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Introduction to Spectral Theory of Differential Operators, SS 2011

Exercises 7, week of May 30, 2011

1 Let $A \in L(H)$ be self-adjoint.
Give a direct proof that the map

$$\Psi_A : \mathbb{C}[z] \ni p \mapsto p(A) \in L(H)$$

induces an isometric $\star$-isomorphism

$$C(\operatorname{spec} A) \to C_A,$$

without using the Gelfand map Γ.

2 Consider in $H = L^2(\mathbb{R})$ the operator

$$A : C_c(\mathbb{R}) \ni f \mapsto xf \in L^2(\mathbb{R}),$$

called the *position operator* in quantum mechanics. Show that A is essentially self-adjoint with closure $\overline{A}$ such that

$$\operatorname{dom} \overline{A} = \{f \in L^2(\mathbb{R}) \ : \ xf \in L^2(\mathbb{R})\}$$

and

$$\overline{A}f = xf, \ f \in \operatorname{dom} A.$$

What is $\operatorname{spec} A$?

3 Let $f \in C_c^\infty(\mathbb{C})$ and let $w \in \mathbb{C}$. Show that

$$f(w) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial f}{\partial \overline{z}}(z)(z-w)^{-1} dxdy, \tag{1}$$

where $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$ if $z = x + iy$.
Hint: Apply Stokes' theorem to a large circle with center w. Deduce from the proof of (1) a generalisation of Cauchy's formula for smooth functions.