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INTRODUCTION TO SPECTRAL THEORY OF DIFFERENTIAL OPERATORS, SS 2011

EXERCISES 4, week of May 9, 2011

In what follows H denotes a Hilbert space.

1 Let $A \in \mathcal{L}(E)$, E a Banach space, $A' \in \mathcal{L}$ is called a *generalised inverse* of A if

$$AA'A = A.$$

Show that $\operatorname{im} A$ and $\operatorname{im} A'$ are closed.
Is a generalised inverse unique?

2 Show that in a Hilbert space every bounded normal operator has empty residual spectrum.

3 Let $A \in \mathcal{F}(H)$ be normal. Show that there is $\epsilon > 0$ such that $B_\epsilon^{\mathbb{C}}(0) \setminus \{0\} \subset \operatorname{res} A$.

4 Let $C = C^\star \in \mathcal{L}(H)$ such that $\langle Cx, x \rangle = 0\ \forall x \in H$. Show that $C = 0$.

5 Show that the following conditions for $P, Q \in \mathcal{P}(H)$ are equivalent.

a) $P(H) \subset Q(H)$,

b) $P \leq Q$,

c) $PQ = QP = P$.

6 *Uniqueness of the square root*
Let $A \geq 0,\ B \geq 0$ satisfy $A^2 = B^2$. Show that then $A = B$.