Bundlemethods

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Problem

$$
\begin{array}{llll}
\text{(P)} & \min\ c^T x, & c, x \in \mathbb{R}^n & \\
& s.t. & & \\
\text{(i)} & Ax = a, & A \in \mathbb{R}^{m' \times n}, & \text{“difficult” constraints} \\
\text{(ii)} & Bx = b, & B \in \mathbb{R}^{m'' \times n}, & \text{“easy” constraints} \\
\text{(iii)} & x \geq 0. & &
\end{array}
$$

Questions:

- find a lower bound for (P),
- calculate a solution x^* of (P).

Lagrangean Relaxation

Lagrangean relaxation of (P):

$$
\begin{array}{llrl}
\text{(L)} & \max\limits_{\lambda \in \mathbb{R}^{m'}} & \min\ c^T x & + \lambda^T (a - Ax) \\
 & & \text{s.t.} & \\
\text{(ii)} & & Bx & = b, \\
\text{(iii)} & & x & \geq 0.
\end{array}
$$

Lagrangean function:

$$
f(\lambda) := \min_{Bx=b, x \geq 0} \overbrace{\lambda^T a + (c^T - \lambda^T A)x}^{L(x;\lambda):=}
$$

Lagrangean problem: $\max_\lambda f(\lambda)$.

Conventional Subgradient Methods

Solve (L): $\max_{\lambda} \min_{Bx=b, x\geq 0} L(x;\lambda)$

1. Initialization:

   Let $i \leftarrow 0$, select Lagrangean multipliers λ_0 and a solution x_0 of $\min_{Bx=b,x\geq 0} L(x;\lambda_0)$.

2. $v_i \leftarrow a - Ax_i$

3. Update of the Lagrangean multipliers:

   $\lambda_{i+1} \leftarrow \lambda_i + s_i v_i / \|v_i\|, \quad s_i \in \mathbb{R}_+$.

4. Solve $\min_{Bx=b,x\geq 0} L(x, \lambda_{i+1})$, let x_{i+1} a solution.

5. $i \leftarrow i + 1$.

6. go to step 2.

Properties of Subgradient Methods

(P) $\min c^t x, \quad$ s. t. $Ax = a, \quad Bx = b, x \geq 0$

(L) $\max_\lambda f(\lambda).$

$f(\lambda) := min_{Bx=b,x\geq 0} L(x;\lambda), \qquad L(x;\lambda) := c^T x - \lambda^T (a - Ax)$

- $(\lambda_i)_{i=0,1,\ldots}$ converges theoretically to an optimal solution of (L), if $(s_i)_{i\in\mathbb{R}}$ is selected appropriately (Polyak 1969, Shor 1985):

$$\lim_{i\to\infty} s_i = 0, \quad \sum_{i=0}^{\infty} s_i = \infty, \quad \left(\text{e. g. } s_i = \frac{1}{i}\right)$$

- Every value of $f(\lambda)$ is a lower bound for (P).
- Numerically not stable (no convergence in practice).
- No solution for the primal problem, i. e., $Ax_i = a$ does not hold in general.

Bundlemethod: Idea

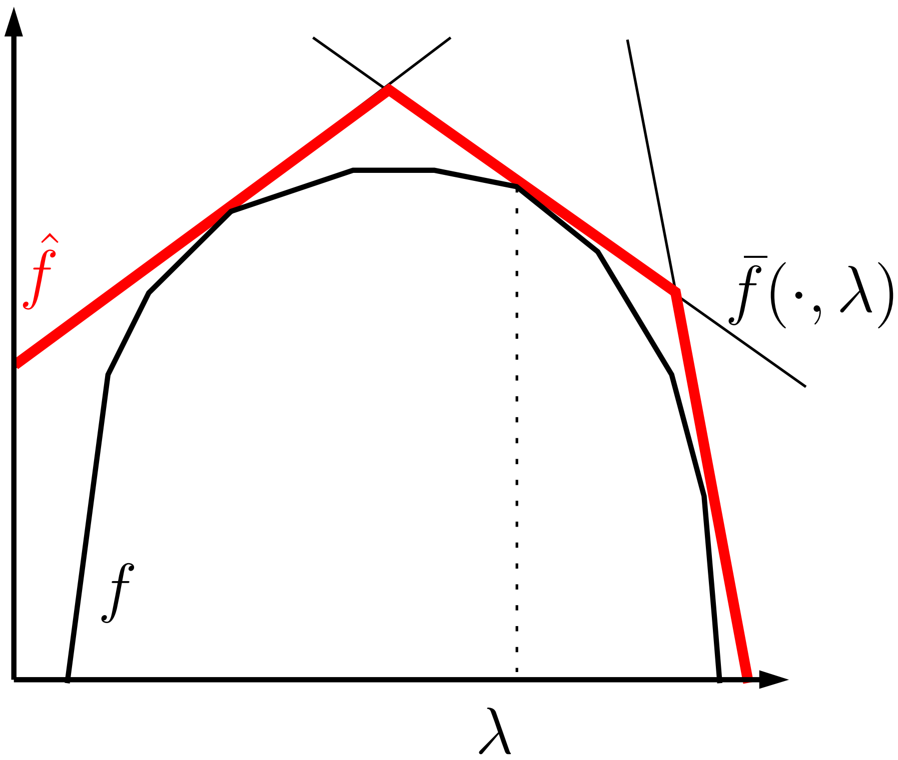
Helmberg, Kiewiel 2002

Idea: Do not use a single subgradient $a - Ax_i$ at each iteration, but a bundle of subgradients.

$$\bar{f}(\mu;\lambda) := f(\mu) + g(\mu)^T(\lambda - \mu)$$

$$\hat{f}_J(\lambda) := \min_{\mu\in J} \bar{f}(\mu;\lambda)$$

$$\tilde{f}_J(\hat{\lambda};\lambda) := \hat{f}_J(\lambda) - \|\lambda - \hat{\lambda}\|^2$$



Bundlemethod: Algorithm

Solve (L): $\max_\lambda \min_{Bx=b, x\geq 0} L(x;\lambda)$

1. Initialization:

   Let $i \leftarrow 0$, select Lagrangean multipliers λ_0 and a solution x_0 of $\min_{Bx=b,x\geq 0} L(x;\lambda_0)$. Let $\hat{\lambda}_0 \leftarrow \lambda_0$.
2. Solve $\lambda = \operatorname{argmax}_\lambda \tilde{f}_J(\hat{\lambda}_i;\lambda)$.
3. Update Lagrangean multipliers: $\lambda_{i+1} \leftarrow \lambda$.
4. Update Stability Center: if $f(\lambda)$ good enough, let $\hat{\lambda}_{i+1} \leftarrow \lambda$, otherwise $\hat{\lambda}_{i+1} \leftarrow \hat{\lambda}_i$.
5. $i \leftarrow i + 1$.
6. go to step 2.

Solving the Quadratic Problem

Problem: $\max_{\lambda} \tilde{f}_J(\hat{\lambda};\lambda)$.

Remember:
$$\tilde{f}_J(\hat{\lambda};\lambda) := \hat{f}_J(\lambda) - \|\lambda - \hat{\lambda}\|^2$$
$$= \min_{\mu \in J} \left[\bar{f}(\mu;\lambda) - \|\lambda - \hat{\lambda}\|^2 \right].$$

Then holds

$$\max_{\lambda} \tilde{f}_J(\hat{\lambda};\lambda) \Longleftrightarrow \qquad \max u - \|\lambda - \hat{\lambda}\|^2,$$
$$\text{s.t. } u - \bar{f}(\mu;\lambda) \leq 0, \quad \forall \mu \in J.$$

and this is equivalent to

$$\max \sum_{\mu \in J} \alpha_\mu \bar{f}(\mu;\lambda) - \|\sum_{\mu \in J} \alpha_\mu g(\mu)\|^2,$$
$$\text{s.t.} \quad \sum_{\mu \in J} \alpha_\mu = 1,$$
$$\alpha_\mu \geq 0, \quad \forall \mu \in J.$$

Properties of the Bundlemethod

- $(f(\hat{\lambda}_i))_{i=0,1,\dots}$ converges to the optimal value of (P).
- One can find a series $(x_i)_{i=0,1,\dots}$ converging to an optimal solution x^* of (P).
- works in applications:
  - Set-partitioning with additional knapsack constraints,
  - multi-commodity-flow,
  - shortest-path-problems with additional linear constraints (RCSP),
  - integrated duty and vehicle scheduling problem.

Implementation

Spectral Bundle Method of Ch. Helmberg for semidefinite programming available at
`http://www-user.tu-chemnitz.de/~helmberg/`

- Manages bundle,
- solves quadratic problem,
- approximates primal solution.

C and C++ Interfaces for SBMETHOD to solve linear programming are also available.

Interface

Solve problem of type

$$
\begin{array}{llll}
\text{(P)} & \min\ c^T x, & c, x \in \mathbb{R}^n & \\
\text{(i)} & \text{s.t. } Ax = a, & A \in \mathbb{R}^{m' \times n}, & \text{"difficult" constraints} \\
\text{(ii)} & Bx = b, & B \in \mathbb{R}^{m'' \times n}, & \text{"easy" constraints} \\
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\end{array}
$$

User has to implement a procedure which solves

$$\min_{Bx=b, x \geq 0} (c - \lambda^T A)x$$

for a given λ and returns a minimizer x^0, a subgradient $a - Ax^0$ and the objective value $(c - \lambda^T A)x^0$.

Extensions

- Special handling of separable functions,
- inequality constraints,
- dynamic addition of constraints,
- ϵ-subgradients.

Running Times

Set partitioning problem with 3,570 rows and 838,500 columns.