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Introduction to Spectral Theory of Differential Operators, SS 2011

Exercises 10, week of June 27, 2011

1 Let $D \in \mathrm{Diff}_k(M, E_1, E_2)$. Show that

$$\hat{D}(\xi) \in \mathcal{L}(E_{1,p}, E_{2,p}),\ p \in M,\ \xi \in T_p^\star M,$$

defined by

$$\hat{D}(\xi)[e] := i^k D\left(\frac{\phi^k}{k!} s\right)(p),$$

is well defined if $e \in E_p$, $\xi \in T_p^\star M$, $s \in C_c^\infty(M.E)$ with $s(p) = e$, $\phi \in C_c^\infty(M)$ with $\phi(p) = 0$, $d\phi(p) = \xi$.

2 Show that for $D_i \in \mathrm{Diff}_{k_i}(M, E_i, E_{i+1}),\ i = 1, 2$, one has

$$\hat{D}_2(\xi) \circ \hat{D}_1(\xi) = \widehat{D_2 \circ D}_1(\xi).$$

3 Assume that M is Riemannian and oriented, and that $E_i \to M$ is smooth with hermitian metric h^{E_i}, $i = 1, 2$, and consider $D \in \mathrm{Diff}_k(E_1, E_2)$. Define for $s_j \in C_c^\infty(M, E_i),\ i, j = 1, 2$,

$$(s_1, s_2)_{L^2(E_i)} := \int_M \langle s_1, s_2 \rangle_{h^{E_i}}(p)\, \mathrm{vol_M}(p);$$

Show that there is $D^\dagger \in \mathrm{Diff}_k(E_2, E_1)$ such that for $s_j \in C_c^\infty(M, E_j)$

$$(Ds_1, s_2)_{L^2(E_2)} = (s_1, D^\dagger s_2)_{L^2(E_1)}.$$

Prove that

$$\widehat{D^\dagger}(\xi) = \hat{D}(\xi)^\star.$$

4 Let M be an orientable Riemannian manifold. For $X \in \tau^1(M)$, define the divergence by

$$\mathrm{div}\, X := \sum_{i=1}^m \langle \nabla^k_{e_i} X e_i, \rangle_{TM},$$

where (e_i) is any local orthonormal frame for TM. Show that for any $X \in \tau^1(M), f \in C^\infty(M)$

a) $\mathrm{div}\, X\, \mathrm{vol_M} = \mathcal{L}_X(\mathrm{vol_M})$, $\mathcal{L}$ the Lie derivative;

b) $\mathrm{div}(fX) = Xf + f\, \mathrm{div}\, X$;

c) $X^\dagger = -Xx\, \mathrm{div}\, X$, if X is considered as a first order differential operator on $C^\infty(M)$;

d) $\left(\nabla_X^E\right)^\dagger = -\nabla_X^E - \mathrm{div}\, X$ for any smooth hermitian bundle $E \to M$ and any metric connection ∇^E.