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Introduction to Spectral Theory of Differential Operators, SS 2011

Exercises 5, week of May 16, 2011

Let H be a separable and not fininte dimensional Hilbert space.

1 Show that any compact subset of $\mathbb{C}$ can be the spectrum of a normal operator. Hint: Consider a fixed o.n.b. $(e_j)_{j\in\mathbb{N}}$ of H and consider the operator

$$A_\alpha e_j = \alpha_j e_j, \quad \alpha = (\alpha_j)_{j\in\mathbb{N}} \subset \ell^\infty(\mathbb{N}).$$

2 Let $\mathcal{B}$ be a commutative B-algebra. Then for $a \in \mathcal{B}$,

$$r_{\text{spec}}(a) := \sup_{\lambda\in\operatorname{spec} a} |\lambda| = \lim_{n\to\infty} \|a^n\|^{1/n} \le \|a\|.$$

If $\|a^2\| = \|a\|^2$ then

(1) $r_{\text{spec}}(a) = \|a\|$.

In particular, if $\mathcal{B}$ is a $C^\star$-algebra then (1) holds for all $a \in \mathcal{B}$.

3 Let E be a B-space and $(a_j)_{j\in\mathbb{Z}_+} \subset E$. Prove Hadamard's theorem:

$$\text{If } \rho := \overline{\lim}_{j\to\infty} \|a_j\|^{1/j} \text{ and } R := \begin{cases} 1/\rho, & \rho \neq 0, \\ \infty, & \rho = 0, \end{cases}$$

then the series

$$f(z) := \sum_{j\ge 0} a_j z^j$$

is absolutely convergent for $|z|\rho < 1$, and divergent for $|z|\rho > 1$.

4 If A is a linear map in a Hilbert space H, then the set

$$W(A) := \{\langle Ax, x\rangle \in \mathbb{C} \;:\; \|x\| = 1\} \subset \mathbb{C}$$

is called the *numerical range* of A.

a) $A \in \mathcal{L}(H) \iff W(A)$ is bounded.

b) If $A \in \mathcal{L}(H)$ then $W(A)$ is convex, and if $\dim H < \infty$, then $W(A)$ is compact.

5 If $A \in \mathcal{K}(E)$, E Banach space, and $\operatorname{im} A$ is closed then $A \in \mathcal{K}_0(E)$.