Efficient Optimization of Aerodynamic Coefficients

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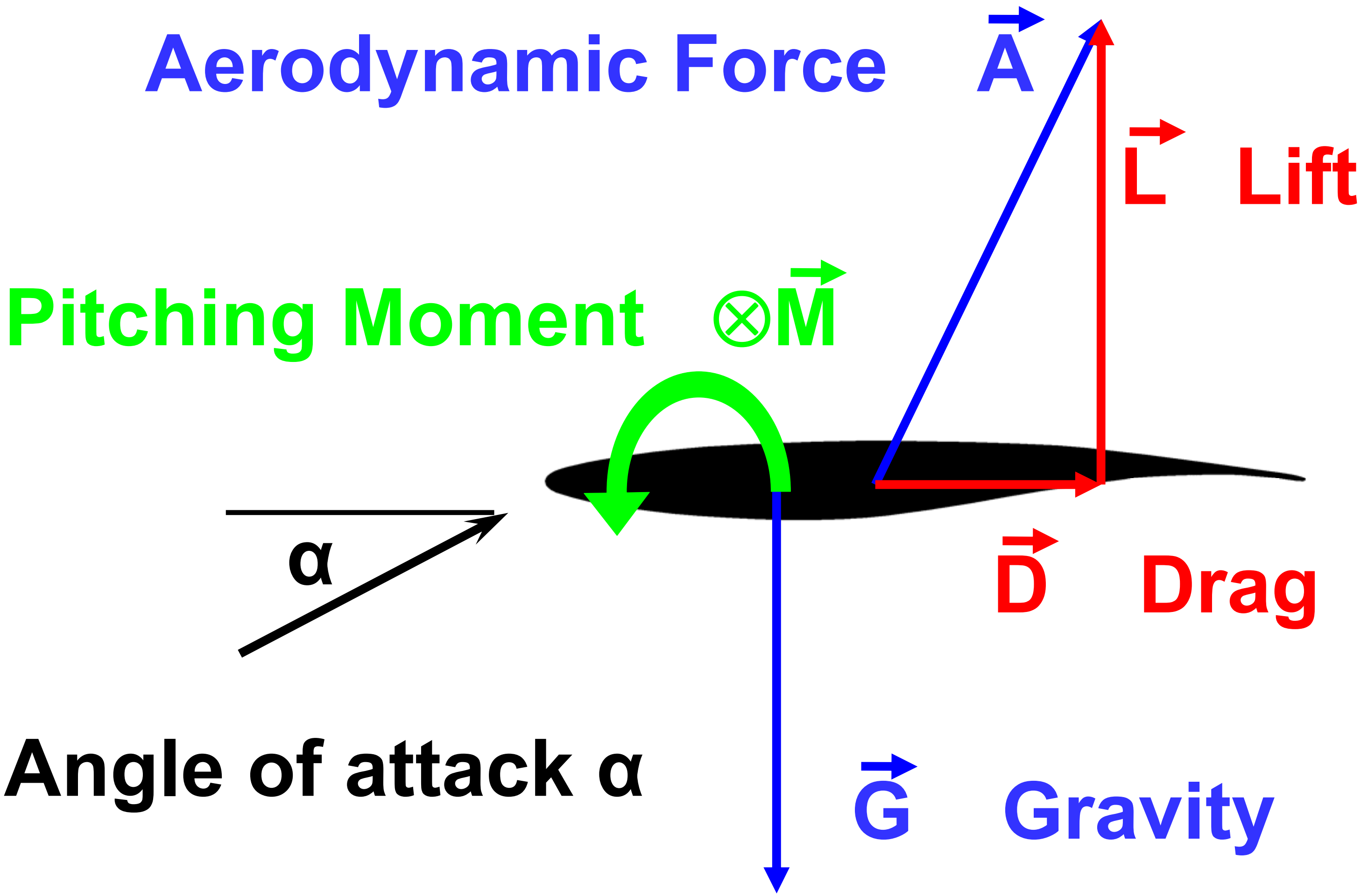
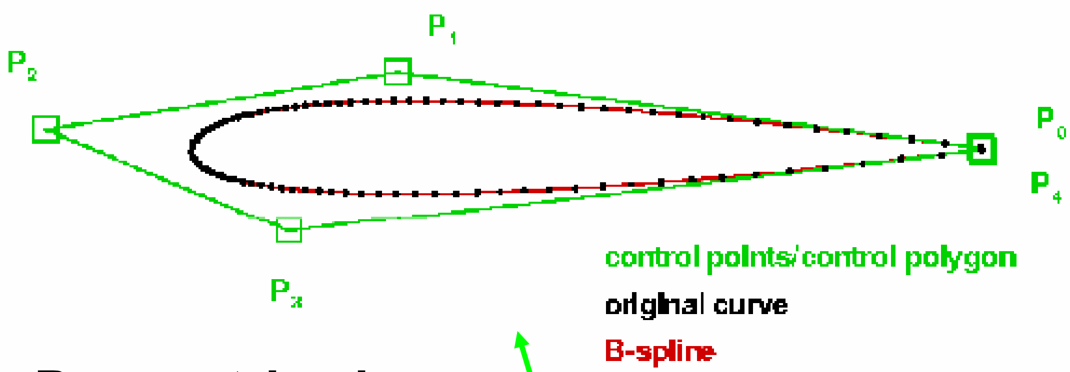
Outline Part 1

- Aerodynamic forces and moments
- Aerodynamic shape optimization
- Compressible Euler equations
- Motivation of adjoint approach
- Continuous adjoint approach
- Derivation of adjoint Euler equations
- Implementation of an adjoint Euler solver
- Validation and application of continuous adjoint approach in 2D and 3D

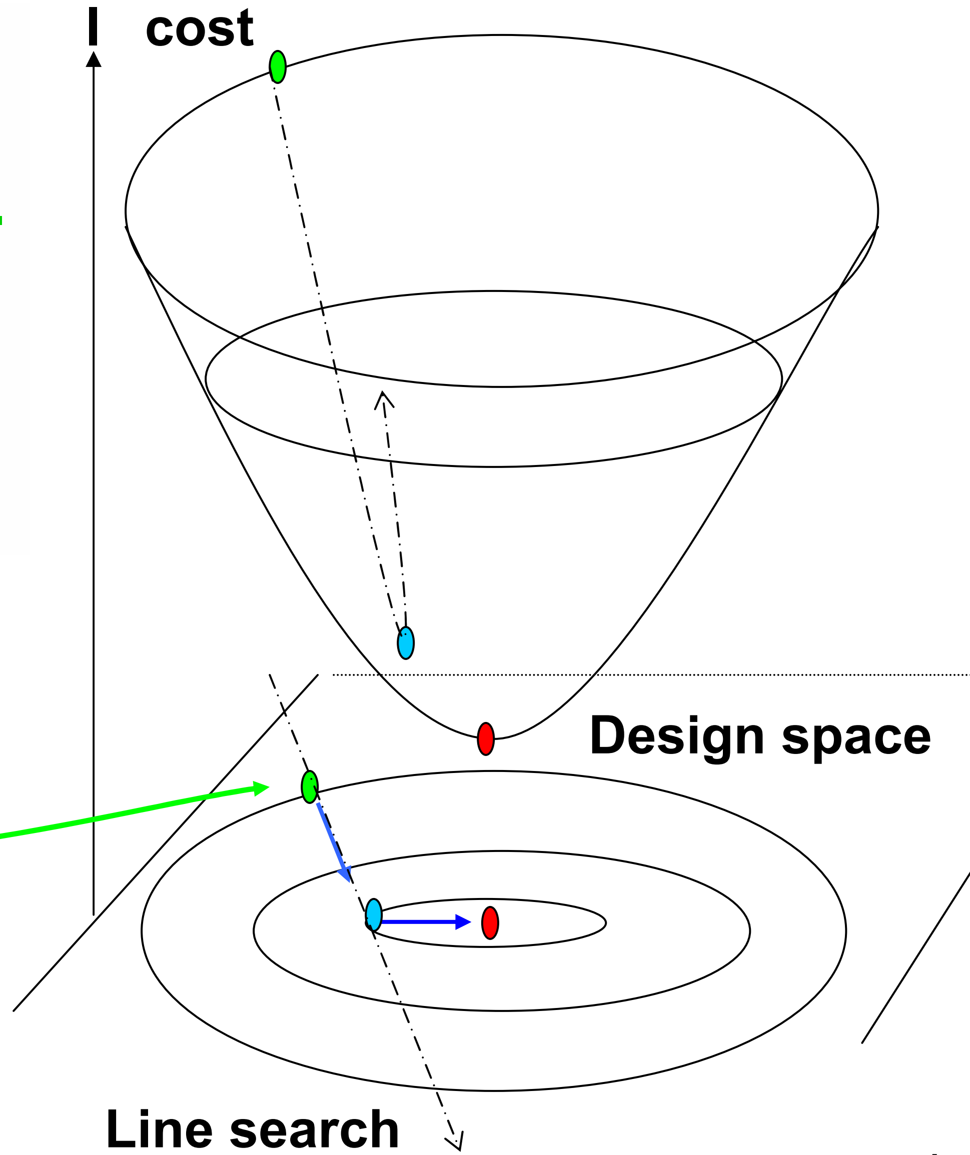
Aerodynamic Forces and Moments



Aerodynamic Shape Optimization



Parametrized airfoil



Search direction

“ ↘ “ : $$-\nabla I = -\left(\dots,\frac{\delta I}{\delta P_i},\dots\right)^T_{i=1,\dots,n}$$

Governing Equations and Aerodynamic Coefficients

Compressible 2D Euler Equations

$$\frac{\partial w}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0$$

while

$$w = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix} \quad , \quad f = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uH \end{pmatrix} \quad , \quad g = \begin{pmatrix} \rho v \\ \rho vu \\ \rho v^2 + p \\ \rho vH \end{pmatrix}$$

Pressure (ideal gas)

$$p = (\gamma - 1)\rho(E - \frac{1}{2}\vec{v}^2)$$

Dimensionless pressure

$$C_p = \frac{2(p - p_\infty)}{\gamma M_\infty^2 p_\infty}$$

Drag, lift, pitching moment coefficients

$$C_D = \frac{1}{C_{ref}} \int_C C_p (n_x \cos\alpha + n_y \sin\alpha) dl$$

$$C_L = \frac{1}{C_{ref}} \int_C C_p (n_y \cos\alpha - n_x \sin\alpha) dl$$

$$C_m = \frac{1}{C_{ref}^2} \int_C C_p (n_y (x - x_m) - n_x (y - y_m)) dl$$

Finite Differences

i-loop
i=1,...,n

Variation of i-th design variable

$$\delta C_D = \frac{2}{\gamma M_\infty^2 p_\infty C_{ref}} \int_C \delta p \,(n_x \cos\alpha + n_y \sin\alpha)\,dl$$

$$+ \frac{1}{C_{ref}} \int_C C_p (\delta n_x \cos\alpha + \delta n_y \sin\alpha)\,dl \,,$$

Metric sensitivities → pressure variation → aerodynamic sensitivity

i-th component of cost function‘s gradient

- **Finite Differences** → **n design variables require n+1 flow calculations**

Motivation of Adjoint Approach

High number of design variables

- **Finite Differences** → **n design variables require n+1 flow calculations**

- **Adjoint Approach** → **n design variables require 1 flow and 1 adjoint flow calculation**

  Independent of number of design variables

  High accuracy

Dual or Adjoint (Linear) Problem

Let be $A \in \mathrm{R}^{n\times m}$, $h \in \mathrm{R}^m$, $\varphi \in \mathrm{R}^m$ and $b \in \mathrm{R}^n$.

We define the primal lineare problem:

$$\begin{aligned} \text{evaluate } I &= h^T\varphi \,, & (1)\\ \text{while } A\varphi &= b. & (2)\end{aligned}$$

Furthermore, $\psi \in \mathbb{R}^n$ fulfills:

$$A^T\psi = h. \tag{3}$$

Then eqs. (2) and (3) imply

$$h^T\varphi = (A^T\psi)^T\varphi = (A^T\psi, \varphi) = (\psi, A\varphi) = \psi^T A\varphi = \psi^T b \quad \forall\varphi, \psi \tag{4}$$

and we have the equivalent dual or adjoint linear problem:

$$\begin{aligned} \text{evaluate } I &= \psi^T b \,, & (5)\\ \text{while } A^T\psi &= h \,. & (6)\end{aligned}$$

The vector $\psi = (\psi_i)_{i\in\{1,\ldots,n\}}$ is called the vector of adjoint variables ψ_i.

Continuous Adjoint

We define now the scalar product

$$(h, \varphi) := \int_\Omega h^T \varphi dx \ . \tag{7}$$

Let φ be the solution of the PDE

$$L\varphi = b \tag{8}$$

in the domain Ω, which fulfills the homogeneous boundary conditions on $\partial\Omega$.

Then L^*, the dual or adjoint operator of L, is defined as:

$$L^* : \quad (\psi, L\varphi) = (L^*\psi, \varphi) \quad \forall \varphi, \psi. \tag{9}$$

Furthermore, ψ, the vector(-field) of adjoint variables, solves the dual or adjoint PDE

$$L^*\psi = h \tag{10}$$

in the domain Ω and again fulfills the homogeneous boundary conditions on $\partial\Omega$.

Then finally we have as before:

$$(h, \varphi) = (L^*\psi, \varphi) = (\psi, L\varphi) = (\psi, b). \tag{11}$$

Examples of Adjoint Operators

Let's take e.g. the convection-diffusion equation

$$L\varphi \equiv \frac{d\varphi}{dx} - \epsilon\frac{d^2\varphi}{dx^2}, \quad 0 < x < 1, \tag{12}$$

with homogeneous boundary conditions $\varphi(0) = \varphi(1) = 0$.

Integration by parts yields ($\varphi, \psi \in \mathcal{C}^2$):

$$(\psi, L\varphi) = \int_0^1 \psi\left(\frac{d\varphi}{dx} - \epsilon\frac{d^2\varphi}{dx^2}\right)\, dx \tag{13}$$

$$= \int_0^1 \left(-\frac{d\psi}{dx} - \epsilon\frac{d^2\psi}{dx^2}\right)\varphi\, dx + \left[\psi\varphi - \epsilon\psi\frac{d\varphi}{dx} + \epsilon\varphi\frac{d\psi}{dx}\right]_0^1 \tag{14}$$

$$= \int_0^1 \underbrace{\left(-\frac{d\psi}{dx} - \epsilon\frac{d^2\psi}{dx^2}\right)}_{=:L^*\psi}\varphi\, dx + \left[-\epsilon\psi\frac{d\varphi}{dx}\right]_0^1. \tag{15}$$

Examples of Adjoint Operators

For the adjoint convection-diffusion equation

$$L^*\psi \equiv -\frac{d\psi}{dx} - \epsilon\frac{d^2\psi}{dx^2}, \tag{16}$$

with homogeneous boundary conditions $\psi(0) = \psi(1) = 0$, the boundary term (15) vanishes and it holds (11):

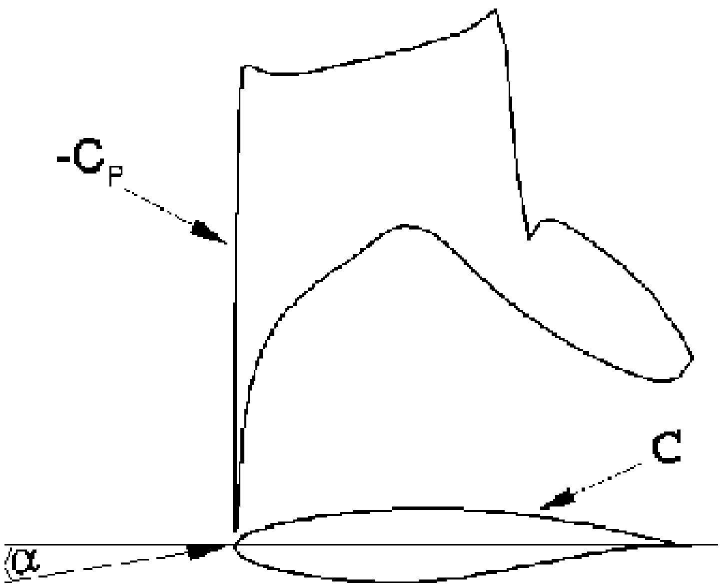
$$(h,\varphi) = (L^*\psi,\varphi) = (\psi, L\varphi) = (\psi, b).$$

Some examples:

	Operator	Adjoint
Convection-Diffusion Eq.	$\frac{d\varphi}{dx} - \epsilon\frac{d^2\varphi}{dx^2}$	$-\frac{d\psi}{dx} - \epsilon\frac{d^2\psi}{dx^2}$
Wave Eq.	$\frac{d\varphi}{dt} - \frac{d^2\varphi}{dx^2}$	$-\frac{d\psi}{dt} - \frac{d^2\psi}{dx^2}$
Convection Eq.	$\frac{d\varphi}{dt} + \frac{d\varphi}{dx}$	$-\frac{d\psi}{dt} - \frac{d\psi}{dx}$

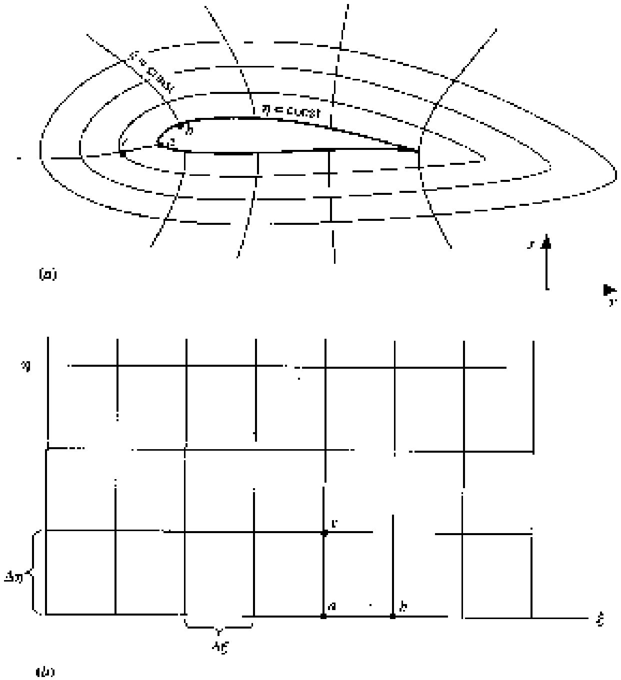
Derivation of the

Adjoint Euler Equations



D	flow field domain
B	far field
C	wall
$\partial D := B \cup C$	flow field boundary
$\vec{S} := \binom{S_x}{S_y}$	normal vector $\perp \partial D$
$\vec{n} := \binom{n_1}{n_2}$	normal unit vector $\perp \partial D$
α	angle of attack
C_D	drag coefficient
C_L	lift coefficient
p	pressure
M	Mach number
$)_\infty$	... at free stream
γ	ratio of specific heats
S_{ref}	area of airfoil
$\frac{2(p-p_\infty)}{\gamma M_\infty^2 p_\infty} =: C_p$	pressure coefficient

2D Euler Equations in body fitted coordinates



Cartesian coordinates:

$$\frac{\partial w}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0$$

$$w = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}, f = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u H \end{pmatrix}, g = \begin{pmatrix} \rho v \\ \rho v u \\ \rho v^2 + p \\ \rho v H \end{pmatrix}$$

$$p = (\gamma - 1)\rho(E - \tfrac{1}{2}(u^2 + v^2)), \qquad \rho H = \rho E + p$$

Body fitted transformation:

$(x, y) \mapsto (\xi(x, y), \eta(x, y))$,

$$J = \det \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}, \qquad \begin{pmatrix} U \\ V \end{pmatrix} = \frac{1}{J} \begin{pmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Body fitted coordinates:

$$\frac{\partial W}{\partial t} + \frac{\partial F}{\partial \xi} + \frac{\partial G}{\partial \eta} = 0$$

$$W = J \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}, F = J \begin{pmatrix} \rho U \\ \rho U u + \frac{\partial \xi}{\partial x} p \\ \rho U v + \frac{\partial \xi}{\partial y} p \\ \rho U H \end{pmatrix}, G = J \begin{pmatrix} \rho V \\ \rho V u + \frac{\partial \eta}{\partial x} p \\ \rho V v + \frac{\partial \eta}{\partial y} p \\ \rho V H \end{pmatrix}$$

Derivation of the continuous adjoint Euler equations

In the case of steady state it holds for the perturbed geometry

$$\frac{\partial}{\partial\xi}(F+\delta F)+\frac{\partial}{\partial\eta}(G+\delta G)=0$$

$$\Rightarrow \quad (1) \quad \frac{\partial}{\partial\xi}(\delta F)+\frac{\partial}{\partial\eta}(\delta G)\overset{!}{=}0.$$

Furthermore

$$(2) \qquad \delta F=\delta\left(J\frac{\partial\xi}{\partial x}\right)f+\delta\left(J\frac{\partial\xi}{\partial y}\right)g+J\frac{\partial\xi}{\partial x}\frac{\partial f}{\partial w}\delta w+J\frac{\partial\xi}{\partial y}\frac{\partial g}{\partial w}\delta w$$

and

$$(3) \qquad \delta G=\delta\left(J\frac{\partial\eta}{\partial x}\right)f+\delta\left(J\frac{\partial\eta}{\partial y}\right)g+J\frac{\partial\eta}{\partial x}\frac{\partial f}{\partial w}\delta w+J\frac{\partial\eta}{\partial y}\frac{\partial g}{\partial w}\delta w.$$

Derivation of the continuous adjoint Euler equations

Together with (1) and the fundamental lemma of variational calculus it holds

$$\int_D \psi^T \left(\frac{\partial}{\partial \xi}(\delta F) + \frac{\partial}{\partial \eta}(\delta G) \right) d\xi d\eta = 0$$

for any Lagrangian multiplier ψ.

If ψ is differentiable one obtains together with Greens formula

$$-\int_D \left(\frac{\partial \psi^T}{\partial \xi} \delta F + \frac{\partial \psi^T}{\partial \eta} \delta G \right) d\xi d\eta + \int_B (n_1 \psi^T \delta F + n_2 \psi^T \delta G) d\xi - \int_C (n_1 \psi^T \delta F + n_2 \psi^T \delta G) d\xi = 0.$$

Derivation of the continuous adjoint Euler equations

Now the variation of the cost function can be expressed as

$$\begin{aligned}\delta C_D \; = \; & \frac{2}{\gamma M_\infty^2 p_\infty S_{ref}} \int_C \delta p (S_x \cos\alpha + S_y \sin\alpha)\, d\xi - \int_D \left(\frac{\partial \psi^T}{\partial \xi} \delta F + \frac{\partial \psi^T}{\partial \eta} \delta G \right) d\xi d\eta \\ & + \int_B (n_1 \psi^T \delta F + n_2 \psi^T \delta G) d\xi - \int_C (\underbrace{n_1 \psi^T \delta F}_{=\,0,\; n_1 = 0} + n_2 \psi^T \delta G) d\xi \\ & + \frac{1}{S_{ref}} \int_C C_p (\delta S_x \cos\alpha + \delta S_y \sin\alpha)\, d\xi. \end{aligned}$$

Along C it holds $V = 0$ and yields

$$G = J \begin{pmatrix} 0 \\ \frac{\partial \eta}{\partial x} p \\ \frac{\partial \eta}{\partial y} p \\ 0 \end{pmatrix}, \qquad \delta G = J \begin{pmatrix} 0 \\ \frac{\partial \eta}{\partial x} \delta p \\ \frac{\partial \eta}{\partial y} \delta p \\ 0 \end{pmatrix} + p \begin{pmatrix} 0 \\ \delta \left(J \frac{\partial \eta}{\partial x} \right) \\ \delta \left(J \frac{\partial \eta}{\partial y} \right) \\ 0 \end{pmatrix}.$$

Derivation of the continuous adjoint Euler equations

Together with (2) and (3) one obtains

$$\begin{aligned}
\delta C_D = & \frac{2}{\gamma M_\infty^2 p_\infty S_{ref}} \int_C \delta p (S_x \cos\alpha + S_y \sin\alpha)\, d\xi \\
& - \int_D \frac{\partial \psi^T}{\partial \xi} \left(\delta \left(J \frac{\partial \xi}{\partial x} \right) f + \delta \left(J \frac{\partial \xi}{\partial y} \right) g + J \frac{\partial \xi}{\partial x} \frac{\partial f}{\partial w} \delta w + J \frac{\partial \xi}{\partial y} \frac{\partial g}{\partial w} \delta w \right) \\
& \qquad + \frac{\partial \psi^T}{\partial \eta} \left(\delta \left(J \frac{\partial \eta}{\partial x} \right) f + \delta \left(J \frac{\partial \eta}{\partial y} \right) g + J \frac{\partial \eta}{\partial x} \frac{\partial f}{\partial w} \delta w + J \frac{\partial \eta}{\partial y} \frac{\partial g}{\partial w} \delta w \right) d\xi d\eta \\
& - \int_C \psi_2 \left(J \frac{\partial \eta}{\partial x} \delta p + p \delta \left(J \frac{\partial \eta}{\partial x} \right) \right) + \psi_3 \left(J \frac{\partial \eta}{\partial y} \delta p + p \delta \left(J \frac{\partial \eta}{\partial y} \right) \right) d\xi \\
& + \int_B n_1 \psi^T \delta F + n_2 \psi^T \delta G d\xi + \frac{1}{S_{ref}} \int_C C_p (\delta S_x \cos\alpha + \delta S_y \sin\alpha)\, d\xi .
\end{aligned}$$

If the adjoint Euler equations

$$\frac{\partial \psi^T}{\partial \xi} \left(J \frac{\partial \xi}{\partial x} \frac{\partial f}{\partial w} + J \frac{\partial \xi}{\partial y} \frac{\partial g}{\partial w} \right) + \frac{\partial \psi^T}{\partial \eta} \left(J \frac{\partial \eta}{\partial x} \frac{\partial f}{\partial w} + J \frac{\partial \eta}{\partial y} \frac{\partial g}{\partial w} \right) = 0 \quad \Leftrightarrow \left(\frac{\partial f}{\partial w} \right)^T \frac{\partial \psi}{\partial x} + \left(\frac{\partial g}{\partial w} \right)^T \frac{\partial \psi}{\partial y} = 0$$

Derivation of the continuous adjoint Euler equations

... are fulfilled in the domain D with the boundary conditions

$$\frac{2}{\gamma M_\infty^2 p_\infty S_{ref}}(S_x \cos\alpha + S_y \sin\alpha) = \underbrace{-S_x\psi_2 - S_y\psi_3}_{-\frac{\partial y}{\partial \xi}\psi_2 + \frac{\partial x}{\partial \xi}\psi_3 = J\frac{\partial \eta}{\partial x}\psi_2 + J\frac{\partial \eta}{\partial y}\psi_3}$$

on the airfoil C **(dependent on the cost function!)** and

$$\delta\left(J\frac{\partial \xi}{\partial x}\right), \ldots, \delta\left(J\frac{\partial \eta}{\partial y}\right) \to 0 \qquad \psi^T J \frac{\partial \xi}{\partial x}\frac{\partial f}{\partial w}\delta w = 0, \ldots, \psi^T J \frac{\partial \eta}{\partial y}\frac{\partial g}{\partial w}\delta w = 0$$

at the far field B one can simplify δC_D to

$$\begin{aligned}\delta C_D \;=\; & -\int_D \frac{\partial \psi^T}{\partial \xi}\left(\delta\left(\frac{\partial y}{\partial \eta}\right) f - \delta\left(\frac{\partial x}{\partial \eta}\right) g\right) + \frac{\partial \psi^T}{\partial \eta}\left(-\delta\left(\frac{\partial y}{\partial \xi}\right) f + \delta\left(\frac{\partial x}{\partial \xi}\right) g\right) d\xi d\eta \\ & - \int_C p\,(\delta S_x \psi_2 + \delta S_y \psi_3)\, d\xi + \frac{1}{S_{ref}} \int_C C_p(\delta S_x \cos\alpha + \delta S_y \sin\alpha)\, d\xi .\end{aligned}$$

Continuous Adjoint Approach

Adjoint Euler-Equations:

$$-\frac{\partial \psi}{\partial t}-\left(\frac{\partial f}{\partial w}\right)^T \frac{\partial \psi}{\partial x}-\left(\frac{\partial g}{\partial w}\right)^T \frac{\partial \psi}{\partial y}=0$$

Ψ: Vector of adjoint variables

Boundary conditions:

Wall: $n_x \psi_2 + n_y \psi_3 = \underline{-d(I)}$

Farfield: $\delta x_\xi ,\ldots, \delta y_\eta = 0, \;\; \delta w = 0$

Adjoint formulation of cost function's gradient:

$$\delta I = -\int_C p(-\psi_2 \delta y_\xi + \psi_3 \delta x_\xi) dl + \underline{K(I)}$$
$$-\int_D \psi_\xi^T (\delta y_\eta f - \delta x_\eta g) + \psi_\eta^T (-\delta y_\xi f + \delta x_\xi g) dA$$

Continuous Adjoint Approach

$$d(C_D) = \frac{2}{\gamma M_\infty^2 p_\infty C_{ref}} (n_x \cos\alpha + n_y \sin\alpha)$$

Drag

$$K(C_D) = \frac{1}{C_{ref}} \int_C C_p (\delta n_x \cos\alpha + \delta n_y \sin\alpha) dl$$

$$d(C_L) = \frac{2}{\gamma M_\infty^2 p_\infty C_{ref}} (n_y \cos\alpha - n_x \sin\alpha)$$

Lift

$$K(C_L) = \frac{1}{C_{ref}} \int_C C_p (\delta n_y \cos\alpha - \delta n_x \sin\alpha) dl$$

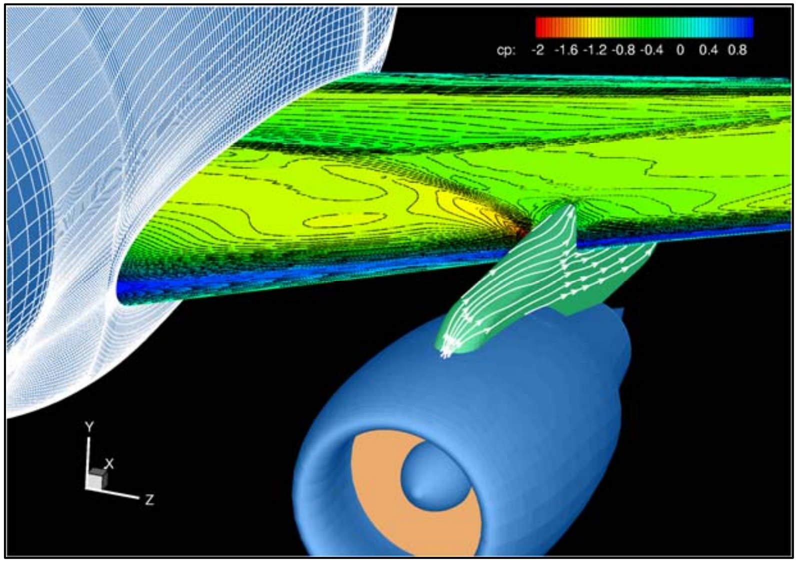
$$d(C_m) = \frac{2}{\gamma M_\infty^2 p_\infty C_{ref}^2} (n_y (x - x_m) - n_x (y - y_m))$$

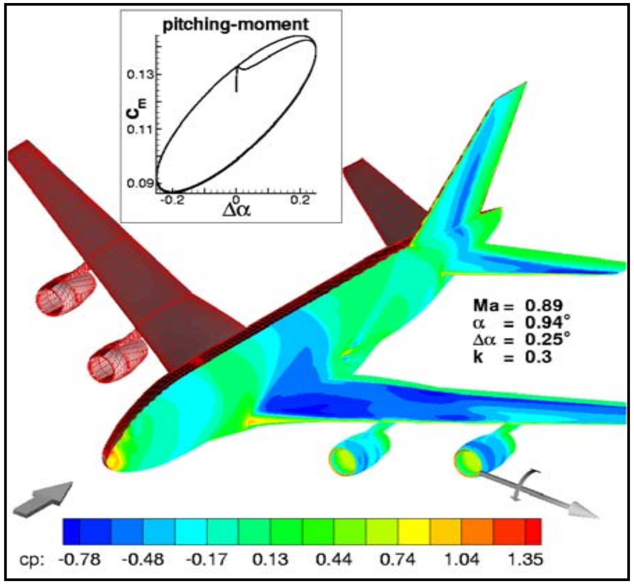
Pitching moment

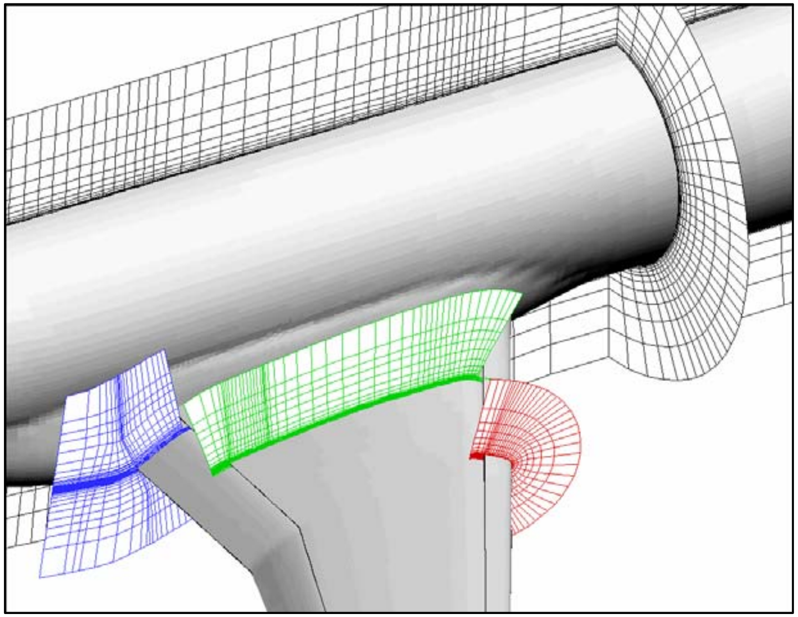
$$K(C_m) = \frac{1}{C_{ref}^2} \int_C C_p \delta (n_y (x - x_m) - n_x (y - y_m)) dl$$

Block-Structured RANS Solver FLOWer
(Reynolds-Averaged Navier-Stokes)

- advanced turbulence and transition models
- 2nd order finite volume discretization (cell centered & cell vertex option)
- central and upwind schemes
- multigrid
- implicit treatment of turbulence equations
- implicit schemes for time accurate flows
- preconditioning for low speed flow
- design option (inverse design, adjoint)



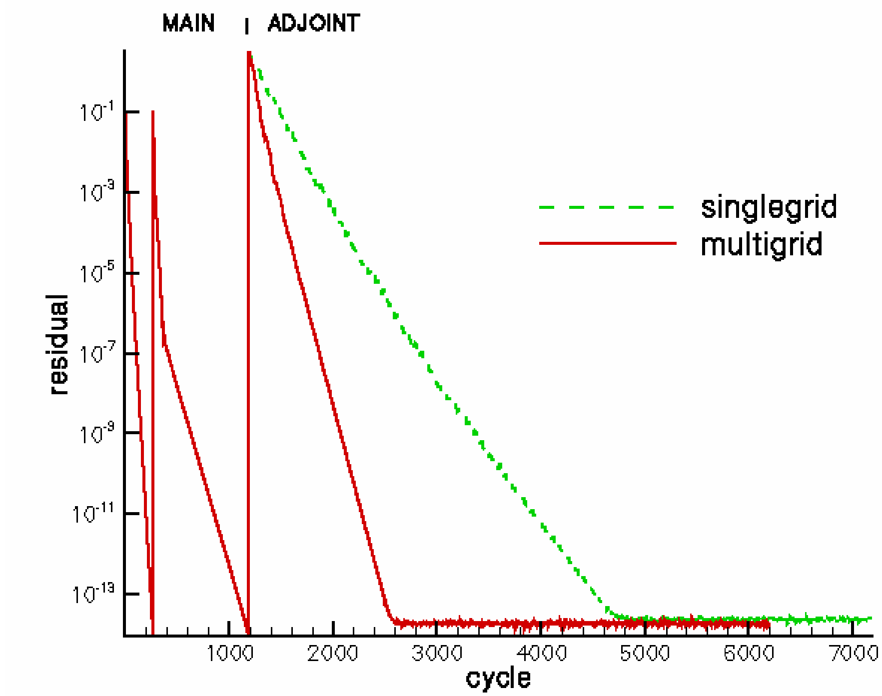




Adjoint Flow Solver

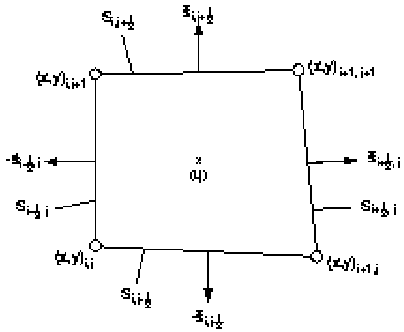
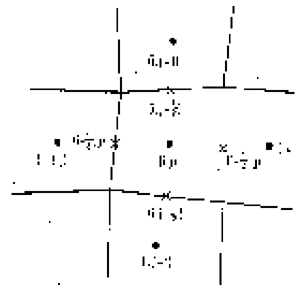
Adjoint solver on block-structured grids

- continuous adjoint approach
- implemented in FLOWer
- cost functions: lift, drag & moment and combinations
- adjoint solver based on multigrid
- Euler & Navier-Stokes (frozen μ)
- AD for turbulence equations (*Fast*Opt)



convergence history, FLOWer

Spatial discretization of the adjoint Euler equations

$$\frac{\partial}{\partial t}\int_{V_{i,j}} \psi dV + \int_{\partial V_{i,j}} \overline{\overline{F}} \cdot \vec{n} ds = 0, \qquad \overline{\overline{F}} = \left[\left(\frac{\partial f}{\partial w}\right)^T \psi, \left(\frac{\partial g}{\partial w}\right)^T \psi \right]$$

$$\psi_{i,j} := \frac{1}{V_{i,j}} \int_{V_{i,j}} \psi dV, \quad \textbf{e.g.} \quad \overline{\overline{F}}_{i,j+\frac{1}{2}} := \left[\left(\frac{\partial f}{\partial w}\right)^T_{i,j} \frac{\psi_{i,j+1}+\psi_{i,j}}{2}, \left(\frac{\partial g}{\partial w}\right)^T_{i,j} \frac{\psi_{i,j+1}+\psi_{i,j}}{2} \right],$$

Adjoint flux: $\tilde{Q}_{i,j} := \overline{\overline{F}}_{i,j+\frac{1}{2}} \vec{S}_{i,j+\frac{1}{2}} - \overline{\overline{F}}_{i,j-\frac{1}{2}} \vec{S}_{i,j-\frac{1}{2}} + \overline{\overline{F}}_{i+\frac{1}{2},j} \vec{S}_{i+\frac{1}{2},j} - \overline{\overline{F}}_{i-\frac{1}{2},j} \vec{S}_{i-\frac{1}{2},j},$

$$\begin{pmatrix} \bar{s}_x^{(j)} \\ \bar{s}_y^{(j)} \end{pmatrix} := \frac{\vec{S}_{i,j+\frac{1}{2}} + \vec{S}_{i,j-\frac{1}{2}}}{2}, \quad \begin{pmatrix} \bar{s}_x^{(i)} \\ \bar{s}_y^{(i)} \end{pmatrix} := \frac{\vec{S}_{i+\frac{1}{2},j} + \vec{S}_{i-\frac{1}{2},j}}{2},$$

$$\Rightarrow Q_{i,j} = \left(\bar{s}_x^{(j)} \left(\frac{\partial f}{\partial w}\right)^T_{i,j} + \bar{s}_y^{(j)} \left(\frac{\partial g}{\partial w}\right)^T_{i,j} \right) \frac{\psi_{i,j+1}-\psi_{i,j-1}}{2} + \left(\bar{s}_x^{(i)} \left(\frac{\partial f}{\partial w}\right)^T_{i,j} + \bar{s}_y^{(i)} \left(\frac{\partial g}{\partial w}\right)^T_{i,j} \right) \frac{\psi_{i+1,j}-\psi_{i-1,j}}{2}$$

$V_{i,j}\frac{\partial \psi_{i,j}}{\partial t} + Q_{i,j} + D_{i,j} = 0,$ $\quad D_{i,j}$: artificial dissipation

Efficient evaluation of the adjoint flux

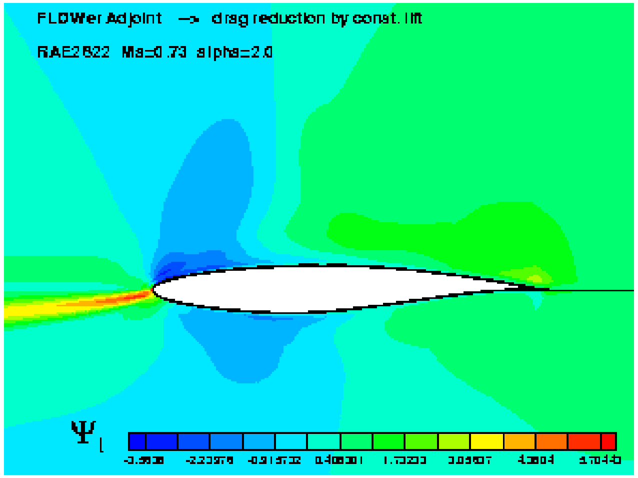
$$Q_{i,j} = \left(\bar{s}_x^{(j)} \left(\frac{\partial f}{\partial w}\right)_{i,j}^T + \bar{s}_y^{(j)} \left(\frac{\partial g}{\partial w}\right)_{i,j}^T \right) \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2} + \left(\bar{s}_x^{(i)} \left(\frac{\partial f}{\partial w}\right)_{i,j}^T + \bar{s}_y^{(i)} \left(\frac{\partial g}{\partial w}\right)_{i,j}^T \right) \frac{\psi_{i+1,j} - \psi_{i-1,j}}{2}$$

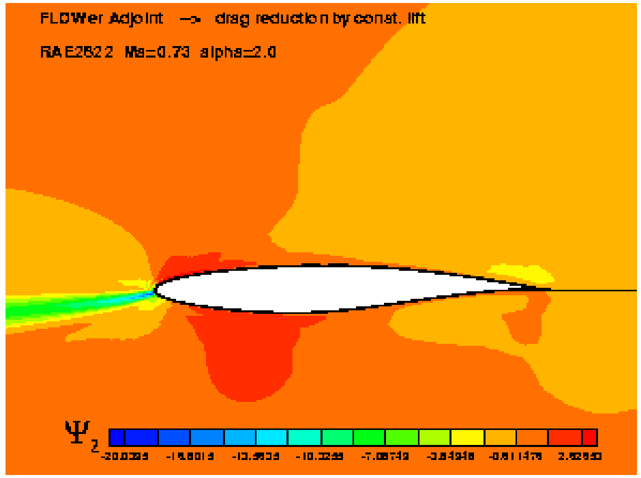
$$T^{-1} \underbrace{T \left(\bar{s}_x^{(j)} \left(\frac{\partial f}{\partial w}\right)_{i,j}^T + \bar{s}_y^{(j)} \left(\frac{\partial g}{\partial w}\right)_{i,j}^T \right) T^{-1}}_{=:Q_{i,j}^{(j)}} T, \quad \hat{\psi}_{i,j}^{(j)} := T \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2},$$

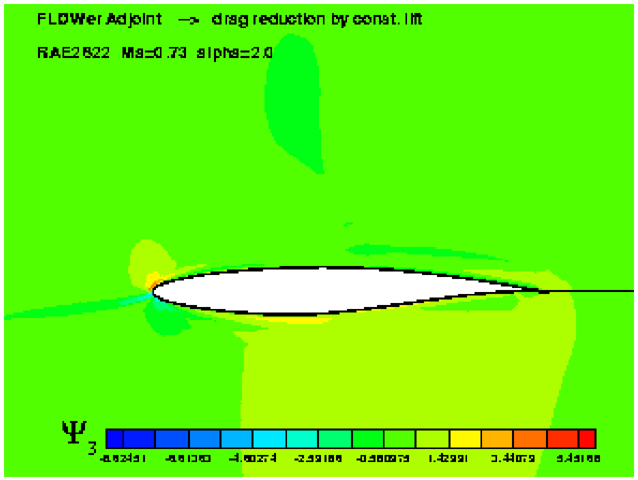
$$Q_{i,j}^{(j)} = \begin{pmatrix} u\bar{s}_x^{(j)} + v\bar{s}_y^{(j)} & 0 & 0 & 0 \\ \bar{s}_x^{(j)} & u\bar{s}_x^{(j)} + v\bar{s}_y^{(j)} & 0 & \frac{\gamma}{\gamma-1}\frac{p}{\rho}\bar{s}_x^{(j)} \\ \bar{s}_y^{(j)} & 0 & u\bar{s}_x^{(j)} + v\bar{s}_y^{(j)} & \frac{\gamma}{\gamma-1}\frac{p}{\rho}\bar{s}_y^{(j)} \\ 0 & (\gamma-1)\bar{s}_x^{(j)} & (\gamma-1)\bar{s}_y^{(j)} & u\bar{s}_x^{(j)} + v\bar{s}_y^{(j)} \end{pmatrix},$$

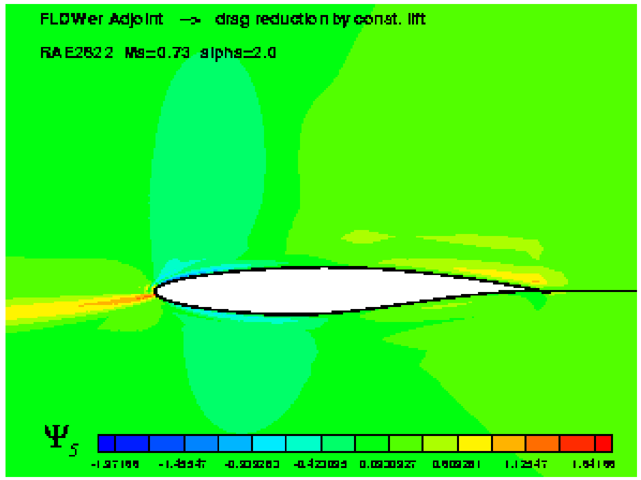
$$T^{-1} = \begin{pmatrix} 1 & -u & -v & \frac{1}{2}(u^2+v^2) \\ 0 & 1 & 0 & -u \\ 0 & 0 & 1 & -v \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \Rightarrow Q_{i,j} = T^{-1}\left(Q_{i,j}^{(j)} \hat{\psi}_{i,j}^{(j)} + Q_{i,j}^{(i)} \hat{\psi}_{i,j}^{(i)} \right)$$

Adjoint vector calculated by FLOWer









Calculation of the gradient

'Grid moving technique' based on J. Reuther

$$\delta C_D = -\int_D \frac{\partial \psi^T}{\partial \xi}\left(\delta y_\eta f - \delta x_\eta g\right) + \frac{\partial \psi^T}{\partial \eta}\left(-\delta y_\xi f + \delta x_\xi g\right) d\xi d\eta$$
$$- \int_C p\left(-\psi_2 \delta y_{\xi_s} + \psi_3 \delta x_{\xi_s}\right) d\xi + \frac{1}{S_{ref}} \int_C C_p (\delta y_{\xi_s} \cos\alpha - \delta x_{\xi_s} \sin\alpha) d\xi$$

Distance function e.g. $R(\eta) = 1 - \frac{\eta}{\eta_B}$

Values at wall C: $\eta = 0 \rightarrow R(0) = 1$

Values at far field B: $\eta = \eta_B \rightarrow R(\eta_B) = 0$

The grid for the perturbed (new) geometry is defined as:

$$x_{new} := x_{old} + R \cdot (x_{new_s} - x_{old_s}),$$

$$y_{new} := y_{old} + R \cdot (y_{new_s} - y_{old_s}).$$

Index s means the values at the surface.

Calculation of the gradient

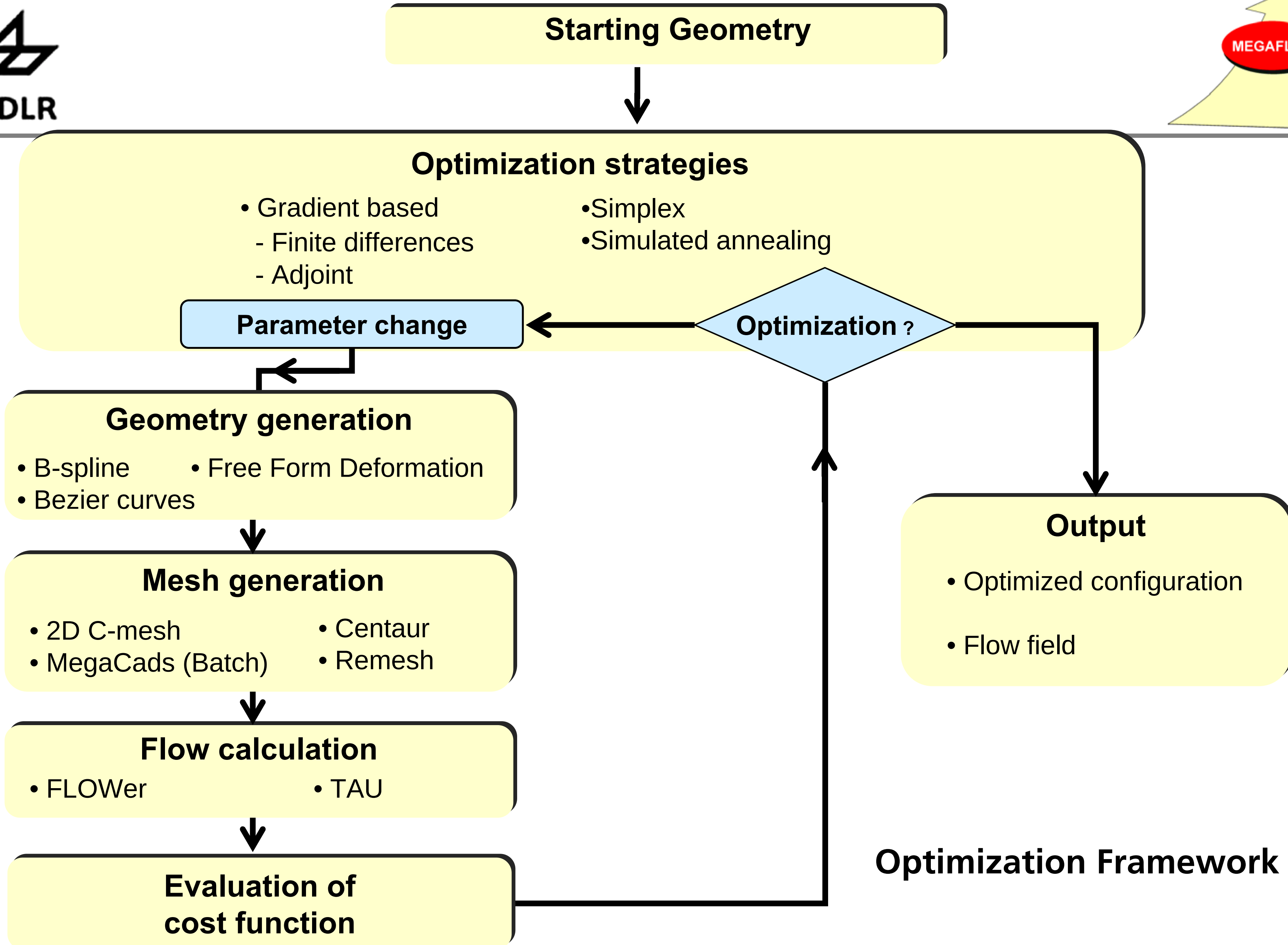
Now the metric sensitivities can be expressed as

$$\delta x_\xi \approx R \cdot \delta x_{\xi_s}, \quad \delta y_\xi \approx R \cdot \delta y_{\xi_s},$$

$$\delta x_\eta \approx R \cdot \delta x_{\eta_s}, \quad \delta y_\eta \approx R \cdot \delta y_{\eta_s}$$

and we get the gradient as an integral in the curve C:

$$\begin{aligned}\delta C_D \approx & -\int_C \left(\delta y_{\eta_s} \int_\eta \frac{\partial \psi^T}{\partial \xi} (R(\eta) \cdot f) d\eta \right) d\xi + \int_C \left(\delta x_{\eta_s} \int_\eta \frac{\partial \psi^T}{\partial \xi} (R(\eta) \cdot g) d\eta \right) d\xi \\ & + \int_C \left(\delta y_{\xi_s} \int_\eta \frac{\partial \psi^T}{\partial \eta} (R(\eta) \cdot f) d\eta \right) d\xi - \int_C \left(\delta x_{\xi_s} \int_\eta \frac{\partial \psi^T}{\partial \eta} (R(\eta) \cdot g) d\eta \right) d\xi \\ & - \int_C p \left(-\psi_2 \delta y_{\xi_s} + \psi_3 \delta x_{\xi_s} \right) d\xi + \frac{1}{S_{ref}} \int_C C_p (\delta y_{\xi_s} \cos \alpha - \delta x_{\xi_s} \sin \alpha) d\xi.\end{aligned}$$



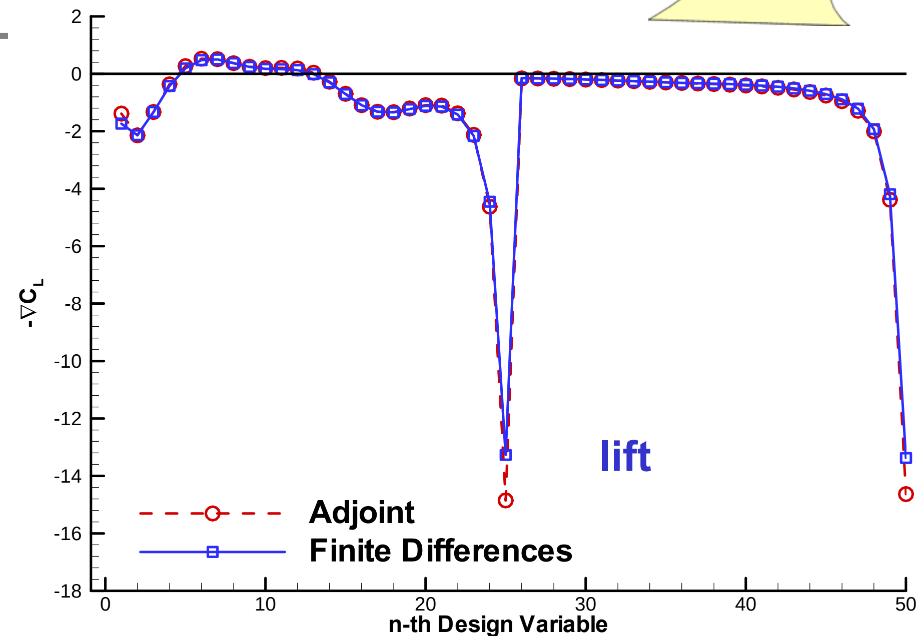
Optimization Framework

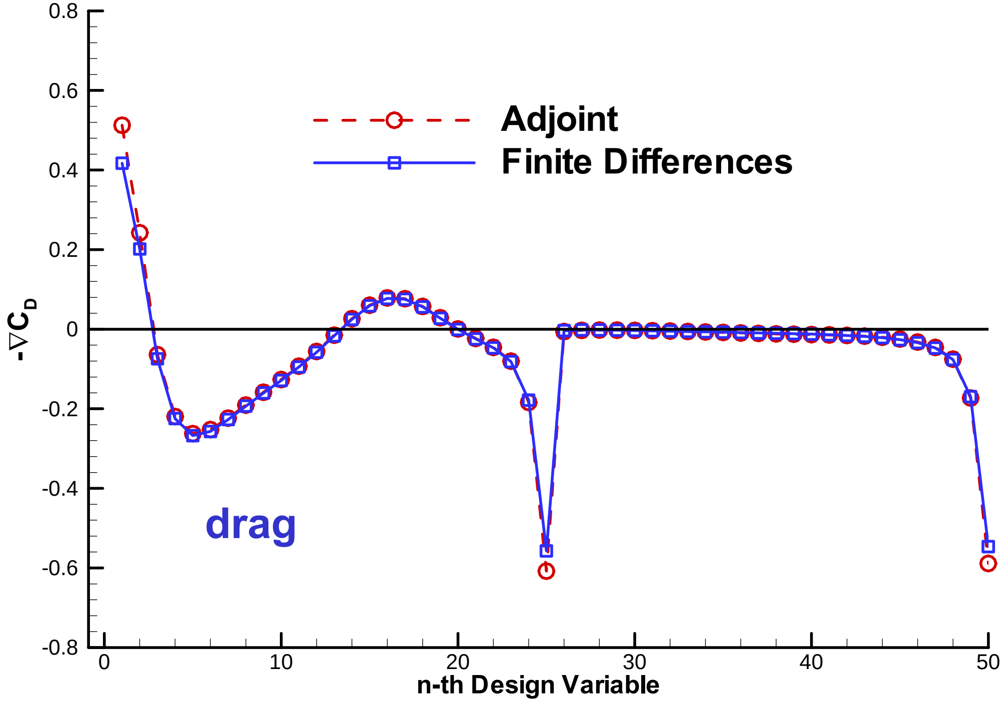
Validation of Euler Adjoint

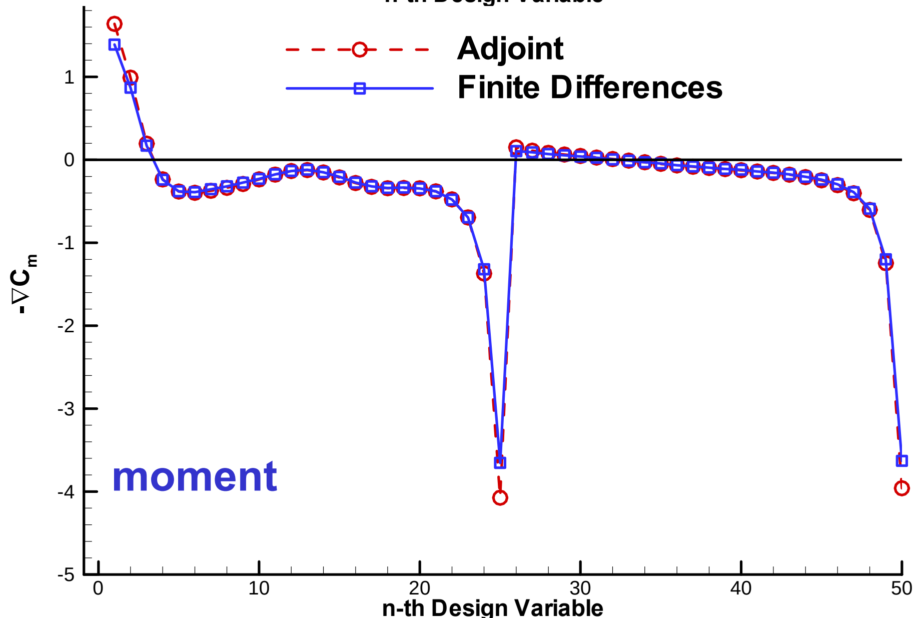
adjoint gradient vs. finite differences' gradient

finite differences:
51 calls of FLOWer MAIN
adjoint approach:
1 call of FLOWer MAIN
3 calls of FLOWer ADJOINT

RAE2822
$M_\infty = 0.73$, $\alpha = 2.0°$
50 design variables
(B-spline)







Validation of adjoint gradient based optimization

Objective function

- Drag reduction for RAE 2822 airfoil
- $M_\infty = 0.73$, $\alpha = 2.00°$

Constraints

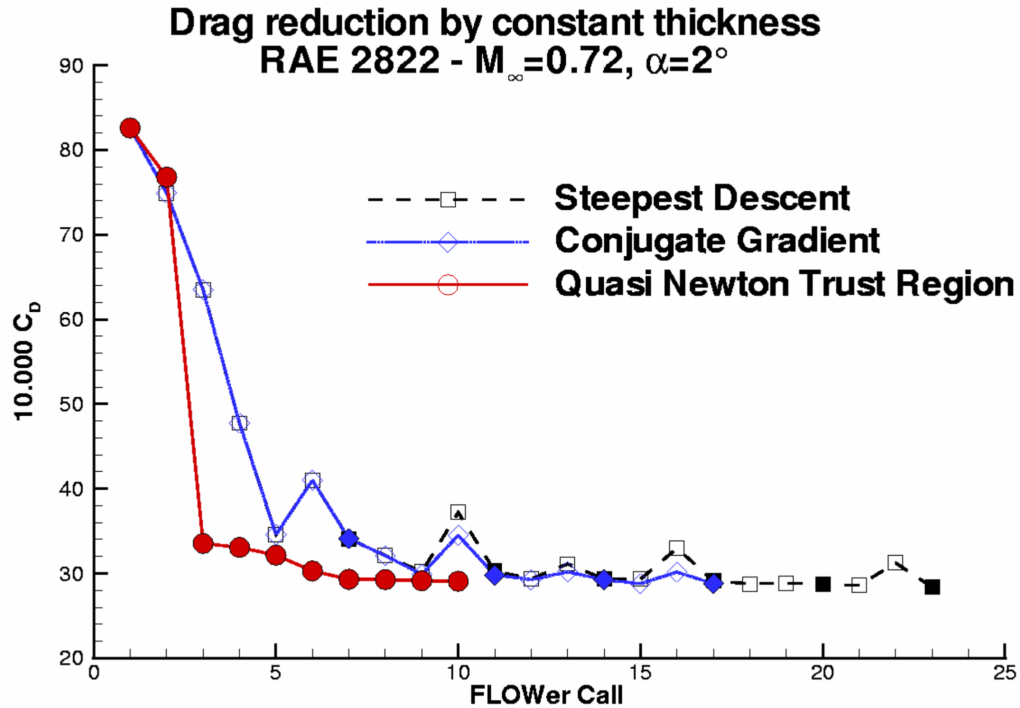
- Constant thickness

Approach

- FLOWer Euler Adjoint
- Deformation of camberline (20 Hicks-Henne functions)

Optimizer

- Steepest Descent
- Conjugate Gradient
- Quasi Newton Trust Region



Validation of adjoint gradient based optimization

Objective function

- Drag reduction for RAE 2822 airfoil
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Constraints

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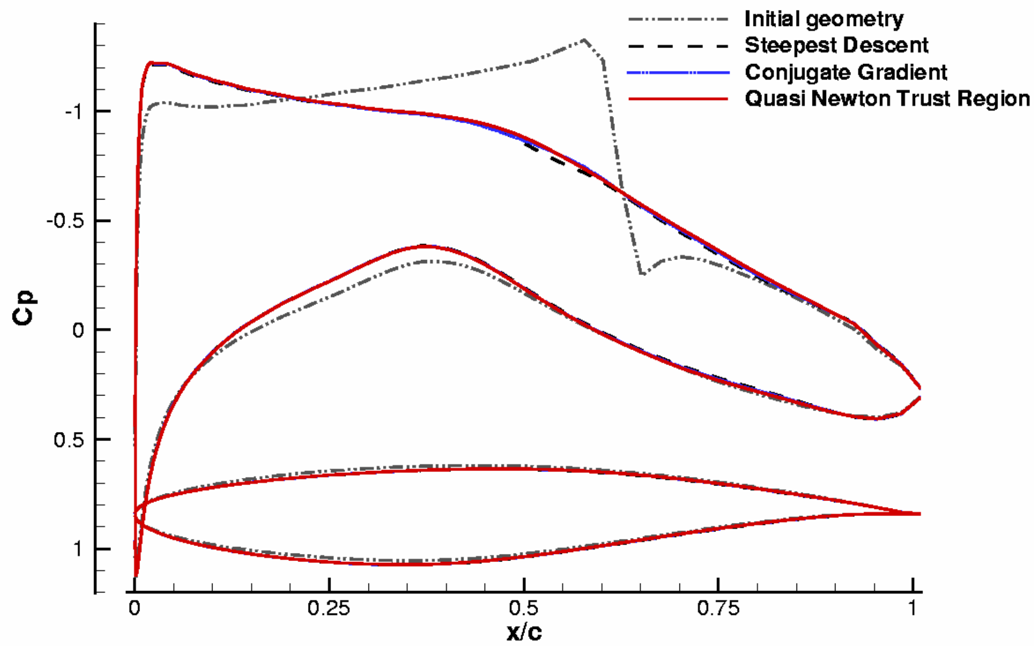
Approach

- FLOWer Euler Adjoint
- Deformation of camberline (20 Hicks-Henne functions)



Optimizer

- Steepest Descent
- Conjugate Gradient
- Quasi Newton Trust Region

Multi-constraint airfoil optimization RAE2822

Objective function

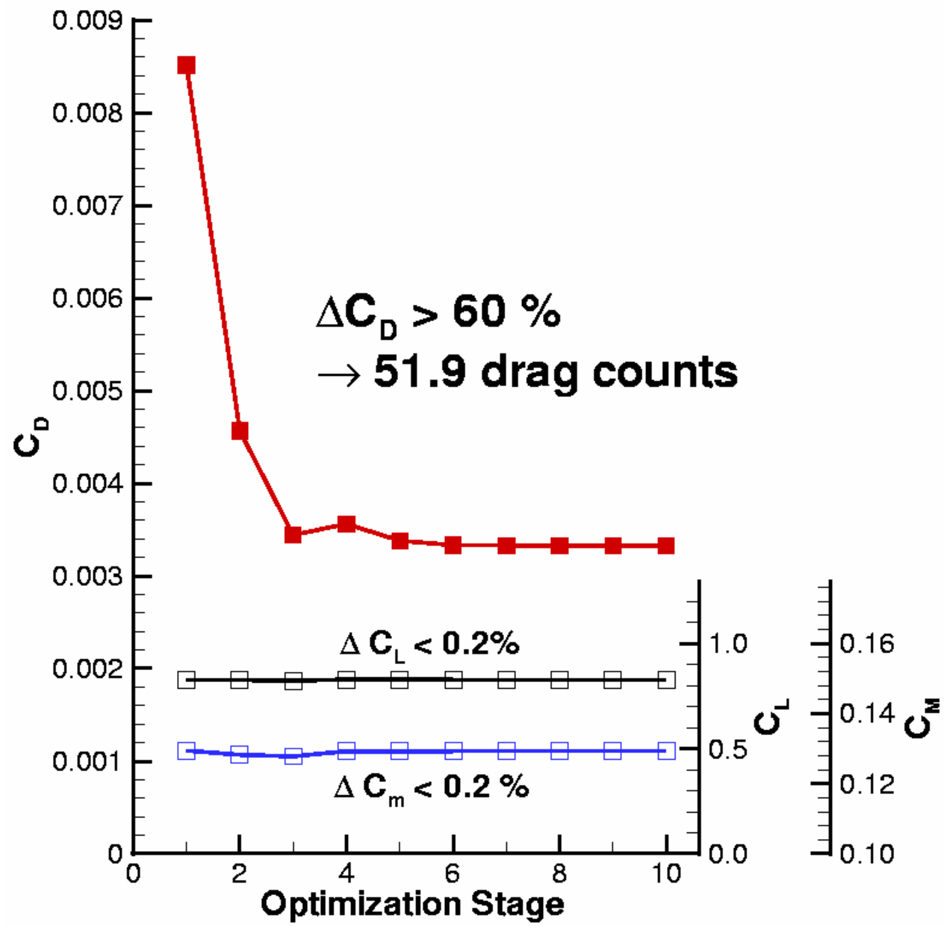
- Drag reduction for RAE 2822 airfoil
- $M_\infty = 0.73$, $\alpha = 2.0°$

Constraints

- Lift, pitching moment and angle of attack held constant
- Constant thickness

Approach

- FLOWer Euler Adjoint
- Constraints handled by feasible direction
- Deformation of camberline



Multi-constraint airfoil optimization RAE2822

Objective function

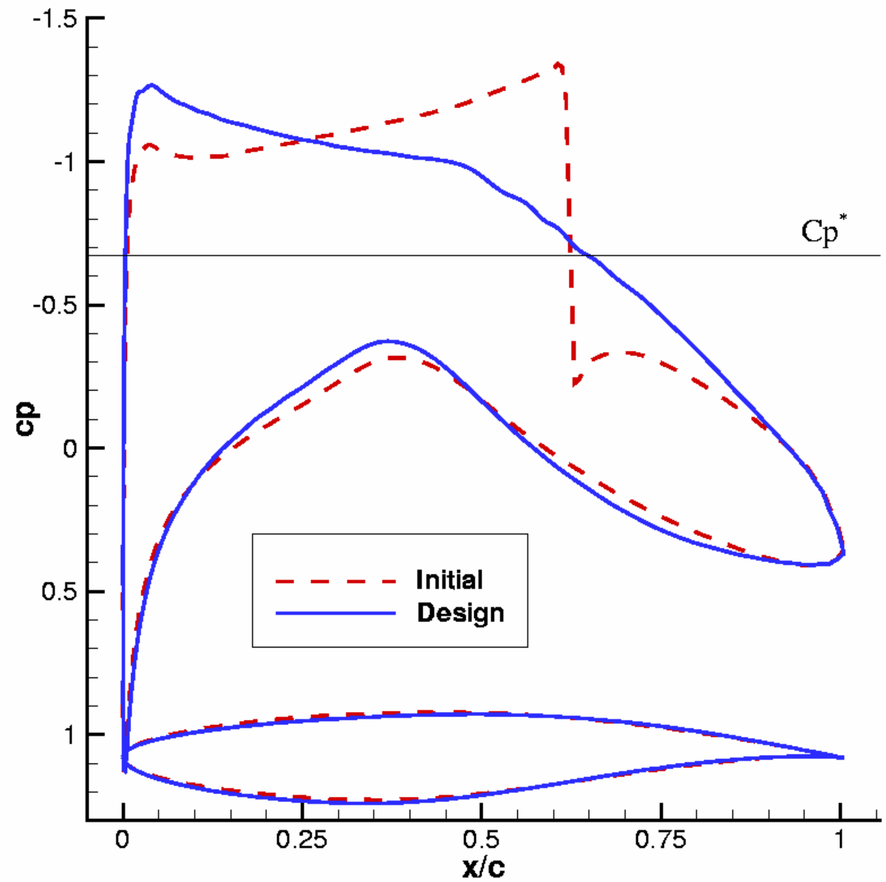
- Drag reduction for RAE 2822 airfoil
- M_∞ =0.73, α=2.0°

Constraints

- Lift, pitching moment and angle of attack held constant
- Constant thickness

Approach

- FLOWer Euler Adjoint
- Constraints handled by feasible direction
- Deformation of camberline



surface pressure distribution

Multipoint airfoil optimization RAE2822

Objective function

- Reduction of drag in 2 design points

$$I = \sum_{i=1}^{2} W_i C_d(\alpha_i, M_i)$$

Design points

- 1 : M_∞=0.734, CL = 0.80 , α = 2.8°, Re=6.5×10^6, xtrans=3%, W_1=2
- 2 : M_∞=0.754, CL = 0.74 , α = 2.8°, Re=6.2×10^6, xtrans=3%, W_2=1

Constraints

- No lift decrease, no change in angle of incidence
- Variation in pitching moment less than 2% in each point
- Maximal thickness constant and at 5% chord more than 96% of initial
- Leading edge radius more than 90% of initial
- Trailing edge angle more than 80% of initial

Multipoint airfoil optimization RAE2822

Parameterization

- 20 design variables changing camberline, Hicks-Henne functions

Optimization strategy

- Constrained SQP
- Navier-Stokes solver FLOWer, Baldwin/Lomax turbulence model
- Gradients provided by FLOWer Adjoint, based on Euler equations

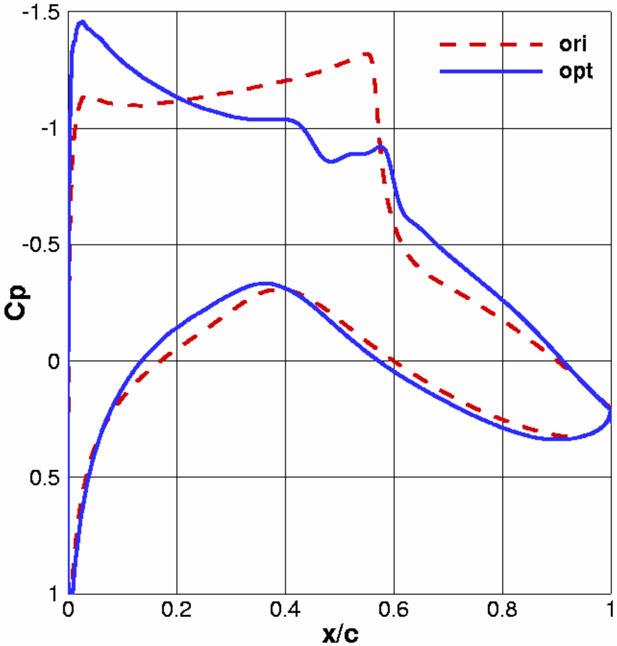
Results

Pt	α	M_i	cl^t	cd^t (.10^{-4})	cl	cd^t (.10^{-4})	$\Delta cd/cd^t$	$\Delta cl/cl^t$	$\Delta cm/cm^t$
1	2.8	0.734	0.811	197.1	0.811	135.5	-31.2%	0%	+1.6%
2	2.8	0.754	0.806	300.8	0.828	215.0	-27.4%	+2.7%	+2.0%

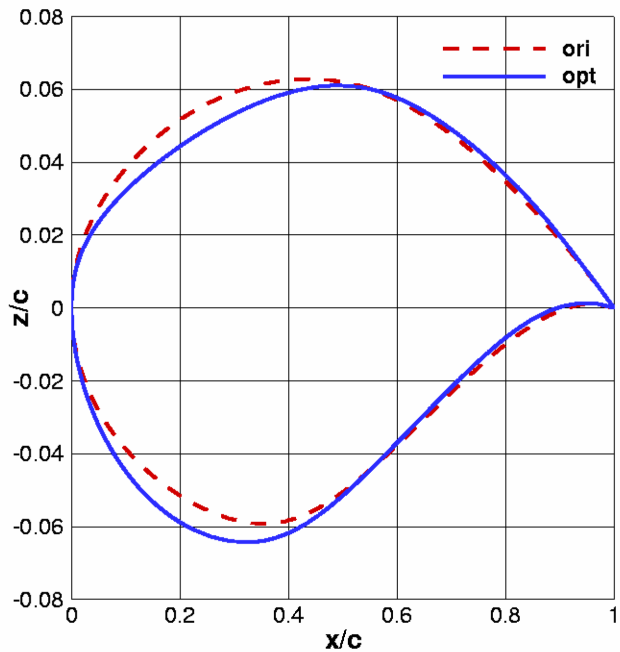
Multipoint airfoil optimization RAE2822

1. design point

M_∞=0.734, α=2.8°



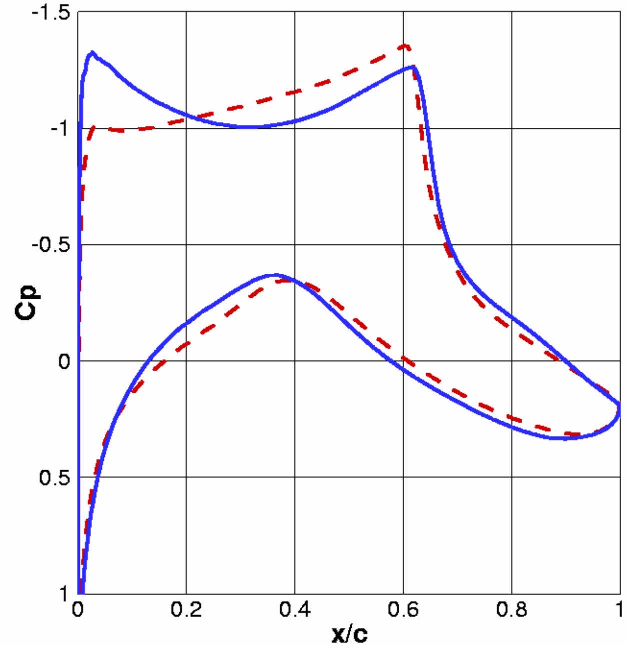
Airfoil Geometry



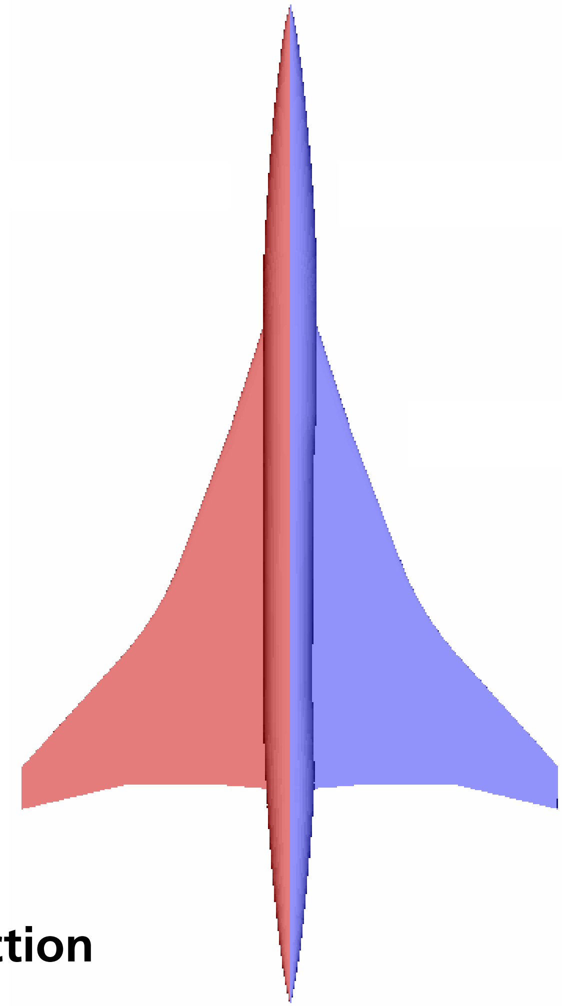
shape geometry

2. design point

M_∞=0.754, α=2.8°



Optimization of SCT Configuration (SCT – Supersonic Cruise Transporter)



Objective function

- drag reduction by constant lift

Design point

- Mach number = 2.0
- lift coefficient = 0.12

Constraints

- fuselage incidence
- minimum fuselage radius
- wing planform unchanged
- minimum wing thickness distribution in spanwise direction

Optimization of SCT Configuration

Approach

- **FLOWer code in Euler mode with target lift option**
- **Lift kept constant by adjusting angle of attack**
- **FLOWer code in Euler adjoint mode**
- **Structured mono-block grid (MegaCads), 230.000 grid points**

Optimization strategy

- **Quasi-Newton Method (BFGS algorithm)**

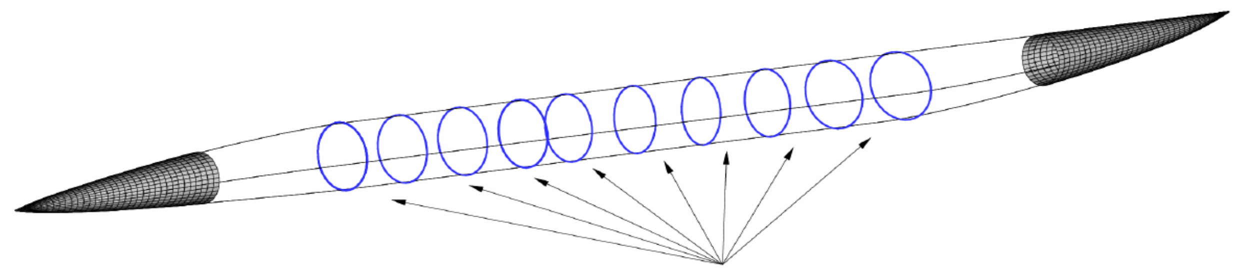
Optimization of SCT Configuration

Design variables

- fuselage: 10 parameters
- twist deformation: 10 parameters
- camberline (8 sections): 32 parameters
- thickness (8 sections): 32 parameters
- angle of attack: 1 parameter

85 parameters

Fuselage



10 sections controlled by Bezier nodes

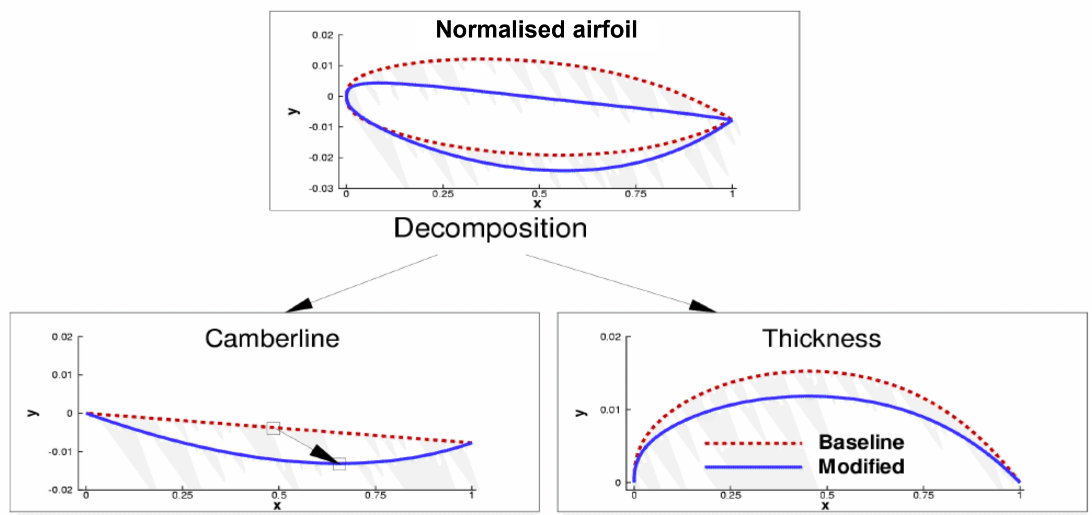
Optimization of SCT Configuration

Design variables

- fuselage: 10 parameters
- twist deformation: 10 parameters
- camberline (8 sections): 32 parameters
- thickness (8 sections): 32 parameters
- angle of attack: 1 parameter

85 parameters

Thickness and camberline



J. Brezillon, N. Gauger

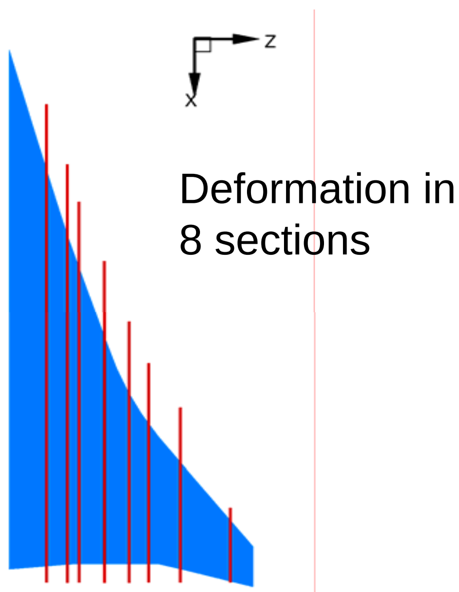
Optimization of SCT Configuration

Design variables

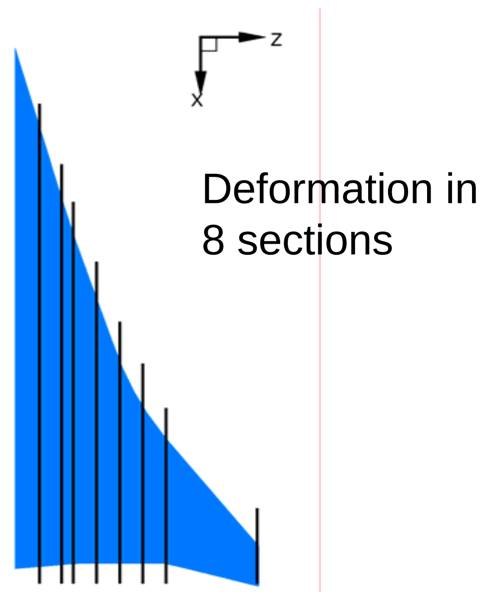
- **fuselage:** **10 parameters**
- **twist deformation:** **10 parameters**
- **camberline (8 sections):** **32 parameters**
- **thickness (8 sections):** **32 parameters**
- **angle of attack:** **1 parameter**

85 parameters

Camberline

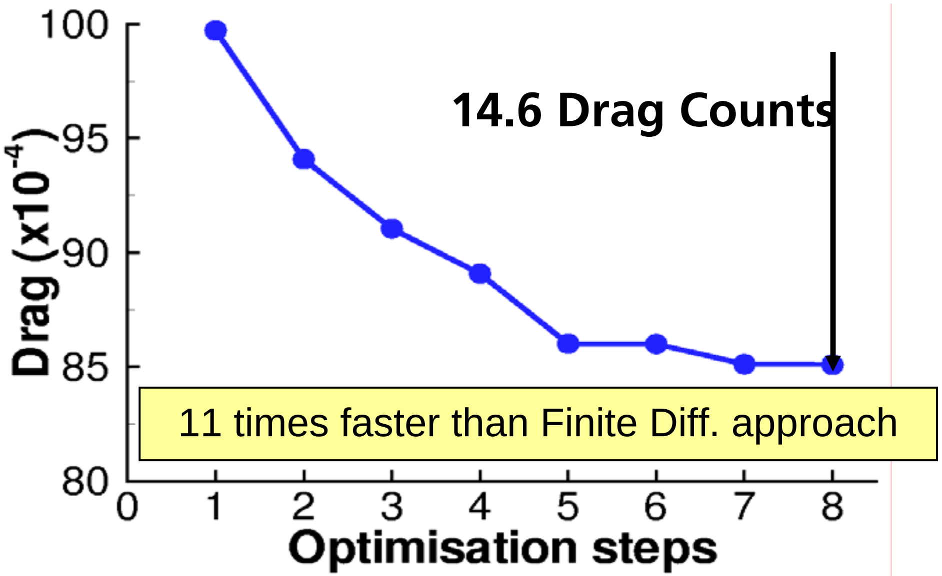


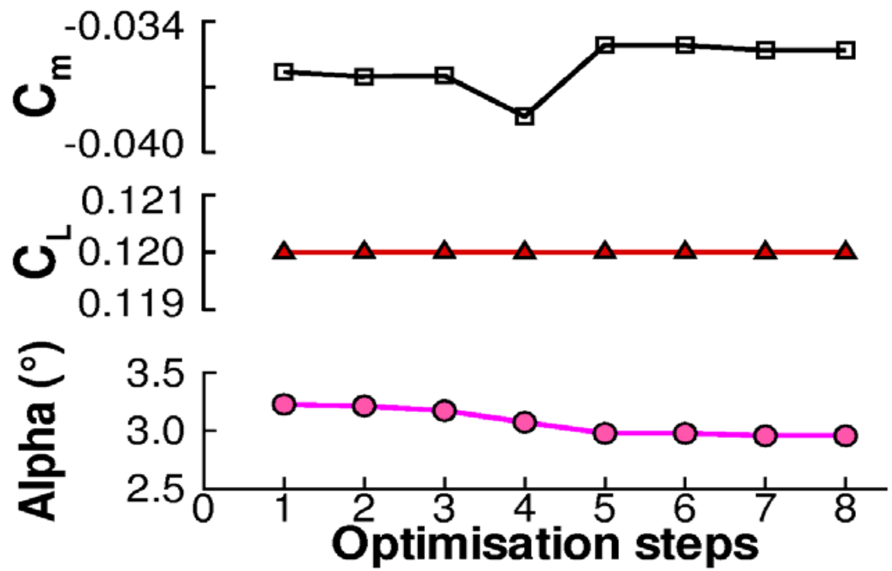
Thickness



Optimization of SCT Configuration

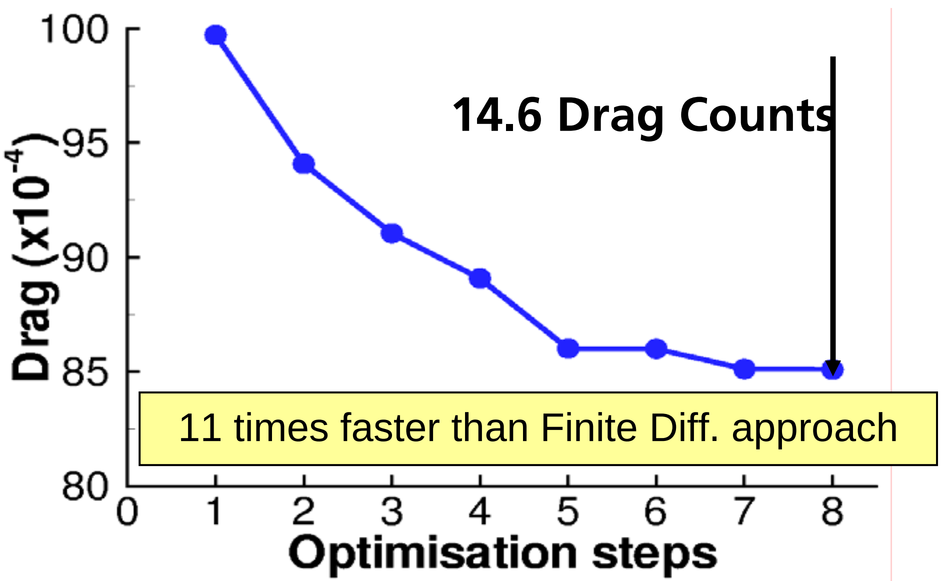
Results

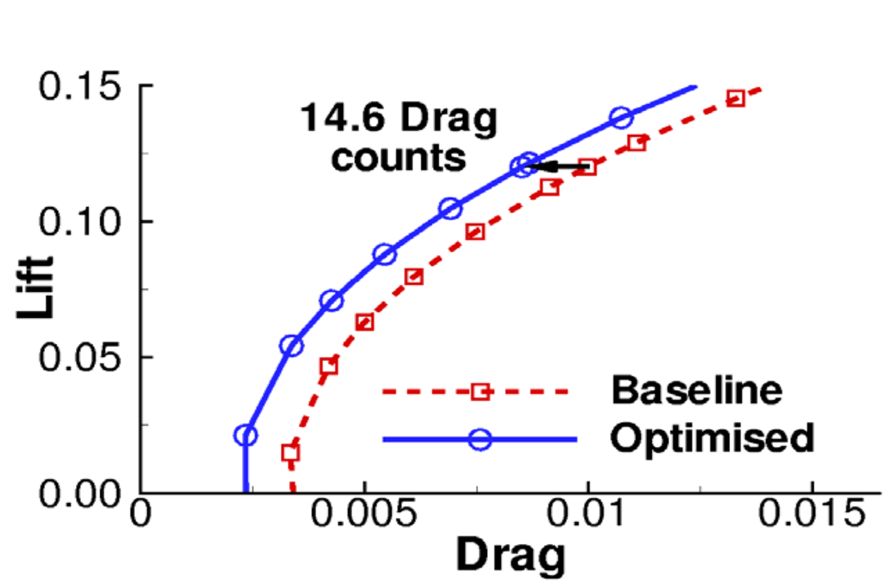




- **54 aerodynamic state computations**
- **7 gradient evaluations**

Results



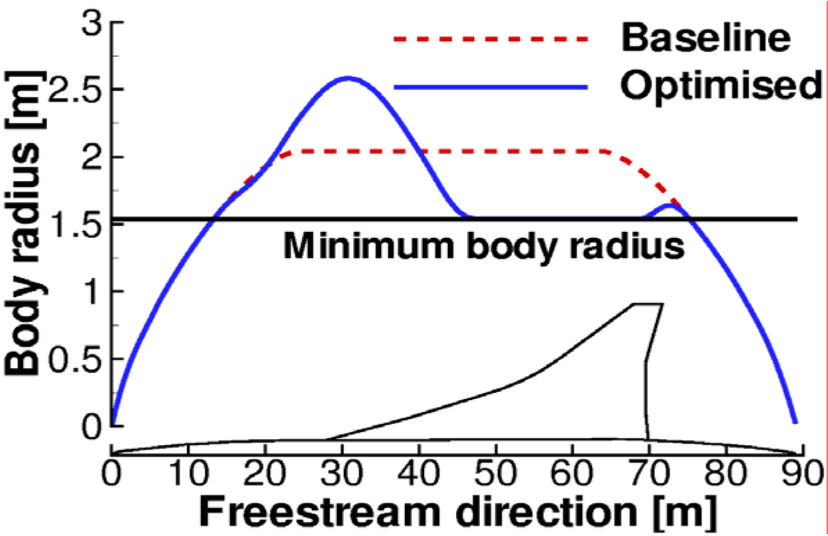


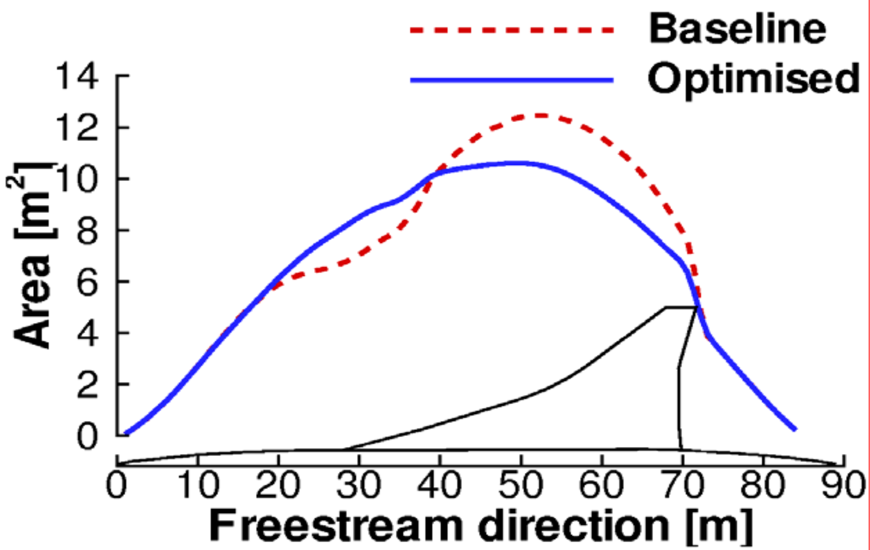
- 54 aerodynamic state computations
- 7 gradient evaluations

Optimization of SCT Configuration

Radius of the fuselage in freestream direction

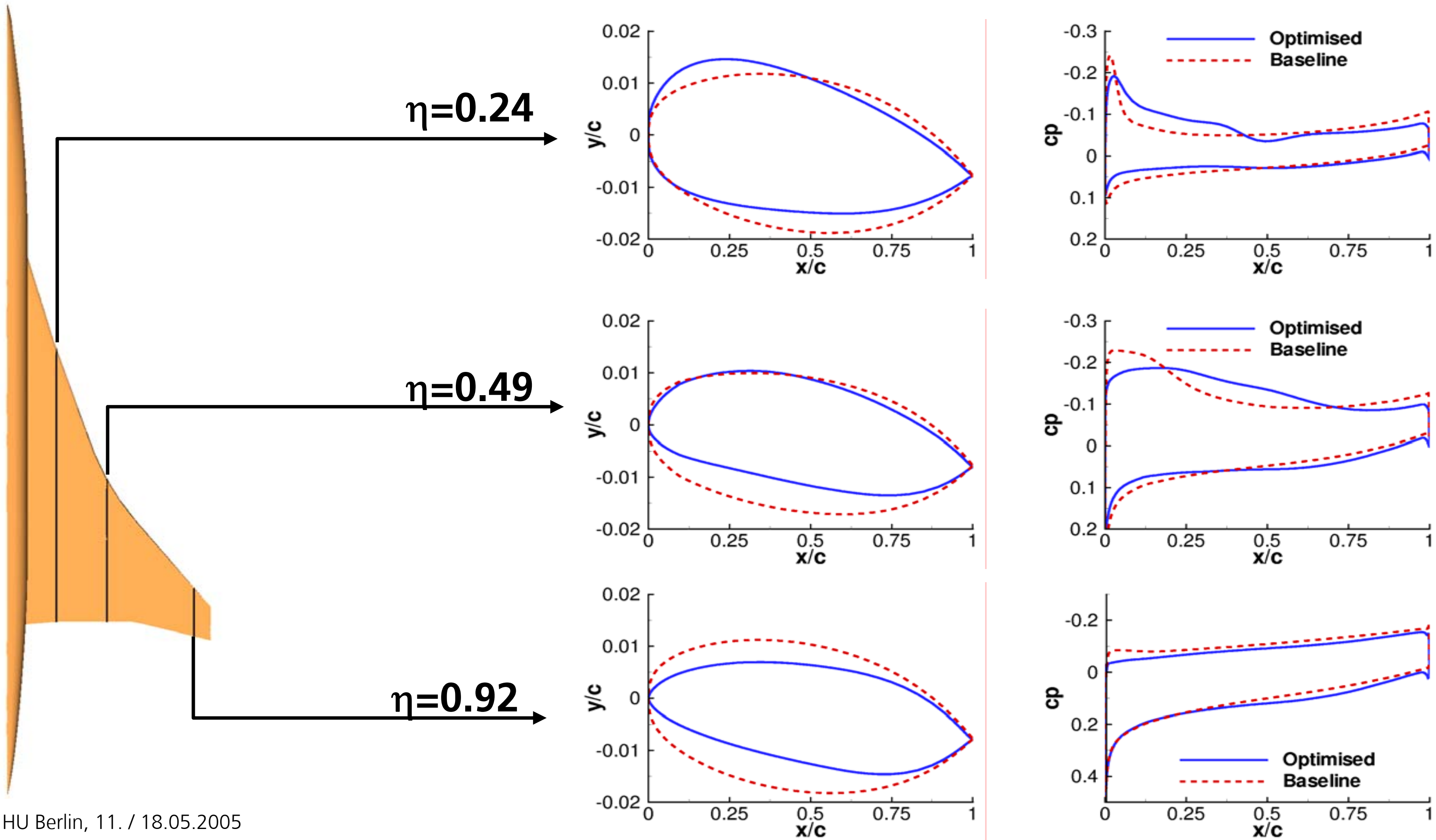
and Area Rule



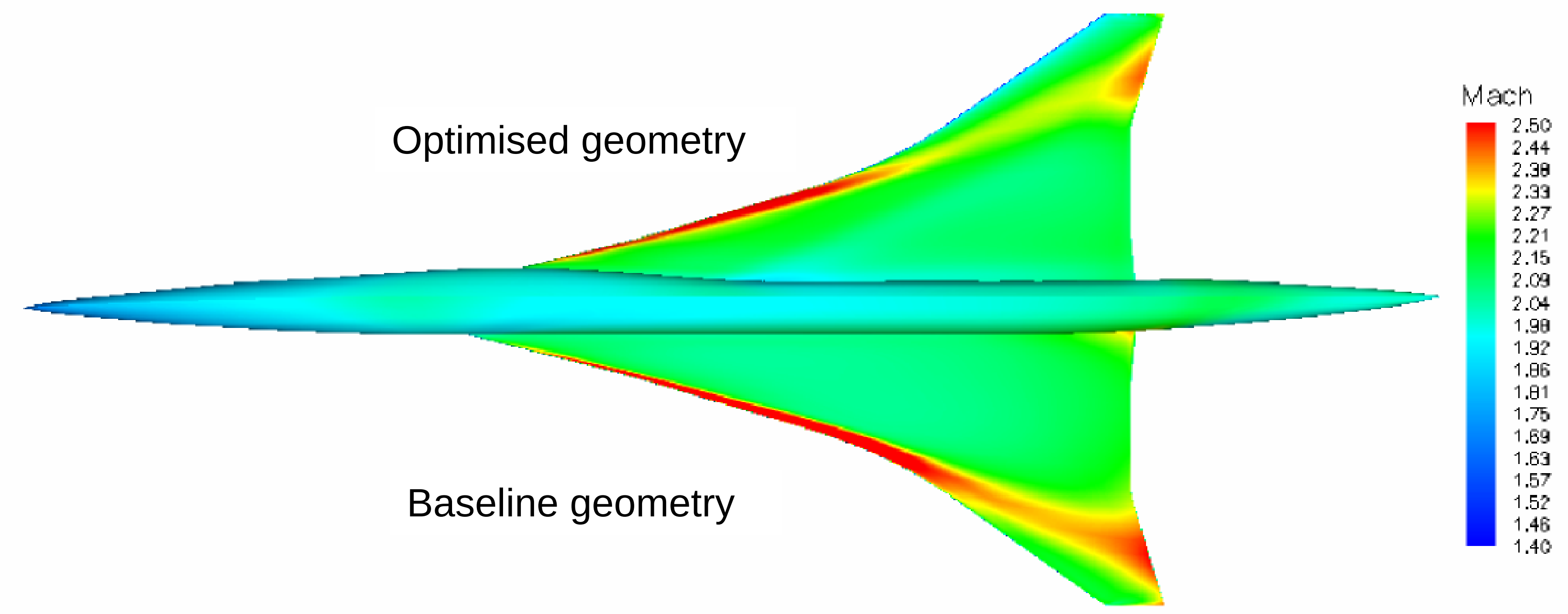


Optimization of SCT Configuration

Wing section and pressure distribution



Mach number distribution on the wing



Part 2

Coupled Aero-Structure Adjoint

Coupled Aero-Structure Adjoint

Motivation

Wing deflection up to 7% of wing span!

Deflected aerodynamic optimal shape can be worse than the initial …

Boeing 737-800 at ground and in cruise (Ma = 0.76)

Coupled Aero-Structure Adjoint

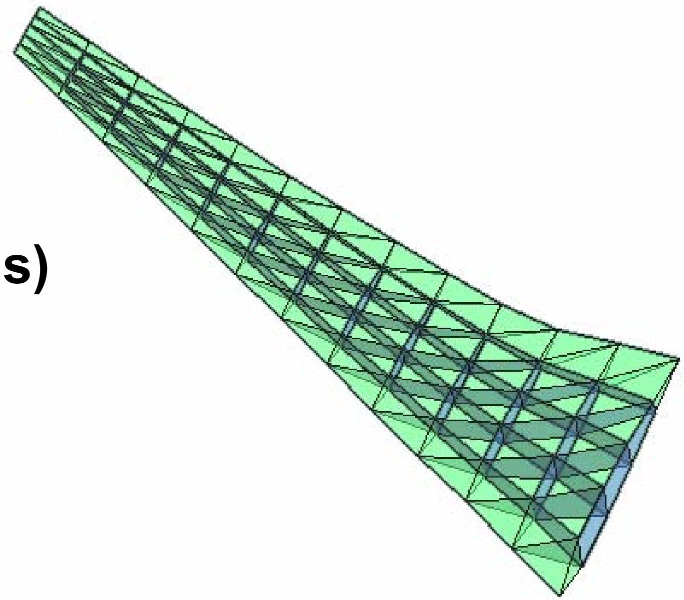
AMP wing

15 design variables (shape bumping functions based on Bernstein polynomials)

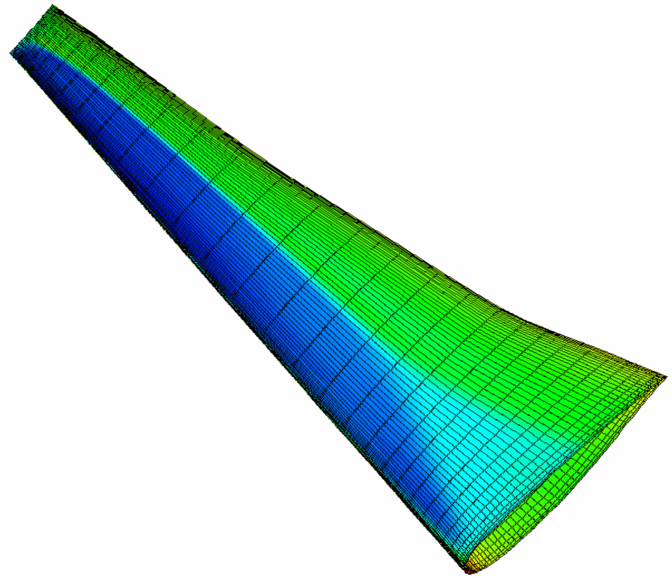
Ma=0.78
alpha=2.83

Drag reduction by constant lift

Taking into account static deformation



NASTRAN
shell/beam model
126 nodes



FLOWer MAIN/ADJOINT
15 design variables
Ma=0.78
alpha=2.83
(300.000 cells)

Coupled Aero-Structure Adjoint

Aerodynamics,
e.g Euler Eqn.: $R_A = 0$

Structure:

$$R_S = Kd - a = 0$$

K: Symmetric stiffness matrix
a: Aerodynamic force
d: Displacement vector
P: Vector of Design variables

ψ_A : Aerodynamic Adjoint

ψ_S : Structure Adjoint

~: Lagged ...

Conventional Gradient:

$$\frac{dC_D}{dP} = \frac{\partial C_D}{\partial P} + \frac{\partial C_D}{\partial w}\frac{\partial w}{\partial P} + \frac{\partial C_D}{\partial d}\frac{\partial d}{\partial P}$$

Aero/Structure Adjoint System:

$$\left(\frac{\partial R_A}{\partial w}\right)^T \psi_A = \frac{\partial C_D}{\partial w} - \left(\frac{\partial R_S}{\partial w}\right)^T \tilde{\psi}_S$$

$$\left(\frac{\partial R_S}{\partial d}\right)^T \psi_S = \frac{\partial C_D}{\partial d} - \left(\frac{\partial R_A}{\partial d}\right)^T \tilde{\psi}_A$$

Adjoint Gradient:

$$\frac{dC_D}{dP} = \frac{\partial C_D}{\partial P} - \psi_A^T \frac{\partial R_A}{\partial P} - \psi_S^T \frac{\partial R_S}{\partial P}$$

Coupled Aero-Structure Adjoint

$\frac{\partial R_A}{\partial d}, \frac{\partial R_A}{\partial P}$ **: perturb shape by d,P → calculate change in CFD residual**

$\frac{\partial C_D}{\partial d}, \frac{\partial C_D}{\partial P}$ **: perturb shape by d,P → calculate change in drag coefficient**

$\frac{\partial C_D}{\partial w}$ **: treat** $\int_C \ldots \frac{\partial p}{\partial w}(n_x \cos\alpha + n_y \sin\alpha)\ldots$ **→ boundary condition**

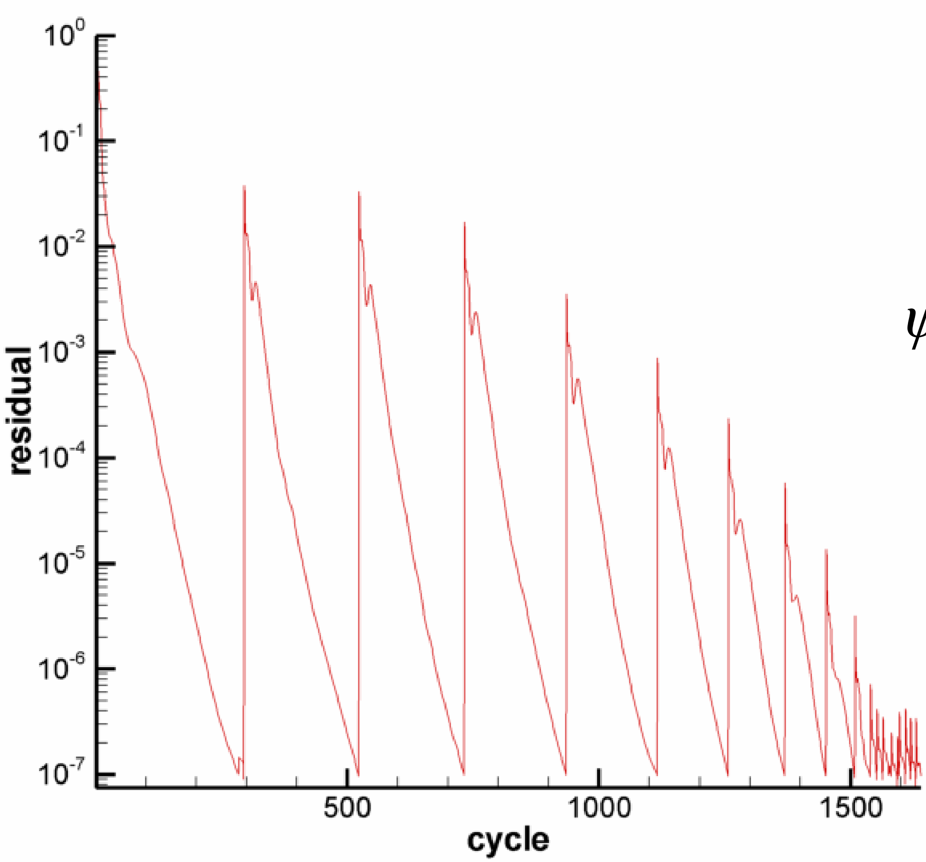
… has been derived in the last lecture!

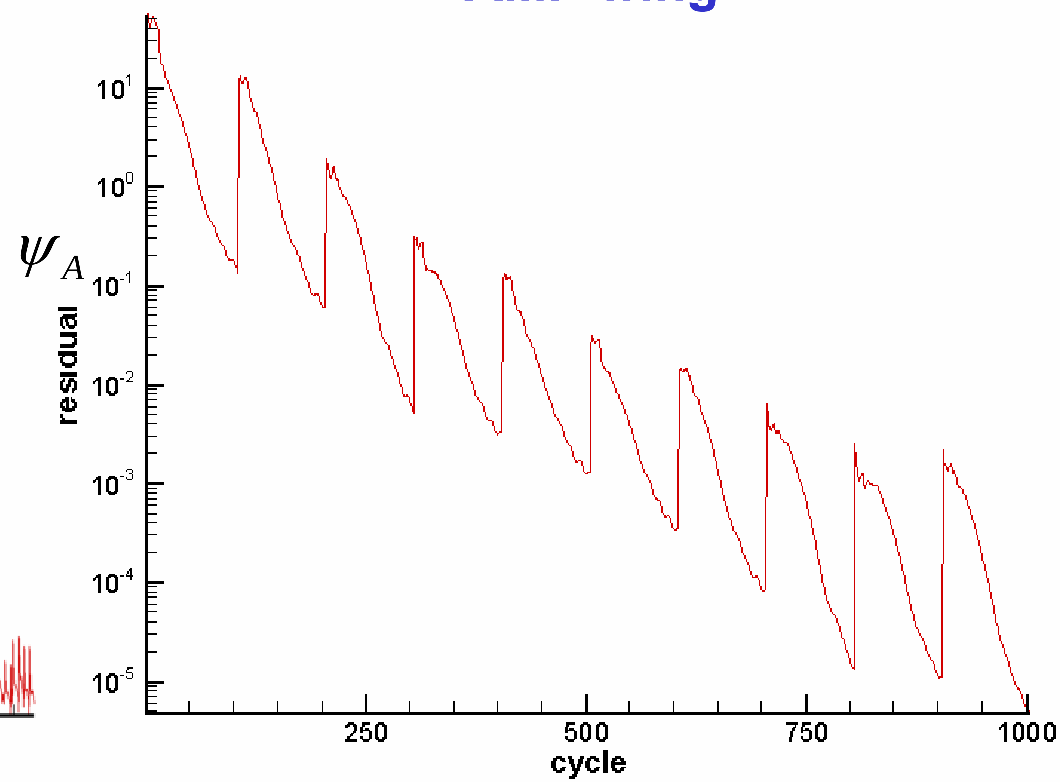
$\frac{\partial R_S}{\partial w} = \frac{\partial (Kd - a)}{\partial w} = -\frac{\partial a}{\partial w}$ **: treat** $\int_C \ldots \frac{\partial p}{\partial w} \ldots$ **→ boundary condition**

$$\frac{\partial R_S}{\partial d} = \frac{\partial (Kd - a)}{\partial d} = K = K^T$$

$$\frac{\partial R_S}{\partial P} = \frac{\partial (Kd - a)}{\partial P} = \frac{\partial K}{\partial P} d - \frac{\partial a}{\partial P}$$

Coupled Aero-Structure Adjoint





Finite Differences:

Perturb the shape by each design variable and converge the aero-elastic loop until stationary behavior

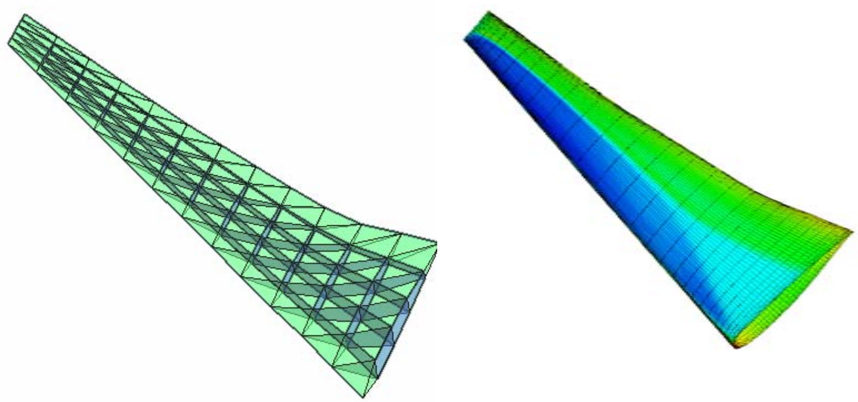
Coupled Aero-Structure Adjoint:

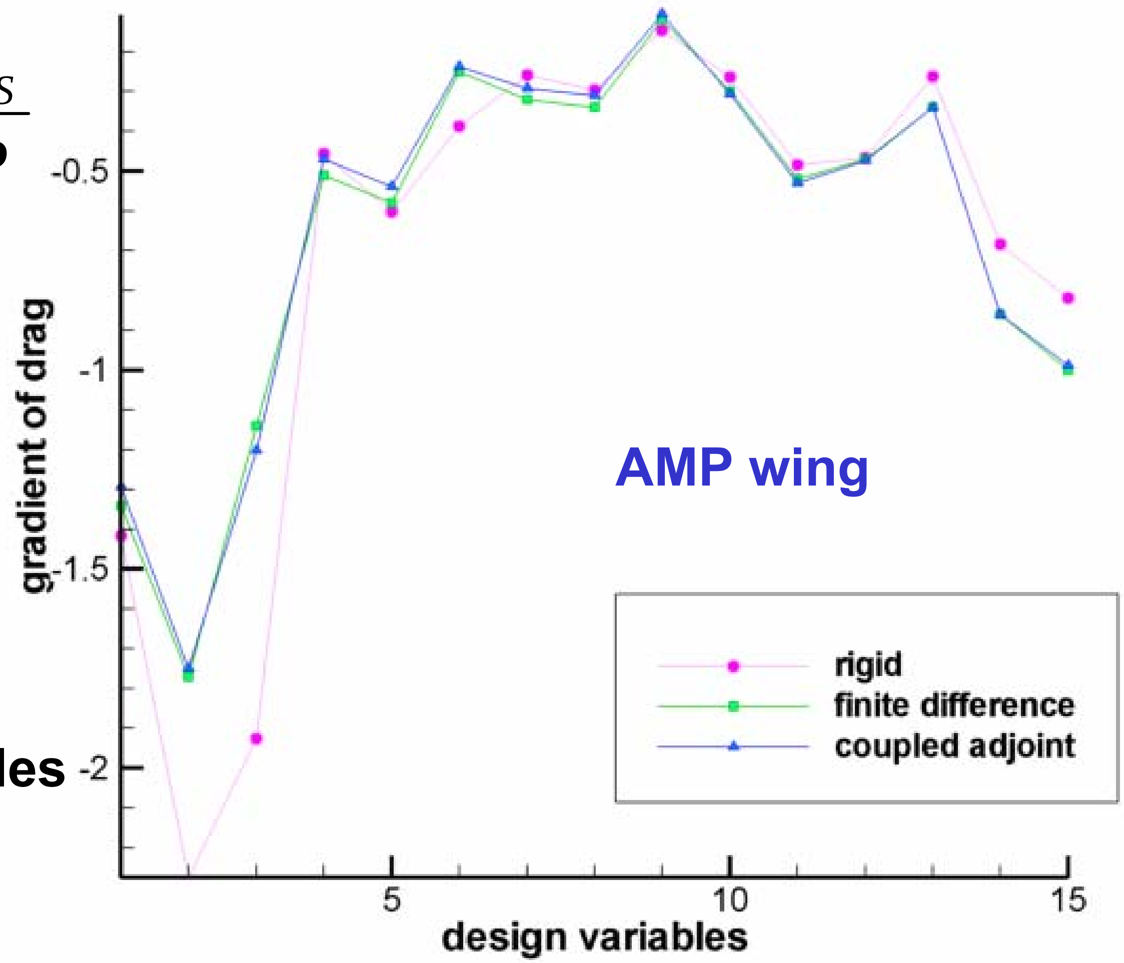
Each 100 iterations the lagged $\tilde{\psi}_S$ is updated ...

Coupled Aero-Structure Adjoint

Validation of Adjoint Gradient

$$\frac{dC_D}{dP} = \frac{\partial C_D}{\partial P} - \psi_A^T \frac{\partial R_A}{\partial P} - \psi_S^T \frac{\partial R_S}{\partial P}$$





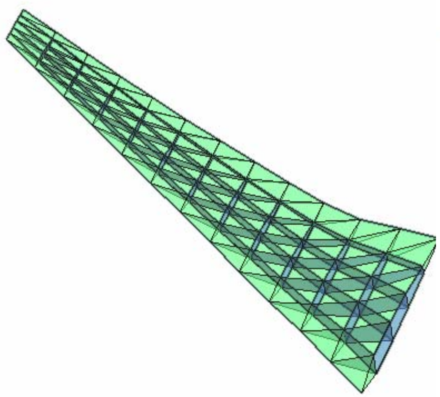
NASTRAN
shell/beam model
126 nodes

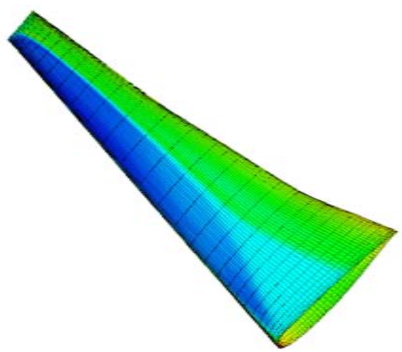
15 design variables
Ma=0.78
alpha=2.83
(300.000 cells)

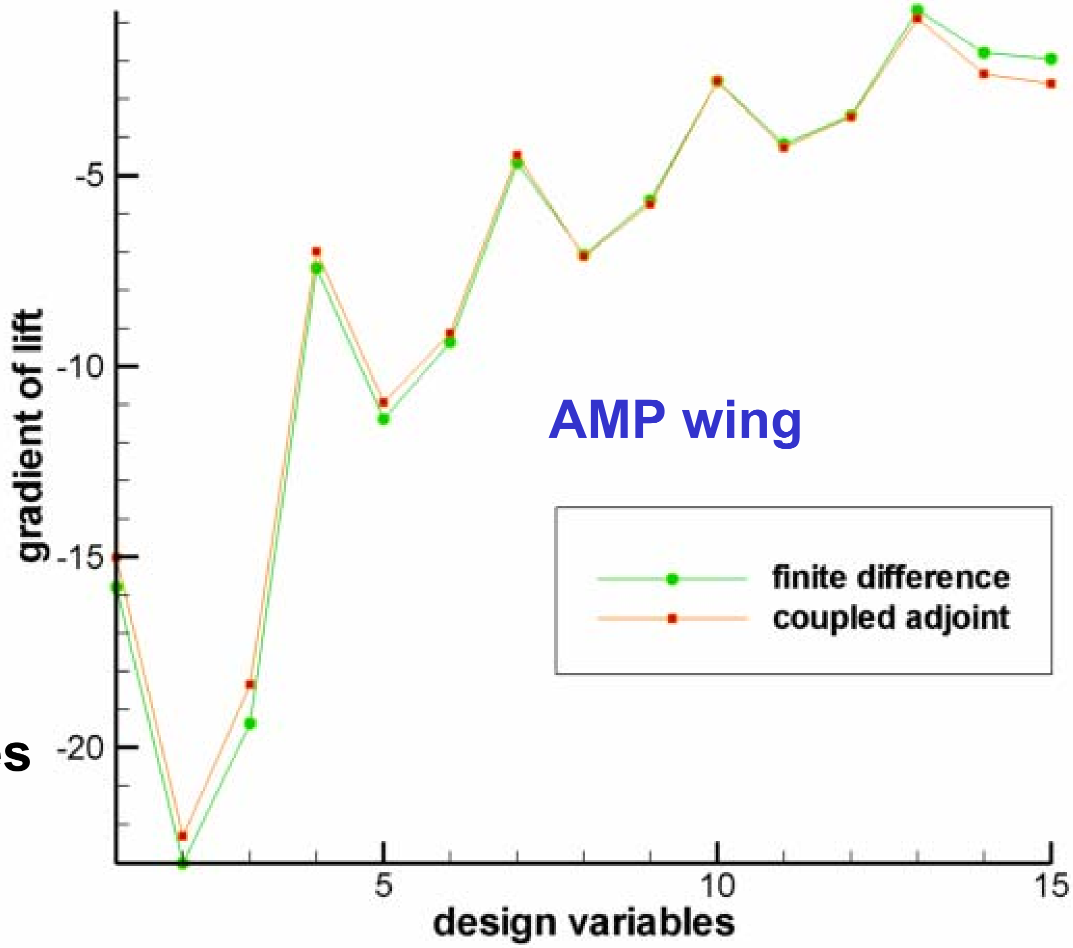
Coupled Aero-Structure Adjoint

Validation of Adjoint Gradient

$$\frac{dC_L}{dP} = \frac{\partial C_L}{\partial P} - \psi_A^T \frac{\partial R_A}{\partial P} - \psi_S^T \frac{\partial R_S}{\partial P}$$







NASTRAN
shell/beam model
126 nodes

15 design variables
Ma=0.78
alpha=2.83
(300.000 cells)

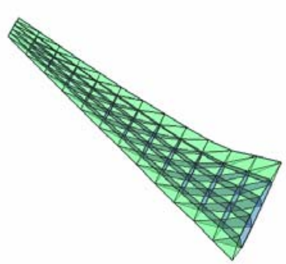
Coupled Aero-Structure Adjoint

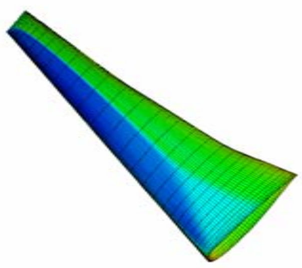
AMP wing

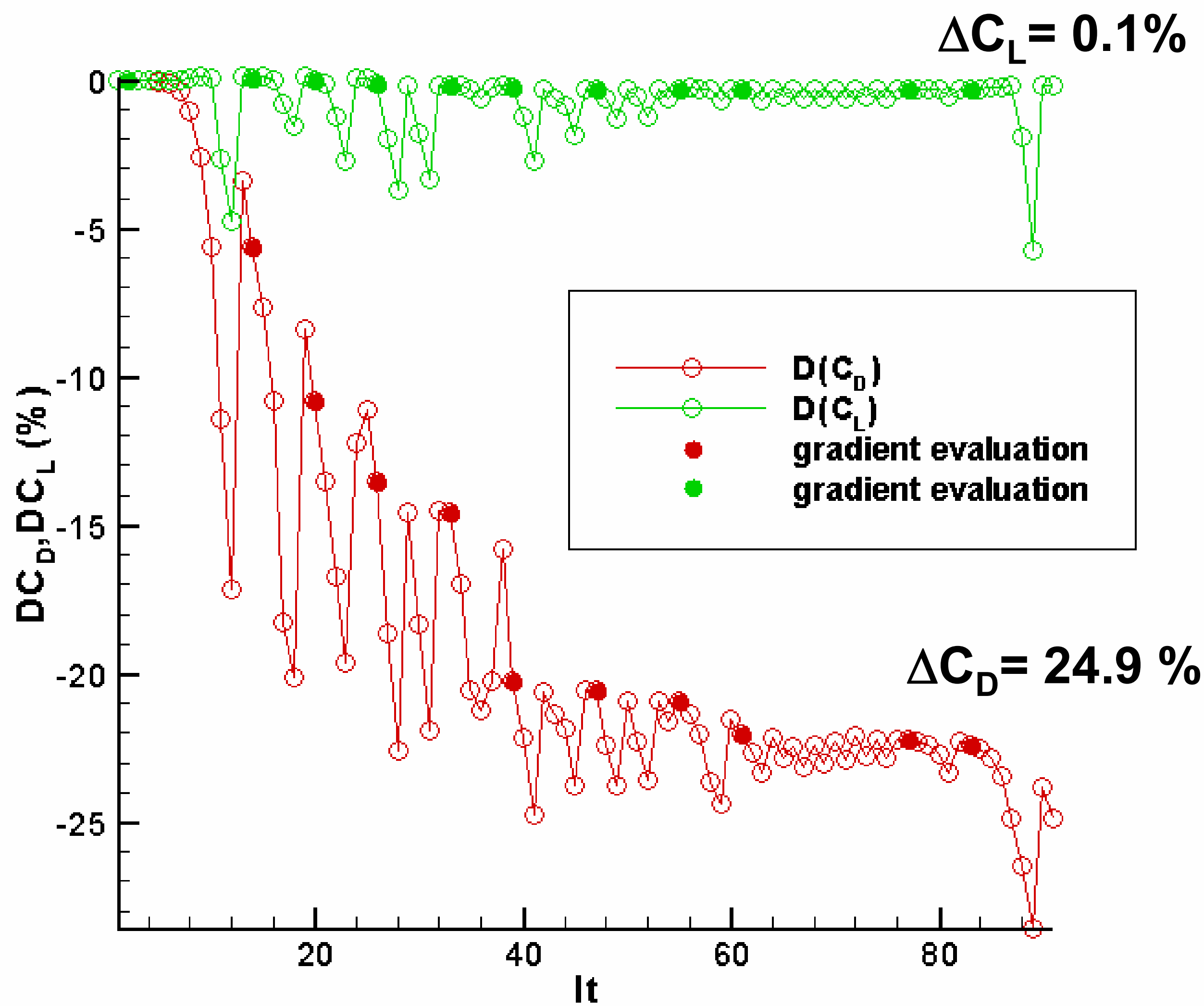
240 design variables (control points free form deformation)

Ma=0.78
alpha=2.83

Drag reduction by constant lift







feasible direction method

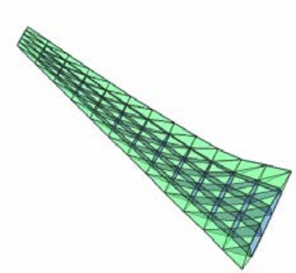
Coupled Aero-Structure Adjoint

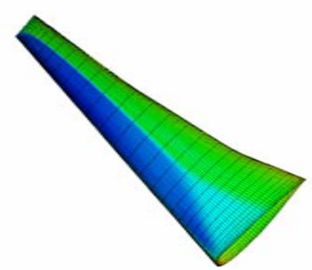
AMP wing

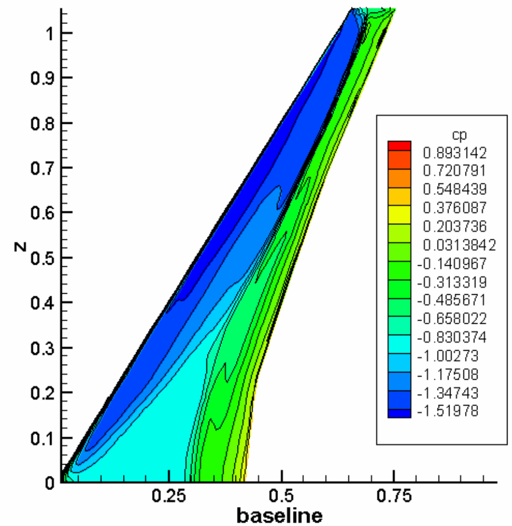
240 design variables (control points free form deformation)

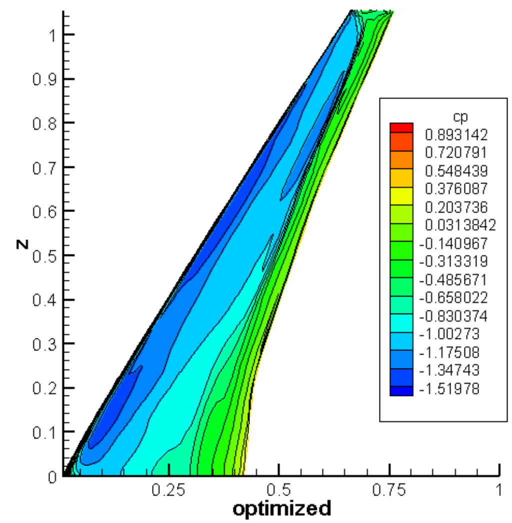
Ma=0.78
alpha=2.83

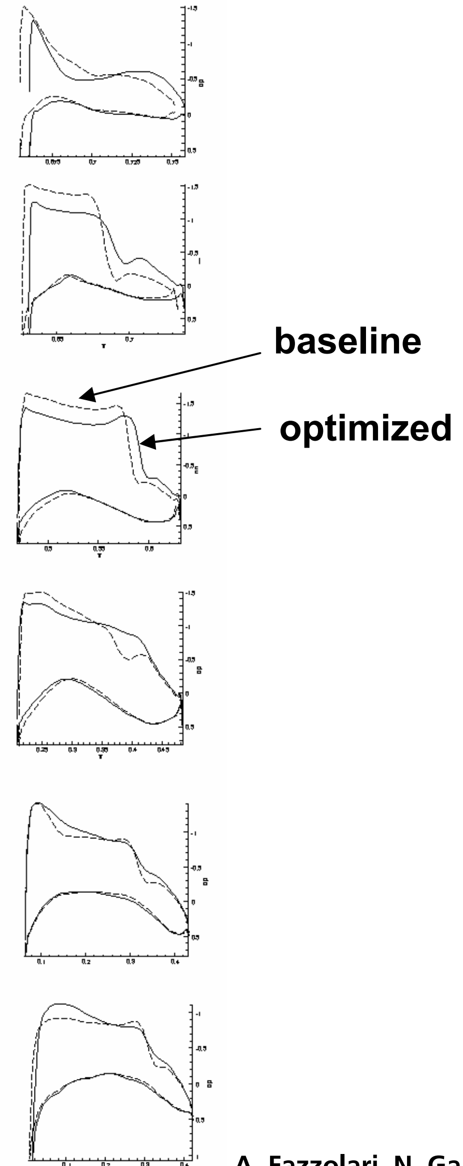
Drag reduction by constant lift











Coupled Aero-Structure Adjoint

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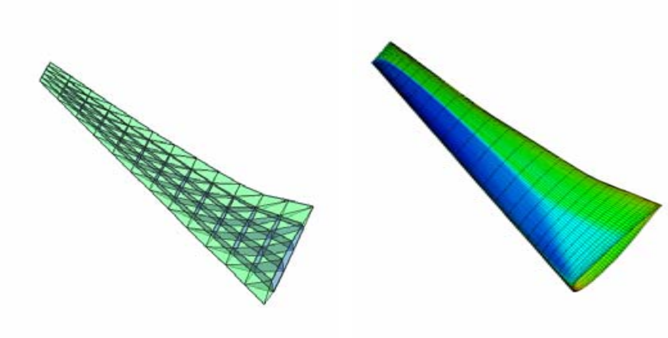


Comparison of numerical effort:

(PC Pentium IV, 2.6 GHz, 2GB RAM)

- **Coupled adjoint: 15 days**
  (11 gradient and 91 state evaluations)
- **Finite differences: 227 days**

Aero-Structure MDO

Range R:

$$R \propto \frac{C_L}{C_D} \ln \frac{W}{W-F} = \frac{C_L}{C_D} \ln \left(\frac{1+\lambda\, ks}{1+\lambda\, ks - \frac{F}{W_0}} \right)$$

Fuel Weight F

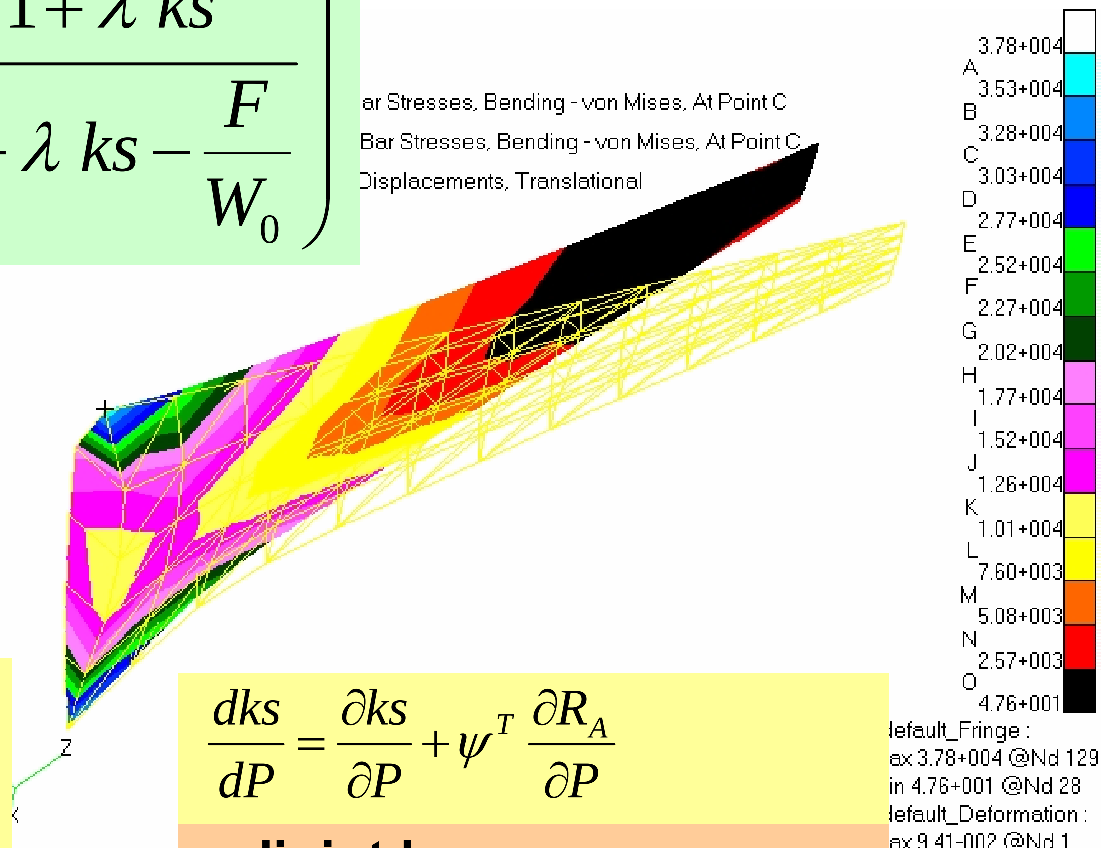
Weight W:

$$W = W_0 (1+\lambda\, ks)$$

Kreisselmeier-Steinhauser:

$$ks = \frac{1}{\beta} \ln\Big(\sum_n \exp\Big(\beta \frac{\sigma_n - \sigma_0}{\sigma_0}\Big)\Big)$$

$\lambda=0.2$, $\sigma_0 = 30.000$ and $\beta=40$



$$\frac{dks}{dP} = \frac{\partial ks}{\partial P} + \psi^T \frac{\partial R_A}{\partial P}$$

adjoint b.c.

$$n_x \psi_2 + n_y \psi_3 + n_z \psi_4 = -\frac{\partial ks}{\partial p}$$

Aero-Structure MDO

AMP wing

240 design variables (control points free form deformation)

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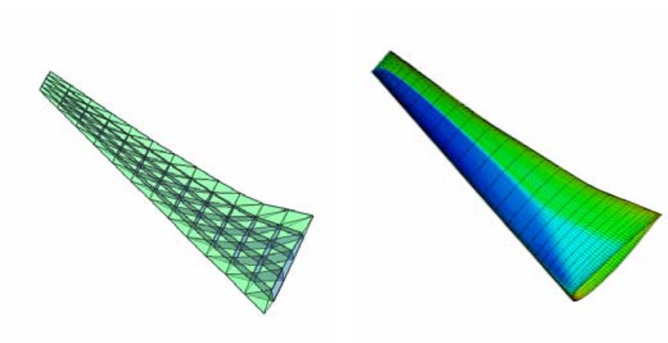
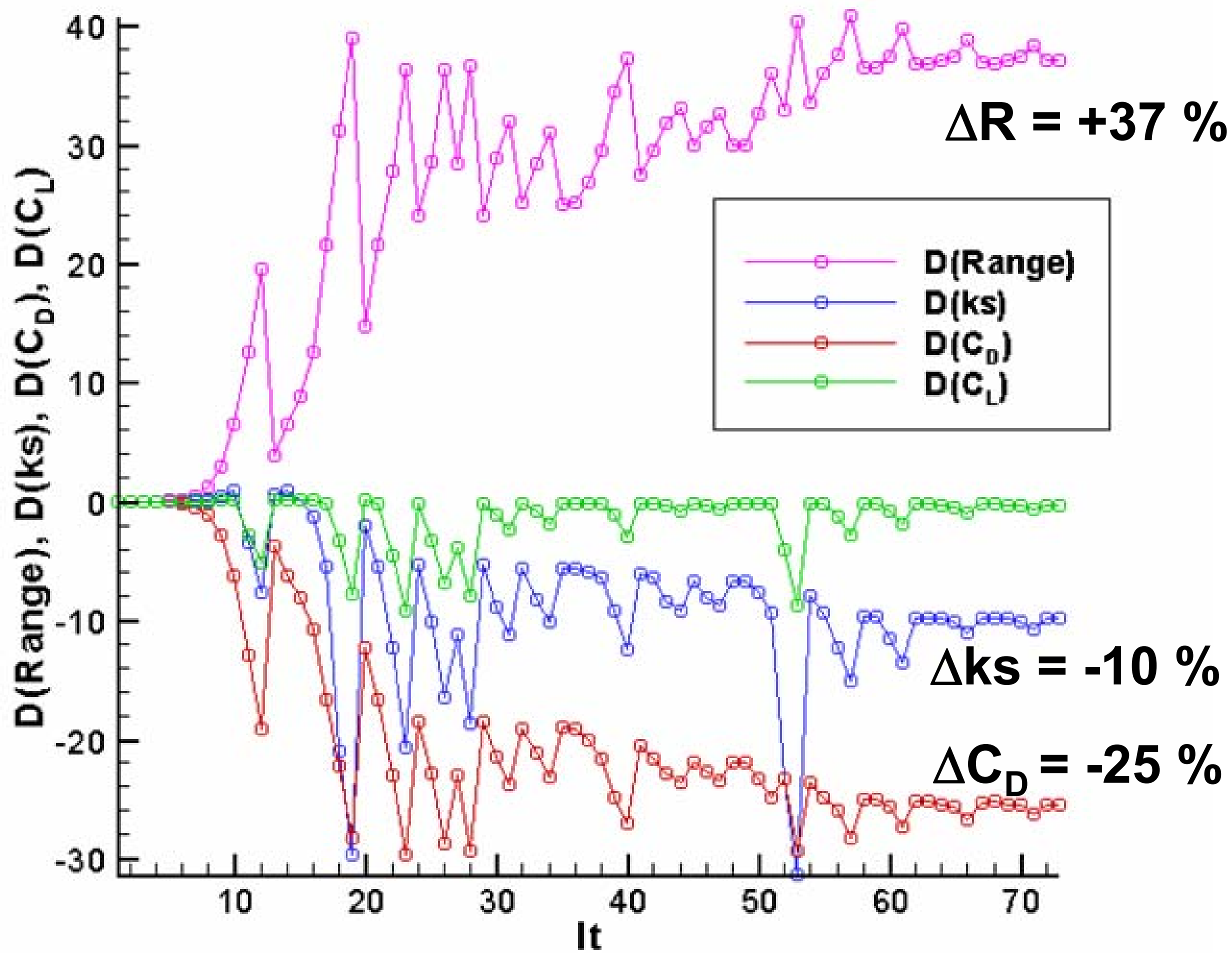
alpha=2.83

Range maximization by constant lift





feasible direction method

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