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Introduction to Spectral Theory of Differential Operators, SS 2011

Exercises 3, week of May 2, 2011

In what follows all spaces are Banach spaces if not stated otherwise.

1 If $A \in \mathcal{L}(E_L, E_R)$ and $\operatorname{codim} A = \dim E_R / \operatorname{im} A < \infty$, then A is closed.

2 Prove the Stability Theorem for Fredholm operators by using the matrix representation.

3 Consider the space

$$\ell^2(\mathbb{N}) := \left\{ a := (a_j)_{j \in \mathbb{N}} \subset \mathbb{C} \ : \ \|a\|^2_{\ell^2(\mathbb{N})} := \sum_{j \in \mathbb{N}} |a_j|^2 < \infty \right\}.$$

a) Show that $\ell^2(\mathbb{N})$ is a Hilbert space.

b) Consider the operator

$$S : \ell^2(\mathbb{N}) \ni (a_j)_{j \in \mathbb{N}} \longmapsto (a_{j+1})_{j \in \mathbb{N}} \in \ell^2(\mathbb{N}).$$

Show that S is Fredholm and compute $S^\star$ and $\operatorname{im} S$.

c) Construct $T_m \in \mathcal{F}\left(\ell^2(\mathbb{N})\right)$ with $\operatorname{ind} T_m = m$ for any $m \in \mathbb{Z}$.

d) Is S normal?

Note: S is called the shift operator.

4 Consider in $L^2[0,1]$ the operator

$$Kf(x) := \int_0^x f(t)dt.$$

a) Show that $K \in \mathcal{K}\left(L^2[0,1]\right)$.

b) Show for an eigenvalue $\lambda \neq 0$ with eigenfunction f the inequality

$$|f(x)| \leq \|f\|_{L^2[0,1]} \left(\frac{x}{|\lambda|} \right)^n \frac{1}{n}.$$

c) Prove that $\operatorname{spec} K = \{0\} = \operatorname{spec}_e K$.

Note: K is called the Volterra operator.