Calculating Values and Gradients of Polyhedral Chance Constraints

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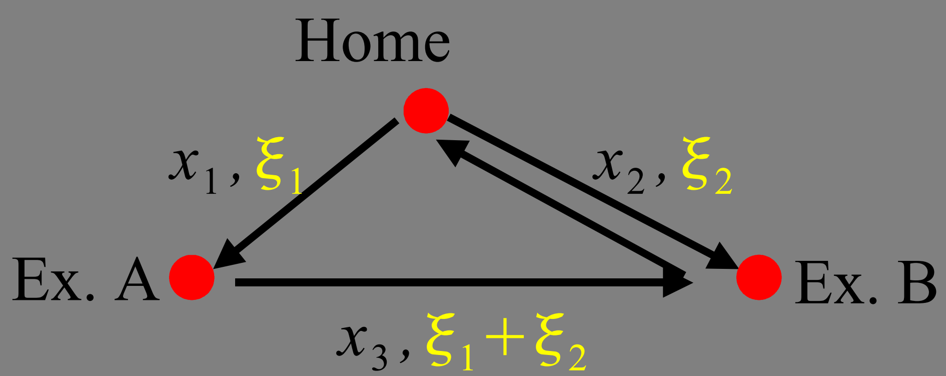
Matheon MF1 Workshop

Optimisation Software

Zuse Institute, June 1, 2005

Organizing a meeting under incomplete information



$$(\xi_1, \xi_2) \sim N(\mu, \Sigma)$$

$$min \; \left\{ c_1 x_1 + c_2 x_2 + c_3 x_3 \;\; s.t. \;\; P\begin{pmatrix} \xi_1 & \le & x_1 \\ \xi_2 & \le & x_2 \\ \xi_1+\xi_2 & \le & x_3 \end{pmatrix} \ge p \right\}$$

probabilistic constraint: $P(A\xi \le x) \ge p$ $\qquad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$

no distribution function!

$$\eta := A\xi \sim N(A\mu, \Sigma' = A\Sigma A^T)$$

$\Rightarrow$ $P(\eta \le x) = F_\eta(x) \ge p$

$$\Sigma' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Inclusion/Exclusion formula for Polyhedra[1,2]

$$M = \{\xi \in \mathbb{R}^n \mid \langle a_i, \xi \rangle \le b_i \ \ (i = 1, \ldots, m)\}$$

$I(\xi) = \{i \mid \langle a_i, \xi \rangle = b_i\}$ (set of active indices)

$I_M = \{I(\xi) \mid \xi \in M\}$ (system of all active index sets)

Full rank assumption: $rank\{a_i \mid (i \in I)\} = card(I) \ \ \forall I \in I_M$

$$P(M) = 1 + \sum_{I \in I_M} (-1)^{card(I)} P(\langle a_i, \xi \rangle > b_i \ \ (i \in I))$$

[1]Naiman/Wynn 1997 [2] Bukszar/H./Hujter/Szantai 2004

Formula for normal distributions of polyhedra

$$\xi \sim N(\mu, \Sigma), \quad \xi^I := -A^I \xi \sim N(-A^I \mu, A^I \Sigma (A^I)^T)$$

nondegenerate

$$P(M) = 1 + \sum_{I \in I_M} (-1)^{card(I)} F_{\xi^I}(-b^I)$$

sum of (many) lowdimensional, nondegenerate normal distribution values

Algorithm

$$I_M^0 := \{ I(\xi) | \xi \in M, card(I(\xi)) = n \} \longrightarrow I_M = \bigcup_{I \in I_M^0} 2^I \quad (*)$$

1. Determine I_M^0 by corner evaluation (e.g. Fukuda's Scdd+ code)

2. Determine I_M from (*) by extraction of all sublists (without copies)

3. Apply normal probability formula using codes for nondegenerate normal distributions (e.g. Genz, Szantai)

Numerical Comparison I

Dim:	**5 x 10**	Dim:	**5 x 15**	Dim:	**5 x 20**	Dim:	**5 x 25**
Prob.:	0.982894	Prob.:	0.942901	Prob.:	0.947281	Prob.:	0.954225
Error:	0.000001	Error:	0.000002	Error:	0.000002	Error:	0.000002
Poly:	1.00	**Poly:**	1.00	**Poly:**	1.00	**Poly:**	1.00
Szantai:	1599.89	Deak:	16200.45	Deak:	7594.09	Szantai:	597.93
Deak:	24579.00	Szantai:	105322.83	Szantai:	27126.86	Deak:	960.76
MC:	1715262.63	MC:	720219.79	MC:	360681.66	MC:	48868.45
Genz:	2236403.28	Genz:	1422991.07	Genz:	614578.55	Genz:	58847.63

Dim:	**10 x 20**	Dim:	**10 x 25**	Dim:	**15 x 20**	Dim:	**15 x 25**
Prob.:	0.974131	Prob.:	0.977193	Prob.:	0.940839	Prob.:	0.987551
Error:	0.000001	Error:	0.000001	Error:	0.000003	Error:	0.000003
Poly:	1.00	**Poly:**	1.00	**Poly:**	1.00	**Szantai:**	1.00
Szantai:	174.41	Szantai:	16.08	Szantai:	2.79	Deak:	55.42
Deak:	217.08	Deak:	67.87	Deak:	4.37	MC:	204.00
MC:	1888.59	MC:	601.72	MC:	16.68	Genz:	373.16
Genz:	3852.22	Genz:	1583.00	Genz:	21.22	Poly:	not available

Numerical Comparison II

Dim:	**5 x 10**	Dim:	**5 x 15**	Dim:	**5 x 20**	Dim:	**5 x 25**
Prob.:	0.094152	Prob.:	0.067328	Prob.:	0.116918	Prob.:	0.086062
Error:	0.000008	Error:	0.000012	Error:	0.000022	Error:	0.000029
Poly:	1.00	**Poly:**	1.00	**Poly:**	1.00	**Poly:**	1.00
Deak:	4138.71	Deak:	424.83	Deak:	61.17	Deak:	3.65
Genz:	4242.29	Genz:	1604.44	MC:	344.26	MC:	25.41
MC:	14950.12	MC:	1949.55	Genz:	407.47	Genz:	26.39
Szantai:	37242.92	Szantai:	6173.88	Szantai:	1162.01	Szantai:	92.22

Dim:	**10 x 15**	Dim:	**10 x 20**	Dim:	**10 x 25**	Dim:	**15 x 20**
Prob.:	0.121896	Prob.:	0.138410	Prob.:	0.091906	Prob.:	0.097084
Error:	0.000044	Error:	0.000064	Error:	0.000065	Error:	0.000040
Genz:	1.00	**Deak:**	1.00	**Deak**	1.00	**Genz:**	1.00
Poly:	1.02	Poly:	2.91	Genz:	2.44	Deak:	2.27
Deak:	1.52	MC:	4.15	MC:	3.40	MC:	5.07
MC:	5.00	Genz:	4.86	Szantai:	12.00	Szantai:	17.46
Szantai:	18.94	Szantai:	11.61	Poly:	41.21	Poly:	56.19

A Gradient Formula

Polyhedron formula for variable right-hand side (b = x):

$$\Phi(x) = P(M(x)) = 1 + \sum_{I \in I_{M(x)}} (-1)^{card(I)} F_{\xi^I}(-x^I)$$

Under full rank assumption $I_{M(x)}$ remains locally constant.

$$\Rightarrow \quad \nabla \Phi(x) = \sum_{I \in I_{M(x)}} (-1)^{card(I)+1} \sum_{j \in I} \frac{\partial F_{\xi^I}}{\partial x_j}(-x^I)$$