converges to 0 as $\sup_i |s_{i+1} - s_i| \to 0$, which is a contradiction. □

3 Criteria for Brownian functionals with vanishing projections of order k

Let us consider a Brownian motion $(B_t)_{t \ge 0}$ and a functional $\Psi \in L^2(\mathcal{B}_\infty)$ with $E[\Psi] = 0$, where $\mathcal{B}_\infty = \sigma(B_t; t \ge 0)$. By Itô's representation theorem, there exists a unique class of predictable processes (ψ_u) satisfying

$$E\left[\int_0^\infty \psi_u^2 du \right] < \infty$$

such that

$$\Psi = \int_0^\infty \psi_u dB_u. \tag{24}$$

Our aim in this section is to formulate conditions on the integrand (ψ_u) which guarantee that Ψ has vanishing projections of order k, in the sense of equation (2). Let us first consider the case $k = 1$.

Proposition 3.1. *The functional* Ψ *in* (24) *satisfies*

$$E[\Psi|B_t] = 0, \tag{25}$$

for all t*, if and only if*

$$E[\psi_u|B_u] = 0, \quad du - a.s.. \tag{26}$$

Proof. (25) is equivalent to: for all t and λ,

$$E\left[\Psi \cdot \exp\left(i\lambda B_t + \frac{\lambda^2 t}{2}\right)\right] = 0.$$

But this expectation is equal to

$$i\lambda E\left[\int_0^t \psi_u \cdot \exp\left(i\lambda B_u + \frac{\lambda^2 u}{2}\right) du\right] = 0.$$

This proves immediately the equivalence between (25) and (26). □

Remark 3.1. Note that finding solutions of (26) is easy, whereas, a priori, finding solutions of (25) looks hard. The reason is that in (25) we look for a variable Ψ which satisfies infinitely many constraints, whereas in (26), we look for a process (ψ_u) which, for (almost) every u, satisfies only one constraint. Here is a construction of a square-integrable predictable process $\psi \not\equiv 0$ which satisfies condition (26). Let $f \not\equiv 0$ be a continuous function on $C[0,1]$ which has zero expectation and finite variance under the law of the Brownian bridge. For each $u \in (0,1]$ define

$$\psi_u = f(X^u)$$

where

$$X_t^u = \frac{1}{\sqrt{u}}\left(B_{tu} - tB_u\right), \qquad (t \leq 1).$$

Since X^u is a standard Brownian bridge which is independent from B_u, we obtain

$$E[\psi_u|B_u] = E[f(X^u)] = 0$$

for any $u \in (0,1]$. Thus, we have shown the existence of a functional $\Psi \in L^2$ with vanishing projections of order 1. Note that, in view of Theorem 1.1, we have to construct a bounded functional with this property. This additional step will be carried out in Section 4.

Remark 3.2. Let us give two further proofs for the sufficiency of condition (26).
1) (25) is satisfied if and only if for all $f \in C_c^\infty(\mathbb{R})$ and for all t,

$$E[\Psi\, f(B_t)] = 0. \tag{27}$$

It is well known that for fixed t, Itô's representation theorem of the random variable $f(B_t)$ is

$$f(B_t) = E[f(B_t)] + \int_0^t P_{t-u}(f')(B_u)dB_u.$$

Therefore, (27) is satisfied if and only if

$$E\left[\int_0^t \psi_u \cdot (P_{t-u}(f'))(B_u)du\right] = 0,$$

and so (26) implies (27), hence (25). We note that, more generally, this argument yields the following identity

$$E[\Psi|B_t] = E\left[\int_0^t E[\psi_u|B_u]dB_u \middle| B_t\right],$$

where, on the right hand side, $E[\psi_u|B_u]$ is chosen in a measurable way.
2) We enlarge the filtration of (B_u) with B_t. Then we get, for $u \le t$,

$$B_u = \beta_u^{(t)} + \int_0^u \frac{B_t - B_s}{t-s}ds,$$

where $(\beta_u^{(t)})_{u\ge 0}$ is a Brownian motion with respect to the enlarged filtration ($\mathcal{B}_u^{(t)} := \mathcal{B}_u \vee \sigma(B_t)$); in particular, $\beta^{(t)}$ is independent from B_t. Therefore,

$$\begin{aligned} E[\Psi|B_t] &= E\left[\int_0^t \psi_u dB_u \middle| B_t\right] \\ &= E\left[\int_0^t \psi_u d\beta_u^{(t)} \middle| B_t\right] + E\left[\int_0^t \frac{\psi_u(B_t - B_u)}{t-u}du \middle| B_t\right] \\ &= \int_0^t \frac{E[\psi_u(B_t - B_u)|B_t]}{t-u}du. \end{aligned}$$

Since

$$\begin{aligned} E[\psi_u(B_t - B_u)|B_t] &= E[(B_t - B_u)E[\psi_u|B_u]|B_t] \\ &= B_t E\left[E[\psi_u|B_u]|B_t\right] - E\left[B_u E[\psi_u|B_u]|B_t\right], \end{aligned} \tag{28}$$

we see that condition (26) implies (25).

In fact, using (26), we can solve equation (25) completely as follows:

Proposition 3.2. *The solutions of* (25) *consist precisely in the variables*

$$\Psi = \int_0^\infty \psi_u dB_u,$$

with

$$\psi_u = \int_0^u \varphi_{s,u}(B_u; (B_h, h \le s)) d\beta_s^{(u)} \tag{29}$$

for a measurable process $\varphi_{s,u}(B_u; (B_h, h \le s))$ *in all "4" variables, such that*

$$E\left[\int_0^\infty \int_0^u \varphi_{s,u}^2(B_u; (B_h, h \le s)) ds du\right] < \infty.$$

Proof. ¿From the equivalence of (25) and (26), and a measurability argument, all we need is to represent ψ_u as a stochastic integral with respect to $(d\beta_s^{(u)})$. In fact, the representation (29) is a particular case of the following representation of any variable $\psi_u \in L^2(\mathcal{B}_u)$ as:

$$\psi_u = E[\psi_u | B_u] + \int_0^u \varphi_{s,u}(B_u; (B_h, h \le s)) d\beta_s^{(u)}.$$

To prove this representation, it suffices to consider variables ψ_u of the form:

$$f(B_u) F(\beta_h^{(u)}, h \le u),$$

which are total in $L^2(\mathcal{B}_u)$. We then use the classical representation result for the filtration of Brownian motion (here: $\beta^{(u)}$), together with the fact that $\mathcal{B}_u = \mathcal{B}_u^{(u)} \vee \sigma(B_u)$. ☐

We need the following extension of Proposition 3.1 to the higher dimensional case.

Proposition 3.3. *For the functional* Ψ *in* (24) *to satisfy the condition*

$$E\left[\Psi | B_{t_1}, \cdots, B_{t_k}\right] = 0, \tag{30}$$

for all $t_1 < t_2 < \cdots < t_k$, *it is necessary and sufficient that*

$$E\left[\psi_t | B_{t_1}, \cdots, B_{t_{k-1}}, B_t\right] = 0, \tag{31}$$

for dt*-almost all* t, *and for all* $t_1 < \cdots < t_{k-1} < t$.

Proof. 1) In order to show that condition (31) is sufficient, it is enough to show

$$u_k := E\left[\Psi \cdot f_1(B_{t_1}) \cdots f_k(B_{t_k})\right] = 0,$$

for all $f_1, \cdots, f_k \in C_c^\infty$, and all $t_1 < \cdots < t_k \leq 1$. We write

$$f_k(B_{t_k}) = P_{t_k - t_{k-1}} f_k(B_{t_{k-1}}) + \int_{t_{k-1}}^{t_k} (P_{t_k - u} f_k')(B_u) dB_u.$$

Therefore,

$$u_k = u_{k-1} + v_k,$$

where

$$u_{k-1} := E\left[\Psi \cdot f_1(B_{t_1}) \cdots \left(f_{k-1} \cdot (P_{t_k - t_{k-1}} f_k)\right)(B_{t_{k-1}})\right], \tag{32}$$

and

$$\begin{aligned} v_k &= E\left[\Psi \cdot f_1(B_{t_1}) \cdots f_{k-1}(B_{t_{k-1}}) \int_{t_{k-1}}^{t_k} (P_{t_k-u} f_k')(B_u) dB_u\right] \\ &= \int_{t_{k-1}}^{t_k} E\left[\psi_u \cdot f_1(B_{t_1}) \cdots f_{k-1}(B_{t_{k-1}}) \cdot (P_{t_k-u} f_k')(B_u)\right] du. \end{aligned} \tag{33}$$

Due to the assumption (31) we have $v_k = 0$, and since

$$u_0 := E\left[\Psi\ E\left[f_1(B_{t_1}) \cdots f_k(B_{t_k})\right]\right] = 0,$$

we obtain $u_1 = \cdots = u_k = 0$ by iterating the argument. Conversely, if, in analogy to the proof of Proposition 3.1, we take

$$f_k(x) = \exp\left(i\lambda x + \frac{\lambda^2 u}{2}\right)$$

in (32), then we see from (33) that condition (31) is also necessary for (30) to hold.
2) For an alternative proof of the sufficiency of (31), we enlarge the filtration $(\mathcal{B}_t)$ with $(B_{t_1}, \cdots, B_{t_k})$. This is easy since we need only make these enlargements with B_{t_1}, then between times $[t_1, t_2]$, with B_{t_2}, etc. Explicitly, we get, for $t_{i-1} \leq u < t_i$,

$$B_u = \beta_u^{(t_i)} + \int_{t_{i-1}}^{u} \frac{B_{t_i} - B_s}{t_i - s} ds,$$

where $(\beta_u^{(t_i)})_{t_{i-1} \leq u \leq t_i}$ is a Brownian motion relative to $(\mathcal{B}_u^{(t_i)} := \mathcal{B}_u \vee \sigma(B_{t_i}))$. Applying a similar argument as in Remark 3.2, we get the desired result. □

Remark 3.3. The existence of a bounded predictable process (ψ_t) which satisfies condition (31) can be shown by iterating the construction in Remark 3.1. Such an iterative construction will be carried out in the proof of Theorem 4.1. There we have to use some additional care since we want to ensure that the resulting function Ψ is not only square-integrable but even bounded.

4 Construction of weak Brownian motions of order k

In this section we are going to prove Theorem 1.1. Let $k \in \mathbb{N}$. First we show that there exists a weak Brownian motion of order k whose law is equivalent to Wiener measure. As pointed out in the introduction, it suffices to construct a bounded functional of Brownian motion with vanishing projections of order k, and we will use the characterization of such functionals given in Proposition 3.3.

Theorem 4.1. *Let $k \in \mathbb{N}$. There exists a bounded nonzero measurable function Ψ on $C[0,\infty)$ such that, for any $0 < t_1 < \cdots < t_k < \infty$,*

$$E\left[\Psi | X_{t_1}, \cdots, X_{t_k}\right] = 0 \tag{34}$$

with respect to Wiener measure $\mathbb{P}$.

Proof. We proceed by induction.
1) The assertion is trivially true for $k = 0$: simply take a bounded nonzero measurable function Ψ such that $E[\Psi] = 0$. Let us now assume that the assertion holds for a given $n \geq 0$. Thus, there exists a nonzero measurable function Φ on $C[0,\infty)$ which satisfies (34) for $k = n$ and is bounded in absolute value by 1. We are going to construct a bounded measurable function Ψ on $C[0,\infty)$ which satisfies (34) for $k = n + 1$.
2) We fix $t_0 > 0$. Consider the Brownian bridge

$$X_t^{t_0} = X_t - \frac{t}{t_0} X_{t_0} \qquad (0 \leq t \leq t_0),$$

the induced Brownian motion

$$B_t^{t_0} = \frac{1+t}{\sqrt{t_0}} X_{t \cdot t_0/(1+t)}^{t_0} \qquad (t > 0),$$

and the bounded random variable

$$\psi = \Phi \circ B^{t_0}.$$

Note that ψ depends on X only via X^{t_0}; in particular (ψ, X^{t_0}) is independent of the σ-field $\hat{\mathcal{F}}_{t_0} = \sigma(X_u; u \geq t_0)$. Due to (34), assumed to hold for $k = n$, we can conclude that

$$\begin{aligned} E\left[\psi \left| X_{t_1}, \cdots, X_{t_n}, \hat{\mathcal{F}}_{t_0}\right.\right] &= E\left[\psi \left| X^{t_0}_{t_1}, \cdots, X^{t_0}_{t_l}, \hat{\mathcal{F}}_{t_0}\right.\right] \\ &= E\left[\psi \left| X^{t_0}_{t_1}, \cdots, X^{t_0}_{t_l}\right.\right] \\ &= E\left[\psi \left| B^{t_0}_{s_1}, \cdots, B^{t_0}_{s_l}\right.\right] \\ &= E\left[\Phi \left| X_{s_1}, \cdots, X_{s_l}\right.\right] \\ &= 0, \end{aligned} \tag{35}$$

for any $t_1 < \cdots < t_n$, where $l = \max\{i : t_i < t_0\}$ and s_i is defined by $s_i \cdot t_0/(1+s_i) = t_i$.

3) Let $c > 0$, define the stopping time

$$T := \inf\{t > t_0 : |X_t - X_{t_0}| \geq c\} \tag{36}$$

and the bounded predictable process

$$\psi_t = \begin{cases} 0 & t \leq t_0, \\ \psi I_{\{T \geq t\}} & t > t_0. \end{cases}$$

For $t > t_0$ and for $0 < t_1 < \cdots < t_n < t$ we have

$$E\left[\psi_t \left| X_{t_1}, \cdots, X_{t_n}, \hat{\mathcal{F}}_{t_0}\right.\right] = I_{\{T \geq t\}} E\left[\psi \left| X_{t_1}, \cdots, X_{t_n}, \hat{\mathcal{F}}_{t_0}\right.\right] = 0$$

due to (35), hence

$$E\left[\psi_t \left| X_{t_1}, \cdots, X_{t_n}, X_t\right.\right] = 0$$

for any $t > 0$ and for $0 < t_1 < \cdots < t_n < t$. Thus, Proposition 3.3 allows us to conclude that the functional

$$\Psi = \int_0^\infty \psi_t dX_t = \psi(X_T - X_{t_0})$$

satisfies

$$E\left[\Psi \left| X_{t_1}, \cdots, X_{t_n}, X_t\right.\right] = 0.$$

Moreover, Ψ is bounded in absolute value by c and $\Psi \not\equiv 0$. □

Corollary 4.1. *For $k \geq 1$ and for any $\epsilon \in (0,1)$ there exists a measurable function Φ_ϵ on $C[0,1]$ with $\| \Phi_\epsilon - 1 \|_\infty \leq \epsilon$ such that the coordinate process $(X_t)_{0 \leq t \leq 1}$ is a weak Brownian motion of order k under $\mathbb{P}_\epsilon = \Phi_\epsilon \cdot \mathbb{P}$.*

Proof. Take $c = \epsilon$ in the preceding construction of Ψ. The functional $\Psi_1 = E[\Psi|\mathcal{F}_1]$ satisfies (34) for $0 < t_1 < \cdots < t_k \leq 1$, and $\Phi_\epsilon = 1 + \Psi_1$, viewed as a measurable function on $C[0,1]$, defines a measure $\mathbb{P}_\epsilon = \Phi_\epsilon \cdot \mathbb{P} \approx \mathbb{P}$ on $C[0,1]$ with the desired properties. □

Note that, due to the Hahn-Banach theorem, we have shown, ipso facto, the following theorem.

Theorem 4.2. *Let $k \in \mathbb{N}$. The set*

$$\Pi_k = \left\{ \prod_{i=1}^{k} f_i(X_{t_i}) : t_1, \cdots, t_k \leq 1; f_i : \text{ bounded, Borel-measurable} \right\}$$

is not total in $L^1(\mathbb{P})$.

Remark 4.1. The existence of a functional $\Psi \in L^2(\mathbb{P})$ with vanishing projections of order k (see Remarks 3.1 and 3.3) implies the non-totality of Π_k in $L^2(\mathbb{P})$. In order to prove the non-totality of Π_k in $L^1(\mathbb{P})$, we need the refined construction in our proof of Theorem 4.1 which guarantees that the resulting functional Ψ is actually bounded.

We may illustrate the construction of weak Brownian motion made in Theorem 4.1 as follows.

Proposition 4.1. *Let $k \in \mathbb{N}$, and $0 < t_0 < 1$. Consider two independent Brownian motions $(W_t)_{0 \leq t \leq t_0}$ and $(\tilde{W}_t)_{t \geq 0}$ starting from 0, as well as a 3-dimensional Bessel process $(R_t)_{t \geq 0}$ which starts from 1 and is independent of W and $\tilde{W}$. Moreover, let φ_k be a Bernoulli random variable with values ± 1, measurable with respect to $\mathcal{F}_{t_0}^W$ and satisfying*

$$E\left[\varphi_k \,|W_{t_1}, \cdots, W_{t_k}\right] = 0, \tag{37}$$

for $t_1 \leq t_2 \leq \cdots \leq t_k \leq t_0$. Then the process $(X_t)_{0 \leq t \leq 1}$ given by

$$\begin{aligned} &i) \quad X_t = W_t, && 0 \leq t \leq t_0, \\ &ii) \quad X_{t_0+t} - X_{t_0} = \varphi_k(R_t - 1), && 0 \leq t \leq S \equiv \inf\left\{ s > 0 : |R_s - 1| = \frac{1}{2} \right\}, \\ &iii) \quad X_{t_0+S+t} - X_{t_0+S} = \tilde{W}_t, && 0 \leq t \leq (1-S) \vee 0, \end{aligned}$$

is a weak Brownian motion of order k, whose law is equivalent to Wiener measure in $C([0,1])$.

Proof. 1) First, we shall construct a probability measure $\tilde{\mathbb{P}}$ equivalent to Wiener measure $\mathbb{P}$ such that the coordinate process (X_t) is a weak Brownian motion under $\tilde{\mathbb{P}}$. To φ_k, a Bernoulli random variable with value ± 1, measurable with respect to $\mathcal{F}_{t_0}^X$ on Wiener space and satisfying (37), we associate

$$\varphi_t^k := \begin{cases} 0, & t \leq t_0, \\ \varphi_k I_{\{\tilde{T} \geq t\}}, & t > t_0, \end{cases}$$

with

$$\tilde{T} := \inf \left\{ t > t_0 : |X_t - X_{t_0}| \geq \frac{1}{2} \right\}.$$

Then

$$M_t := 1 + \int_0^t \varphi_s^k dX_s$$

is bounded and bounded away from 0. We can therefore define a probability measure $\tilde{\mathbb{P}} \sim \mathbb{P}$ via

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} := M_t = \exp\left[\int_0^t b_s dX_s - \frac{1}{2}\int_0^t b_s^2 ds\right],$$

where we set

$$b_t := \frac{\varphi_t^k}{M_t}.$$

Under $\tilde{\mathbb{P}}$ the coordinate process satisfies

$$X_t = \begin{cases} W_t, & 0 \leq t < t_0, \\ W_t + \displaystyle\int_{t_0}^{t \wedge \tilde{T}} \frac{\varphi_k \, ds}{1 + \varphi_k (X_s - X_{t_0})}, & t \geq t_0, \end{cases} \tag{38}$$

where W is a Wiener process under $\tilde{\mathbb{P}}$. Since for $t_1 \leq \cdots \leq t_i \leq t_0 \leq t_{i+1} \leq \cdots \leq t_k = t$,

$$E[\varphi_t^k | X_{t_1}, \cdots, X_{t_{k-1}}, X_t] = I_{\{\tilde{T} \geq t\}} E\left[E\left[\varphi_t | X_{t_1}, \cdots, X_{t_i}\right] | X_{t_1}, \cdots, X_{t_{k-1}}, X_t\right] = 0.$$

we get that X is a weak Brownian motion of order k under $\tilde{\mathbb{P}}$ due to Proposition 2.2 and condition (37). From(38) we get i).
2) Writing

$$Y_t := X_{t_0+t} - X_{t_0},$$

we obtain

$$Y_t = (W_{t_0+t} - W_{t_0}) + \int_0^{t\wedge(\tilde{T}-t_0)} \frac{\varphi_k\, ds}{1+\varphi_k Y_s}. \tag{39}$$

Multiplying both sides of equation (39) by φ_k, we get

$$\varphi_k Y_t = \bar{W}_t + \int_0^{t\wedge(\tilde{T}-t_0)} \frac{ds}{1+\varphi_k Y_s}, \tag{40}$$

where $\bar{W}_t := \varphi_k(W_{t_0+t} - W_{t_0})$ is a Brownian motion independent of $(W_t)_{t\le t_0}$. Furthermore, from (40), we obtain that the process

$$R_t := 1 + \varphi_k Y_t$$

is a BES(3) process starting from 1 and up to

$$\tilde{T} - t_0 = \inf\left\{t : |R_t - 1| = \frac{1}{2}\right\} \equiv S.$$

Hence we get the desired result. □

Since a weak Brownian motion of order $k \ge 4$ has the same quadratic variation as Brownian motion, one might suspect that its law is even equivalent to Wiener measure. The following theorem shows that this is not necessarily so.

Theorem 4.3. *For any $k \ge 1$, there exists a weak Brownian motion of order k whose law $\tilde{\mathbb{P}}$ on $C[0,1]$ is singular to Wiener measure $\mathbb{P}$.*

Proof. Let $\tilde{\mathbb{P}}_0 \approx \mathbb{P}$, $\tilde{\mathbb{P}}_0 \neq \mathbb{P}$ be the law of a weak Brownian motion of order k which is not a Brownian motion (cf. Theorem 4.1). Take an infinite collection $(Y_t^{(i)})_{0\le t\le 1}$ of independent copies of this weak Brownian motion defined on some common probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, and a sequence $0 = s_0 < s_1 < \cdots$ converging to 1. The idea is to construct a new weak Brownian motion by patching together rescaled versions of the weak Brownian motions $Y^{(i)}$, using them as increments on the different time intervals $[s_i, s_{i+1})$. Thus, let us define a continuous process $(Z_t)_{0\le t<1}$ such that, for $s_{n-1} \le t < s_n$,

$$Z_t = \sum_{i=1}^{n-1} \sqrt{s_i - s_{i-1}}\, Y_1^{(i)} + \sqrt{s_n - s_{n-1}}\, Y^{(n)}_{\frac{t-s_{n-1}}{s_n-s_{n-1}}}.$$

Note that the sequence

$$Z_{s_n} = \sum_{i=1}^{n-1} \sqrt{s_i - s_{i-1}}\, Y_1^{(i)} \qquad (n = 1, 2, \cdots)$$

is a martingale with bounded L^2-norm, hence a.s. convergent to a Gaussian random variable Z_1. We denote by $\tilde{\mathbb{P}}$ the distribution of $(Z_t)_{0\le t\le 1}$ on $C[0,1]$.

1) In order to check that the coordinate process $(X_t)_{0\le t\le 1}$ is a weak Brownian motion under $\tilde{\mathbb{P}}$, we have to show that

$$\tilde{E}\left[\prod_{j=1}^{k} f_j(X_{t_j})\right] = E_{\mathbb{Q}}\left[\prod_{j=1}^{k} f_j(Z_{t_j})\right] = E\left[\prod_{j=1}^{k} f_j(B_{t_j})\right], \tag{41}$$

for $0 < t_1 < \cdots < t_n \le 1$ and $f_j \in C_c(\mathbb{R}^1)$, where $\tilde{E}$ and E denote the expectation under $\tilde{\mathbb{P}}$ and $\mathbb{P}$, respectively. By continuity, it is enough to consider the case $t_n < 1$. We claim that, for any $l \ge 1$, condition (41) holds if $0 < t_1 < \cdots < t_k \le s_l$. This is clear if $l = 1$. Suppose that condition (41) holds for $l = n$, and let us check it for $l = n+1$. For $0 < t_1 < \cdots < t_k < s_{n+1}$, let

$$l := \max\{j : t_j \le s_n\} < k.$$

We have

$$E_{\mathbb{Q}}\left[\prod_{j=1}^{k} f_j(Z_{t_j})\right] = E_{\mathbb{Q}}\left[\prod_{j=1}^{l} f_j(Z_{t_j}) \prod_{j=l+1}^{k} f_j(Z_{s_n} + \sqrt{s_{n+1}-s_n}\, Y^{(n+1)}_{\frac{t_j - s_n}{s_{n+1}-s_n}})\right].$$

Since $Y^{(n+1)}$ is a weak Brownian motion of order k which is independent of $\sigma(Z_u; u \le s_n)$, the right hand side takes the form

$$E_{\mathbb{Q}}\left[\prod_{j=1}^{l} f_j(Z_{t_j}) g(Z_{s_n})\right],$$

where

$$g = P_{t_{l+1}-s_n}\left(f_{l+1}\cdots f_{k-1}\left(P_{t_k - t_{k-1}} f_k\right)\right),$$

and by our induction hypothesis this is equal to

$$E\left[\prod_{j=1}^{l} f_j(X_{t_j}) g(X_{s_n})\right] = E\left[\prod_{j=1}^{k} f_j(X_{t_j})\right].$$

2) Since $\tilde{\mathbb{P}}_0 \neq \mathbb{P}$, there exists a bounded measurable φ on $C[0,1]$ such that

$$\int \varphi \, d\tilde{\mathbb{P}}_0 \neq \int \varphi \, d\mathbb{P}.$$

The random variables

$$\varphi^{(i)} = \varphi\left(\frac{1}{\sqrt{s_i - s_{i-1}}}\left(X_{s_{i-1}+t(s_i-s_{i-1})} - X_{s_{i-1}}\right); 0 \leq t \leq 1\right)$$

are independent and identically distributed, both under $\tilde{\mathbb{P}}$ and under $\mathbb{P}$. By the law of large numbers, $\tilde{\mathbb{P}}$ is concentrated on the set

$$A = \left\{\frac{1}{n}\sum_{i=1}^{n} \varphi^{(i)} \longrightarrow \int \varphi \, d\tilde{\mathbb{P}}_0\right\},$$

while $\mathbb{P}(A) = 0$. Thus $\tilde{\mathbb{P}}$ is singular to $\mathbb{P}$. □

5 Construction of weak Brownian motions which coincide with Brownian motion outside of any small interval

In this section we give a different construction of weak Brownian motions of order k. The resulting processes have an even stronger resemblance to Brownian motion: their law $\tilde{\mathbb{P}}$ coincides with Wiener measure $\mathbb{P}$ outside of any interval of length $\varepsilon = (k+1)^{-1}$.

For $J \subseteq [0,1]$ we use the notation $\mathcal{F}_J = \sigma(X_t : t \in J)$, where $(X_t)_{0 \leq t \leq 1}$ denotes the coordinate process on $\Omega = C[0,1]$.

Theorem 5.1. *For any $\varepsilon \in (0,1)$, there exists a probability measure $\tilde{\mathbb{P}} \neq \mathbb{P}$ on $C[0,1]$ which is equivalent to Wiener measure $\mathbb{P}$ and satisfies*

$$\tilde{\mathbb{P}} = \mathbb{P} \qquad \text{on } \mathcal{F}_J \tag{42}$$

for any $J \subseteq [0,1]$ such that J^c contains some interval of length ε.

Proof. We take $n \geq 2\varepsilon^{-1}$ and partition the interval $[0,1]$ into the intervals $I_k = [(k-1)/n, k/n] (k = 1, \cdots, n)$. For each $k \in \{1, \cdots, n\}$, there exists a random variable $\varphi_k \neq 0$ bounded in absolute value by 1 such that

$$\varphi_k \text{ is } \mathcal{F}_{I_k}\text{-measurable} \tag{43}$$

and

$$E[\varphi_k|\mathcal{F}_{I_k^c}] = 0. \tag{44}$$

For example, we can take

$$\varphi_k = \operatorname{sign}\left(X_{\frac{k-1}{n}+\frac{1}{2n}} - \frac{1}{2}\left(X_{\frac{k-1}{n}} + X_{\frac{k}{n}}\right)\right) \tag{45}$$

which is $\mathcal{F}_{I_k}$-measurable, has expectation 0, and is independent of $\mathcal{F}_{I_k^c}$. Let $\tilde{\mathbb{P}}$ denote the probability measure on $C[0,1]$ defined by the density

$$\Psi = 1 + \frac{1}{2}\prod_{k=1}^{n} \varphi_k > 0 \tag{46}$$

with respect to Wiener measure $\mathbb{P}$. Take any $J \subseteq [0,1]$ such that J^c contains some interval of length ε. In particular, J is contained in I_l^c for some $l \in \{1, \cdots, n\}$. But

$$E[\Psi|\mathcal{F}_{I_l^c}] = 1 + \frac{1}{2}\left(\prod_{k \neq l} \varphi_k\right) E[\varphi_l|\mathcal{F}_{I_l^c}] = 1$$

due to (43) and (44). Thus, $\tilde{\mathbb{P}} = \mathbb{P}$ on $\mathcal{F}_{I_l^c}$, and this implies $\tilde{\mathbb{P}} = \mathbb{P}$ on $\mathcal{F}_J$. □

Remark 5.1. (1) The measure $\tilde{\mathbb{P}}$ constructed in Theorem 5.1 defines a weak Brownian motion of order k for any $k \leq \varepsilon^{-1} - 1$. In fact, for any choice of $0 \leq t_1 < \cdots < t_k \leq 1$, the complement of $J = \{t_1, \cdots, t_k\}$ contains an interval of length ε, and so we have $\tilde{\mathbb{P}} = \mathbb{P}$ on $\mathcal{F}_{\{t_1,\cdots,t_k\}}$.
(2) It is interesting to describe the weak Brownian motions constructed in this section in terms of the Lévy-Ciesielski construction of Wiener measure $\mathbb{P}$ as a random field $(X_\alpha)_{\alpha \in A}$ of i.i.d. Gaussian random variables indexed by a binary tree. Our construction of $\tilde{\mathbb{P}}$ introduces interactions in the random field, and it can be modified in such a way that $\tilde{\mathbb{P}}$ is singular to $\mathbb{P}$. The details will be discussed in a separate paper.

Note that our proof of Theorem 5.1 does not involve the special properties of Brownian motion. It is valid whenever $\mathbb{P}$ is the law of some non-degenerate Markov process $(X_t)_{0 \leq t \leq 1}$ with state space $(S, \mathcal{S})$. We only need the property that, for any $0 \leq s < t \leq 1$, the conditional distributions $\mathbb{P}[\cdot|\mathcal{F}_{(s,t)^c}]$ are non-degenerate in the sense that there exists some $A \in \mathcal{F}_{(s,t)}$ such that

$$\mathbb{P}[A|\mathcal{F}_{(s,t)^c}] = \mathbb{P}[A|X_s, X_t]$$

is not constant $\mathbb{P}$-a.s.. Then

$$\varphi = I_A - \mathbb{P}[A|X_s, X_t]$$

has the properties (43) and (44) with respect to the interval $I = (s,t)$. Thus, the proof of Theorem 5.1 yields the existence of some measure $\tilde{\mathbb{P}} \neq \mathbb{P}$ such that $\tilde{\mathbb{P}} \approx \mathbb{P}$ and $\tilde{\mathbb{P}} = \mathbb{P}$ on $\mathcal{F}_J$ whenever the complement of J contains an interval of length $\varepsilon > 0$. In particular, we can make sure that, for a given $k \geq 1$, $\tilde{\mathbb{P}}$ and $\mathbb{P}$ have the same marginals of order k. As a special case, we could consider a Poisson process P and thus recover and strengthen a result of Szász [21] which solved a problem posed by Rényi [17].

6 Criteria in terms of Volterra kernels

In this section we consider a continuous Gaussian semimartingale $(X_t)_{t\geq 0}$ defined as follows in terms of a Brownian motion W and a kernel l:

$$X_t = W_t - \int_0^t \int_0^u l(u,v) dW_v du. \tag{47}$$

We assume that l is a continuous Volterra kernel, i.e., l satisfies

$$l(u,v) = 0, \qquad \text{for } 0 \leq u < v \leq 1,$$

and the function

$$\tilde{l}(u,v) := \begin{cases} l(u,v), & \text{for } u \leq v, \\ \\ l(v,u), & \text{for } u > v, \end{cases}$$

is continuous on $(0,1) \times (0,1)$. We further assume that

$$\int_0^t \left(\int_0^u l^2(u,v) dv \right)^{1/2} du < \infty, \tag{48}$$

so that (47) is well-defined as the semimartingale decomposition of (X_t) in the filtration $(\mathcal{F}_t^W)$. Clearly X has quadratic variation $\langle X \rangle_t = t$.

Remark 6.1. Note that the representation (47) is in general not unique. For example, the process X satisfying

$$X_t = W_t - \int_0^t \frac{W_u}{u} du,$$

with a Wiener process W, is again a Brownian motion. Thus X admits two different Volterra representations: one with $l_X(u,v) \equiv 0$, the other with $l_W(u,v) = 1/u$ for $v \leq u$. But if we add the condition $l \in L^2([0,1] \times [0,1])$, the representation (47) is indeed unique; see, e.g., Hida-Hitsuda [8].

In Section 6.1 we will characterize the case where the process X defined by (47) is a Brownian motion. In Section 6.2 we obtain criteria for X to be a weak Brownian motion of order 1. In particular we construct examples of continuous Gaussian semimartingales which are weak Brownian motions of order 1 but not Brownian motions.

6.1 Brownian motions defined in terms of Volterra kernels

Hitsuda [9] shows that the law of a centered Gaussian process (X_t) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is equivalent to Wiener measure if and only if X admits a Volterra representation (47) with a square-integrable Volterra kernel $l(t,s)$. Hence, from the uniqueness of the Doob-Meyer decomposition we know that if X is a Brownian motion admitting a Volterra representation (47), then the associated Volterra kernel l is not square-integrable unless $l \equiv 0$. For the case $l \not\equiv 0$, we can conclude that $(\mathcal{F}_t^X) \subsetneqq (\mathcal{F}_t^W)$, i.e., the filtration generated by X is strictly smaller than the one generated by W. Otherwise, the representation (47) would be the Doob-Meyer decomposition of X as a semimartingale in its own filtration. Uniqueness of the Doob-Meyer decomposition would imply $l \equiv 0$, which is obviously a contradiction. But is it possible to find a Volterra representation for Brownian motion, where the kernel l is not square-integrable? If so, how does the associated Volterra kernel look like? The following theorem will provide a characterization of Brownian motions with Volterra representation (47).

Theorem 6.1. *A process $(X_t)_{t\geq 0}$ defined by (47) is a Brownian motion if and only if the Volterra kernel $l(t,s)$ is self-reproducing, i.e., $l(t,s)$ satisfies*

$$l(t,s) = \int_0^s l(t,v)l(s,v)dv, \tag{49}$$

for all t and for all $s \leq t$. In this case, $\{X_s; s \leq t\}$ is independent of $\int_0^t l(t,u)dW_u$ for any $t > 0$.

Proof. It follows from Lemma 2.3 in Föllmer-Wu-Yor [5] that (X_t) is a Brownian motion if and only if

$$E\left[X_s\int_0^t l(t,u)dW_u\right]=0, \tag{50}$$

for all $s\le t$; i.e., if the Gaussian family $\{X_s; s\le t\}$ is independent of $\int_0^t l(t,u)dW_u$. Since l is continuous and

$$E\left[X_s\int_0^t l(t,u)dW_u\right]=\int_0^s l(t,u)du-\int_0^s\int_0^u l(t,v)l(u,v)dvdu.$$

due to (47), equation (50) is equivalent to

$$\int_0^s l(t,u)du=\int_0^s\int_0^u l(t,v)l(u,v)dvdu$$

for all $s\le t$, hence to (49) for all t and for all $s\le t$. □

Remark 6.2. The terminology "self-reproducing" is used in Neveu [14] in a different context.

Remark 6.3. If $l(t,s)$ satisfies (49), then it satisfies the following properties:

(a) $l(t,t)\ge 0$.

(b) $l(t,s)\le\sqrt{l(t,t)l(s,s)}$.

(c) If $l(t,s)\not\equiv 0$, then $l(t,t)\notin L^1(0,1)$, and this implies $l\notin L^2((0,1)\times(0,1))$. This is consistent with the above discussion. In particular we see that a non-zero self-reproducing Volterra kernel l is not square-integrable.

Proof. Taking $s=t$, we have

$$l(t,t)=\int_0^t l^2(t,u)du, \tag{51}$$

which leads to assertion (a). Then it follows from Hölder's inequality that

$$\begin{aligned} l(t,s) &\le \left(\int_0^s l^2(t,v)dv\right)^{\frac12}\left(\int_0^s l^2(s,v)dv\right)^{\frac12}\le\left(\int_0^t l^2(t,v)dv\right)^{\frac12}\left(\int_0^s l^2(s,v)dv\right)^{\frac12}\\ &= \sqrt{l(t,t)l(s,s)}. \end{aligned}$$

This gives (b). As for (c), assume $l \not\equiv 0$. Since l is continuous, we see that $l(t,t) \neq 0$ for some $t \in (0,1)$ due to (51). Let us write

$$\{s \in (0,1) : l(s,s) \neq 0\} = \bigcup_i (a_i, b_i),$$

with disjoint intervals (a_i, b_i). Substituting (b) in (49), we obtain

$$l(t,s) \leq \sqrt{l(t,t)l(s,s)} \int_0^s l(v,v)dv.$$

This implies

$$l(s,s) \leq l(s,s) \int_0^s l(v,v)dv,$$

for all s. Since $l(s,s) = 0$ for $s \leq a := \inf_i a_i$, we obtain

$$\int_a^s l(v,v)dv = \int_0^s l(v,v)dv \geq 1 \tag{52}$$

for all $s \in \bigcup_i [a_i, b_i]$. Either we have $a = a_i$ for some i or a is an accumulation point of (a_i). In both cases, (52) implies $l(u,u) \notin L^1(0,1)$. And this implies

$$\int_0^1 \int_0^u l^2(u,v)dvdu = \int_0^1 l(u,u)du = \infty.$$

□

In order to illustrate Theorem 6.1 more explicitly, let us consider some special cases:

$$l(t,s) := a(t)b(s),$$

where a and b are two deterministic continuous functions satisfying:
(C1) $a \in L^1[0,t]$ for all t and for all $t_0 > 0$, $a(t) \not\equiv 0$ on the interval (t_0, ∞).
(C2) $b \in L^2[0,t]$ for all $t > 0$ and

$$\int_0^t \frac{|b(u)|}{(\int_0^u b^2(v)dv)^{1/2}} du < \infty$$

for all $t > 0$.

Corollary 6.1. *Let the process (X_t) admit the representation*

$$X_t = W_t - \int_0^t a(u) \int_0^u b(v)dW_v du,$$

with some deterministic functions a and b satisfying the conditions $(C1)$ and $(C2)$. Then the process X is a Brownian motion if and only if it is of the form

$$X_t = W_t - \int_0^t \frac{b(u)}{\int_0^u b^2(v)dv} \int_0^u b(r)dW_r du. \tag{53}$$

Proof. In view of Theorem 6.1, it is sufficient to prove that a Volterra kernel $l(t,s)$ of the form $a(t)b(s)$ satisfies (49) if and only of

$$a(t) = \frac{b(t)}{\displaystyle\int_0^t b^2(u)du}. \tag{54}$$

Substituting $l(t,s) = a(t)b(s)$ into (49), we have

$$a(t)b(s) = a(t)a(s) \int_0^s b^2(u)du.$$

According to the condition (C1) we get the desired result. □

As an example, we take $b(t) = t^m$ for $m > -\frac{1}{2}$. Then the process X given by

$$X_t = W_t - (2m+1) \int_0^t \int_0^u u^{-m-1} v^m dW_v du,$$

is a Brownian motion. This special case has been discussed in Lévy [12], [13], Chiu [2] and Hibino-Hitsuda-Muraoka [7]. For $m = 0$ we recover the result of Deheuvels [3] that

$$X_t = W_t - \int_0^t \frac{W_u}{u} du$$

is again a Brownian motion.

6.2 Weak Brownian motions defined in terms of Volterra kernels

We are now going to show that, in the class of Gaussian semimartingales with Volterra representation (47), weak Brownian motions of order 1 can be characterized by an integrated form of (49).

Theorem 6.2. *A process given by* (47) *is a weak Brownian motion of order* 1 *if and only if*

$$\int_0^t l(t,v)dv = \int_0^t \int_0^s l(t,v)l(s,v)dvds, \tag{55}$$

for all t.

Proof. Let X be defined by (47). Since X is a centered Gaussian process, X is a weak Brownian motion of order 1 if and only if

$$E[X_t^2] = t, \tag{56}$$

Since $\langle X \rangle_t = t$, we have

$$X_t^2 = 2\int_0^t X_u dX_u + t$$

by Itô's formula. Condition (56) is equivalent to

$$E\left[\int_0^t X_u dX_u\right] = 0,$$

hence to

$$E\left[\int_0^t X_u \int_0^u l(u,v)dW_v du\right] = 0 \tag{57}$$

due to (47). But the validity of (57) for all $t \geq 0$ is equivalent to the condition

$$0 = E\left[X_u \int_0^u l(u,v)dW_v\right] = \int_0^u l(u,v)dv - \int_0^u \int_0^v l(u,r)l(v,r)drdv,$$

for all u.

□

Corollary 6.2. *Suppose that* $l(t,s) = a(t)b(s)$ *for some deterministic continuous functions* a, b *satisfying* $(C1)$ *and* $(C2)$. *Then a process* X *given by* (47) *is a weak Brownian motion of order* 1 *if and only if it is of the form* (53). *In other words, if the process* X *given by* (47) *is a weak Brownian motion of order* 1, *it is also a Brownian motion.*

Finally, we want to construct a Gaussian weak Brownian motion of order 1 which is not a Brownian motion. Consider the Volterra kernel of the form

$$l(u,v) = \frac{1}{u}\varphi\left(\frac{v}{u}\right).$$

Then l satisfies (49) if and only if

$$\varphi(x) = \int_0^1 \varphi(zx)\varphi(z)dz. \tag{58}$$

On the other hand, l satisfies (55) if and only if

$$\int_0^1 \varphi(x)dx = \int_0^1 \int_0^1 \varphi(zx)\varphi(z)dzdx. \tag{59}$$

Clearly, there are many functions φ which satisfy (59), without satisfying (58). To be quite explicit, consider $\varphi(x) = ce^{-ax}$, and we then see that, given a, (59) is satisfied if and only if

$$c = \frac{(1-e^{-a})}{\displaystyle\int_0^a e^{-u}(1-e^{-u})\frac{du}{u}},$$

whereas (58) is never satisfied for any $c \neq 0$. Therefore, we obtain the following theorem.

Theorem 6.3. *There exist continuous Gaussian semimartingales $(X_t)_{t\geq 0}$ with quadratic variation $\langle X\rangle_t = t$ which are weak Brownian motions of order* 1*, but not Brownian motions.*

7 Weak martingales and weak Brownian motions of order 1

In the class of continuous semimartingales of the form

$$X_t = B_t + \int_0^t v_s ds, \tag{60}$$

weak Brownian motions of order 1 can be characterized by a weak martingale property, in analogy to Lévy's characterization of Brownian motions. To this end, we introduce the definition of weak martingales.

Definition 7.1. A continuous semimartingale X is called a *weak martingale* if it satisfies

$$E\left[\int_0^t f(X_s)dX_s\right] = 0, \tag{61}$$

for all bounded Borel-measurable functions f, and for all $t > 0$.

Remark 7.1. 1) This notion is different from Kazamaki's weak martingales ([10], [11]).
2) A different notion is that of a *weak increment martingale*, i.e., a process (X_t) which satisfies

$$E[X_t - X_s | X_s] = 0, \qquad \text{for } s < t.$$

This implies that (X_t) is a weak martingale, but the converse is not true.

To make Definition 7.1 more precise, if $X_t = M_t + V_t$ is the semimartingale decomposition of X, we assume

$$E[\sqrt{\langle M \rangle_t}] < \infty, \qquad \text{and} \qquad E\left[\int_0^t |dV_s|\right] < \infty, \tag{62}$$

for any $t \geq 0$. Below, we shall use several times the important fact: if X is a weak martingale, and Y is independent of X and $\mathbb{R}^k$-valued, then for every bounded measurable $\varphi : \mathbb{R}^k \times \mathbb{R} \longrightarrow \mathbb{R}$, and every t,

$$E\left[\int_0^t \varphi(Y_s, X_s) dX_s\right] = 0. \tag{63}$$

This follows easily from (61) and (62), and the monotone class theorem. In particular, we may use (63) for $Y_s = s$. Here is a stability property of weak martingales.

Proposition 7.1. *If (X_t) and (Y_t) are two independent weak martingales, then both $(X_t + Y_t)$ and $(X_t Y_t)$ are weak martingales.*

Proof. For the second assertion, we have

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t.$$

Since X and Y are independent, hence

$$E\left[\int_0^t f(X_s Y_s) d(X_s Y_s)\right] = E\left[\int_0^t f(X_s Y_s) X_s dY_s\right] + E\left[\int_0^t f(X_s Y_s) Y_s dX_s\right] = 0.$$

□

Proposition 7.2. *If the process (X_t) is a weak martingale of bounded variation with $X_0 = 0$, then $X \equiv 0$.*

Proof. Take $f(x) = \mathrm{sgn}(x)$, then

$$E\left[|X_t|\right] = E\left[\int_0^t f(X_u)dX_u\right] = 0.$$

This implies $X \equiv 0$. □

Proposition 7.3. *If (X_t) is a weak martingale such that $X_0 = 0$, and its local time at 0 is equal to 0, then $X \equiv 0$.*

Proof. Use Tanaka's formula. □

Theorem 7.1. *Assume that a semimartingale (X_t) can be represented in the form*

$$X_t = M_t + \int_0^t v_s ds, \tag{64}$$

where M is a martingale with respect $(\mathcal{F}_t^X)$, then (X_t) is a weak martingale if and only if

$$E[v_s|X_s] = 0, \qquad ds\text{-a.s.}$$

Proof. For every bounded, Borel function f, the expectation

$$E\left[\int_0^t f(X_s)dX_s\right] = 0$$

holds if and only if

$$0 = E\left[\int_0^t f(X_s)v_s ds\right] = \int_0^t E[f(X_s)v_s]ds,$$

which immediately yields the equivalence. □

The next proposition provides a characterization of weak Brownian motions of order 1 in terms of the weak martingale property.

Proposition 7.4. *Assume that (X_t) is a continuous semimartingale of the form (60) with $\int_0^t E[|v_u|]du < \infty$ for all t. Then X is a weak Brownian motion of order 1 if and only if it is a weak martingale.*

Proof. 1) Since (X_t) satisfies (61), and since $\langle X\rangle_t = t$, then for all $c \in \mathbb{R}$, we obtain, with the help of Itô's formula

$$\begin{aligned} E[\exp(icX_t)] &= 1 + icE\left[\int_0^t \exp(icX_s)dX_s\right] - \frac{c^2}{2}E\left[\int_0^t \exp(icX_s)ds\right] \\ &= 1 - \frac{c^2}{2}\int_0^t E[\exp(icX_s)]ds. \end{aligned}$$

Thus, we have

$$E[\exp(icX_t)] = \exp\left(-\frac{c^2}{2}t\right).$$

This shows that X has the same one-dimensional marginals as Brownian motion.

2) Suppose that (X_t) is a weak Brownian motion of order 1. Then we get, for $F \in C_c^2(\mathbb{R})$, again as a consequence of Itô's formula:

$$\begin{aligned} E\left[\int_0^t F'(X_s)dX_s\right] &= E[F(X_t)] - F(0) - \frac{1}{2}E\left[\int_0^t F''(X_s)ds\right] \\ &= E[F(B_t)] - F(0) - \frac{1}{2}\int_0^t E[F''(B_s)]ds \\ &= E\left[\int_0^t F'(B_s)dB_s\right] = 0. \end{aligned} \tag{65}$$

For a bounded measurable function f on $\mathbb{R}$, we can choose $F_n \in C_c^2(\mathbb{R})$, such that the derivatives $f_n := F_n'$ satisfy

$$\lim_{n\to\infty} E\left[\int_0^1 (f_n - f)^2(X_s)ds + \int_0^1 |f_n - f|^2(X_s)d|V|_s\right] = 0.$$

This implies

$$\lim_{n\to\infty} E\left[\int_0^t (f_n - f)(X_s)dX_s\right] = 0,$$

hence

$$E\left[\int_0^t f(X_s)dX_s\right] = 0$$

due to (65). □

We now show that weak Brownian motions of order 1 which are stable under stochastic integration are necessarily Brownian motions.

Proposition 7.5. *Let (X_t) be a continuous semimartingale of the form* (60) *such that every predictable process (ϵ_t), with values ± 1, $(\int_0^t \epsilon_s dX_s)_{t\geq 0}$ is a weak Brownian motion of order* 1*. Then (X_t) is a Brownian motion.*

Proof. Define

$$\epsilon_t = \begin{cases} dV_t/|dV_t|, & \text{if } dV_t \neq 0, \\ 1, & \text{if } dV_t = 0. \end{cases}$$

If the process

$$X_t^{(\epsilon)} := \int_0^t \epsilon_s dX_s = \int_0^t \epsilon_s dB_s + \int_0^t |dV_s|$$

is a weak Brownian motion of order 1, then it has expectation 0. This implies $E[\int_0^t |dV_s|] = 0$, hence $X = B$. □

We now present examples of weak Brownian motions $X_t = B_t + V_t$ of order 1 where dV_t is singular with respect to dt.

Proposition 7.6. *Let $\epsilon_u = F(B_s, s \leq u)$ be a predictable process, taking values ± 1, and let $g_t = \sup\{s < t : B_s = 0\}$. If we assume*

$$F(-B_s, s \leq u) = -F(B_s, s \leq u), \tag{66}$$

then the process $\tilde{B}_t^\epsilon \equiv \epsilon_{g_t}|B_t|$ is a weak Brownian motion of order 1, and it is not a Brownian motion.

Proof. (1) From the balayage formula and Tanaka's formula (see Revuz-Yor [18]), we know that

$$\tilde{B}_t^\epsilon = \int_0^t \epsilon_{g_s} d|B_s| = \int_0^t \epsilon_{g_s} d\beta_s + \int_0^t \epsilon_s dL_s, \tag{67}$$

where $d\beta_s = \operatorname{sgn}(B_s)dB_s$. The fact that $(\tilde{B}_t^\epsilon)$ is a weak Brownian motion of order 1 follows from

$$\begin{aligned} E[\int_0^t f(\tilde{B}_s^\epsilon)d\tilde{B}_s^\epsilon] &= E\left[\int_0^t \epsilon_{g_s} f(\tilde{B}_s^\epsilon)d\beta_s\right] + E\left[\int_0^t \epsilon_s f(\tilde{B}_s^\epsilon)dL_s\right] \\ &= E\left[\int_0^t \epsilon_s f(\tilde{B}_s^\epsilon)dL_s\right] = f(0)E\left[\int_0^t \epsilon_s dL_s\right] = 0, \end{aligned}$$

due to (66).
(2) $(\tilde{B}_t^\epsilon)$ is not a Brownian motion, since (67) is really its canonical decomposition, which easily follows by remarking that $\epsilon_{g_t} = \operatorname{sgn}(\tilde{B}_t^\epsilon)$, and $|\tilde{B}_t^\epsilon| = |B_t|$. □

Here is another example, due to M. Émery.

Example 7.1. If $(B_t)_{t\geq 0}$ is a standard Brownian motion, then the process $(\tilde{B}_t)$ given by

$$\tilde{B}_t = \begin{cases} B_t, & t \leq 1 \\ \\ \mathrm{sgn}(B_1)|B_t|, & t > 1 \end{cases} \tag{68}$$

is a weak Brownian motion of order 1.

Example 7.2. Consider the process $(\hat{B}_t)$ given by

$$\hat{B}_t = \mathrm{sgn}(B_1)|B_t|, \qquad (t > 0)$$

Then $(\hat{B}_t)$ is a semimartingale and a weak Brownian motion of order 1 in its own filtration $(\mathcal{F}_t^{\hat{B}})$. But this example is not genuine, since $\mathcal{F}_{0+}^{\hat{B}} = \sigma(\mathrm{sgn}(B_1))$ is not trivial.

8 Related equations and studies

8.1. Throughout this paper, Brownian motion is our process of reference. Obviously, we could also address the analogous question for a Poisson process; in this case, an analogue of Theorem 1.1 was already proved by Szász [21], whose construction is discussed in Stoyanov ([20], Section 24.3, pp.284-285). More generally, one could consider a Lévy process or even more general processes; see the last paragraph of Section 5.

8.2. Gyöngy [6] shows that the one-dimensional marginals of an Itô process

$$\xi_t = \int_0^t (\delta(s,\omega)dW_s + \beta(s,\omega)ds)$$

are those of a weak solution X_t of

$$X_t = \int_0^t \sigma(s, X_s)dW_s + \int_0^t b(s, X_s)ds,$$

where $\sigma^2(s,x) = E[\delta^2(s)|\xi_s = x]$, $b(s,x) = E[\beta(s)|\xi_s = x]$. Our study goes in the reverse direction: that is, given a diffusion, to find other processes which admit the same 1- (or more generally k-) marginals.

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