# Stability of Linear Stochastic Difference Equations in Strategically Controlled Random Environments

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**Abstract**

We consider the stochastic sequence $\{Y_t\}_{t\in\mathbb{N}}$ defined recursively by the linear relation $Y_{t+1} = A_t Y_t + B_t$ in a random environment. The environment is described by the stochastic process $\{(A_t, B_t)\}_{t\in\mathbb{N}}$ and is under the simultaneous control of several agents playing a discounted stochastic game. We formulate sufficient conditions on the game which ensure the existence of Nash equilibrium in Markov strategies which has the additional property that, in equilibrium, the process $\{Y_t\}_{t\in\mathbb{N}}$ converges in distribution to a stationary regime.

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# 1 Introduction

This paper considers the stochastic sequence $\{Y_t\}_{t\in\mathbb{N}}$ defined recursively by the linear relation

$$Y_{t+1} = A_t Y_t + B_t \qquad (t \in \mathbb{N}) \tag{1}$$

in the random environment $\{(A_t, B_t)\}_{t\in\mathbb{N}}$. The dynamics of the environment is under the simultaneous control of several agents who play a discounted stochastic game. We formulate sufficient conditions on the game which guarantee the existence of Nash equilibria in Markov strategies which have the additional property that, in equilibrium, the solution to (1) converges in distribution as $t \to \infty$.

Stochastic sequences of the form (1) have been extensively investigated under a mean contraction condition and under the assumption that the driving sequence $\{(A_t, B_t)\}_{t\in\mathbb{N}}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is stationary under the *exogenous* measure $\mathbb{P}$. For example, Vervaat (1979) considers the case where the environment consists of i.i.d. random variables. Brandt (1986) assumes that the driving sequence is stationary and ergodic under the law $\mathbb{P}$; see also Borovkov (1998). Horst (2001) studies the case where the environment is asymptotically stationary in the sense that the process $\{(A_t, B_t)\}_{t\in\mathbb{N}}$ is stationary under some law $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ which coincides with $\mathbb{P}$ on the tail-field generated by $\{(A_t, B_t)\}_{t\in\mathbb{N}}$.

In view of many applications in economics it seems natural to extend the analysis of stochastic difference equations to situations in which the environment is under the simultaneous control of several agents. In such a situation the measure $\mathbb{P}$ is no longer exogenous, but is derived through a game-theoretic solution concept. For example, the process $\{Y_t\}_{t\in\mathbb{N}}$ could be sequence of temporary equilibrium prices of a risky asset generated by the microeconomic interaction of investors who are active on a financial market. In such a model, the sequence $\{(A_t, B_t)\}_{t\in\mathbb{N}}$ may be driven by the changes in the behavioral characteristics of the agents. If the investors are 'large', i.e., if their behavior influences the dynamics of the random environment, then it seems natural to assume that the agents anticipate their impact on the formation of stock prices and, therefore, interact in a strategic manner. In this context, the process $\{Y_t\}_{t\in\mathbb{N}}$ may be viewed as the state sequence associated to a stochastic game, and so the probabilistic structure of the random environment specified by the measure $\mathbb{P}$ is no longer exogenous. Instead, it is defined through an equilibrium strategy implemented by the individual investors. In order to analyze the dynamics of the price process in equilibrium, it is now desirable to have sufficient conditions which guarantee the existence of Nash equilibria which ensure that the sequence $\{Y_t\}_{t\in\mathbb{N}}$ settles down in the long run.

In this paper we consider discounted stochastic games with weakly interacting players in which the conditional distribution of the random variable $(A_t, B_t)$ only depends on the average action taken by the players. We formulate conditions on the game which guarantee the existence of a Nash equilibrium in Markov strategies $\tau$ such that under the induced measure $\mathbb{P}^\tau$ on $(\Omega, \mathcal{F})$ the solution to (1) converges as $t \to \infty$. In a first step we show that stochastic games in which the state sequence follows a linear dynamics and in which the interaction between different agents is sufficiently weak have stationary equilibria in Markov strategies that depend in a Lipschitz continuous manner on the current state. This part of the paper is inspired by the work of Curtat (1996) and uses a perturbation of a *Moderate Social Influence* condition introduced in Horst and Scheinkman (2002). Under a suitable mean contraction condition on the random environment we then prove that the sequence $\{(A_t, B_t)\}_{t\in\mathbb{N}}$ defined on $(\Omega, \mathcal{F}, \mathbb{P}^\tau)$ has a nice tail structure in the sense of Horst (2001). This allows us to show that the shifted sequence $\{Y_{t+T}\}_{t\in\mathbb{N}}$ converges in law to a uniquely determined stationary process as $T \to \infty$.

The remainder of this paper is organized as follows. In Section 2 we formulate our main results. Section 3 proves the existence of a Lipschitz continuous equilibrium in Markov strategies. The convergence result for the solution to (1) is proved in Section 4.

## 2 Assumptions and the main results

Let $\psi := \{(A_t, B_t)\}_{t\in\mathbb{N}}$ be a sequence of $\mathbb{R}^2$-valued random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{Y_t\}_{t\in\mathbb{N}}$ be the sequence in (1) driven by the "input" $\psi$. In this section we specify a probabilistic framework which allows us to analyze the asymptotic behavior of the solution to the linear stochastic difference equation (1) in a situation in which the evolution of the random environment is controlled by several strategically interacting agents.

The long run behavior of the sequence $\{Y_t\}_{t\in\mathbb{N}}$ has been intensively investigated under a mean contraction condition and under the assumption that $\psi$ is stationary under some *exogenous* measure $\mathbb{P}$; see, e.g., Brandt (1986) or Vervaat (1979). Horst (2001) assumes that the environment is asymptotically stationary and that is has a nice tail structure the sense of the following definition.

**Definition 2.1** *(Horst (2001)) Let $\hat{\mathcal{F}}_t := \sigma\left(\{(A_t, B_t)\}_{s \geq t}\right)$ and let*

$$\mathcal{T}_\psi := \bigcap_{t \in \mathbb{N}} \hat{\mathcal{F}}_t \tag{2}$$

*be the tail-$\sigma$-algebra generated by $\psi$. A driving sequence $\psi$ is called* nice *with respect to a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ if the following properties are satisfied:*

*(i) $\psi$ is stationary and ergodic under $\mathbb{Q}$ and satisfies*

$$\mathbb{E}_\mathbb{Q} \ln |A_0| < 0 \quad \text{and} \quad \mathbb{E}_\mathbb{Q} (\ln |B_0|)^+ < \infty \tag{3}$$

*where $\mathbb{E}_\mathbb{Q}$ denotes the expectation with respect to the measure $\mathbb{Q}$.*

*(ii) The asymptotic behavior of $\psi$ is the same under $\mathbb{P}$ and $\mathbb{Q}$, i.e.,*

$$\mathbb{P} = \mathbb{Q} \quad \text{on} \quad \mathcal{T}_\psi. \tag{4}$$

**Remark 2.2** *We denote by $\| \cdot \|_\mathcal{E}$ the total variation of a signed measure on a measurable space $(E, \mathcal{E})$. Since*

$$\lim_{t \to \infty} \|\mathbb{P} - \mathbb{Q}\|_{\hat{\mathcal{F}}_t} = \|\mathbb{P} - \mathbb{Q}\|_{\mathcal{T}_\psi}, \tag{5}$$

*a driving sequence $\psi$ satisfies (4) if and only if it converges to a stationary regime. Horst (2001) shows that (5) holds if, for example, $\psi$ is driven by an underlying Markov chain that converges in total variation norm to a stationary distribution or if $\psi$ coupling converges to a stationary sequence in the sense of Borovkov (1998). Note that (5) is equivalent to the existence of a sequence $\{c_t\}_{t \in \mathbb{N}}$ satisfying $\lim_{t \to \infty} c_t = 0$ and*

$$\sup_{l \geq t} ||\mathbb{P} - \mathbb{Q}||_{\hat{\mathcal{F}}_{t,l}} \leq c_t, \tag{6}$$

*where $\hat{\mathcal{F}}_{t,l} := \sigma\left(\{(A_t, B_t)\}_{t \leq s \leq l}\right)$. Here, both (5) and (6) follow from the continuity of the total variation distance along increasing and decreasing $\sigma$-algebras.*

In the sequel it will be convenient to denote by $\mathrm{Law}(Y, \mathbb{P})$ the law of a random variable $Y$ on $(\Omega, \mathcal{F}, \mathbb{P})$ and to write $\xrightarrow{w}$ for weak convergence of probability measures.

Let us turn to the solution $\{Y_t\}_{t \in \mathbb{N}}$ of (1). For any initial value $Y_0 = y \in \mathbb{R}$, we have the explicit representation

$$Y_t = y_t(y, \psi) := \sum_{j=0}^{t-1} \left( \prod_{i=t-j}^{t-1} A_i \right) B_{t-j-1} + \left( \prod_{i=0}^{t-1} A_i \right) y \qquad (t \in \mathbb{N}). \tag{7}$$

In the stationary setting $\mathbb{P} = \mathbb{Q}$ analyzed by Brandt (1986), we may as well assume that the driving sequence is defined for all $t \in \mathbb{Z}$, due to Kolmogorov's extension theorem. Under the *mean contraction condition* (3), there exists a unique stationary solution to (1) under $\mathbb{Q}$ driven by $\psi$. That is, there is a unique stationary process $\{Y_t^*\}_{t\in\mathbb{Z}}$ which satisfies the recursive relation (1) for all $t \in \mathbb{Z}$. The random variable $Y_t^*$ is $\mathbb{Q}$-a.s. finite, takes the form

$$Y_t^* = \sum_{j=0}^{\infty} \left( \prod_{i=t-j}^{t-1} A_i \right) B_{t-j-1} \qquad (t \in \mathbb{Z}), \tag{8}$$

and, for any initial value $y \in \mathbb{R}$ the solution $\{y_t(y,\psi)\}_{t\in\mathbb{N}}$ to (1) converges almost surely to the stationary solution in the sense that

$$\lim_{t\to\infty} |y_t(y,\psi) - Y_t^*| = 0 \qquad \mathbb{Q}\text{-a.s.}$$

In the non-stationary case $\mathbb{P} \neq \mathbb{Q}$ studied in Horst (2001) the shifted sequence $\{y_{t+T}(y,\psi)\}_{t\in\mathbb{N}}$ driven by a nice input $\psi$ converges in distribution to the unique stationary solution $\{Y_t^*\}_{t\in\mathbb{N}}$ to (1) under $\mathbb{Q}$:

$$\mathrm{Law}(\{y_{t+T}(y,\psi)\}_{t\in\mathbb{N}}, \mathbb{P}) \xrightarrow{w} \mathrm{Law}(\{Y_t^*\}, \mathbb{Q}) \qquad (T \to \infty). \tag{9}$$

So far, the asymptotics of the sequence $\{Y_t\}_{t\in\mathbb{N}}$ have only been analyzed in situations where the probabilistic structure of the random environment $\psi$ is described by an *exogenous* measure. Our aim is study to dynamics of the solution to the linear stochastic difference equation (1) in a situation in which the evolution of $\psi$ is controlled by strategically interacting agents who play a stochastic game.

### 2.1 Financial price fluctuations in the presence of big players

In order to motivate our subsequent analysis, this section discusses an extension of the microstructure model for asset price fluctuations by Föllmer and Schweizer (1993) where the class of stochastic games that we analyze in this paper arises rather naturally.

We consider financial market with a finite set $\mathbb{A} = \{a_1, \ldots, a_N\}$ of *small* investors trading a single risky stock. In reaction to a proposed price $p$ in period $t \in \mathbb{N}$ the agent $a \in \mathbb{A}$ forms a random excess demand $e_t^a(p,\omega)$. The actual stock price $P_t(\omega)$ will be determined by the market clearing condition of zero total excess demand. Thus, we obtain a random sequence

$\{P_t(\omega)\}_{t\in\mathbb{N}}$ of temporary price equilibria on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ defined by the implicit equation

$$\sum_{a\in\mathbb{A}} e_t^a(P_t(\omega), \omega) = 0. \tag{10}$$

Following Föllmer and Schweizer (1993), we consider demand functions of the log-linear form

$$e_t^a(p, \omega) = \log \hat{S}_t^a(\omega) - \log p + \eta_t^a(\omega). \tag{11}$$

Here $\hat{S}_t^a(\omega)$ denotes the agent's reference level for the following period. The $N(0,1)$-distributed random variable $\eta_t^a(\omega)$ may be viewed as his current liquidity demand. To simplify our analysis we assume that all the agents are fundamentalists in the sense of Example 2.1 in Föllmer and Schweizer (1993). This means that an agent's price expectation for the period $t \in \mathbb{N}$ is based on the idea that the logarithmic stock price $S_t := \log P_t$ will move closer to his current subjective perception $F_t^a$ of the stock's fundamental value. This idea is captured by reference levels of the form

$$\log \hat{S}_t^a(\omega) = S_{t-1}(\omega) + \beta_t(\omega) \left[ F_t^a(\omega) - S_{t-1}(\omega) \right].$$

The random variables $\beta_t \in (0,1)$ specify the speeds of price adjustments. They are independent and identically distributed, and independent of all the other random variables. Specifically, we assume that the agents are either optimistic or pessimistic. That is, they associate either a low ($F_t^a = 0$) or a high ($F_t^a = 1$) fundamental value to the stock. We refer the reader to Föllmer and Schweizer (1993) and Föllmer, Horst and Kirman (2003) for microstructure models with more general reference levels.

In order to describe the dynamics of asset prices we denote by $\pi_t := \frac{1}{|\mathbb{A}|}\sum_{a\in\mathbb{A}} \mathbf{1}_{\{F_t^a=1\}}$ the fraction of optimistic agents at time $t$ and put $\eta_t := \frac{1}{|\mathbb{A}|}\sum_{a\in\mathbb{A}} \eta_t^a$. The explicit form (11) of individual excess demand permits us to solve the market clearing condition (10) for the logarithmic equilibrium stock price to obtain

$$S_t = \frac{1}{|\mathbb{A}|}\sum_{a\in\mathbb{A}} \log \hat{S}_t^a + \eta_t = (1-\beta_t)S_{t-1} + \beta_t \pi_t + \eta_t =: A_t S_{t-1} + B_t. \tag{12}$$

Thus, the actual logarithmic stock price is a weighted average of individual logarithmic price assessments and liquidity demands. In the benchmark model where both the agents' assessments of fundamentals and the average liquidity demands fluctuate in an i.i.d. manner through time, the random environment $\{(A_t, B_t)\}_{t\in\mathbb{N}}$ for the evolution of the stock price process is given by a sequence of independent and identically distributed random variables. In this case it follows from, e.g., Theorem 1 in Brandt (1986) that $\mathrm{Law}(S_t, \mathbb{P}) \xrightarrow{w} \mu$ for some law $\mu$ on $\mathbb{R}$. Thus, in this simple benchmark model the asymptotic distribution of stock prices is uniquely determined.

We are now going to study the additional effects that arise the presence of strategically interacting market participants. To this end, we assume that the small investors choose their reference levels at random in reaction to the actions $\tau_t = (\tau_t^i)_{i=1}^M \in [0,1]^M$ taken by some "big players" $i \in \{1, 2 \ldots, M\}$. More precisely, we assume that

$$F_t^a \sim Z(\tau_t; \cdot) \quad \text{for a stochastic kernel } Z \text{ from } \ [0,1]^M \text{ to } \{0,1\}.$$

One may, for example, think of a central bank that tries to keep the "mood of the market" from becoming too optimistic and, if necessary, warns the market participants of emerging bubbles. One may also think of financial experts whose recommendations tempt the agents into buying or selling the stock. These market participants influence the stock price process through their impact on the behavior of small investors, but without actively trading the stock themselves. It seems natural to assume that the big players anticipate the feedback effect their actions have on the evolution of stock prices. Hence it is natural to assume that these players interact in a strategic manner.

We assume that the big players choose their actions according to homogeneous Markov strategies $\tau^i : \mathbb{R} \to [0,1]$. This means that the players take their respective actions $\tau_t^i = \tau^i(S_t)$ in reaction to the current logarithmic stock price. In this case

$$F_t^a \sim Z(\tau(S_t); \cdot) \quad \text{where} \quad \tau(S_t) = \{\tau^i(S_t)\}_{i=1}^M$$

denotes the action profile in period $t$. Thus, given a Markov strategy $\tau$, the conditional law $Q^\tau(S_t; \cdot)$ of the fraction of optimistic agents at time $t$ is described by a suitable stochastic kernel $Q^\tau$ from $\mathbb{R}$ to the finite set of all empirical distributions on $\{1, 2, \ldots, M\}$ :

$$\pi_t \sim Q^\tau(S_t; \cdot).$$

In view of (12), and because the random variables $\beta_t$ and $\eta_t$ ($t \in \mathbb{N}$) are independent and independent and identically distributed, the evolution of temporary logarithmic price equilibria can be described by a linear recursive relation of the form

$$S_{t+1} = A_t S_t + B_t \quad \text{where} \quad (A_t, B_t) \sim Q(\tau(S_t); \cdot), \tag{13}$$

and where $Q$ is a suitable stochastic kernel from $[0,1]^M$ to $(0,1) \times \mathbb{R}$. In particular, the environment for the evolution of asset prices is no longer described by an i.i.d. sequence of random variables. Instead it is under the simultaneous control of strategically interacting players. It is now desirable to have sufficient conditions which guarantee the existence of an equilibrium $\tau^*$ which preserves the main qualitative feature of the original model, namely asymptotic stability of stock prices. That is, it is desirable to have an equilibrium $\tau^*$ such that

$$\mathrm{Law}(S_t, \mathbb{P}^{\tau^*}) \xrightarrow{w} \mu^{\tau^*}.$$

However, in our more general model the long run distribution of stock prices depends on the equilibrium strategy. Thus, the asymptotics of the asset price process is not necessarily uniquely determined. Thus, the presence of strategically interacting market participants can be viewed as an additional source of uncertainty.

## 2.2 The stochastic game

The infinite-horizon discounted stochastic games $\Sigma = (I, X, (U^i), \beta, Q, y)$ which we consider in this paper are defined in terms of the following objects:

- $I = \{1, 2, \ldots, M\}$ is a finite set of *players*.
- $X \subset \mathbb{R}$ is a common compact and convex *action space* for the players.
- $U^i : \mathbb{R} \times \prod_{i \in I} X \to \mathbb{R}$ is the *utility function* for player $i \in I$.
- $\beta \in (0,1)$ is a common discount factor.
- $Q$ is a stochastic kernel from $X$ to $\mathbb{R}^2$.
- $y \in \mathbb{R}$ is the starting point of the state sequence $\{Y_t\}_{t \in \mathbb{N}}$.

A typical action of player $i \in I$ is denoted $x^i \in X$. The actions taken by his competitors are denoted $x^{-i} \in X^{-i} := \{x^{-i} = (x^j)_{j \in I \setminus \{i\}}\}$, and $\underline{\overline{X}} := \{\underline{\overline{x}} = (x^i)_{i \in I} : x^i \in X\}$ is the compact set of all *action profiles*. To each action profile $\underline{\overline{x}} \in \underline{\overline{X}}$, we associate the average action $x := \frac{1}{M} \sum_{i \in I} x^i$.

**Remark 2.3** *The assumption that all the players share a common discount factor is made merely to ease notational complexity. This condition can be dropped without altering the model's qualitative features.*

At each time $t \in \mathbb{N}$, the players observe the current position $Y_t$ of the state sequence $\{Y_t\}_{t\in\mathbb{N}}$. They take their actions $x_t^i = \tau^i(Y_t)$ independently of each other according to a *stationary Markov strategy* $\tau^i : \mathbb{R} \to X$ and the selected action profile $\underline{\overline{x}}_t = (x_t^i)_{i\in I} = (\tau^i(Y_t))_{i\in I}$ along with the present state $Y_t$ yields the instantaneous payoff $U^i(Y_t, \underline{\overline{x}}_t) = U^i(Y_t, x_t^i, x_t^{-i})$ to the agent $i \in I$. We assume that the law of motion depends only on the average action taken by the individual players. More precisely,

$$Y_{t+1} = A_t Y_t + B_t \text{ with } (A_t, B_t) \sim Q\left(cx_t; \cdot\right) \text{ for some } c > 0 \text{ where } x_t := \frac{1}{M}\sum_{i=1}^{M} x_t^i. \tag{14}$$

In this sense we assume that the indirect interaction between different agents, i.e., the interaction through the state sequence $\{Y_t\}_{t\in\mathbb{N}}$, is *global*. Thus, in games with many players or in games with a small $c > 0$, the impact of an individual agent on the dynamics of the random environment $\{(A_t, B_t)\}_{t\in\mathbb{N}}$ for the evolution of the state sequence $\{Y_t\}_{t\in\mathbb{N}}$ is weak.

**Remark 2.4** *For our subsequent analysis it will be essential that the impact of an individual player on the law of the random environment is sufficiently weak. Assuming that the conditional distribution of the random variable $(A_t, B_t)$ depends on the current action profile $\underline{\overline{x}}_t$ only through $cx_t = \frac{c}{M}\sum_{i\in I} x_t^i$ simplifies the formulation of an appropriate weak interaction condition. Of course, weak dependence assumptions can also be formulated differently.*

A stationary Markov strategy $\tau = (\tau^i)_{i\in I}$ along with an initial distribution $\mu$ for the starting point of the state sequence and together with the law of motion $Q$ induces a probability measure $\mathbb{P}_\mu^\tau$ on $(\Omega, \mathcal{F})$ in the canonical way. Under the measure $\mathbb{P}_\mu^\tau$ the state sequence $\{Y_t\}_{t\in\mathbb{N}}$ is a Markov chain on the state space $\mathbb{R}$, and the *expected discounted reward* to player $i \in I$ is given by

$$J^i(y, \tau) = J^i(y, \tau^i, \tau^{-i}) := \mathbb{E}_\mu^\tau\left[\sum_{t=0}^{\infty} \beta^t\, U^i(Y_t, \underline{\overline{x}}_t)\right]. \tag{15}$$

Here the expectation is taken with respect to the measure $\mathbb{P}_\mu^\tau$ and $\tau^{-i} = (\tau^j)_{j\neq i}$. In what follows we write $\mathbb{P}_y^\tau$ for $\mathbb{P}_{\delta_y}^\tau$.

**Definition 2.5** *A stationary Markov strategy profile $\tau$ is a Nash equilibrium for $\Sigma$ if no player can increase his payoff by unilateral deviation from $\tau$, i.e., if*

$$J^i(y, \tau) \geq J^i(y, \sigma^i, \tau^{-i}) \tag{16}$$

*for all Markov strategies $\sigma^i : \mathbb{R} \to X$ and each $i \in I$.*

Our objective is to formulate conditions which guarantee the existence of a Nash equilibrium in Markov strategies $\tau$ such that the Markov chain $(\{Y_t\}, \mathbb{P}_y^\tau)$ converges in law to a unique limiting distribution. To this end, we need to assume strong concavity of an agent's utility function with respect to his own action, and we have to place a quantitative bound on the dependence of the instantaneous utility for player $i \in I$ on the actions taken by his competitors.

**Assumption 2.6** *(i) Uniformly in $i \in I$, the utility functions $U^i : \mathbb{R} \times \underline{X} \to \mathbb{R}$ are bounded, Lipschitz continuous and twice continuously differentiable.*

*(ii) There exists constants $L^{i,i}(y) > 0$ and $L > 0$ such that*

$$\frac{\partial^2}{\partial (x^i)^2} U^i(y, x^i, x^{-i}) \leq -L^{i,i}(y) \leq -L < 0. \tag{17}$$

*In particular, the function $U^i(y, \cdot, x^{-i})$ is strongly concave on $X$.*

*(iii) The law $Q(x; \cdot)$ has a density $q(x, \cdot)$ with respect to the Lebesgue measure $\lambda^2$ on $\mathbb{R}^2$. The maps $x \mapsto q(x, \eta)$ are uniformly bounded, twice continuously differentiable on an open set containing $X$ and uniformly Lipschitz continuous:*

$$|q(x_1, \eta) - q(x_2, \eta)| \leq \bar{L}|x_1 - x_2| \qquad (\bar{L} < \infty).$$

*Moreover, $\left|\frac{\partial}{\partial x} q(x, \eta)\right| \leq \varrho_1(\eta)$ and $\left|\frac{\partial^2}{\partial x^2} q(x, \eta)\right| \leq \varrho_2(\eta)$ for some functions $\varrho_1, \varrho_2 : \mathbb{R}^2 \to \mathbb{R}$ with are integrable with respect to $\lambda^2$.*

The Lipschitz continuity condition on the conditional densities $q(x; \cdot)$ translates into a norm-continuity assumption on the transition probabilities $Q(x; \cdot)$:

$$\|Q(x_1; \cdot) - Q(x_2; \cdot)\|_{\mathcal{B}} \leq \bar{L}|x_1 - x_2| \quad \text{and so} \quad \|Q(x_n; \cdot) - Q(x; \cdot)\|_{\mathcal{B}} \overset{n \to \infty}{\longrightarrow} 0 \quad \text{if} \quad x_n \overset{n \to \infty}{\longrightarrow} x,$$

where $\mathcal{B}$ denotes the Borel-$\sigma$-field on $\mathbb{R}^2$. Norm-continuity conditions have also been imposed by, e.g., Nowak (1985) and Duffie, Geanakopolos, MasColell, and McLennan (1994).

Let us consider an example where our assumptions on the densities $q(x; \cdot)$ can be verified.

**Example 2.7** *Let $f : X \to \mathbb{R}^2$ be a twice continuously differentiable function and let $\varphi_m(\cdot)$ be the density of the two-dimensional standard normal distribution with mean $m = (m_1, m_2)$ with respect to $\lambda^2$. It is easily seen that $q(x, \eta) := \varphi_{f(x)}(\eta)$ satisfies Assumption 2.6 (iii).*

In order to establish the existence of stationary Lipschitz continuous Nash equilibria for $\Sigma$, we have to control the strength of interactions between different players. To this end, we introduce the constants

$$L^{i,j}(y) = \sup_{\underline{x} \in \underline{X}} \left| \frac{\partial^2}{\partial x_i \partial x_j} U^i(y, \underline{x}) \right| \quad (i \neq j) \quad \text{and} \quad L^i = \sup_{y \in \mathbb{R}, \underline{x} \in \underline{X}} \left| \frac{\partial^2}{\partial x_i \partial y} U^i(y, \underline{x}) \right|. \tag{18}$$

The quantity $L^{i,j}(y)$ may be viewed as a measure for the dependence of agent $i$'s instantaneous utility on the choice of player $j$, given the current state $y$. By analogy, $L^i$ measures the dependence of his one-period utility on the current position of the state sequence. We put

$$\hat{L} := \sup_{x \in X} \frac{c^2}{M^2} \left\| \frac{\partial^2}{\partial x^2} q(x, \cdot) \right\|_{L^1} \tag{19}$$

where $\| \cdot \|_{L^1}$ denotes the $L^1$-norm with respect to $\lambda^2$. In order to guarantee the existence of Lipschitz continuous equilibria in the discounted stochastic game $\Sigma$ we need to assume that the interaction between different agents is not too strong. Since the players interact both through their instantaneous utility functions and through their individual impacts on the evolution of the state sequence, we have to control both the dependence of an action of player $j$ on the instantaneous utility of the agent $i$ and the dependence of the law of motion on the actions taken by an individual player. We formulate this condition in terms of a perturbation of the *Moderate Social Influence* condition in Horst and Scheinkman (2002).

**Assumption 2.8** *There exists $\gamma < 1$ such that the following holds for all $i \in I$:*

$$\sum_{j \neq i} L^{i,j}(y) + \frac{c^2}{M} R \leq \gamma L^{i,i}(y) \quad \textit{where} \quad R := \sup_x \frac{\beta \|U^i\|_\infty}{1 - \beta} \left\| \frac{\partial^2}{\partial x^2} q(x, \cdot) \right\|_{L^1}. \tag{20}$$

Thus, *Moderate Social Influence* prevails if an agents marginal utility is less affected by a change in his own action then by changes in all the other agents' choices. Let us consider a case study where our Assumption 2.8 can easily be verified.

**Example 2.9** *Consider the law of motion*

$$Q\left( \frac{c \sum_{i \in I} x_i}{M}; \cdot \right) = \frac{c \sum_{i \in I} x^i}{M} Q_1(\cdot) + \left( 1 - \frac{c \sum_{i \in I} x^i}{M} \right) Q_2(\cdot). \tag{21}$$

*If $Q_i(\cdot)$ has a density $q_i(\cdot)$ with respect to $\lambda^2$, then our Moderate Social Influence condition translates into an assumption on the marginal rates of substitution. Indeed, if the stochastic* *kernel $Q$ takes the linear form (21), then* $\frac{\partial^2}{\partial x^2} q^i(x;\cdot) = 0$, *and Moderate Social Influence prevails if*

$$\sum_{j \neq i} \sup_{\underline{x}} \left| \frac{\frac{\partial^2}{\partial x_i \partial x_j} U^i(y, \underline{x})}{\frac{\partial^2}{\partial^2 x_i} U^i(y, \underline{x})} \right| \leq \gamma \tag{22}$$

*for all $i \in I$, and for some $\gamma < 1$. In addition, (17) requires* $\inf_y \left| \frac{\partial^2}{\partial^2 x_i} U^i(y, \underline{x}) \right| > 0$. *The proof of Theorem 2.10 below shows that stochastic games in which the law of motion takes the linear form (21) have Lipschitz continuous equilibria for all $c > 0$.*

Observe that (22) is necessary for (20). On the other hand, if (22) holds, then we can always choose a small enough $c > 0$ or a large enough $M$ such that Assumption 2.8 is satisfied. In this sense, for games in which the impact of an individual agent on the dynamics of the state sequence is sufficiently weak, (20) reduces to the *Moderate Social Influence* assumption $\sum_{j \neq i} L^{i,j}(y) \leq \gamma L^{i,i}(y)$ introduced in Horst and Scheinkman (2002).

Let us now formulate sufficient conditions for the existence of Lipschitz continuous equilibria in the stochastic game $\Sigma$. The following result will be proved in Section 3 below.

**Theorem 2.10** *Assume that the stochastic game $\Sigma$ satisfies Assumptions 2.6 and 2.8 and*

$$\hat{q} := \sup_{x \in X} \int |A| Q(x; dA, dB) < \infty, \quad q := \sup_{x \in X} \int |A| \left| \frac{\partial^2}{\partial x^2} q(x, A, B) \right| \lambda(dA, dB) < \infty. \tag{23}$$

*Then there exists $C^* > 0$ such that, for all $c < C^*$ the game $\Sigma$ has a stationary equilibrium in Markov strategies $\tau$ which is Lipschitz continuous. That is, there exists $L^* < \infty$ such that*

$$|\tau^i(y_1) - \tau^i(y_2)| \leq L^* |y_1 - y_2| \qquad (i \in I).$$

## 2.3 Convergence of the state sequence

Let us now return to the solution to the stochastic difference equation (1). For a given stationary Nash equilibrium in Markov strategies $\tau$, we denote by $\mathbb{P}^\tau_y$ the law on $(\Omega, \mathcal{F})$ induced by $\tau$ and by the starting point $y \in \mathbb{R}$. In order to guarantee asymptotic stability of the process $\{Y_t\}_{t \in \mathbb{N}}$ under $\mathbb{P}^\tau_y$, we need to assume that the following condition is satisfied.

**Assumption 2.11** *(i) The set $M_x := \{\eta \in \mathbb{R}^2 : q(x, \eta) > 0\}$ is convex.*

*(ii) There exists a constant $r < 1$ such that*

$$\sup_x \int |A| Q(x; dA, dB) \leq r \quad \text{and} \quad \sup_x \int |B| Q(x; dA, dB) < \infty. \tag{24}$$

*(iii) There is a measure $\tilde{\nu}$ on $\mathbb{R}^2$ and a constant $\varrho > 0$ such that*

$$Q(x;\cdot) \geq \varrho\tilde{\nu}(\cdot) \quad \textit{and} \quad \int |A|\tilde{\nu}(dA, dB) \leq r.$$

We are now ready to formulate the main results of this paper. Their proofs will be given in Sections 3.2 and 4 below.

**Theorem 2.12** *Let $\tau$ be a stationary Lipschitz continuous equilibrium in Markov strategies for the stochastic game $\Sigma$. If Assumption 2.11 is satisfied, then the following holds:*

*(i) There is a probability measure $\mathbb{Q}^\tau$ on $(\Omega, \mathcal{F})$ such that the driving sequence $\psi$ defined on $(\Omega, \mathcal{F}, \mathbb{P}_y^\tau)$ is nice with respect to $\mathbb{Q}^\tau$.*

*(ii) Under $\mathbb{Q}^\tau$ the process $\{Y_t\}_{t\in\mathbb{N}}$ defined by (1) is stationary and ergodic and*

$$\mathrm{Law}(\{Y_{t+T}\}_{t\in\mathbb{N}}, \mathbb{P}_y^\tau) \xrightarrow{w} \mathrm{Law}(\{Y_t\}_{t\in\mathbb{N}}, \mathbb{Q}^\tau) \quad \textit{for all } y \in \mathbb{R} \textit{ as } T \to \infty.$$

The proof of Theorem 2.12 turns out to be an immediate consequence of the following stability result for Markov chains. Its proof is given in Section 4.

**Theorem 2.13** *Let $\widetilde{Q}$ be a stochastic kernel from $\mathbb{R}$ to $\mathbb{R}^2$ and assume that $\widetilde{Q}(y;\cdot)$ has a density $\tilde{q}(y,\cdot)$ with respect to $\lambda^2$ that satisfies*

$$|\widetilde{q}(y_1,\eta) - \widetilde{q}(y_2,\eta)| \leq \widetilde{L}|y_1 - y_2| \qquad (\widetilde{L} < \infty). \tag{25}$$

*If $\tilde{Q}$ satisfies Assumption 2.11, then the Markov chain $\{\widetilde{Y}_t\}_{t\in\mathbb{N}}$ defined by the recursive relation*

$$\widetilde{Y}_{t+1} = A_t\widetilde{Y}_t + B_t \quad \textit{where} \quad (A_t, B_t) \sim \widetilde{Q}(\widetilde{Y}_t;\cdot)$$

*has a unique stationary distribution $\mu^*$. Moreover, with $\mathbb{P}^*(\cdot) := \int \mathbb{P}_x(\cdot)\mu^*(dx)$, we have that*

$$\mathrm{Law}(\{\widetilde{Y}_{t+T}\}_{t\in\mathbb{N}}, \mathbb{P}_y) \xrightarrow{w} \mathrm{Law}(\{\widetilde{Y}_t\}_{t\in\mathbb{N}}, \mathbb{P}^*) \quad \textit{for all } y \in \mathbb{R} \textit{ as } T \to \infty.$$

**Remark 2.14** *(i) The existence of Lipschitz continuous equilibria can be shown without restricting the random variables $(A_t, B_t)$ to take values in $\mathbb{R}^2$. However, the proof of Theorem 2.13 uses Horst's (2001) stability result for linear stochastic difference equations in non-stationary random environments. This result is proven for the case $(A_t, B_t) \in \mathbb{R}^2$. Since our focus is on the convergence of the state sequence, it is convenient to start right away with the case $(A_t, B_t) \in \mathbb{R}^2$.*

(ii) *Our Theorem 2.13 may be viewed as a version of Theorem 2.1 in Barnsley, Demko, Elton, and Geronimo (1988). Using Theorem 2.1 in Barnsley* et al. *(1988) one can show that Theorem 2.10 carries over to situations in which the random variables* $(A_t, B_t)$ *take only finitely many values. In fact, this case is much simpler to analyze and we leave this modification to the reader.*

**Example 2.15** *Let us return to the financial market model described in Section 2.2 and assume that the kernel* $\hat{Q}$ *in (13) takes the linear form (21). If the small investors are optimistic (pessimistic) with probability at least* $\delta > 0$*, under the assumption that law of the random variable* $\beta_0$ *is equivalent to* $\lambda$ *on the convex set* $(0,1)$ *and under suitable regularity conditions on the big players' instantaneous utility functions, Theorem 2.12 yields an equilibrium* $\tau^*$ *such that the induced stock price process settles down in the long run.*

We close with section with two more case studies where the assumptions on $Q$ in Theorem 2.12 can indeed be verified.

**Example 2.16** (i) *Let us return to the law of motion* $Q$ *introduced in Example 2.9. In this case, (23) is clearly satisfied. Assumption 2.11 holds if, for instance,* $Q_i(\cdot)$ *has a strictly positive density* $q_i$ *which a convex domain, if* $Q_1(\cdot)$ *and* $Q_2(\cdot)$ *satisfy (24) and if each* $Q_i$ *is of the form* $Q_i(\cdot) = (1-\varrho)\widehat{Q}_i(\cdot) + \varrho\tilde{\nu}(\cdot)$ *for a suitable measure* $\tilde{\nu}$ *on* $\mathbb{R}^2$*.*

(ii) *Consider now the densities* $q(x,\eta) = \varphi_{f(x)}(\eta)$ *where* $\varphi_{f(x)}(\eta)$ *is introduced in Example 2.7. We assume* $f(x) = (f_1(x), x)$ *for some* $f_1 : X \to \mathbb{R}$*. Since* $\eta \mapsto \varphi_m(\eta)$ *is twice continuously differentiable and because* $X \subset \mathbb{R}$ *is compact, it is easy to show that (23) holds. Moreover, there exists* $r_1 > 0$*, such that the mean-contraction condition (24) holds whenever* $f_1(x) \leq r_1$*.*

# 3 Existence of Lipschitz continuous Nash equilibria

This section is devoted to the proof of Theorem 2.10. Since the state space of our stochastic game $\Sigma$ is not compact, we cannot prove the theorem directly. Instead, we shall first analyze 'truncated' games

$$\Sigma^N = \big(I, X, (U^i), \beta, Q, y, [-N, N]\big) \qquad (N \in \mathbb{N})$$

with compact state spaces $[-N, N]$. To this end, we fix a family $G^N$ $(N \geq 2)$ of *truncating functions*. That is, a family of *strictly increasing* functions $G^N : \mathbb{R} \to [-N, N]$ which are Lipschitz continuous with Lipschitz constant one and that satisfy $G^N(x) = x$ for $x \in [-N + 1, N - 1]$. We consider stochastic games $\Sigma^N$ in which the conditional dynamics of the new state $Y_{t+1}$, given $Y_t$ and the action profile $\underline{\overline{x}}_t \in \underline{\overline{X}}$ is of the form

$$Y_{t+1} = v^N(Y_t, A_t, B_t) := G^N(A_t Y_t + B_t) \quad \text{and} \quad (A_t, B_t) \sim Q(cx_t; \cdot).$$

Using our *Moderate Social Influence* assumption we show that the games $\Sigma^N$ have Lipschitz continuous Nash equilibria in Markov strategies $\tau^N$ whose Lipschitz constants do not depend on $N \in \mathbb{N}$. A Lipschitz continuous equilibrium for $\Sigma$ will then be determined as a suitable accumulation point of the sequence $\{\tau^N\}_{N \in \mathbb{N}}$.

### 3.1 Lipschitz continuous equilibria in truncated games

The aim of this section is to establish the existence of Lipschitz continuous Nash equilibria in the truncated games $\Sigma^N$.

**Proposition 3.1** *Under the assumptions of Theorem 2.10 there exists $C^* > 0$ such that for all $c \leq C^*$ the following holds:*

*(i) For all $N \in \mathbb{N}$, the truncated game $\Sigma^N$ has a stationary Lipschitz continuous Nash equilibrium $\tau^N$ in Markov strategies.*

*(ii) The Lipschitz constant of $\tau^N$ $(N \in \mathbb{N})$ does not depend on $N \in \mathbb{N}$.*

The proof of Proposition 3.1 needs some preparation. We put $u := \max_i \|U^i\|_\infty$ and denote by $\mathcal{L}_{L,u}([-N, N], \mathbb{R}^M)$ the class of all Lipschitz continuous functions $f : [-N, N] \to \mathbb{R}^M$ with Lipschitz constant $L$ that satisfy $\|f\|_\infty \leq u$. To each such *average continuation function* $f = (f_i)_{i \in I}$ we associate the *reduced one-shot game* $\Sigma_f^N := (I, X, (U_f^{i,N}), \beta, y, [-N, N])$ with individual payoff functions

$$U_f^{i,N}(y, \underline{\overline{x}}) = (1 - \beta) U^i(y, \underline{\overline{x}}) + \beta \int f_i \circ v^N(y, A, B) q(x, A, B) \lambda(dA, dB) \tag{26}$$

viewed as functions from $[-N, N] \times \underline{\overline{X}}$ to $\mathbb{R}$.

**Remark 3.2** *Observe that $U_f^{i,N}(y, \underline{\overline{x}})$ is the payoff to player $i \in I$ in the discounted stochastic game $\Sigma^N$ if the game terminates after the first round, if the players receive rewards according* *to the payoff functions $f_i$ in the second period, and if first period payoffs are discounted at the rate $1-\beta$.*

The following Lemma shows that the conditional best reply $\tau_f^{i,N}(y, x^{-i})$ of player $i \in I$ in the game $\Sigma_f^N$ is uniquely determined and that $\Sigma_f^N$ has a unique equilibrium $\tau_f^N(y)$. The map $\tau_f^N : [-N, N] \to \underline{X}$ is Lipschitz continuous. Its Lipschitz constant can be specified in terms of the Lipschitz constant of $f$, the discount factor, the bounds for the utility functions and the quantities $L^{i,j}(y), L^i$ and $\hat{L}$ introduced in (18) and (19), respectively. In particular, it can be chosen *independently* of the specific average continuation function $f$ and independently of $N$. This turns out to be the key to the proof of Proposition 3.1.

**Lemma 3.3** *For $N \in \mathbb{N}$, let $\Sigma_f^N$ be the reduced game with average continuation function $f \in \mathcal{L}_{L,u}([-N, N], \mathbb{R}^M)$. Under the assumptions of Theorem 2.10 the following holds:*

*(i) The conditional best rely $\tau_f^{i,N}(y, x^{-i})$ of player $i \in I$ is uniquely determined and depends in a Lipschitz continuous manner on the actions of his competitors. More precisely,*

$$|\tau_f^{i,N}(y, x_1^{-i}) - \tau_f^{i,N}(y, x_2^{-i})| \leq \frac{(1-\beta)L^{i,j}(y) + \beta u \hat{L}}{(1-\beta)L^{i,i}(y) - \beta u \hat{L}} |x_1^j - x_2^j| \tag{27}$$

*if $x_1^k = x_2^k$ for all $k \neq j$. Moreover,*

$$|\tau_f^{i,N}(y_1, x^{-i}) - \tau_f^{i,N}(y_2, x^{-i})| \leq \tilde{L}|y_1 - y_2| \tag{28}$$

*for all $y_1, y_2 \in [-N, N]$ and each $x^{-i} \in \underline{X}^{-i}$. Here*

$$\tilde{L} := \sup_{y,i} \frac{(1-\beta)L^i + \beta L(f) q}{(1-\beta)L^{i,i}(y) - \beta u \hat{L}},$$

*$L(f)$ denotes the Lipschitz constant of $f$ and the quantity $q$ is defined in (23).*

*(ii) The reduced game $\Sigma_f^N$ has a unique equilibrium $\tau_f^N(y) = \{\tau_f^{i,N}(y)\}_{i \in I} \in \underline{X}$.*

*(iii) The map $y \mapsto \tau_f^{i,N}(y)$ is Lipschitz continuous uniformly in $i$, and the Lipschitz constant can be chosen independently of both $N \in \mathbb{N}$ and the average continuation function $f$.*

*(iv) The map $f \mapsto \tau_f^{i,N}(\cdot)$ is continuous.*

PROOF:

(i) Let us fix an average continuation function $f \in \mathcal{L}_{L,u}([-N,N],\mathbb{R}^M)$, an action profile $x^{-i} \in \underline{\overline{X}}^{-i}$ and a state $y \in [-N,N]$. By Assumption 2.6, the map $U_f^{i,N}(y,\cdot,x^{-i})$ is two times continuously differentiable on an open set containing $X$ and our *Moderate Social Influence* condition yields

$$\frac{\partial^2}{\partial (x^i)^2} U_f^{i,N}(y,x^i,x^{-i}) \le -(1-\beta)L^{i,i}(y) + \beta u \hat{L} < 0.$$

Thus, an agent's utility function is strongly concave with respect to his own action, and so his conditional best reply given the choices of his competitors is uniquely determined. To establish the quantitative bound (27) on the dependence of player $i$'s best reply on the behavior of all the other agents, we fix a player $j \neq i$ and action profiles $x_1^{-i}$ and $x_2^{-i}$ which differ only at the $j$-th coordinate. Under the assumptions of Theorem 2.10 we have

$$\left| \frac{\partial}{\partial x^i} U_f^{i,N}(y,x^i,x_1^{-i}) - \frac{\partial}{\partial x^i} U_f^{i,N}(y,x^i,x_2^{-i}) \right| \le \left\{ (1-\beta)L^{i,j}(y) + \beta u \hat{L} \right\} |x_1^j - x_2^j|. \quad (29)$$

Thus, (27) follows from Theorem A.1. In view of (29) and because

$$\left| \frac{\partial}{\partial x^i} U_f^{i,N}(y_1,x^i,x^{-i}) - \frac{\partial}{\partial x^i} U_f^{i,N}(y_2,x^i,x^{-i}) \right| \le \left\{ (1-\beta)L^i + \beta L(f) q \right\} |x_1^j - x_2^j| \quad (30)$$

our estimate (28) also follows from Theorem A.1.

(ii) The existence of an equilibrium in pure strategies for the static game $\Sigma_f^N$ follows from strict concavity of the utility functions $U_f^{i,N}$ with respect to the player's own actions along with compactness of the action spaces using standard fixed points arguments. In order to prove uniqueness, we proceed as in the proof of Proposition 4.21 in Horst and Scheinkman (2002). In view of (17), our *Moderate Social Influence* condition yields

$$\overline{L} := \sup_{i,y} \sum_{j \neq i} \frac{(1-\beta)L^{i,j}(y) + \beta u \hat{L}}{(1-\beta)L^{i,i}(y) - \beta u \hat{L}} < 1.$$

Thus, given action profiles $x_1^{-i}$ and $x_2^{-i}$ of player $i$'s competitors, (27) shows that

$$|\tau_f^{i,N}(y,x_1^{-i}) - \tau_f^{i,N}(y,x_2^{-i})| \le \overline{L} \max_j |x_1^j - x_2^j|.$$

For $x_1 \neq x_2$, we therefore obtain

$$\max_i |\tau_f^{i,N}(y,x_1^{-i}) - \tau_f^{i,N}(y,x_2^{-i})| < \max_i |x_1^i - x_2^i|.$$

Thus, the map $\underline{\overline{x}} \mapsto (\tau_f^{i,N}(y,x^{-i}))_{i \in I}$ has at most one fixed point. This proves uniqueness of equilibria in $\Sigma_f^N$.

(iii) Let $\tau_f^N(y)$ be an equilibrium. Then $\tau_f^{i,N}(y) = \tau_f^{i,N}(y, \{\tau_f^{j,N}(y)\}_{j\neq i})$, and so

$$\begin{aligned}
|\tau_f^{i,N}(y_1) - \tau_f^{i,N}(y_2)| &\leq |\tau_f^{i,N}(y_1, \{\tau_f^{j,N}(y_1)\}_{j\neq i}) - \tau_f^{i,N}(y_1, \{\tau_f^{j,N}(y_2)\}_{j\neq i})| \\
&\quad + |\tau_f^{i,N}(y_1, \{\tau_f^{j,N}(y_2)\}_{j\neq i}) - \tau_f^{i,N}(y_2, \{\tau_f^{j,N}(y_2)\}_{j\neq i})| \\
&\leq \overline{L} \sum_{j\neq i} |\tau_f^{j,N}(y_1) - \tau_f^{j,N}(y_2)| + \tilde{L}|y_1 - y_2|.
\end{aligned}$$

This yields

$$|\tau_f^{i,N}(y_1) - \tau_f^{i,N}(y_2)| \leq \frac{\tilde{L}}{(1-\overline{L})}|y_1 - y_2|. \tag{31}$$

Hence the equilibrium mapping $\tau_f^N : [-N, N] \to \underline{\overline{X}}$ is Lipschitz continuous which a constant that does not depend on $N \in \mathbb{N}$ and which depends on the average continuation function $f$ only through its Lipschitz constant $L(f)$.

(iv) We fix $y \in [-N, N]$ and $x^{-i} \in \underline{\overline{X}}^{-i}$ and apply Theorem A.1 to the map

$$(x^i, f) \mapsto U_f^{i,N}(y, x^i, x^{-i}).$$

Due to Assumption 2.6 (ii) there exists a constant $q^* < \infty$ such that

$$\left|\frac{\partial}{\partial x^i} U_{f_1}^{i,N}(y, x^i, x^{-i}) - \frac{\partial}{\partial x^i} U_{f_1}^{i,N}(y, x^i, x^{-i})\right| \leq q^* \|f_1 - f_2\|_\infty,$$

for all $f, g \in \mathcal{L}_{L,u}([-N, N], \mathbb{R}^M)$, and so Theorem A.1 shows that there is $L_0 < \infty$ such that

$$|\tau_{f_1}^{i,N}(y, x^{-i}) - \tau_{f_2}^{i,N}(y, x^{-i})| \leq L_0 \|f_1 - f_2\|_\infty.$$

Thus, similar arguments as in the proof of (iii) yield the assertion.

□

Our *Moderate Social Influence* conditions appears to be rather strong. However, for the proof of Theorem 2.10 it will be essential that the maps $\tau_f^N : [-N, N] \to \underline{\overline{X}}$ are Lipschitz continuous with a constant that depends on the average continuation function $f$ only through its Lipschitz constant. For that, we need uniqueness of equilibria in the reduced one-shot games $\Sigma_f^N$. We guarantee this by assuming that the utility functions $U_f^{i,N}$ inherit enough concavity in the player's own actions from the original reward functions $U^i$.

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1: Let $B_u([-N,N],\mathbb{R}^M)$ be the class of all measurable functions $f:[-N,N]\to\mathbb{R}^M$ with $\|f\|_\infty\le u$. We introduce an operator $T$ from $\mathcal{L}_{L,u}([-N,N],\mathbb{R}^M)$ to $B_u([-N,N],\mathbb{R}^M)$ by

$$(Tf)_i(y)=(1-\beta)U^i(y,\tau_f^N(y))+\beta\int f_i\circ v^N(y,A,B)q\left(\frac{c}{M}\sum_{i\in I}\tau_f^{i,N}(y),A,B\right)\lambda(dA,dB).$$

A standard argument in discounted dynamic programming shows that for any fixed point $F^N$ of $T$, the action profile $\tau_{F^N}^N(y)$ is an equilibrium in the non-zero sum stochastic game $\Sigma^N$. The equilibrium payoff to player $i\in I$ is given by $\frac{F^N(y)}{1-\beta}$, and the map $\tau_{F^N}^N:[-N,N]\to\underline{X}$ is Lipschitz continuous, due to Lemma 3.3. In order to establish our assertion it is therefore enough find $L<\infty$ and $C^*>0$ such that the operator $T$ has a fixed point in $\mathcal{L}_{L,u}([-N,N],\mathbb{R}^M)$ for all $c<C^*$. For this, we proceed in three steps:

(i) Let $f\in\mathcal{L}_{L,u}([-N,N],\mathbb{R}^M)$. Lipschitz continuity of the utility functions and the densities along with (23) yields a constant $c_1<\infty$ such that

$$|Tf_i(y_1)-Tf_i(y_2)|\le c_1\left(|y_1-y_2|+\|\tau_f^N(y_1)-\tau_f^N(y_2)\|_\infty\right). \tag{32}$$

Thus, Lipschitz continuity of the map $\tau_f^N(\cdot)$ yields Lipschitz continuity of $Tf(\cdot)$.

(ii) For any sequence $\{f^n\}_{n\in\mathbb{N}}$, $f^n\in\mathcal{L}_{L,u}([-N,N],\mathbb{R}^M)$, that converges to $f$ in the topology of uniform convergence, Lemma 3.3 (v) yields $\lim_{n\to\infty}\|\tau_{f^n}^N-\tau_f^N\|_\infty=0$. Thus, Lipschitz continuity of the reward functions and the densities gives us constants $c_2$ and $c_3$ such that

$$|Tf_i^n(y)-Tf_i(y)|\le c_2\|\tau_{f^n}^N-\tau_f^N\|_\infty+c_3\left\{\|f_i^n-f_i\|_\infty+\|\tau_{f^n}^N-\tau_f^N\|_\infty\right\},$$

and so $\lim_{n\to\infty}\|Tf^n-Tf\|_\infty=0$. Thus, $T$ is continuous.

(iii) Lemma 3.3 and (31) show that there are constants $d_1,d_2$ and $d_3$ such that, for all $N\in\mathbb{N}$ and each $f\in\mathcal{L}_{L,u}([-N,N],\mathbb{R}^M)$, the equilibrium mappings $\tau_f^{i,N}:[-N,N]\to\overline{X}$ are Lipschitz continuous and that the Lipschitz constant $L^*=L^*(L)$ takes the form

$$L^*=\frac{d_1+c\frac{d_2}{M}L}{d_3}.$$

In particular, (32) yields

$$|Tf_i(y_1)-Tf_i(y_2)|\le c_1\left[1+\frac{d_1+c\frac{d_2}{M}L}{d_3}\right]|y_1-y_2|.$$

If we choose $C^* \leq \frac{d_3 M}{c_1 d_2}$, then for all sufficiently large $L \in \mathbb{N}$ and for each $c < C^*$, the operator $T$ maps the set $\mathcal{L}_{L,u}([-N,N],\mathbb{R}^M)$ continuously into itself. Since $\mathcal{L}_{L,u}([-N,N],\mathbb{R}^M)$ is a compact set with respect to the topology of uniform convergence, $T$ has a fixed point by Kakutani's theorem.

This finishes the proof $\square$

## 3.2 Lipschitz continuous equilibria in discounted stochastic games and stability of the equilibrium process

We are now prepared to prove the existence of Lipschitz continuous Nash equilibria in Markov strategies for discounted stochastic games with affine state sequences.

Proof of Theorem 2.10: For $N \in \mathbb{N}$, let $\tau^N$ and $F^N$ be a stationary Lipschitz continuous Nash equilibrium and the associated payoff function for the truncated game $\Sigma^N$, respectively. In view of Proposition 3.1 we may assume that these maps are Lipschitz continuous with a common Lipschitz constant, and so the sequences $\{\tau^N\}_{N\in\mathbb{N}}$ and $\{F^N\}_{N\in\mathbb{N}}$ are equicontinuous. Thus, by the theorem of Ascoli and Arzela, there exists a subsequence $(N_k)$ and Lipschitz continuous functions $\tau : \mathbb{R} \to \underline{\overline{X}}$ and $F : \mathbb{R} \to \mathbb{R}$ such that

$$\tau^{N_k}_{F_{N_k}}(y) \to \tau(y) \quad \text{and} \quad F^{N_k}(y) \to F(y) \quad \text{uniformly on compact sets as } k \to \infty.$$

Let us now fix $y \in \mathbb{R}$. For any compact set $K \subset \mathbb{R}^2$ we have, by definition of our truncation functions that

$$\lim_{n\to\infty} \sup_{(A,B)\in K} \left| F^N \circ v^N(y,A,B) - F(Ay+B) \right| = 0.$$

We also have $\lim_{N\to\infty} \left\| Q\left(\frac{c}{M}\sum_{i\in I} \tau^{i,N}(y);\cdot\right) - Q\left(\frac{c}{M}\sum_{i\in I}\tau^i(y);\cdot\right)\right\|_{\mathcal{B}} = 0$. Since the sequence $\{F^N \circ v^N\}_{N\in\mathbb{N}}$ is uniformly bounded, we obtain

$$F_i(y) = (1-\beta)U^i(y,\tau(y)) + \beta \int F_i(Ay+B)q\left(\frac{c}{M}\sum_{i\in I}\tau^i(y), A, B\right)\lambda(dA,dB).$$

Now, it is easily seen that $\tau : \mathbb{R} \to \underline{\overline{X}}$ is a stationary Nash equilibrium in Markov strategies for the game $\Sigma$. By construction, $\tau$ is Lipschitz continuous. $\square$

The proof of Theorem 2.12 is now straightforward.

Proof of Theorem 2.12: For any Lipschitz continuous equilibrium $\tau$, the stochastic kernel $Q^\tau$ from $\mathbb{R}$ to $\mathbb{R}^2$ defined by $Q^\tau(y;\cdot) = Q\left(\frac{c}{M}\sum_{i=1}^M \tau^i(y);\cdot\right)$ satisfies the assumptions of Theorem 2.13. Thus, the assertion follows from that theorem. $\square$

# 4 A Stability Result for a Class of Markov Chains

This section is devoted to the proof of Theorem 2.13. Throughout, we denote by $K$ the transition kernel of the Markov chain $\{\widetilde{Y}_t\}_{t\in\mathbb{N}}$:

$$K(y;\cdot) = \widetilde{Q}\left(y;\{A,B\} : Ay + B \in \cdot\right).$$

Since the densities $\tilde{q}(y,\cdot)$ satisfy (25), it is easy to show that $K$ has the Feller property.

The proof of Theorem 2.13 needs some preparation. In a first step we show that the Feller process $\{\widetilde{Y}_t\}_{t\in\mathbb{N}}$ has a stationary distribution if Assumption 2.11 holds. In a second step, we use a convergence result for Markov chains with *compact* state spaces to show that $K$ has at most one invariant measure.

The proofs of Proposition 4.1 and Lemma 4.4 below use arguments which, in a modified form, also appear in Föllmer, Horst and Kirman (2003).

**Proposition 4.1** *Under the assumptions of Theorem 2.13, the Markov chain $\{\widetilde{Y}_t\}_{t\in\mathbb{N}}$ on $\mathbb{R}$ has a stationary distribution.*

PROOF: Let $\widetilde{\mu}_t^y$ be the law of the random variable $\widetilde{Y}_t$ under $\mathbb{P}_y$ and assume that the Feller process $\{\widetilde{Y}_t\}_{t\in\mathbb{N}}$ is tight. In this case the sequence

$$\{\widetilde{\nu}_t\}_{t\in\mathbb{N}} \quad \text{where} \quad \widetilde{\nu}_t := \frac{1}{t}\sum_{i=1}^{t} \widetilde{\mu}_t^y$$

is tight, too. Thus, by Prohorov's theorem it has an accumulation point $\nu^*$. A modification of standard arguments for Markov chains on compact state spaces given in, e.g., Liggett (1985), Proposition 1.8, shows that $\nu^*$ is a stationary measure for the Feller process $\{\widetilde{Y}_t\}_{t\in\mathbb{N}}$. It is thus enough to show that the sequence $\{\widetilde{Y}_t\}_{t\in\mathbb{N}}$ is tight. For this, it suffices to show that

$$\lim_{c\to\infty} \sup\left\{\mathbb{P}_y[|\widetilde{Y}_t| \geq c], t \in \mathbb{N}\right\} = 0. \tag{33}$$

By Tchebychev's inequality, (33) follows from

$$\sup\left\{\mathbb{E}_y|\widetilde{Y}_t|, t \in \mathbb{N}\right\} < \infty. \tag{34}$$

In order to prove (34), we put

$$a := \sup_y \int |A|\widetilde{Q}(y; dA, B) < 1 \quad \text{and} \quad b := \sup_y \int |B|\widetilde{Q}(y; dA, dB) < \infty$$

and consider the deterministic sequence $\{\overline{Y}_t^y\}_{t\in\mathbb{N}}$ defined by the recursive relation

$$\overline{Y}_{t+1}^y = a\overline{Y}_t^y + b \qquad (t \in \mathbb{N},\ Y_0 = y).$$

Since $a < 1$, we have $\lim_{t\to\infty} \overline{Y}_t^y = \frac{b}{1-a} < \infty$. In particular, $\sup_t \overline{Y}_t^y < \infty$, and so it suffices to show that

$$\mathbb{E}_y|\widetilde{Y}_t| \leq \overline{Y}_t^y \qquad \text{for all} \quad y \geq 0. \tag{35}$$

Clearly, (35) holds for $t = 0$. We proceed by induction and assume that (35) holds for all $t \leq T$. For the induction step, note that

$$\begin{aligned}
\mathbb{E}_y|\widetilde{Y}_{T+1}| &\leq \mathbb{E}_y[|A_T\widetilde{Y}_T|] + \mathbb{E}_y[|B_T|] \\
&\leq \mathbb{E}_y\left[|\widetilde{Y}_T|\mathbb{E}_y[|A_T|\,|\,|\widetilde{Y}_T|]\right] + b \\
&\leq a\mathbb{E}_y|\widetilde{Y}_T| + b \\
&\leq a\overline{Y}_T + b = \overline{Y}_{T+1}.
\end{aligned}$$

This finishes the proof. □

In a second step, we are now going to show that the Feller process $\{\widetilde{Y}_t\}_{t\in\mathbb{N}}$ has at most one stationary distribution. For this, we need the following result.

**Lemma 4.2** *Under the assumptions of Theorem 2.13 the following holds:*

*(i) The support $I_y \subset \mathbb{R}$ of $K(y;\cdot)$ is an interval and $K(y;\cdot)$ is equivalent to $\lambda$ on $I_y$:*

$$dK(y;\cdot) = g_y d\lambda \quad \textit{where} \quad g_y(\cdot) > 0 \quad \textit{on} \quad I_y.$$

*(ii) There exists $m < \infty$ such that*

$$I_y \cap (-\infty, y) \neq \emptyset \quad \textit{for} \quad y \geq m \quad \textit{and} \quad I_y \cap (y, \infty) \neq \emptyset \quad \textit{for} \quad y \leq -m.$$

Proof:

(i) Let us fix $y \in \mathbb{R}$. The support $I_y$ of the measure $K(y;\cdot)$ is given by the topological closure of the set $\{\hat{y} \in \mathbb{R} : \hat{y} = Ay + B$ for some $(A,B) \in M_y\}$. Convexity of $M_y$ yields convexity of $I_y$. In order to prove equivalence of $K(y;\cdot)$ and $\lambda$ on $I_y$, we introduce a bijection $T_y$ on $M_y$ by $T_y(A,B) := (A, Ay+B)$. By the transformation formula for Lebesgue integrals the law of $T_y(A,B)$ has the density $\tilde{q}(y,\cdot) \circ T_y^{-1}$ with respect to $\lambda^2$. Since $\tilde{q}(y;\cdot) > 0$ on $M_y$, the distribution of $Ay + B$ has a strictly positive density $g_y$ with respect to $\lambda$ on $I_y$.

(ii) In view of (i) it is enough to establish the existence of a constant $m < \infty$ that satisfies

$$\widetilde{Q}\,(y; \{(A,B) : |Ay + B| \le |y|\}) > 0 \qquad (|y| \ge m).$$

To this end, we assume to the contrary that, for all $m < \infty$ there is $|y_m| \ge m$ such that

$$\widetilde{Q}\,(y_m; \{(A,B) : |Ay_m + B| \le |y_m|\}) = 0. \tag{36}$$

However, by Assumption 2.11 (ii) we have $\widetilde{Q}\,(y_m; \{(A,B) : |Ay_m| \le r|y_m|\}) > 0$, and so (36) yields

$$\widetilde{Q}\,(y_m; \{(A,B) : |B| \ge (1-r)|y_m|\}) = 1$$

for all $m < \infty$. Hence

$$(1-r)m \le \sup_{|y| \ge m} \int |B| \widetilde{Q}(y; dA, dB) \le \sup_{y} \int |B| \widetilde{Q}(y; dA, dB).$$

For $m \to \infty$, this contradicts Assumption 2.11 (i).

$\square$

In order to prove that the Markov chain $\{\widetilde{Y}_t\}_{t \in \mathbb{N}}$ has at most one invariant distribution we recall the definition of the truncation function $G^N$ from the previous section and introduce Feller processes $\{\widetilde{Y}_t^N\}_{t \in \mathbb{N}}$ with transition kernels $K^N$ on the respective compact states spaces $[-N, N]$ by

$$\widetilde{Y}_{t+1}^N = G^N(A_t \widetilde{Y}_t^N + B_t) \quad \text{where} \quad (A_t, B_t) \sim \widetilde{Q}(\widetilde{Y}_t^N; \cdot).$$

The following lemma shows that the Markov chain $\{\widetilde{Y}_t^N\}_{t \in \mathbb{N}}$ on $[-N, N]$ converges in distribution to a unique limiting measure $\mu^N$. From this we deduce below that the original process $\{\widetilde{Y}_t\}_{t \in \mathbb{N}}$ converges to a stationary regime.

**Lemma 4.3** *(i) For all $N \in \mathbb{N}$ there exists probability measure $\mu^N$ on $[-N, N]$ such that*

$$\mathrm{Law}(\widetilde{Y}_t^N, \mathbb{P}_y) \xrightarrow{w} \mu^N \quad \text{for all } y \in [-N, N] \text{ as } t \to \infty.$$

*(ii) For all sufficiently large $N \in \mathbb{N}$ and for each $y \in (-N, N)$, the measure $K^N(y; \cdot)$ is absolutely continuous with respect to $K(y; \cdot)$.*

PROOF:

(i) The stochastic kernel $\widetilde{Q}$ satisfies Assumptions 2.11 (ii) and (iii) and, due to (25) the Lipschitz continuity condition $\|\widetilde{Q}(y_1;\cdot) - \widetilde{Q}(y_2;\cdot)\| \leq \widetilde{L}|y_1 - y_2|$. Thus, the assertion follows from Theorem 4.1.1 in Norman (1972) because the state space of the process $\{\widetilde{Y}_t^N\}_{t\in\mathbb{N}}$ is compact.

(ii) Suppose that, for all sufficiently large $N$ and for each $y \in (-N, N)$ we have $G^N(I_y) \subset I_y$. In this case it follows from strict monotonicity of the truncating functions $G^N$, from Lemma 4.2 (i) and from the transformation formula for Lebesgue integrals that

$$K^N(y;\cdot) \approx \lambda \quad \text{on} \quad G^N(I_y) \quad \text{and so} \quad K^N(y;\cdot) \ll K(y;\cdot).$$

Hence it is enough to prove $G^N(I_y) \subset I_y$. Since $G^N$ is strictly increasing and Lipschitz continuous with constant one, because $G^N(x) = x$ on $(-N+1, N-1)$ and because $I_y$ is convex,

$$G^N(I_y) \subset I_y \quad \text{or} \quad G^N(I_y) \cap I_y = \emptyset \quad \text{and} \quad G^N(I_y) \cap I_y = \emptyset \Leftrightarrow (-N, N) \cap I_y = \emptyset.$$

Let us now assume to the contrary that for all $N \in \mathbb{N}$ there exists $y_N \in (-N, N)$ such that the intervals $I_{y_N} := [a_N, b_N]$ satisfy $[a_N, b_N] \cap (-N, N) = \emptyset$. In this case we may with no loss of generality assume that $a_N \geq N$ for all $N \in \mathbb{N}$; the case $b_N \leq -N$ along some subsequence can be analyzed similarly. Because of the Feller property of $K$ the sequence $\{y_N\}_{N\in\mathbb{N}}$ cannot have an accumulation point. Therefore we may also assume that $y_N \to \infty$; the case $y_N \to -\infty$ follows by similar means. Hence by Lemma 4.2 (ii) there exists $N^* \in \mathbb{N}$ such that $[a_N, b_N] \cap (-\infty, y_N) \neq \emptyset$ for all $N \geq N^*$. Since $y_N \in (-N, N)$ this yields $[a_N, b_N] \cap (-\infty, N) \neq \emptyset$ contradicting $a_N \geq N$.

□

Now we are ready to prove that the Feller process $\{\widetilde{Y}_t\}_{t\in\mathbb{N}}$ has at most one invariant distribution.

**Lemma 4.4** *Under the assumption of Theorem 2.13, the Markov chain $\{\widetilde{Y}_t\}_{t\in\mathbb{N}}$ has at most one invariant distribution.*

Proof: Let us assume to the contrary that $|\{\mu : \mu K = \mu\}| > 1$. In this case there exist at least two mutually singular ergodic invariant measures $\mu_1, \mu_2$ for $K$. In particular there are two disjoint sets $C_1, C_2$ which are stochastically closed with respect to $K$, i.e., which satisfy

$$K(y; C_i) = \delta_{i,j} \quad \text{for all} \quad y \in C_j;$$

see, e.g., Breiman (1968). We are now going to show that this contradicts Assumption 2.11.

(i) We put $C_i^N = C_i \cap [-N, N]$. Choosing $N \in \mathbb{N}$ sufficiently large, $K^N(y; \cdot) \ll K(y; \cdot)$ for each $y \in (-N, N)$, due to Lemma 4.3 (ii) and $\mu_i([-N, N]) > 0$. Thus, using absolute continuity of $K^N(y; \cdot)$ with respect to $K(y; \cdot)$, it is easy to show that the sets $C_i^N$ are non-empty, disjoint and stochastically closed with respect to $K^N$.

Let $y_n \in C_1$ and assume that $\lim_{n\to\infty} y_n = y \in \overline{C}_1$. Since the transition kernel $K$ of the Markov chain $\{\widetilde{Y}_t\}_{t\in\mathbb{N}}$ has the Feller property, the Portmonteau-Theorem yields

$$K(y; \overline{C}_1) \geq \limsup_{n\to\infty} K(y_n; \overline{C}_1) = 1;$$

see, e.g., Ethier and Kurtz (1986). Hence the *topologically* closed sets $\overline{C}_i$ are *stochastically* closed with respect to $K$. Similar arguments show that the sets $\overline{C}_i^N$ are *stochastically* closed with respect to the transition kernel $K^N$ of the Markov chain $\{\widetilde{Y}_t^N\}_{t\in\mathbb{N}}$.

(ii) If $\overline{C}_1 \cap \overline{C}_2 = \emptyset$, then $\overline{C}_1^N$ and $\overline{C}_2^N$ are two disjoint topologically closed subsets of $[-N, N]$ that are stochastically closed with respect to $K^N$. In view of Assumption 2.8 (ii) and (iii), this contradicts Lemma 3.4.2 and Theorem 4.1.1 in Norman (1972). Thus there exists $y \in \overline{C}_1 \cap \overline{C}_2$. Since $C_i \subset \overline{C}_i$ dense, there are sequences $\{y_n^i\}_{n\in\mathbb{N}}$ in $C_i$ such that $y_n^i \to y$ as $n \to \infty$. By the Feller property of $K$ and because of Lemma 4.2 (i) this yields $\lambda(I_{y_n^1} \cap I_{y_n^2}) > 0$ for all sufficiently large $n \in \mathbb{N}$. Since $K(y_i; \cdot) \approx \lambda$ on $I_{y_n^i}$ we obtain $K(y_n^i; I_{y_n^1} \cap I_{y_n^2}) > 0$. This, however, contradicts $K(y_n^1; \cdot) \perp K(y_n^2; \cdot)$. Thus, $\mu_1 = \mu_2$.

□

In view of Lemma 4.2 and Lemma 4.4 we have the following result.

**Corollary 4.5** *For any initial value $y \in \mathbb{R}$, the Markov chain $\{\widetilde{Y}_t\}_{t\in\mathbb{N}}$ converges in distribution to a unique stationary measure $\mu^*$.*

To prove the stability result stated in Theorem 2.13 it remains to prove the following proposition.

**Proposition 4.6** *The driving sequence $\{(A_t, B_t)\}_{t\in\mathbb{N}}$ is nice with respect to $\mathbb{P}^*$.*

Proof: Since the Markov chain $\{\widetilde{Y}_t\}_{t\in\mathbb{N}}$ converges in law to the unique limiting distribution $\mu^*$, the sequence $\{\widetilde{Y}_t\}_{t\in\mathbb{N}}$ is stationary and ergodic under $\mathbb{P}^*$. Thus, we need to show that the asymptotic distribution of the random environment $\{(A_t,B_t)\}_{t\in\mathbb{N}}$ is the same under $\mathbb{P}_y$ and under $\mathbb{P}^*$. To this end, let $L<\infty$ be the Lipschitz of the stochastic kernel $\widetilde{Q}$:

$$\left\|\widetilde{Q}(y_1;\cdot)-\widetilde{Q}(y_2;\cdot)\right\|_{\mathcal{B}} \le L|y_1-y_2|. \tag{37}$$

Let $\psi_t := \{(A_1,B_1),\ldots,(A_t,B_t)\}$ and let $B_t \in \mathcal{B}_t$ be the Borel-$\sigma$-field on $\mathbb{R}^{2t}$. Our aim is to show that there is a constant $L^*<\infty$ such that

$$|\mathbb{P}_{y_1}[\psi_t \in B_t] - \mathbb{P}_{y_2}[\psi_t\in B_t]| \le L^*|y_1-y_2| \quad \text{for all } y_1,y_2\in\mathbb{R}. \tag{38}$$

For this, we proceed by induction. For $t=1$, (38) follows from (37). Let us therefore assume that there is $\alpha_t$ such that

$$|\mathbb{P}_{y_1}[\psi_t \in B_t] - \mathbb{P}_{y_2}[\psi_t\in B_t]| \le \alpha_t|y_1-y_2| \quad \text{for all } y_1,y_2\in\mathbb{R}.$$

For the induction step observe first that it suffices to prove (38) for sets $B_t$ of the form $B_t = B_t^1\times\cdots\times B_t^t$ with $B_t^l\in\mathcal{B}_1$. The induction hypothesis along with our mean contraction condition yields

$$\begin{aligned}
& |\mathbb{P}_{y_1}[\psi_{t+1}\in B_{t+1}] - \mathbb{P}_{y_2}[\psi_{t+1}\in B_{t+1}]| \\
= \;& \Big|\mathbf{1}_{B^1_{t+1}}(\{A,B\})\mathbb{P}_{Ay_1+B}[\psi_t\in B_t]\widetilde{Q}(y_1;dA,dB) \\
& -\mathbf{1}_{B^1_{t+1}}(\{A,B\})\mathbb{P}_{Ay_2+B}[\psi_t\in B_t]\widetilde{Q}(y_2;dA,dB)\Big| \\
\le \;& \int |\mathbb{P}_{Ay_1+B}[\psi_t\in B_t] - \mathbb{P}_{Ay_2+B}[\psi_t\in B_t]| \;\; \widetilde{Q}(y_1;dA,dB) \\
& + \sup_{B_1} |\mathbb{P}_{y_1}[\psi_1\in B_1] - \mathbb{P}_{y_2}[\psi_1\in B_1]| \\
\le \;& L|y_1-y_2| + \alpha_t \int |Ay_1 - Ay_2| \;\; \widetilde{Q}(y_1;dA,dB) \\
\le \;& (L + r\alpha_t)\,|y_1-y_2|
\end{aligned}$$

where $r<1$ is defined in (24). Thus, $\alpha_{t+1}\le C + r\alpha_t$, and this yields (38) with $L^* := \frac{C}{1-r}$.

Since the maps $y\mapsto \mathbb{P}_y[\psi_t\in B_t]$ are Lipschitz continuous, and because the Lipschitz constants do not depend on $t\in\mathbb{N}$ nor on $B_t\in\mathbb{B}_t$,

$$\|\mathbb{P}_y - \mathbb{Q}\|_{\hat{F}_{t,l}} \le L^* d(\mu_t^y,\mu^*) \qquad \text{for all } l\in\mathbb{N}.$$

Here $\mu_t^y$ denotes the law of $\widetilde{Y}_t$ under $\mathbb{P}_y$ and $d(\cdot,\cdot)$ is the Vasserstein metric that induces the weak topology on the set of probability measure on $\mathbb{R}$. In view of Remark 2.2, this proves that $\psi$ is nice with respect to $\mathbb{P}^*$. $\square$

It is now straightforward to finish the proof of Theorem 2.13.

Proof of Theorem 2.13: By Proposition 4.6 the random environment $\psi$ defined on $(\Omega, \mathcal{F}, \mathbb{P}_y)$ is nice with respect to the measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F})$. Thus, the assertion follows from Theorem 2.4 in Horst (2001). $\square$

# A Parameter dependent optimization problems

The proof of Lemma 3.3 uses a result on Lipschitz continuous dependence of solutions to parameter dependent optimization problem. The result we use is a special case of Theorem 3.1 in Montrucchio (1987):

**Theorem A.1** *Let $X \subset \mathbb{R}$ be closed and convex and let $(Y, \|\cdot\|)$ be a normed space. Let $F : X \times Y \to \mathbb{R}$ be a finite function which satisfies the following conditions:*[1]

*(i) For all $y \in Y$, the map $x \mapsto F(x, y)$ is concave and two times continuously differentiable on $X$. Moreover, there exists $\alpha > 0$ such that $\frac{\partial^2}{\partial x^2} F(x, y) \leq -\alpha$.*

*(ii) The derivative $\frac{\partial}{\partial x} F(x, y)$ of $F$ at $(x, y)$ satisfies the Lipschitz continuity condition*

$$\left| \frac{\partial}{\partial x} F(x, y_1) - \frac{\partial}{\partial x} F(x, y_2) \right| \leq L \|y_1 - y_2\|$$

*for all $y_1, y_2 \in Y$ and all $x \in X$.*

*Under the above assumptions there exists a unique map $\theta : Y \to X$ that satisfies $\sup_{x \in X} = F(x, y) = F(\theta(y), y)$. Moreover, $\theta$ is Lipschitz continuous and*

$$|\theta(y_1) - \theta(y_2)| \leq \frac{L}{\alpha} \|y_1 - y_2\| \quad \text{for all } y_1, y_2 \in Y.$$

# References

Barnsley, M. F., S. G. Demko, J. H. Elton, and J. S. Geronimo (1988): "Invariant Measures for Markov Processes Arising from Iterated Function Systems with Place-Dependent Probabilities," *Ann. Inst. Henri Poincare*, 24, 367–394.

[1] Montrucchio (1987) formulated this theorem under the additional assumption of $Y$ being a closed and convex subset of a Hilbert space. His proof, however, shows that this assumptions is redundant.

BOROVKOV, A. A. (1998): *Ergodicity and Stability of Stochastic Processes.* John Wiley & Sons, New York.

BRANDT, A. (1986): "The Stochastic Equation $Y_{n+1} = A_n Y_n + B_n$ with Stationary Coefficients," *Adv. Appl. Probab.*, 18, 211–220.

BREIMAN, L. (1968): *Probability.* Addison-Wesley, Reading.

CURTAT, L. (1996): "Markov Equilibria in Stochastic Games with Complementaries," *Games and Economic Behavior*, 17, 177–199.

DUFFIE, D., J. GEANAKOPOLOS, A. MASCOLELL, AND A. MCLENNAN (1994): "Stationary Markov Equilibria," *Econometrica*, 62 (4), 745–781.

ETHIER, S. N. AND T. G. KURTZ (1986): *Markov Processes: Characterisation and Convergence.* John Wiley & Sons, New York.

FÖLLMER, H., U. HORST, AND A. KIRMAN (2003): "Financial Price Fluctuation in a Simple Learning Model," Working paper, Humboldt-Universität zu Berlin.

FÖLLMER, H., AND M. SCHWEIZER (1993): "A Microeconomic Approach to Diffusion Models for Stock Prices," *Mathematical Finance*, 3, 1–23.

HORST, U. (2001): "The Stochastic Equation $Y_{t+1} = A_t Y_t + B_t$ with Non-Stationary Coefficients," *J. Appl. Prob.*, 38, 80–95.

HORST, U. AND, J. SCHEINKMAN (2002): "Equilibria in Systems of Social Interactions," *Working paper.* Princeton University.

LIGGETT, T. (1985): *Interacting Particle Systems*, Springer Verlag, Berlin.

MONTRUCCHIO, L. (1987): "Lipschitz Continuous Policy Functions for Strongly Concave Optimization Problems," *Journal of Mathematical Economics*, 16, 259–273.

NORMAN, F. M. (1972): *Markov Processes and Learning Models.* Academic Press, New York.

NOWAK, A. (1985): "Existence of Equilibrium Stationary Strategies in Discounted Non-Cooperative Stochastic Games with Uncountable State Space," *J. Optim. Theory Appl.*, 45, 591–602.

Vervaat, W. (1979): "On a Stochastic Difference Equation and a Representation of Non-Negative Infinitely Diversible Random Variables," *Adv. Appl. Probab.*, 11, 750–783.