Global flows for stochastic differential equations without global Lipschitz conditions

Shizan Fang
Institut de Mathématiques
Université de Bourgogne
B.P. 47870
21078 Dijon
France

Peter Imkeller
Institut für Mathematik
Humboldt-Universität zu Berlin
Unter den Linden 6
10099 Berlin
Germany

Tusheng Zhang
School of Mathematics
University of Manchester
Oxford Road
Manchester M13 9PL
England

December 7, 2005

Abstract

We consider stochastic differential equations driven by Wiener processes. The vector fields are supposed to satisfy only local Lipschitz conditions. The Lipschitz constants of the drift vector field, valid on balls of radius R, are supposed to grow not faster than $\log R$, those of the diffusion vector fields not faster than $\sqrt{\log R}$. We regularize the stochastic differential equations by associating with them approximating ordinary differential equations obtained by discretization of the increments of the Wiener process on small intervals. By showing that the flows associated with the regularized equations converge uniformly to the solution of the stochastic differential equation, we at the same time establish the existence of a global flow for the stochastic equation under local Lipschitz conditions.

2000 AMS subject classifications: primary 60 H 10, 34 F 05; secondary: 60 G 48, 37 C 10, 37 H 10.
Key words and phrases: stochastic differential equation; global flow; local Lipschitz conditions; moment inequalities; martingale inequalities; approximation by ordinary differential equation; uniform convergence.
Short title: Global flow of homeomorphisms

Introduction

Let $A_0, A_1, \cdots, A_N$ be $N+1$ vector fields on the Euclidean space $\mathbf{R}^d$ and $(w_t)_{t\geq 0}$ be a $\mathbf{R}^N-$ valued standard Brownian motion. Consider the following Stratonovich stochastic differential equation

$$dx_t = \sum_{i=1}^{N} A_i(x_t) \circ dw_t^i + A_0(x_t)\,dt, \quad x_0 = x, \tag{0.1}$$

where w_t^i denotes the ith component of w_t. If the coefficients are sufficiently smooth, for example, if $A_1, \cdots, A_N$ are $\mathcal{C}^2$ and A_0 is $\mathcal{C}^1$, the stochastic differential equation (0.1) has a unique solution (x_t). In this case, (x_t) solves also the following Itô stochastic differential equation

$$dx_t = \sum_{i=1}^{N} A_i(x_t) dw_t^i + \tilde{A}_0(x_t)\,dt, \quad x_0 = x \tag{0.2}$$

where

$$\tilde{A}_0 = A_0 + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{d} \frac{\partial A_i}{\partial x_j} A_i^j. \tag{0.3}$$

Under global Lipschitz conditions on the coefficients $\tilde{A}_0, A_1, \cdots, A_N$, Kunita [7] proved that the Itô stochastic differential equation (0.2) defines a global flow of homeomorphisms. On the other hand, under the hypothesis that the coefficients $A_1, \cdots, A_N$ are $\mathcal{C}^2$, bounded, and with bounded derivatives of first and second order, and A_0 is $\mathcal{C}^1$, also bounded with bounded derivative, J.M. Moulinier [10] proved that almost surely the solutions (x_t^n) of the following regularized ordinary differential equations

$$dx_t^n = \sum_{i=1}^{N} A_i(x_t^n)\dot{w}_t^{n,i}\,dt + A_0(x_t^n)\,dt, \quad x_0^n = x, \tag{0.4}$$

where

$$\dot{w}_t^n = 2^n(w_{(k+1)2^{-n}} - w_{k2^{-n}}), \text{ for } t \in [k2^{-n}, (k+1)2^{-n}[, \quad k \geq 0, \tag{0.5}$$

converge to (x_t), uniformly with respect to (t, x) in each compact subset of $\mathbf{R}_+ \times \mathbf{R}^d$. This gives another approach to the existence of global flows. For related works, we refer to Bismut [1], Carverhill-Elworthy[2], Ikeda-Watanabe [4], Malliavin [9], Stroock-Varadhan [11].

The main aim of this paper is to remove the global Lipschitz conditions from the hypotheses needed to arrive at these conclusions. Based on moment estimates for the one-point and two-point motions with explicit dependence on the Lipschitz constants, we still obtain the smooth approximation to the solution of (0.1). Consequently, we will prove the following result

Theorem A. Let $A_1, \cdots, A_N$ be in the class $\mathcal{C}^2$ and A_0 in $\mathcal{C}^1$. Suppose that (i) the growth of the coefficients $A_1, \cdots, A_N$ and their first and second order derivatives is dominated by $\sqrt{\log |x|}$, (ii) the growth of A_0 and its first order derivatives is controlled by $\log |x|$, as $|x| \to \infty$. Then the Stratonovich stochastic differential equation (0.1) defines a global flow of homeomorphisms, that is, for each $t > 0$ the map $x \to x_t(x)$ is a homeomorphism of $\mathbf{R}^d$ almost surely.

The moment estimates we propose in order to derive this Theorem are in the spirit of Imkeller, Scheutzow [5] and Imkeller [6]. Their starting point is a specification of the constant $c_p (p \geq 1)$ in an inequality of the type

$$\mathbf{E}(\sup_{0 \leq t \leq 1} |x_t(x) - x_t(y)|^p) \leq c_p \, |x - y|^p, \quad x, y \in \mathbf{R}^d.$$

In [5], under global Lipschitz conditions on the vector fields, it is seen to be essentially given by $c_p = \exp(cp^2)$ with a universal c. Here we shall work with similar ideas. The essential novelty is the following observation. Assume that only local Lipschitz conditions are given, which on large balls of radius m centered at the origin are given by L_m. For each m, replace the original vector fields by vector fields with global Lipschitz conditions and Lipschitz constant essentially equal to L_m. Then the global two-point motions $(x_t(x), x_t(y))$ are related to the two-point motions $(x_t^m(x), x_t^m(y))$ associated with the modified vector fields through the following key equality

$$|x_t(x) - x_t(y)|^p = \sum_{m=1}^{\infty} |x_t^m(x) - x_t^m(y)|^p \; 1_{\{m-1 \leq Y_1(x) \vee Y_1(y) < m\}},$$

where $t \in [0, 1]$, and $Y_1(x) = \sup_{0 \leq t \leq 1} |x_t(x)|, x \in \mathbf{R}^d$. So, the two-point motions of the global flow will be controlled by the two-point motions of the modified flows and the growth behavior of the one-point motions of the global flow. This idea is exploited in section 1 below (Theorems 1.7 and 1.8). Section 2 is devoted to giving moment estimates of the same type for regularized ordinary differential equations obtained by discretizing the increments of the Wiener process on dyadic time intervals. Again, this is done for one- and two-point motions separately. But the discretization procedure will produce a bad term e^{α_n} (see theorem 2.6) where α_n is an exponential function of Lipschitz constants. In order to get the desired result, in section 3, we truncate the vector fields and at the same time we discretize the Wiener process. In this way, the bad term e^{α_n} can be handled. We show by using our moment inequality techniques, that the flows of the regularized ordinary differential equations converge to the flows of the original stochastic differential equation, under local Lipschitz conditions (Theorem 3.4). The Lipschitz constants on balls of radius R centered at zero are of the order $\log R$ for the drift vector field and $\sqrt{\log R}$ for the diffusion vector fields. These conditions constitute hypothesis (H). Hence Theorem 3.4 implies Theorem A in the usual way (see for example [4]).

We should mention that the existence of global flows of homeomorphisms for *one dimensional* stochastic differential equations was established by Yamada and Ogura

[12], under local Lipschitz and linear growth conditions on the coefficients. For the multi-dimensional case, the situation is quite different; in fact if we denote by τ_x the life-time of the solution $(x_t(x))$ to the stochastic differential equation (0.1), the linear growth (even boundedness) of coefficients is not sufficient to ensure that

$$P(\tau_x = +\infty, \ \textit{for all } x \in \mathbf{R}^d) = 1. \tag{0.6}$$

In the case where the diffusion coefficients are in $\mathcal{C}^{2+\delta}$ and the drift is $\mathcal{C}^{1+\delta}$ with $\delta > 0$, using local flows of derivatives of solutions, Xue-Mei Li [8] proved (0.6) for the stochastic differential equation (0.1), as well as for its dual equation (see [7] for this notion), under the same growth condition on the local Lipschitz constants as ours in theorem A; therefore by theorem 6.1 or theorem 7.3 in [7] she obtains a global flow of diffeomorphisms. Note that even for Itô stochastic differential equations, smoothness of coefficients with $\delta > 0$ was needed to apply theorem 6.1 of [7]. For a study of stochastic differential equations under non-(local) Lipschitz conditions, we refer to [3].

1 Moment estimates for one- and two-point motions

Let $(x_t(x))$ be the solution of the Itô stochastic differential equation (0.2). The growth of the moments of $(x_t(x))$ in the spatial parameter will crucially depend on the growth behavior of the diffusion coefficients $A_1, \cdots, A_N$. In order to capture well the growth of the local Lipschitz constants for estimating moments of the two-point motions $\mathbf{E}(|x_t(x) - x_t(y)|^p)$, we shall distinguish between the following hypotheses

$(H1)$ there are constants C_1 and $C_2 > 0$ such that

$$\sum_{i=1}^N |A_i(x)|^2 \le C_1^2, \quad |\tilde{A}_0(x)| \le C_2(1+|x|);$$

$(H2)$ there are constants C_3 and $C_4 > 0$ such that

$$\sum_{i=1}^N |A_i(x)|^2 \le C_3^2(1+|x|^2), \quad |\tilde{A}_0(x)| \le C_4(1+|x|).$$

Let us remark at this place that our setting could be extended to the case of infinitely many vector fields and correspondingly an infinite dimensional Wiener process, by noting that (H1) and (H2) only concern Euclidean norms, and could be stated for Hilbert-Schmidt norms instead. In what follows, universal positive constants appearing in the inequalities are denoted by C and allowed to change from place to place.

1.1 Precise L^p–estimates for the one-point motion

Denote $Y_t(x) = \sup_{0\le s\le t} |x_s(x)|$. We shall first give the explicit estimate of $||Y_1(x)||_p$ as a function of p.

Proposition 1.1 *Under the condition* $(H1)$, *we have for any* $p > 1$,

$$||Y_1(x)||_p \leq (1 + CC_1\sqrt{p})e^{C_2}\,(1 + |x|). \tag{1.1}$$

Proof. Fix $x \in \mathbf{R}^d$. For $0 \leq t \leq 1$, put $\varphi(t) = ||Y_t(x)||_p$ and $M_t = \sum_{i=1}^{N} \int_0^t A_i(x_s(x))\, dw_s^i$. By the inequality of Burkholder, Davis and Gundy (see [5]), for any $0 \leq T \leq 1$,

$$\mathbf{E}\Big(\sup_{0 \leq t \leq T} |M_t|^p\Big) \leq C\sqrt{p}^p\, \mathbf{E}\Big[\Big(\int_0^T \sum_{i=1}^{N} |A_i(x_s(x))|^2\, ds\Big)^{p/2}\Big] \leq C\, C_1^p \sqrt{p}^p,$$

or

$$||\sup_{0 \leq t \leq T} |M_t|^p||_p \leq CC_1\sqrt{p}.$$

Using equation (0.2), we get the inequality

$$\varphi(T) \leq |x| + CC_1\sqrt{p} + C_2 \int_0^T (1 + \varphi(s))\, ds.$$

Dividing both sides by the term $1 + |x|$ and applying Gronwall's lemma to the function $\varphi(T) + 1/(1 + |x|)$, we get $\frac{\varphi(1)}{1+|x|} \leq (1 + CC_1\sqrt{p})\, e^{C_2}$ and the estimate (1.1) follows. ∎

The preceding moment inequality implies the following exponential inequality.

Corollary 1.2 *Suppose that* $(H1)$ *holds. For* $R > 0$, *there is* $\delta_0 = \delta_0(C_1, C_2, R) > 0$ *such that*

$$\sup_{|x| \leq R} \mathbf{E}\Big(e^{\delta_0 Y_1^2(x)}\Big) < +\infty. \tag{1.2}$$

Proof. By (1.1), there is a constant β such that $||Y_1(x)||_p \leq \beta\sqrt{p}\,(1+|x|)$. Let $\delta > 0$. We have

$$\mathbf{E}\Big(e^{\delta Y_1^2(x)}\Big) = 1 + \sum_{p=1}^{+\infty} \frac{\delta^p \mathbf{E}(Y_1^{2p}(x))}{p!} \leq 1 + \sum_{p=1}^{+\infty} \frac{\delta^p \beta^{2p} (2p)^p (1 + |x|)^{2p}}{p!}.$$

By Stirling's formula: $\dfrac{p^p}{p!} \sim \dfrac{e^p}{\sqrt{2\pi p}}$ as $p \to +\infty$. Hence, if $|x| \leq R$, the above expression is dominated by

$$C\,\Big(1 + \sum_{p=1}^{+\infty} (2\delta\beta^2\, e\,(1 + R)^2)^p\Big) = \frac{C}{1 - 2\delta\beta^2\, e\,(1 + R)^2}$$

which is finite if $\delta < 1/(2\beta^2\, e\,(1 + R)^2)$. So we get (1.2). ∎

In the following proposition, we shall investigate estimates under (H2).

Proposition 1.3 *Under condition* $(H2)$, *there are constants* β_1 *and* $\beta_2 > 0$ *such that for all* $p > 1$ *and* $x \in \mathbf{R}^d$

$$||Y_1(x)||_p \leq \beta_1\, e^{\beta_2 p}\,(1 + |x|). \tag{1.3}$$

Proof. Let M and φ be defined as in the proof of proposition 1.1. Under $(H2)$, we have

$$|| \sup_{0\le t\le T} |M_t| \, ||_p \le CC_1\sqrt{p}\Big[\int_0^T (1+\varphi^2(s))\, ds\Big]^{1/2}.$$

Therefore in this case, the inequality

$$\varphi(T) \le |x| + CC_1\sqrt{p}\Big[\int_0^T (1+\varphi^2(s))\, ds\Big]^{1/2} + C_2\int_0^T (1+\varphi^2(s))\, ds \tag{1.4}$$

follows. To apply Gronwall's lemma, we have to square the two sides of (1.4), with the effect

$$\varphi^2(T) \le 3\Big(|x|^2 + (C^2C_1^2p + 2C_2^2)\int_0^T (1+\varphi^2(s))\, ds\Big).$$

It follows that

$$\frac{\varphi^2(T)+1}{(1+|x|)^2} \le 3\, \exp\{3(C^2C_1^2p+2C_2^2)T\},$$

from which we deduce (1.3). ∎

In the same spirit, we can treat the time variation of the one-point motion moments.

Corollary 1.4 *Under hypothesis* $(H1)$ *or* $(H2)$*, for any* $p>1$*, there is a constant* $C_p>0$ *(which depends on* C_1 *and* C_2*, or on* C_3 *and* C_4 *respectively) such that for* $x\in \mathbf{R}^d, s,t\ge 0$

$$\mathbf{E}(|x_t(x)-x_s(x)|^{2p}) \le C_p\, |t-s|^p\, (1+|x|)^{2p}. \tag{1.5}$$

Proof. We have for $s<t, x\in\mathbf{R}^d$,

$$x_t(x)-x_s(x) = \sum_{i=1}^N \int_s^t A_i(x_u(x))\, dw_u^i + \int_s^t \tilde{A}_0(x_u(x))\, du.$$

Hence there exists a constant $\beta_p>0$ such that

$$\mathbf{E}\Big(|x_t(x)-x_s(x)|^{2p}\Big) \le \beta_p\Big\{\mathbf{E}\Big[\Big(\int_s^t\sum_{i=1}^N |A_i(x_u(x))|^2\, du\Big)^p\Big] + \mathbf{E}\Big[\Big(\int_s^t |\tilde{A}_0(x_u(x))|\, du\Big)^{2p}\Big]\Big\}.$$

So we see that for some constant $C_p>0$ big enough, the right hand side of the above inequality is dominated by

$$C_p(t-s)^p(1+\mathbf{E}(Y_1(x)^{2p})).$$

Now we obtain (1.5) for an eventually different C_p by using (1.1) or (1.3). ∎

1.2 Precise L^p–estimates for the two-point motion under global Lipschitz conditions

Here we shall work under the following global Lipschitz condition

(L) there exist constants L_1 and $L_2 > 0$ such that

$$\sum_{i=1}^{N} |A_i(x) - A_i(y)|^2 \le L_1^2\,|x-y|^2, \quad |\tilde{A}_0(x) - \tilde{A}_0(y)| \le L_2\,|x-y|, \quad x, y \in \mathbf{R}^d.$$

Set $Y_T(x,y) = \sup_{0\le t\le T} |x_t(x) - x_t(y)|$. We shall give the explicit dependence on L_1 and L_2 for L^p– estimates of $Y_1(x,y)$.

Proposition 1.5 *Under hypothesis* (L),*we have for any* $p > 1$, *all* $x, y \in \mathbf{R}^d$

$$\mathbf{E}(Y_1(x,y)^p) \le 2^p\,|x-y|^p\,e^{C\,L_1^2 p^2 + L_2^2 p}. \tag{1.6}$$

Proof. Put $\varphi(T) = ||Y_T(x,y)||_p$. As in the estimates above, we have

$$\varphi(T) \le |x-y| + C\,L_1\sqrt{p}\Big[\int_0^T \varphi^2(s)\,ds\Big]^{1/2} + L_2\int_0^T \varphi(s)\,ds. \tag{1.7}$$

Squaring the two sides of (1.7) results in

$$\varphi^2(T) \le 2\Big(2|x-y|^2 + (2C^2L_1^2 p + L_2^2)\int_0^T \varphi^2(s)\,ds\Big), \quad T \le 1.$$

It follows that for an eventually different constant $C > 0$

$$\varphi(1) \le 2\,|x-y|\,e^{CL_1^2 p + L_2^2},$$

from which we get (1.6). ∎

Remark. In squaring the two sides of (1.7), the control on the Lipschitz constant L_2 was lost. In order to recapture it, we shall now only consider $\mathbf{E}(|x_t(x) - x_t(y)|^p)$.

Proposition 1.6 *Assume* (L). *Then for any* $p \ge 2$, *all* $x, y \in \mathbf{R}^d$, $t \in [0,1]$

$$\mathbf{E}(|x_t(x) - x_t(y)|^{2p}) \le |x-y|^{2p}\,e^{2p^2 L_1^2 + 2pL_2}. \tag{1.8}$$

Proof. Let $\xi_t = |x_t(x) - x_t(y)|^2$. By Itô's formula, we have

$$\begin{aligned} d\xi_t &= 2\sum_{i=1}^{N} \langle x_t(x) - x_t(y), A_i(x_t(x)) - A_i(x_t(y))\rangle\,dw_t^i \\ &\quad + 2\langle x_t(x) - x_t(y), \tilde{A}_0(x_t(x)) - \tilde{A}_0(x_t(y))\rangle\,dt \\ &\quad + \sum_{i=1}^{N} |A_i(x_t(x)) - A_i(x_t(y))|^2\,dt. \end{aligned}$$

The Itô stochastic contraction $d\xi_t \cdot d\xi_t$ is dominated by

$$4\sum_{i=1}^N \langle x_t(x) - x_t(y), A_i(x_t(x)) - A_i(x_t(y))\rangle^2 \le 4L_1^2 \xi_t^2.$$

Again by Itô formula,

$$\begin{aligned} d\xi_t^p &= 2p\sum_{i=1}^N \xi_t^{p-1} \langle x_t(x) - x_t(y), A_i(x_t(x)) - A_i(x_t(y))\rangle \, dw_t^i \\ &\quad +2p\xi_t^{p-1}\langle x_t(x) - x_t(y), \tilde{A}_0(x_t(x)) - \tilde{A}_0(x_t(y))\rangle \, dt \\ &\quad +p\xi_t^{p-1}\sum_{i=1}^N |A_i(x_t(x)) - A_i(x_t(y))|^2 \, dt + \frac{p(p-1)}{2}\xi_t^{p-2}\, d\xi_t \cdot d\xi_t, \end{aligned}$$

which is less than

$$dM_t + (2pL_2 + 2p^2 L_1^2)\, \xi_t^p \, dt$$

where M_t is the martingale part of ξ_t^p. Taking expectations, we get

$$\mathbf{E}(\xi_t^p) \le |x-y|^{2p} + (2pL_2 + 2p^2 L_1^2) \int_0^t \mathbf{E}(\xi_s^p)\, ds.$$

Now Gronwall's lemma gives

$$\mathbf{E}(\xi_t^p) \le |x-y|^{2p}\, e^{2pL_2 + 2p^2 L_1^2}, \quad t \in [0,1],$$

which is nothing but (1.8). ∎

1.3 Precise L^p–estimates for the two-point motion under local Lipschitz conditions

We shall next assume that the vector fields $\tilde{A}_0, A_1, \cdots, A_N$ are only locally Lipschitz. We shall describe growth conditions in m for the Lipschitz coefficients L_m valid on Euclidean balls of radius m that lead to L^p– moment estimates for the two-point motion of the flow. For this purpose, set

$$L_{m,1}^2 = \sum_{i=1}^N \sup_{|x|\le m} ||A_i'(x)||^2, \quad L_{m,2} = \sup_{|x|\le m} ||\tilde{A}_0'(x)|| \tag{1.9}$$

where A_i' denotes the Jacobian of the mapping $x \to A_i(x)$. Then for any $x, y \in B(m) := \{z \in \mathbf{R}^d;\ |z| \le m\}$ we have

$$\sum_{i=1}^N |A_i(x) - A_i(y)|^2 \le L_{m,1}^2\, |x-y|^2, \quad |\tilde{A}_0(x) - \tilde{A}_0(y)| \le L_{m,2}\, |x-y|.$$

Now consider a family of smooth functions $\varphi_m : \mathbf{R}^d \to \mathbf{R}$ satisfying $0 \le \varphi_m \le 1$ and

$$\varphi_m(x) = 1 \text{ for } |x| \le m,\ \varphi_m(x) = 0 \text{ for } |x| > m+2,\ \sup_m \sup_{x\in\mathbf{R}^d} |\varphi'_m(x)| \le 1. \tag{1.10}$$

Define $A_{m,i} = \varphi_m A_i$ for $i = 1, \cdots, N$ and $A_{m,0} = \varphi_m \tilde{A}_0$. Then we have

$$\sup_{x\in\mathbf{R}^d} |A'_{m,i}(x)|^2 \le 2\Big(\sup_{|x|\le m+2} |A_i(x)|^2 + \sup_{|x|\le m+2} ||A'_i(x)||^2\Big), \tag{1.11}$$

$$\sup_{x\in\mathbf{R}^d} |A'_{m,0}(x)| \le \sup_{|x|\le m+2} |\tilde{A}_0(x)| + \sup_{|x|\le m+2} ||\tilde{A}'_0(x)||. \tag{1.12}$$

Set

$$\tilde{L}^2_{m,1} = \sum_{i=1}^N \sup_{x\in\mathbf{R}^d} ||A'_{m,i}(x)||^2, \quad \tilde{L}_{m,2} = \sup_{x\in\mathbf{R}^d} ||A'_{m,0}(x)||.$$

Let $(x_t^m(x))$ be the solution of the following stochastic differential equation

$$dx_t^m = \sum_{i=1}^N A_{m,i}(x_t^m)\, dw_t^i + A_{m,0}(x_t^m)\, dt, \quad x_0^m = x.$$

Applying (1.8), we get for $p \ge 2$,

$$\mathbf{E}(|x_t^m(x) - x_t^m(y)|^{2p}) \le |x-y|^{2p}\, e^{2p^2\tilde{L}^2_{m,1} + 2p\tilde{L}_{m,2}}, \quad t \in [0,1]. \tag{1.13}$$

We have

$$\begin{aligned} |x_t(x) - x_t(y)|^p &= \sum_{m=1}^{+\infty} |x_t(x) - x_t(y)|^p\, \mathbf{1}_{\{m-1 \le Y_1(x)\vee Y_1(y) < m\}} \\ &= \sum_{m=1}^{+\infty} |x_t^m(x) - x_t^m(y)|^p\, \mathbf{1}_{\{m-1 \le Y_1(x)\vee Y_1(y) < m\}}. \end{aligned}$$

According to (1.13) and the Cauchy-Schwarz inequality, we obtain

$$\mathbf{E}(|x_t(x) - x_t(y)|^p) \le |x-y|^p \sum_{m=1}^{+\infty} e^{p^2\tilde{L}^2_{m,1} + p\tilde{L}_{m,2}} \sqrt{P(Y_1(x)\vee Y_1(y) \ge m-1)}. \tag{1.14}$$

With the aid of this inequality, we are able to formulate growth conditions on the Lipschitz constants ensuring global moment estimates for the flow. In the following Theorems, this will be done consecutively under (H1) and (H2).

Theorem 1.7 *Assume (H1). Let $p \ge 2$. Suppose that $L_{m,1} \le \alpha\, m$, $L_{m,2} \le \beta\, m^2$. For $R > 0$, let δ_0 be given according to Corollary 1.2. Suppose*

$$p^2\alpha^2 + p\beta < \delta_0/2. \tag{1.15}$$

Then for any $R > 0$, there exists a constant $C_{p,R} > 0$ such that

$$\mathbf{E}(|x_t(x) - x_t(y)|^p) \le C_{p,R}\, |x-y|^p, \quad \text{for } x, y \in B(R),\ t \in [0,1]. \tag{1.16}$$

In particular if for some $\varepsilon > 0$ *and constants* β_1, β_2 *we have*

$$L_{m,1} \le \beta_1 m^{1-\varepsilon}, \quad L_{m,2} \le \beta_2 m^{2-\varepsilon},$$

then for any $p \ge 2$, *there exists* $C_p > 0$ *such that* (1.16) *holds.*

Proof. Let $C_R = \sup_{|x|\le R} \mathbf{E}(e^{\delta_0 Y_1^2(x)})$. Then for $m \ge 1$, and $x, y \in B(R)$

$$\sqrt{P(Y_1(x) \vee Y_1(y) \ge m-1)} \le \sqrt{2C_R}\, e^{-\delta_0 (m-1)^2/2}.$$

On the other hand, by (1.11) and (1.12), we have

$$\tilde{L}_{m,1}^2 \le 2NC_1^2 + 2\alpha^2(m+2)^2, \quad \tilde{L}_{m,2} \le \beta m^2 + (C_2 + 2\beta)m + 3C_2 + 4\beta.$$

Therefore there exists a constant $\gamma_p > 0$, independent of m, such that

$$e^{p^2 \tilde{L}_{m,1}^2 + p\tilde{L}_{m,2}} \le \gamma_p\, e^{(p^2\alpha^2 + p\beta)m^2}\, e^{(2\alpha^2 + C_2 + 4\beta)m}.$$

Now using (1.14), we get

$$\mathbf{E}(|x_t(x) - x_t(y)|^p) \le \gamma_p \sqrt{2C_R}\, |x-y|^p \sum_{m=1}^{+\infty} e^{-\delta_0(m-1)^2/2}\, e^{(p^2\alpha^2 + p\beta)m^2}\, e^{(2\alpha^2 + C_2 + 4\beta)m}.$$

It is clear that if $p^2\alpha^2 + p\beta < \delta_0/2$, the above series converges, so that (1.16) follows. ∎

Remark: One can specify the R–dependence of the constant $C_{p,R}$ by looking at the proof of Corollary 1.2. It is seen that there is a subtle tradeoff between R and the parameter β appearing in the bound for the Lipschitz constants $L_{m,2}$ which in our setting is expressed through the value of $\delta_0 = \delta_0(C_1, C_2, R)$.

Under (H2), the growth of the diffusion vector fields has to be counterbalanced by a slower growth of the local Lipschitz constants. We shall formulate them implicitly through conditions on the $\tilde{L}_{m,1}, \tilde{L}_{m,2}$.

Theorem 1.8 *Assume* (*H*2) *and the existence of constants* β_1, β_2 *such that*

$$\tilde{L}_{m,1}^2 \le \beta_1 \log m, \quad \tilde{L}_{m,2} \le \beta_2 \log m. \tag{1.17}$$

Then for any $p \ge 2, R > 0$, *there exists a constant* $C_{p,R} > 0$ *such that*

$$\mathbf{E}(|x_t(x) - x_t(y)|^p) \le C_{p,R}\, |x-y|^p, \quad \textit{for } x, y \in B(R),\ t \in [0,1]. \tag{1.18}$$

Proof. Let $q \ge 2$. By (1.3), $\alpha_{q,R} = \sup_{|x|\le R} \mathbf{E}(Y_1(x)^q)$ is finite. Then for any $|x| \le R$ and $m \ge 2$,

$$P(Y_1(x) \ge m-1) \le \alpha_{q,R} \frac{1}{(m-1)^q}.$$

On the other hand, under the condition (1.17),

$$e^{p^2 \tilde{L}_{m,1}^2 + p\tilde{L}_{m,2}} \le (m+2)^{\beta_1 p^2 + p\beta_2}.$$

Therefore if we take $\frac{q}{2} > \beta_1 p^2 + \beta_2 p + 2$, the following series

$$\sum_{m\ge 2} \frac{1}{(m-1)^{q/2}} \cdot (m+2)^{\beta_1 p^2 + p\beta_2}$$

converges. Now using (1.14), we get the desired result (1.18). ∎

2 Moment estimates for regularized ordinary differential equations

Let $n \geq 1$ be an integer. Define $(w_t^n)_{t\in[0,1]}$ by $w_0^n = 0$ and

$$\dot{w}_t^n = 2^n(w_{(\ell+1)2^{-n}} - w_{\ell 2^{-n}}), \qquad \text{for } t \in [\ell 2^{-n}, (\ell+1)2^{-n}[. \tag{2.1}$$

Let $x_t^n(x)$ be the solution of the following ordinary differential equation

$$dx_t^n = \sum_{i=1}^N A_i(x_t^n)\dot{w}_t^{n,i}\,dt + A_0(x_t^n)\,dt, \quad x_0^n = x. \tag{2.2}$$

The aim of this section is to prove moment estimates for one- and two-point motions of these regularized ordinary differential equations, uniformly in the discretization parameter n. For this purpose, we shall use the techniques presented in the previous section, involving the specification of Lipschitz constants.

2.1 Uniform moment estimates for the one-point motions

Define $Y_n(t,x) = \sup_{0\leq s\leq t} |x_s^n(x)|$. Set

$$B_{i,k} = \sum_{j=1}^d \frac{\partial A_i}{\partial x_j} A_k^j, \qquad \text{for } i = 1,\cdots,N \text{ and } k = 0,1,\cdots,N. \tag{2.3}$$

For the first uniform boundedness result, we shall work under growth assumptions very close to (H1) of the previous section.

Proposition 2.1 *Assume that*

$$\sum_{i=1}^N |A_i(x)|^2 \leq C_1^2, \quad |A_0(x)| \leq C_2(1+|x|), \tag{2.4}$$

and

$$|B_{ik}(x)| \leq C_3(1+|x|) \qquad \textit{for all } i,k. \tag{2.5}$$

Then there exist positive constants α_1 and α_2, independent of n and p such that

$$\mathbf{E}(Y_n(1,x)^p) \leq (1+|x|)^p\,\alpha_1^p e^{\alpha_2 p^2}. \tag{2.6}$$

Proof. For $t \in [0,1]$, define $t_n = k2^{-n}$ if $t \in [k2^{-n},(k+1)2^{-n}[$ and $t_n^+ = t_n + 2^{-n}$. Then we have for fixed but arbitrary $t \in [0,1]$

$$\begin{aligned} x_t^n &= x + \sum_{i=1}^N \int_0^t A_i(x_{s_n}^n)\dot{w}_s^{n,i}\,ds + \int_0^t A_0(x_s^n)\,ds \\ &\quad + \sum_{i=1}^N \int_0^t (A_i(x_s^n) - A_i(x_{s_n}^n))\dot{w}_s^{n,i}\,ds \\ &= x + M_n(t) + \int_0^t A_0(x_s^n)\,ds + R_n(t), \end{aligned}$$

accordingly. Consider $Y_i(s) = A_i(x^n_{s_n})$ for $s < t_n$ and $Y_i(s) = (t - t_n)2^n A_i(x^n_{t_n})$ for $t_n \le s \le t$. Then $M_n(t) = \sum_{i=1}^N \int_0^{t_n^+} Y_i(s)\, dw^i_s$. We have

$$\int_0^{t_n^+} |Y_i(s)|^2\, ds = \int_0^{t_n} |Y_i(s)|^2\, ds + 2^{-n}\,(t-t_n)^2 2^{2n} |A_i(x^n_{t_n})|^2 \le \int_0^t |A_i(x^n_{s_n})|^2\, ds.$$

and by Burkholder's inequality

$$(i) \qquad \mathbf{E}(|M_n(t)|^p) \le C\,\sqrt{p}^p\,\mathbf{E}\Big[\Big(\int_0^{t_n^+} \sum_{i=1}^N |Y_i(s)|^2\, ds\Big)^{p/2}\Big] \le CC_1^p \sqrt{p}^p.$$

Remark that for n fixed, $t \to M_n(t)$ is not a martingale. Only $k \to M_n(k2^{-n})$ is a $\mathcal{F}_{k2^{-n}}$-martingale. Let $t \in [\ell 2^{-n}, (\ell+1)2^{-n}[$. According to (i) and by Doob's maximal inequality, we have

$$(ii) \qquad \mathbf{E}\Big(\sup_{0\le k\le \ell} |M_n(k2^{-n})|^p\Big) \le 2e\,\mathbf{E}(|M_n(t_n)|^p) \le 2eCC_1^p\sqrt{p}^p.$$

Here e is Euler's constant, resulting from the simple estimate

$$(\frac{p}{p-1})^p \le 2e, \quad p > 1.$$

Now for $s \in [k2^{-n}, (k+1)2^{-n}[$,

$$(iii) \qquad M_n(s) = M_n(k2^{-n}) + (s - k2^{-n}) \sum_{i=1}^N A_i(x^n_{k2^{-n}})(w^i_{(k+1)2^{-n}} - w^i_{k2^{-n}})\, 2^n.$$

Then $|M_n(s)| \le |M_n(k2^{-n})| + C_1 2^{-n/2}\, \Gamma_n(k2^{-n})$, where

$$(2.7) \qquad \Gamma_n(s) = 2^{n/2} \sum_{i=1}^N |w^i_{s_n^+} - w^i_{s_n}|.$$

Therefore

$$(iv) \qquad \sup_{0\le s\le t} |M_n(s)| \le \sup_{0\le k\le \ell} |M_n(k2^{-n})| + C_1 \sup_{0\le k\le \ell}\Big(2^{-n/2}\Gamma_n(k2^{-n})\Big).$$

Now using lemma 2.2 below, we have, for $p \ge 2$,

$$\begin{aligned}\mathbf{E}\Big[\sup_{0\le k\le \ell}\Big(2^{-n/2}\Gamma(k2^{-n})\Big)^p\Big] \quad &\le \quad \sum_k 2^{-np/2}\mathbf{E}(\Gamma_n(k2^{-n})^p) \\ (v) \quad &\le 2^n \cdot 2^{-np/2}\,(CN)^p\,\sqrt{p}^p \le (CN)^p\,\sqrt{p}^p.\end{aligned}$$

So combining $(iv), (ii)$ and (v), we finally obtain

$$(2.8) \qquad ||\sup_{0\le s\le t} |M_n(s)|^p\,||_p \le CC_1\sqrt{p}.$$

The remainder term R_n is more delicate to estimate. Using the vector fields defined in (2.3), we may express R_n by

$$R_n(t) = \sum_{i,k=1}^N \int_0^t \Big[\int_{s_n}^s B_{ik}(x^n_\sigma)\dot w^{n,k}_\sigma \dot w^{n,i}_s \, d\sigma\Big]\, ds + \sum_{i=1}^N \int_0^t \Big[\int_{s_n}^s B_{i0}(x^n_\sigma)\dot w^{n,i}_s \, d\sigma\Big]\, ds.$$

Let $R_{n,1}$ and $R_{n,2}$ be the two consecutive terms on the right side of the preceding equation. Using hypothesis (2.4), for $\sigma \in [s_n, s[$ we obtain

$$|x^n_\sigma| \le |x^n_{s_n}| + C_1\, 2^{-n} \sum_{i=1}^N |\dot w^{n,i}_{s_n}| + C_2 \int_{s_n}^\sigma (1 + |x^n_s|)\, ds.$$

Hence Gronwall's lemma implies with universal constants C_1, C_2

$$1 + |x^n_\sigma| \le \Big(|x^n_{s_n}| + 1 + C_1\, 2^{-n/2}\, \Gamma_n(s_n)\Big)\, e^{C_2 2^{-n}}. \tag{2.9}$$

Using (2.9) and hypothesis (2.5), we have

$$|R_{n,2}(t)| \le C_3\, e^{C_2 2^{-n}} \Big[\int_0^t (|x^n_{s_n}| + 1)\Gamma_n(s_n)\, ds + C_1 \int_0^t \Gamma_n(s_n)^2\, ds\Big]. \tag{2.10}$$

By independence of $x^n_{s_n}$ and $\Gamma_n(s_n)$, we have

$$\mathbf{E}\Big((|x^n_{s_n}| + 1)^p \Gamma_n(s_n)^p\Big) \le \mathbf{E}((1 + Y_n(s,x))^p)\, \mathbf{E}(\Gamma_n(s_n)^p). \tag{2.11}$$

Combining (2.10) and (2.11) and using (2.14) in Lemma 2.2 again, we get

$$|| \sup_{0\le s\le t} |R_{n,2}(s)|\, ||_p \le C_3\, e^{C_2}\Big(CN\sqrt{p} \int_0^t (1 + ||Y_n(s,x)||_p)\, ds + C_1\, C^2 N^2 p\Big). \tag{2.12}$$

In the same way

$$|| \sup_{0\le s\le t} |R_{n,1}(s)|\, ||_p \le C_3\, e^{C_2}\Big(C^2 N^2 p \int_0^t (1 + ||Y_n(s,x)||_p)\, ds + C_1\, C^3 N^3 p^{3/2}\Big), \tag{2.13}$$

where C_3 is another universal constant, and C results from Lemma 2.2. Now denote $\psi(t) = ||Y_n(t,x)||_p$. Combining (2.8), (2.12) and (2.13), we finally obtain

$$\begin{aligned} \psi(t) + 1 \ &\le\ |x| + 1 + CC_1\sqrt{p} + C_3\, e^{C_2}(C_1 C^2 N^2 p + C_1 C^3 N^3 p^{3/2}) \\ &\quad + C_3\, e^{C_2}(CN\sqrt{p} + C^2 N^2 p) \int_0^t (1 + \psi(s))\, ds. \end{aligned}$$

¿From the structure of the bound just obtained we see that there are two constants $\alpha_1, \alpha_2 > 0$ independent of n and p such that $\psi(1) \le (|x| + 1)\, \alpha_1\, e^{\alpha_2 p}$ holds. The result (2.6) follows. ∎

Lemma 2.2 *There is a constant $C > 0$ such that*

$$||\Gamma_n(s)||_q \le CN\sqrt{q}, \qquad \textit{for all } s \in [0,1[,\ n \ge 1,\ q \ge 2. \tag{2.14}$$

Proof. Let $s \in [k2^{-n}, (k+1)2^{-n}[$ be given. Put $\gamma_i = 2^{n/2}(w^i_{(k+1)2^{-n}} - w^i_{k2^{-n}})$. Then $\gamma_1, \cdots, \gamma_N$ are independent standard Gaussian random variables. For any $1 \le i \le N$

$$\mathbf{E}(|\gamma_i|^q) = 2\int_0^{+\infty} s^q\, e^{-s^2/2}\,\frac{ds}{\sqrt{2\pi}} = \frac{2^{q/2}}{\sqrt{\pi}}\int_0^{+\infty} s^{(q+1)/2-1}\, e^{-s}\, ds.$$

By well known properties of the Gamma function, the above quantity is dominated by $C\, q^{q/2}$ with a universal constant $C > 0$. Now

$$||\Gamma_n(s)||_q \le \sum_{i=1}^N ||\gamma_i||_{L^q} \le CN\sqrt{q}.$$

We obtain (2.14). ∎

We next discuss the case where condition (2.4) is replaced by

$$\sum_{i=1}^N |A_i(x)|^2 \le C_1^2\,(1+|x|^2), \quad |A_0(x)| \le C_2(1+|x|). \tag{2.15}$$

(2.15) combined with (2.5) resembles (H2) of the previous section.

Proposition 2.3 *Assume* (2.15) *and* (2.5). *Then for any $p \ge 2$, there exists a constant $C_p > 0$ such that*

$$\sup_{0\le t\le 1} \mathbf{E}(|x^n_t(x)|^p) \le C_p\,(1+|x|^p), \qquad \textit{for any } n \ge 1. \tag{2.16}$$

Proof. We resume the computation done in the proof of the previous Proposition, taking into account the linear growth of coefficients $A_1, \cdots, A_N$. Let $t \in [0,1]$ be fixed, and set $M_n(t) = \sum_{i=1}^N \int_0^t A_i(x^n_{s_n})\dot w^{n,i}_s\, ds$. By the computations done previously, we see that for some constant $C_p > 0$

$$\mathbf{E}(|M_n(t)|^p) \le C_p \int_0^t (1 + \mathbf{E}(|x^n_{s_n}|^p))\, ds. \tag{i}$$

Moreover, for $\sigma \in [s_n, s_n^+[$, we have

$$|x^n_\sigma| \le |x^n_{s_n}| + C_1\Big(\int_{s_n}^\sigma (1+|x^n_s|)\, ds\Big) \sum_{i=1}^N |\dot w^{n,i}_{s_n}| + C_2\int_{s_n}^\sigma (1+|x^n_s|)\, ds.$$

So Gronwall's lemma gives with some universal constants C_1, C_2

$$|x^n_\sigma| + 1 \le (|x^n_{s_n}| + 1)\, e^{2^{-n}(C_2 + C_1\sum_{i=1}^N |\dot w^{n,i}_{s_n}|)}.$$

It follows that

$$|x^n_\sigma| + 1 \le e^{C_2}\,(|x^n_{s_n}| + 1)\, e^{C_1\Gamma_n(s_n)}, \quad \sigma \in [s_n, s_n^+[. \tag{2.17}$$

Replacing (2.9) by (2.17) in the estimate of $R_n(t)$, we have with another universal constant C_3

$$|R_{n,2}(t)| \le C_3\, e^{C_2} \int_0^t (|x^n_{s_n}| + 1)\, e^{C_1\Gamma_n(s_n)}\, \Gamma_n(s_n)\, ds,$$

$$|R_{n,1}(t)| \le C_3\, e^{C_2} \int_0^t (|x^n_{s_n}| + 1)\, e^{C_1\Gamma_n(s_n)}\, \Gamma_n(s_n)^2\, ds.$$

By a direct calculation,

$$(ii) \qquad \mathbf{E}(e^{2pC_1\Gamma_n(s)}) \le 2^N e^{4p^2C_1^2N/2}.$$

Now using the independence of $|x^n_{s_n}|$ and $\Gamma(s_n)$, (ii) and (2.14), we see that there is a constant $C_p > 0$ such that

$$(iii) \qquad \mathbf{E}(|R_n(t)|^p) \le C_p \int_0^t (1 + \mathbf{E}(|x^n_{s_n}|^p))\, ds.$$

Therefore, (i) and (iii) imply

$$\mathbf{E}(|x^n_t|^p) \le C_p \Big(|x|^p + \int_0^t (1 + \mathbf{E}(|x^n_{s_n}|^p))\, ds + \int_0^t (1 + \mathbf{E}(|x^n_s|^p))\, ds \Big).$$

Finally consider $\psi(t) = \sup_{0\le s\le t} \mathbf{E}(|x^n_s|^p) + 1$. The inequality just derived implies that

$$\psi(t) \le C_p\,(|x|^p + 1) + 2C_p \int_0^t \psi(s)\, ds.$$

So, a final application of Gronwall's lemma yields another constant C_p such that

$$\sup_{0\le t\le 1} \mathbf{E}(|x^n_t|^p) \le C_p\,(1 + |x|^p). \ ∎$$

Using the same techniques, we may also derive uniform moment estimates for the time fluctuations of the approximate ordinary differential equations.

Proposition 2.4 *Assume* (2.15) *and* (2.5). *Then for any $p \ge 2$, there exists a constant $C_p > 0$, independent of n, such that*

$$(2.18) \qquad \mathbf{E}(|x^n_s(x) - x^n_t(x)|^p) \le C_p(1 + |x|^p)\, |s - t|^{p/2}.$$

We finally derive a result describing a bound for the maximal growth of the one-point motions of the regularizing ordinary differential equations, uniformly in n.

Theorem 2.5 *Assume* (2.15) *and* (2.5). *Then for any $p \ge 2$ there exists a constant $C_p > 0$ such that*

$$(2.19) \qquad \mathbf{E}(Y_n(1, x)^p) \le C_p\,(1 + |x|^p), \quad \textit{for any } n \ge 1.$$

Proof. Let $\gamma > 0$ be a parameter such that $0 < \gamma < 1/2$ and $q \ge 2$ be an integer such that $2q\gamma > 1$, $2q(\frac{1}{2} - \gamma) > 1$. Then it is known from the regularity lemma of Garsia, Rodemich and Rumsey that

$$\sup_{0 \le t \le 1} |\psi(t)|^{2q} \le C_{q,\gamma} \int_0^1 \int_0^1 \frac{|\psi(s) - \psi(t)|^{2q}}{|t - s|^{1+2q\gamma}} \, dsdt.$$

Therefore we have

$$\mathbf{E}\Big(\sup_{0 \le t \le 1} |x_t^n(x)|^{2qp}\Big) \le C_{q,\gamma}^p \int_0^1 \int_0^1 \frac{\mathbf{E}(|x_s^n(x) - x_t^n(x)|^{2qp})}{|t - s|^{(1+2q\gamma)p}} dsdt.$$

But by (2.18), this bound is dominated by $C_p\,(1 + |x|^p)^{2q}$, since

$$\int_0^1 \int_0^1 |t - s|^{qp-(1+2q\gamma)p} \, dsdt \le 1.$$

So we get (2.19). ∎

2.2 Uniform moment estimates for the two-point motions

For vector fields satisfying global Lipschitz conditions, and regularizations as considered here, Bismut [1] or Moulinier [10] proved that $\mathbf{E}(|x_t^n(x) - x_t^n(y)|^p) \le C_p\,|x - y|^p$ for all $x, y \in \mathbf{R}^d$, where C_p is independent of n. However, the dependence of C_p on the Lipschitz continuity properties of the vector fields is not specified. In what follows, we shall make this functional dependence explicit.

Theorem 2.6 *Assume that for $x, y \in \mathbf{R}^d$*

(2.20) $$\sum_{i=1}^N |A_i(x) - A_i(y)|^2 \le L_1^2\,|x - y|^2, \quad |A_0(x) - A_0(y)| \le L_2\,|x - y|.$$

and for all $1 \le i \le N, 1 \le k \le N$

(2.21) $$|B_{ik}(x) - B_{ik}(y)| \le K_1\,|x - y|, \quad |B_{i0}(x) - B_{i0}(y)| \le K_2\,|x - y|.$$

Let C be the constant appearing in Lemma 2.2. Define

$$\begin{aligned}\alpha_n &= 2p((2p-1)L_1^2 + K_1)(4C^2N^2 2^N e^{8p^2N2^{-n}L_1^2})e^{2p2^{-n}L_2} \\ &\quad + 2^{-n/2}2p((2p-1)L_1L_2 + K_2)(2CN2^N e^{8p^2N2^{-n}L_1^2})e^{2p2^{-n}L_2}.\end{aligned}$$

Then

$$\mathbf{E}(|x_t^n(x) - x_t^n(y)|^{2p}) \le |x - y|^{2p}\,e^{2pL_2}\,e^{\alpha_n} \le |x - y|^{2p}\,e^{2pL_2}\,e^{\alpha_1}.$$

Proof. For n, x, y, t fixed, we have

$$\begin{aligned}x_t^n(x) - x_t^n(y) &= x - y + \sum_{i=1}^N \int_0^t (A_i(x_s^n(x)) - A_i(x_s^n(y)))\dot{w}_s^{n,i}\, ds \\ &\quad + \int_0^t (A_0(x_s^n(x)) - A_0(x_s^n(y)))\, ds.\end{aligned}$$

Set $\xi_t = |x_t^n(x) - \xi_t^n(y)|^2$. Then

$$\begin{aligned} d\xi_t &= 2\sum_{i=1}^N \langle x_t^n(x) - x_t^n(y),\ A_i(x_t^n(x)) - A_i(x_t^n(y))\rangle\, \dot{w}_t^{n,i}\, dt \\ &\quad +2\langle x_t^n(x) - x_t^n(y),\ A_0(x_t^n(x)) - A_0(x_t^n(y))\rangle\, dt. \end{aligned}$$

Set

$$Q_i(t) = \langle x_t^n(x) - x_t^n(y),\ A_i(x_t^n(x)) - A_i(x_t^n(y))\rangle, \qquad \text{for } i = 0, 1, \cdots, N.$$

Then $d\xi_t$ has the decomposition $d\xi_t = 2\sum_{i=1}^N Q_i(t)\dot{w}_t^{n,i}\, dt + 2Q_0(t)\, dt$. For $p \geq 2$, we have

$$\begin{aligned} d\xi_t^p &= 2p\sum_{i=1}^N \xi_t^{p-1} Q_i(t)\dot{w}_t^{n,i}\, dt + 2p\xi_t^{p-1}Q_0(t)\, dt \\ &= 2p\sum_{i=1}^N \xi_{t_n}^{p-1} Q_i(t_n)\dot{w}_t^{n,i}\, dt + 2p\xi_t^{p-1}Q_0(t)\, dt \\ &\quad +2p\sum_{i=1}^N \Big(\xi_t^{p-1}Q_i(t) - \xi_{t_n}^{p-1}Q_i(t_n)\Big)\dot{w}_t^{n,i}\, dt. \end{aligned} \tag{2.22}$$

Let $M_t = 2p\sum_{i=1}^N \int_0^t \xi_{s_n}^{p-1} Q_i(s_n)\dot{w}_s^{n,i}\, ds$. Then $\mathbf{E}(M_t) = 0$. Moreover, we have

$$2p\int_0^t |\xi_s^{p-1}Q_0(s)|\, ds \leq 2pL_2\int_0^t \xi_s^p\, ds. \tag{2.23}$$

To estimate the third term $R(t) = 2p\sum_{i=1}^N \int_0^t (\xi_s^{p-1}Q_i(s) - \xi_{s_n}^{p-1}Q_i(s_n))\, \dot{w}_s^{n,i}\, ds$ appearing on the right hand side of (2.22), we compute the derivative of $\xi_s^{p-1}Q_i(s)$. We get

$$\Big(\xi_s^{p-1}Q_i(s)\Big)' = (p-1)\xi_s^{p-2}\xi_s' Q_i(s) + \xi_s^{p-1}Q_i'(s).$$

Computing $Q_i'(s)$ and using our Lipschitz continuity hypotheses, we get

$$|Q_i'(s)| \leq (K_1 + L_1^2)\xi_s\sum_{k=1}^N |\dot{w}_s^{n,k}| + (L_1L_2 + K_2)\xi_s.$$

Therefore

$$\Big|\Big(\xi_s^{p-1}Q_i(s)\Big)'\Big| \leq ((2p-1)L_1L_2 + K_2)\xi_s^p + ((2p-1)L_1^2 + K_1)\xi_s^p\sum_{k=1}^N |\dot{w}_s^{n,k}|. \tag{2.24}$$

To estimate the contribution of ξ_s^p, note first that for $\sigma \in [s_n, s_n^+[$ we have

$$\begin{aligned} |x_\sigma^n(x) - x_\sigma^n(y)| &\leq |x_{s_n}^n(x) - x_{s_n}^n(y)| + L_1\Big(\int_{s_n}^\sigma |x_u^n(x) - x_u^n(y)|\, du\Big)\sum_{i=1}^N |\dot{w}_{s_n}^{n,i}| \\ &\quad +L_2\int_{s_n}^\sigma |x_u^n(x) - x_u^n(y)|\, du. \end{aligned}$$

Now apply Gronwall's lemma. This leads to

$$|x^n_\sigma(x) - x^n_\sigma(y)| \leq |x^n_{s_n}(x) - x^n_{s_n}(y)| \cdot e^{2^{-n}L_1 \sum_{i=1}^N |\dot{w}^{n,i}_{s_n}|} \, e^{2^{-n}L_2}.$$

Therefore for $\sigma \in [s_n, s_n^+[$,

$$\xi^p_\sigma \leq \xi^p_{s_n} \cdot e^{2p2^{-n/2}L_1\Gamma_n(s_n)} \, e^{2p2^{-n}L_2}. \tag{2.25}$$

Hence by (2.24)

$$\begin{aligned}
|R(t)| \quad \leq \quad & 2p \sum_{i=1}^N \int_0^t \int_{s_n}^s |(\xi^{p-1}_\sigma Q_i(\sigma))'| \, |\dot{w}^{n,i}_s| \, d\sigma ds \\
& \leq 2p \Big\{ ((2p-1)L_1^2 + K_1) \int_0^t \int_{s_n}^s \xi^p_\sigma (\sum_{i=1}^N |\dot{w}^{n,i}_\sigma|)(\sum_{k=1}^N |\dot{w}^{n,k}_s|) \, d\sigma ds \\
& \quad +((2p-1)L_1L_2 + K_2) \int_0^t \int_{s_n}^s \xi^p_\sigma (\sum_{i=1}^N |\dot{w}^{n,i}_\sigma|) \, d\sigma ds \Big\},
\end{aligned}$$

which, according to (2.25), is dominated by

$$\begin{aligned}
& 2p \Big\{ ((2p-1)L_1^2 + K_1) e^{2p2^{-n}L_2} \int_0^t \xi^p_{s_n} \Gamma_n(s_n)^2 \, e^{2p2^{-n/2}L_1\Gamma_n(s_n)} \, ds \\
& \quad +2^{-n/2}((2p-1)L_1L_2 + K_2) e^{2p2^{-n}L_2} \int_0^t \xi^p_{s_n} \Gamma_n(s_n) \, e^{2p2^{-n/2}L_1\Gamma_n(s_n)} \, ds \Big\}.
\end{aligned}$$

We next employ the independence of ξ_{s_n} and $\Gamma_n(s_n)$. Therefore

$$\mathbf{E}\Big(\xi^p_{s_n} \Gamma_n(s_n)^2 \, e^{2p2^{-n/2}L_1\Gamma_n(s_n)}\Big) = \mathbf{E}(\xi^p_{s_n}) \mathbf{E}\Big(\Gamma_n(s_n)^2 \, e^{2p2^{-n/2}L_1\Gamma_n(s_n)}\Big).$$

By estimates derived before, using Lemma 2.22 we have

$$\mathbf{E}\Big(\Gamma_n(s_n)^2 \, e^{2p2^{-n/2}L_1\Gamma_n(s_n)}\Big) \leq 4C^2N^2 2^N e^{8p^2N2^{-n}L_1^2},$$

and

$$\mathbf{E}\Big(\Gamma_n(s_n) \, e^{2p2^{-n/2}L_1\Gamma_n(s_n)}\Big) \leq 2CN2^N e^{8p^2N2^{-n}L_1^2}.$$

Summarizing, the definition

$$\begin{aligned}
\alpha_n \quad = \quad & 2p((2p-1)L_1^2 + K_1)(4C^2N^2 2^N e^{8p^2N2^{-n}L_1^2}) e^{2p2^{-n}L_2} \\
\text{(2.26)} \quad & +2^{-n/2} 2p((2p-1)L_1L_2 + K_2)(2CN2^N e^{8p^2N2^{-n}L_1^2}) e^{2p2^{-n}L_2}
\end{aligned}$$

implies the inequality for $\mathbf{E}(|R(t)|)$:

$$\mathbf{E}(|R(t)|) \leq \alpha_n \int_0^t \mathbf{E}(\xi^p_{s_n}) \, ds.$$

Substituting all the estimates obtained so far in (2.22), we obtain

$$\mathbf{E}(\xi^p_t) \leq |x-y|^{2p} + 2pL_2 \int_0^t \mathbf{E}(\xi^p_s) \, ds + \alpha_n \int_0^t \mathbf{E}(\xi^p_{s_n}) \, ds.$$

Finally, let $\psi_u = \sup_{0\le s\le u} \mathbf{E}(\xi_s^p)$. For $T > 0$ and any $0 \le t \le T$, the above inequality then leads to

$$\mathbf{E}(\xi_t^p) \le |x-y|^{2p} + 2pL_2 \int_0^T \psi_s\, ds + \alpha_n \int_0^t \psi_s\, ds,$$

in other terms $\psi_T \le |x-y|^{2p} + (2pL_2 + \alpha_n) \int_0^T \psi_s\, ds$. So Gronwall's lemma implies that for any $0 \le t \le 1$

$$\mathbf{E}(\xi_t^p) \le |x-y|^{2p}\, e^{2pL_2} e^{\alpha_n}.$$

We have the desired result. ∎

3 Limit theorem without global Lipschitz conditions

The expression (2.26) for α_n is quite complicated. But it gives the explicit dependence of our uniform moment estimates on the Lipschitz constants for the vector fields of the underlying stochastic differential equation. We shall exploit this fact in the present section, to derive a Theorem about the convergence of the ordinary differential equation regularizations given in the preceding section to the solution of the stochastic differential equation. The explicit form of the dependence allows us to relax the global Lipschitz conditions to suitable local ones. For this purpose the techniques explained in the first section will be applied. Let us first formulate convenient local Lipschitz conditions.

Let $A_1, \cdots, A_N$ be $\mathcal{C}^2$–vector fields on $\mathbf{R}^d$, A_0 is a $\mathcal{C}^1$–vector field. Suppose for $x, y \in B(n)$

$$(3.1) \qquad \sum_{i=1}^N |A_i(x) - A_i(y)|^2 \le L_{n,1}^2\, |x-y|^2, \quad |A_0(x) - A_0(y)| \le L_{n,2}|x-y|,$$

with positive constants $L_{n,1}, L_{n,2}$. Choose a family of smooth functions $\varphi_n : \mathbf{R}^d \to \mathbf{R}$ satisfying $0 \le \varphi_n \le 1$ and

$$(3.2) \quad \varphi_n = 1 \text{ on } B(n),\ \varphi_n = 0 \text{ on } B(n+2)^c,\ \sup_n ||\varphi_n'||_\infty \le 1,\ \sup_n ||\varphi_n''||_\infty \le C < +\infty$$

where $||\cdot||_\infty$ denotes the uniform norm. Introduce the vector fields

$$A_{n,i} = \varphi_n A_i, \qquad \text{for } i = 0, 1, \cdots, N.$$

Put

$$(3.3) \qquad \tilde{L}_{n,1}^2 = \sum_{i=1}^N \sup_{x\in\mathbf{R}^d} ||A_{n,i}'(x)||^2, \quad \tilde{L}_{n,2} = \sup_{x\in\mathbf{R}^d} ||A_{n,0}'(x)||.$$

Define

$$B_{ik}^n = \sum_{j=1}^d \frac{\partial A_{n,i}}{\partial x_j} A_{n,k}^j \qquad \text{for } i = 1, \cdots, N \text{ and } k = 0, 1, \cdots, N,$$

and set

$$K_{n,1} = \sup_{i,k} \sup_{x\in\mathbf{R}^d} ||(B^n_{ik})'(x)||, \quad K_{n,2} = \sup_{i} \sup_{x\in\mathbf{R}^d} ||(B^n_{i0})'(x)||. \tag{3.4}$$

For $n \in \mathbf{N}$, let $(z^n_t(x))$ be the solution of the following ordinary differential equation

$$dz^n_t = \sum_{i=1}^N A_{n,i}(z^n_t)\,\dot w^{n,i}_t\,dt + A_{n,0}(z^n_t)\,dt, \quad z^n_0 = x, \tag{3.5}$$

with $\dot w^{n,i}$ as defined in (2.1). We can apply Theorem 2.6 to obtain the estimate

$$\mathbf{E}(|z^n_t(x) - z^n_t(y)|^{2p}) \le |x-y|^{2p}\,e^{2p\tilde L_{n,2}}\,e^{\tilde\alpha_n} \tag{3.6}$$

where

$$\begin{aligned}\tilde\alpha_n \quad &= \quad 2p((2p-1)\tilde L^2_{n,1} + K_{n,1})(4C^2N^2 2^N e^{8p^2N2^{-n}\tilde L^2_{n,1}})e^{2p2^{-n}\tilde L_{n,2}} \\ (3.7) \quad &+2^{-n/2}2p((2p-1)\tilde L_{n,1}\tilde L_{n,2} + K_{n,2})(2CN2^N e^{8p^2N2^{-n}\tilde L^2_{n,1}})e^{2p2^{-n}\tilde L_{n,2}}.\end{aligned}$$

Now suppose that with positive constants $\tilde\beta_i$, $1\le i\le 4$, we have

$$\tilde L^2_{n,1} \le \tilde\beta_1 \log n,\ \ \tilde L_{n,2} \le \tilde\beta_2 \log n,\ \ K_{n,1} \le \tilde\beta_3 \log n,\ \ K_{n,2} \le \tilde\beta_4 (\log n)^{3/2}. \tag{3.8}$$

Under these conditions, it is easy to see from the definition of $\tilde\alpha_n$ that there is a constant C_p, independent of n, such that

$$\tilde\alpha_n \le C_p\,(\tilde L^2_{n,1} + K_{n,1} + 1). \tag{3.9}$$

Therefore (3.6) implies

$$\mathbf{E}(|z^n_t(x) - z^n_t(y)|^{2p}) \le |x-y|^{2p}\,e^{C_p}\,e^{2p\tilde L_{n,2}} e^{C_p(\tilde L^2_{n,1} + K_{n,1})}.$$

Our aim is to get an estimate which is uniform relative to n. For this purpose, we shall again use the cut-off functions φ_m introduced in (3.2). For the sake of simplicity, we shall formulate conditions only on the coefficients $A_0, A_1, \cdots, A_N$. For $m \ge 1$ set

$$C^2_{m,1} = \sum_{i=1}^N \Big(\sup_{|x|\le m} |A_i(x)|^2\Big), \quad C_{m,2} = \sup_{|x|\le m} |A_0(x)|,$$

$$J_{m,1} = \sup_{i,k\ne 0}\Big(\sup_{|x|\le m} ||B'_{ik}(x)||^2\Big), \quad J_{m,2} = \sup_i \sup_{|x|\le m} ||B'_{i0}(x)||.$$

We shall work under the following hypotheses

$$(H) \qquad \left\{\begin{array}{lllllll} C^2_{m,1} & \le & \gamma_1 \log m, & & C_{m,2} & \le & \gamma_2 \log m, \\ L^2_{m,1} & \le & \beta_1 \log m, & & L_{m,2} & \le & \beta_2 \log m, \\ J_{m,1} & \le & \delta_1 \log m, & & J_{m,2} & \le & \delta_2 (\log m)^{3/2}. \end{array}\right.$$

Recall that $A_{n,i} = \varphi_n A_i$. Under the hypothesis (H), we have

$$\sum_{i=1}^N |A_{n,i}|^2 \le \gamma_1 \log(n+2), \quad |A_{n,0}| \le \gamma_2 \log(n+2).$$

$$\sum_{i=1}^N ||A'_{n,i}||^2 \le 2(\gamma_1+\beta_1)\log(n+2), \quad ||A'_{n,0}|| \le (\gamma_2+\beta_2)\log(n+2).$$

Since $B^n_{ik} = \sum_{j=1}^d \frac{\partial \varphi_n}{\partial x_j}\varphi_n A_i A^j_k + \varphi_n^2 B_{ik}$, hypothesis (H) moreover implies

$$||(B^n_{ik})'|| \le \tilde{\delta}_1 \log(n+2), \quad ||(B^n_{i0})'|| \le \tilde{\delta}_2 (\log(n+2))^{3/2}$$

for some constants $\tilde{\delta}_1$ and $\tilde{\delta}_2$. Therefore hypothesis (H) implies conditions (3.8), so that (3.9) is validated. Now let $m \ge 1$. Consider

$$A_{m,n,i} = \varphi_m A_{n,i}, \qquad \text{for } i = 0, 1, \cdots, N.$$

We have

$$\sum_{i=1}^N |A_{m,n,i}|^2 \le \gamma_1 \log(m \wedge n+2), \quad |A_{m,n,0}| \le \gamma_2 \log(m\wedge n+2), \tag{3.10}$$

$$\sum_{i=1}^N ||A'_{m,n,i}||^2 \le \tilde{\beta}_1 \log(m\wedge n+2), \quad ||A'_{m,n,0}|| \le \tilde{\beta}_2 \log(m\wedge n+2), \tag{3.11}$$

and

$$||(B^{mn}_{ik})'|| \le \tilde{\delta}_1 \log(m\wedge n+2), \quad ||(B^{mn}_{i0})'|| \le \tilde{\delta}_2 (\log(m\wedge n+2))^{3/2}. \tag{3.12}$$

Let $(z^{mn}_t(x))$ be the solution of

$$dz^{mn}_t = \sum_{i=1}^N A_{m,n,i}(z^{mn}_t)\,\dot{w}^{n,i}_t\,dt + A_{m,n,0}(z^{mn}_t)\,dt, \quad z^{mn}_0 = x. \tag{3.13}$$

Using (3.10) – (3.12) to estimate $\tilde{\alpha}_m$ in (3.7), we have for $m \le n$

$$\tilde{\alpha}_m \le C_p((\tilde{\beta}_1 + \tilde{\gamma}_1)\log(m+2)+1).$$

We conclude

$$\begin{aligned}\mathbf{E}(|z^{mn}_t(x) - z^{mn}_t(y)|^{2p}) \quad &\le \quad e^{C_p}\, e^{2p\tilde{\beta}_2 \log(m+2)}\, e^{C_p(\tilde{\beta}_1+\tilde{\delta}_1)\log(m+2)}\,|x-y|^{2p} \\ (3.14) \quad &= e^{C_p}(m+2)^{2p\tilde{\beta}_2 + C_p(\tilde{\beta}_1+\tilde{\delta}_1)}\,|x-y|^{2p}.\end{aligned}$$

Extrapolating in m by means of the techniques presented in section 1, we obtain the following moment estimate for the two-point motion, uniformly in the regularization parameter.

Theorem 3.1 *Under the hypothesis (H), for any $p \geq 2$ and $R > 0$, there is a constant $C_{p,R} > 0$, independent of n, such that*

$$\mathbf{E}(|z_t^n(x) - z_t^n(y)|^p) \leq C_{p,R}\ |x-y|^p, \qquad \text{for } x, y \in B(R). \tag{3.15}$$

Proof. It is clear that (H) implies the growth conditions (2.15) and (2.5). Let $Y_n(x) = \sup_{0 \leq t \leq 1} |z_t^n(x)|$. We have

$$\begin{aligned} |z_t^n(x) - z_t^n(y)|^p &= \sum_{m \geq 1} |z_t^n(x) - z_t^n(y)|^p\, \mathbf{1}_{\{m-1 \leq Y_n(x) \vee Y_n(y) < m\}} \\ &= \sum_{m \geq 1} |z_t^{mn}(x) - z_t^{mn}(y)|^p\, \mathbf{1}_{\{m-1 \leq Y_n(x) \vee Y_n(y) < m\}}. \end{aligned}$$

Let $q \geq 2$. By (2.16), there is a constant $C_{q,R} > 0$ such that for all $|x| \leq R, |y| \leq R$,

$$P(Y_n(x) \vee Y_n(y) \geq m-1) \leq C_{q,R} \frac{1}{m^q}.$$

Using (3.14), we have

$$\begin{aligned} &\mathbf{E}\Big(|z_t^{mn}(x) - z_t^{mn}(y)|^p\, \mathbf{1}_{\{m-1 \leq Y_n(x) \vee Y_n(y) < m\}}\Big) \\ &\leq e^{C_p} (m+2)^{p\tilde{\beta}_2 + C_p(\tilde{\beta}_1 + \tilde{\delta}_1)/2} \cdot \sqrt{C_{q,R} \frac{1}{m^{q/2}}}\, |x-y|^p. \end{aligned}$$

Now taking $q/2 \geq p\tilde{\beta}_2 + \frac{1}{2} C_p(\tilde{\beta}_1 + \tilde{\delta}_1) + 2$ gives (3.15). ∎

The following Proposition states a similar uniform moment estimate for the time fluctuations of the solutions of the regularized equations.

Proposition 3.2 *Assume hypothesis (H) is satisfied. For any $p \geq 2$ and $R > 0$, there exists a constant $C_{p,R} > 0$, independent of n, such that*

$$\mathbf{E}(|z_t^n(x) - z_s^n(x)|^p) \leq C_{p,R}\ |t-s|^{p/2}, \quad |x| \leq R,\ s, t \in [0,1]. \tag{3.16}$$

Proof. The coefficients $A_{n,i}$ and B_{ik}^n satisfy (2.5) and (2.15). So we can apply Corollary 2.4 to get (3.16). ∎

We are finally in a position to prove the convergence of the ordinary differential equations' regularizations (z_t^n) to the solution of the stochastic differential equation (x_t) in the L^p sense, uniformly in space and time. To state this result, we first establish it in a weaker sense.

Lemma 3.3 *Let $R > 0$ and $p \geq 2$. Then*

$$\lim_{n \to +\infty} \sup_{|x| \leq R} \sup_{0 \leq t \leq 1} \mathbf{E}(|z_t^n(x) - x_t(x)|^p) = 0. \tag{3.17}$$

Proof. Let $Y_n(x) = \sup_{0\le t\le 1} |z_t^n(x)|$ and $Y(x) = \sup_{0\le t\le 1} |x_t(x)|$. Let $m \geq 1$. We have

$$\begin{aligned}\mathbf{E}(|z_t^n(x) - x_t(x)|^p) &= \mathbf{E}(|z_t^n(x) - x_t(x)|^p \,\mathbf{1}_{\{Y_n(x)\vee Y(x)\le m\}}) \\ &\quad +\mathbf{E}(|z_t^n(x) - x_t(x)|^p \,\mathbf{1}_{\{Y_n(x)\vee Y(x)> m\}}).\end{aligned}$$

Due to (1.4) and (2.16), the second term is majorized by

$$C_p\mathbf{E}((Y_n(x)^p + Y(x)^p)\,\mathbf{1}_{\{Y_n(x)\vee Y(x)> m\}}) \le C_{p,R}\,\frac{1}{\sqrt{m}}.$$

To get (3.17), it is therefore sufficient to prove that

$$\lim_{n\to+\infty} \sup_{|x|\le R} \sup_{0\le t\le 1} \mathbf{E}(|z_t^n(x) - x_t(x)|^p \mathbf{1}_{\{Y_n(x)\vee Y(x)\le m\}}) = 0. \tag{3.18}$$

Let $n > m+2$. By uniqueness of solutions, on the subset $\{w;\ Y_n(x) \le m\}$, $z_t^n(x) = x_t^{mn}(x)$ for all $t \in [0,1]$, where $x_t^{nm}(x)$ is the solution of the following ordinary differential equation

$$dx_t^{nm} = \sum_{i=1}^N (\varphi_m A_i)(x_t^{nm}(x))\,\dot{w}_t^{n,i}\,dt + (\varphi_m A_0)(x_t^{nm}(x))\,dt, \quad x_0^{nm} = x.$$

On the other hand, let $\tau_m(x) = \inf\{t > 0,\ |x_t(x)| \ge m\}$. Then $x_{t\wedge\tau_m(x)}(x)$ satisfies the following Itô stochastic differential equation

$$dx_t^m(x) = \sum_{i=1}^N (\varphi_m A_i)(x_t^m(x))\,dw_t^i + \Big(\varphi_m A_0 + \frac{1}{2}\sum_{i,j} \frac{\partial(\varphi_m A_i)}{\partial x_j}(\varphi_m A_i^j)\Big)\,dt, \quad x_0^m(x) = x.$$

It follows that on the subset $\{Y(x) \le m\}$ or $\{\tau_m(x) \ge 1\}$, we have $x_t^m(x) = x_t(x)$ for all $t \in [0,1]$. Therefore

$$\begin{aligned}\mathbf{E}(|z_t^n(x) - x_t(x)|^p \mathbf{1}_{\{Y_n(x)\vee Y(x)\le m\}}) &= \mathbf{E}(|x_t^{nm}(x) - x_t^m(x)|^p \mathbf{1}_{\{Y_n(x)\vee Y(x)\le m\}}) \\ &\le \mathbf{E}(|x_t^{nm}(x) - x_t^m(x)|^p).\end{aligned}$$

We are now in the classical situation. Therefore Moulinier's [10] result applies to get (3.18). The proof of (3.17) is completed. ∎

We finally strengthen the previous result to moment convergence, uniformly in space and time.

Theorem 3.4 *Assume hypothesis (H). For any $p \ge 2$,*

$$\lim_{n\to+\infty} \mathbf{E}\Big(\sup_{0\le t\le 1} \sup_{|x|\le R} |z_t^n(x) - x_t(x)|^p\Big) = 0. \tag{3.19}$$

Proof. Let $p \ge 2$ be given. By (3.15),(3.16) and the Kolmogoroff modification theorem, there exists $\beta > 0$ such that for $|x| \le R$, $|y| \le R$ and $t, s \in [0,1]$,

$$|z_t^n(x) - z_s^n(y)| \le F_n \cdot (|x-y|^\beta + |t-s|^\beta), \quad n \ge 1, \tag{3.20}$$

where $\{F_n;\ n \geq 1\}$ is a family of measurable functions bounded in L^p for any p. In the same way, according to Corollary 1.4 and Proposition 1.8, there exists $F \in L^p$ such that

$$|x_t(x) - x_s(y)| \leq F \cdot (|x-y|^\beta + |t-s|^\beta). \tag{3.21}$$

Let $\varepsilon_n = \sup_{0\leq t\leq 1} \sup_{|x|\leq R} \mathbf{E}(|z_t^n(x) - x_t(x)|^p)$. By lemma 3.3, $\lim_{n\to+\infty} \varepsilon_n = 0$. Let $\sigma_n > 0$. Then there exists $N_n \leq C\,(\frac{1}{\sigma_n})^{d+1}$ points $x_1, \cdots, x_{N_n}$ in the ball $B(R)$ and $t_1, \cdots, t_{N_n} \in [0,1]$ such that

$$[0,1] \times B(R) \subset \cup_{i=1}^{N_n} [t_i - \sigma_n, t_i + \sigma_n] \times \{x;\ |x - x_i| \leq \sigma_n\}.$$

Let $(t,x) \in [0,1] \times B(R)$. There exists one i such that $|t - t_i| \leq \sigma_n$ and $|x - x_i| \leq \sigma_n$. We have, according to (3.20) and (3.21)

$$\begin{aligned} |z_t^n(x) - x_t(x)| &\leq |z_t^n(x) - z_{t_i}^n(x_i)| + |z_{t_i}^n(x_i) - x_{t_i}(x_i)| + |x_{t_i}(x_i) - x_t(x)| \\ &\leq 2(F_n + F)\sigma_n^\beta + |z_{t_i}^n(x_i) - x_{t_i}(x_i)|. \end{aligned}$$

It follows that

$$\begin{aligned} \sup_{0\leq t\leq 1} \sup_{|x|\leq R} |z_t^n(x) - x_t(x)|^p &\leq C_p\Big\{(F_n^p + F^p)\sigma_n^{\beta p} + \sup_{1\leq i\leq N_n} |z_{t_i}^n(x_i) - x_{t_i}(x_i)|^p \\ &\leq C_p\Big\{(F_n^p + F^p)\sigma_n^{\beta p} + \sum_{1\leq i\leq N_n} |z_{t_i}^n(x_i) - x_{t_i}(x_i)|^p\Big\} \end{aligned}$$

with a constant C_p depending only on p. Therefore for another such constant $\hat{C}_p > 0$, we have

$$\begin{aligned} \mathbf{E}\Big(\sup_{0\leq t\leq 1} \sup_{|x|\leq R} |z_t^n(x) - x_t(x)|^p\Big) &\leq \hat{C}_p\,\sigma_n^{\beta p} + N_n\,\varepsilon_n \\ &\leq \hat{C}_p\,\sigma_n^{\beta p} + C\,(\frac{1}{\sigma_n})^{d+1} \cdot \varepsilon_n. \end{aligned}$$

Now taking $\sigma_n = \varepsilon_n^{1/2(d+1)}$ gives the result (3.19). ∎

Due to the hypothesis (H), Theorem 3.4 finally implies Theorem A following a procedure in Chapter V in [4].

References

[1] J.-M. Bismut, *Mécanique aléatoire*, Lect. notes in Math., Vol. 866 (1981), Springer-Verlag, Berlin, New-York.

[2] A.P. Carverhill, K.D. Elworthy, Flows of stochastic dynamical systems: the functional analytic approach. Z. W. 65 (1983), 245-267.

[3] S. Fang, T. Zhang, *A study of a class of differential equations with non-Lipschitzian coefficients.* Probab. Theory Relat. Fields 132 (2005), 356-390.

[4] N. Ikeda, S. Watanabe, *Stochastic differential equations and Diffusion processes.* North-Holland, Amsterdam, 1981.

[5] P. Imkeller, M. Scheutzow, *On the spatial asymptotic behavior of stochastic flows in Euclidean space.* Ann. Probability 27 (1999), 109-129.

[6] P. Imkeller, *On the existence of global flows for stochastic differential equations without global Lipschitz conditions.* Unpublished manuscript, HU Berlin 1999.

[7] H. Kunita, *Stochastic Differential Equations and Stochastic Flow of diffeomorphisms.* École d'Été de Probabilités de Saint-Flour XII (1982), Lect. Notes in Math. 1097, Springer-Verlag.

[8] X.M. Li, *Strong p-completeness of stochastic differential equations and the existence of smooth flows on non-compact manifolds.* Probab. Theory Relat. Fields, 100 (1994), 485-511.

[9] P. Malliavin, *Stochastic Analysis*, Grunlehren des Math. 313, Springer, Berlin, 1997.

[10] J. M. Moulinier, *Théorème limite pour les équations différentielles stochastiques.* Bull. Sci. Math. tome 112 (1988), 185-209.

[11] D. Stroock, S. Varadhan, *On the support of diffusion processes with application to the strong maximum principle.* Proc. Sixth Berkeley Symp. Math. Stat. Probab. III. Univ. of California Press (1972), 333-368.

[12] T. Yamada, Y. Ogura, *On the strong comparison theorems for solutions of stochastic differential equations.* Z. W. verw. Gebiete 56 (1981) 3-19.