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Lecture **Analysis on Manifolds**, WS 2010/11

Exercises no. 2

to be submitted by Nov 9th

1 Let X, Y be topological spaces with canonical projections $\pi_{X(Y)} : X \times Y \to X(Y)$. Show that

$$\pi_{x,\star} \times \pi_{y,\star} : \pi(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

is a group isomorphism.

2 Show that $\mathbb{R}^m \setminus \{0\}$ is homotopy equivalent to S^{m-1}.

3 *(Connectedness)*

1. Show that X is connected iff there is no continuous surjection

   $$f : X \to \mathbb{N}_2,$$

   where $\mathbb{N}_2$ carries the discrete topology.

2. Define the components of X in such a way that

   (1) X is decomposed as the union of its connected components,

   (2) each component C is connected (as a subspace) and closed in X.

3. Denote by $\pi_0(X) \subset \mathcal{P}X$ the set of connected components of X. Show that $\sharp\pi_0(X)$ is a topological invariant.

4. Give an example of a space X with $\pi_0(X) = (\{x\})_{x\in X}$; such spaces are called *totally disconnected*.

5. Are the components in general open in X?

4 Prove the Lebesgue Theorem on open covers of compact metric spaces.

5 *(Properly discontinuous group actions)*
Let X be a locally path-connected topological space and $G \subset Aut\, X$ a group with the property that each $x \in X$ has a neighbourhood U_x such that

$$U_x \cap gU_x = \emptyset \text{ for } g \in G \setminus \{e\}.$$

Show the following statements:

1. G acts freely on X.
2. $\pi : X \ni x \mapsto G_x \in X/G$ is a covering map.
3. $G(X, X/G) \simeq G$.