INTRODUCTION TO THE SPECTRAL THEORY OF DIFFERENTIAL OPERATORS
EXERCISES 1, WEEK FROM APRIL 11

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We use the notation introduced in the Lecture. E is a complex vector space of finite dimension, $A \in \mathcal{L}(E)$ is a linear endomorphism. We define

$$\operatorname{spec} A := \chi_A^{-1}(0) = (\lambda_j)_1^{k(A)}, \ \lambda_j \in \mathbb{C}, \ k(A) \leq \dim E,$$
$$P_j := -(2\pi i)^{-1} \int_{\partial B_\varepsilon(0)} R_A(z) dz,$$
$$D_j := -(2\pi i)^{-1} \int_{\partial B_\varepsilon(0)} z R_A(z) dz.$$

P_j and D_j are called the projection and the nilpotent part of A at λ_j, respectively. We also write

$$R_{A,j}(z) = \sum_{i \in \mathbb{Z}} (z - \lambda_j)^i A_{j,i} =: R_{A,j}(z)^{\text{prin}} + \sum_{i \in \mathbb{Z}_+} (z - \lambda_j)^i A_{j,i},$$

and we call $R_{A,j}(z)^{\text{prin}}$ the principal part of A at λ_j.

1. Show that

$$r_{\text{spec}}(R_{A,j}^{\text{prin}}(z) = |z - \lambda_j|, \ z \neq \lambda_j.$$

2. Show the following relations:

$$P_j P_k = P_k P_j = \delta_{jk} P_k,$$
$$\sum_{j=1}^{k(A)} P_j = I_E;$$
$$P_j A = A P_j.$$

3. Show that for any simply closed and positively oriented rectifiable curve c in $\mathbb{C}$ enclosing the domain Int$c$ and not intersecting $\operatorname{spec} A$ we have

$$-(2\pi i)^{-1} \int_c R_A(z) dz = \sum_{\lambda_j \in \text{Int}c} P_j.$$

Date: April 15, 2011.

4. Show that

$$A = \sum_j \lambda_j P_j + \sum_j D_j =: S + D,$$

where S is semisimple and D is nilpotent, and

$$[S, D] = 0.$$

Show in addition that this decomposition is unique.

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