The conjugacy of stochastic and random differential equations and the existence of global attractors

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Abstract

We consider stochastic differential equations in d-dimensional Euclidean space driven by an m-dimensional Wiener process, determined by the drift vector field f_0 and the diffusion vector fields $f_1, \cdots, f_m$, and investigate the existence of global random attractors for the associated flows ϕ. For this purpose ϕ is decomposed into a stationary diffeomorphism Φ given by the stochastic differential equation on the space of smooth flows on $\mathbf{R}^d$ driven by m independent stationary Ornstein Uhlenbeck processes $z^1, \cdots, z^m$ and the vector fields $f_1, \cdots, f_m$, and a flow χ generated by the non-autonomous ordinary differential equation given by the vector field $(\frac{\partial \Phi_t}{\partial x})^{-1}[f_0(\Phi_t) + \sum_{i=1}^m f_i(\Phi_t)\, z_t^i]$. In this setting, attractors of χ are canonically related with attractors of ϕ. For χ, the problem of existence of attractors is then considered as a perturbation problem. Conditions on the vector fields are derived under which a Lyapunov function for the deterministic differential equation determined by the vector field f_0 is still a Lyapunov function for χ, yielding an attractor this way. The criterion is finally tested in various prominent examples.

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Introduction

The motivation for the study of the present paper was the desire to understand better the bifurcation behaviour of noisy non-linear systems as for example the Duffing- van der Pol oscillator with white noise. The oscillator without noise is well known to exhibit a Hopf bifurcation when the bifurcation (damping) parameter crosses 0. If noise is turned on, the picture changes drastically. Instead of one there are now two bifurcation points, i.e. points at which the set of random invariant measures of the system undergoes a qualitative change. Simulations show (see Ochs [10]), that in the intervals determined by these points the picture turns into a very complex, but interesting one. To understand this picture mathematically, one has to get a hand on the invariant measures which are supported by the random attractors.

For this reason we decided to look for general mathematical concepts appropriate for deciding whether a given system has a random attractor. Keller, Schmalfuss [11], Crauel, Flandoli [7] and Crauel, Debussche, Flandoli [6] contain the origins of the basic idea of this paper, which we try to mold into a general concept verifying the existence of attractors for random dynamical systems which originate from stochastic differential equations. The idea is this: a random stationary coordinate change induces an isomorphism of attractors (Theorem 2.1).

Why would one want to perform a random coordinate change on the flow generated by a stochastic differential equation? The reason is this. In the framework of the stochastic integration theory basic to Itô's calculus, attractors are by far harder to describe than in the framework of classical calculus for deterministic systems. So the question is: can one, at the expense of some random coordinate change, pass from one to the other, even if this means taking into account non-autonomous ordinary differential equations?

Random coordinate changes can be realized for example by decompositions of the flow. To be more precise, consider a stochastic differential equation on $\mathbf{R}^d$ given by

$$dx_t = f_0(x_t)\,dt + \sum_{i=1}^{m} f_i(x_t) \circ dW_t^i, \tag{1}$$

with vector fields $f_0, \cdots, f_m$ smooth enough so that the subsequently considered flows exist globally, and an m-dimensional Wiener process W. A decomposition of the flow ϕ generated by (1) can be found in papers by Bismut and Michel [3], [4], Ocone and Pardoux [16] in different frameworks, and for different aims. It rests upon the Itô-Ventzell formula (see Ventzell [23], or Sznitman [22]). The decomposition is given as follows. If ψ is the flow of the pure diffusion part

$$dy_t = \sum_{i=1}^{m} f_i(y_t) \circ dW_t^i, \tag{2}$$

$\chi_\cdot(x)$ the solution of the non-autonomous ordinary differential equation

$$dz_t = (\frac{\partial \psi_t}{\partial x})^{-1}(z_t)\, f_0(\psi_t(z_t))\, dt \tag{3}$$

starting at $x \in \mathbf{R}^d$, then we have

$$\phi_t(x) = \psi_t(\chi_t(x)), \quad t \geq 0. \tag{4}$$

As mentioned above, *stationary* coordinate changes induce isomorphisms of attractors. But the change described by ψ need not be stationary. In fact, being deterministic (equal to the identity) at time 0, it does not have many chances. So a second question arises: can one modify (2) such that the flow on the space of smooth diffeomorphisms of $\mathbf{R}^d$ canonically associated with it has a stationary state, say Φ? If it exists, we obtain the desired *conjugation* relation

$$\phi_t(x) = \Phi_t(\chi_t(\Phi^{-1}(x))),$$

where Φ_t is just Φ, applied to the Wiener paths shifted canonically by time t.

This leads to the general concept of the paper. First decompose the flow as indicated, then look for a stationary solution of (2) lifted into the space of diffeomorphisms, and finally investigate random attractors of (3) to obtain attractors of (1) this way.

For the second step in this program, we take a more pragmatic point of view, based upon the fact that most of the prominent systems investigated in stochastic dynamics consist of rather complicated drift part, but of mostly simple diffusion terms. In fact, in the examples most frequently investigated, the random Duffing-van der Pol oscillator with different sources of noise, noisy harmonic oscillators in potential wells, or the random Lorenz equation, we either have $m = 1$ or $f_1, \cdots, f_m$ commute in the sense of Lie algebras. In this case, the well known Doss-Sussmann approach of stochastic differential equations allows to represent the solutions as smooth functions of the path of the driving noise. So in order to get a stationary diffeomorphism as solution of (2) lifted to the space of diffeomorphisms, we drive (2) with m independent stationary OU processes instead of the m-dimensional Wiener process W. This settles our problem, at the expense of introducing an auxiliary drift in (3) to pass from W to an OU process.

For the third part of the program, we consider (3) as a *perturbated* deterministic equation

$$dx_t = f_0(x_t)\, dt. \tag{5}$$

We assume that (5) has a Lyapunov function V, which canonically yields a deterministic attractor, and ask the question, under which additional conditions on the *perturbation*, i.e. the vector fields $f_1, \cdots, f_m$, and the parameter μ in the OU process

$$dz_t^i = dW_t^i - \mu\, z_t^i\, dt, \quad 1 \le i \le m, \tag{6}$$

V still remains a Lyapunov function of the system described by (3), this way yielding again canonically a random attractor. We derive sufficient conditions under which this is seen to be the case.

The paper is organized as follows.

In section 1, we discuss conditions under which a flow associated with a stochastic differential equation (1) is conjugate to a flow associated with a non-autonomous random ordinary differential equation of the form (3). In Theorem 1.3 and its corollaries this is done for commuting non-linear, linear and affine vector fields. Let us remark at this point that conjugation seems to be a more general phenomenon: at least for vector fields the Lie algebra of which is nilpotent it promises to remain generally true.

In section 2 we discuss general conditions on the vector fields under which a Lyapunov function for (5) remains a Lyapunov function for (3). This way we obtain in Theorems 2.2, 2.3 and their corollaries different sufficient conditions for the existence of a global attractor for χ and thus ϕ.

In section 3, we finally discuss examples to which the theory of section 2 may be applied: the Duffing-van der Pol oscillator with different sources of noise, the noisy harmonic oscillator in a potential well, in particular a double well, and the stochastic Lorenz system with different sources of noise.

Notations and preliminaries

Our basic probability space is the $m-$dimensional canonical Wiener space $(\Omega, \mathbf{F}, P)$, enlarged such as to carry an $m-$dimensional *Wiener process* indexed by $\mathbf{R}$. More precisely, $\Omega = C(\mathbf{R}, \mathbf{R}^m)$ is the set of continuous functions on $\mathbf{R}$ with values in $\mathbf{R}^m$, $\mathbf{F}$ the $\sigma-$algebra of Borel sets with respect to uniform convergence on compacts of $\mathbf{R}$, P the probability measure on $\mathbf{F}$ for which the *canonical Wiener process* $W_t = (W_t^1, ..., W_t^m), t \in \mathbf{R}$, satisfies that both $(W_t)_{t\geq 0}$ and $(W_{-t})_{t\geq 0}$ are usual $m-$dimensional Brownian motions. The natural filtration $\{\mathbf{F}_s^t = \sigma(W_u - W_v : s \leq u, v \leq t) : \mathbf{R} \ni s \leq t \in \mathbf{R}\}$ of W is assumed to be completed by the $P-$completion of $\mathbf{F}$. For $t \in \mathbf{R}$, let $\theta_t : \Omega \to \Omega, \omega \mapsto \omega(t + \cdot) - \omega(t)$, the *shift* on Ω by t. It is well known that θ_t preserves Wiener measure P for any $t \in \mathbf{R}$ and is even ergodic for $t \neq 0$. Hence $(\Omega, \mathbf{F}, P, (\theta_t)_{t\in\mathbf{R}})$ is an ergodic *metric dynamical system* (see Arnold [1]). As usual, we use a "$\circ$" to denote Stratonovich integrals with respect to Wiener process.

For a random vector X, we denote by P_X the law of X with respect to P. ∇ is used as a symbol for the gradient of vector fields on $\mathbf{R}^d$.

Let us briefly recall the notion of a random attractor. For more details consult Crauel, Debussche, Flandoli [6] or Keller, Schmalfuss [11]. Note first that under the smoothness conditions assumed from section 1 on for the vector fields, the completion result of Arnold, Scheutzow [2] implies that the flows of diffeomorphisms generated by our stochastic differential equations in fact generate *random dynamical systems* (see Arnold [1]). More precisely, the flow $(\phi_t)_{t\geq 0}$ of diffeomorphisms on $\mathbf{R}^d$ generated by a stochastic differential equation is called *random dynamical system* on the metric dynamical system $(\Omega, \mathbf{F}, P, (\theta_t)_{t\in\mathbf{R}})$ if the following *cocycle property* is satisfied:

$$\phi_{s+t}(\omega) = \phi_t(\theta_s\omega) \circ \phi_s(\omega), \quad \phi_0(\omega) = id_{\mathbf{R}^d},$$

for $\omega \in \Omega, s, t \geq 0$. An obvious modification gives the notion of a random dynamical system for flows with parameter space $\mathbf{R}$ instead of $\mathbf{R}_+$. Whenever we speak of a flow, we shall, as our hypotheses on the vector fields allow, tacitly assume that it is a random dynamical system.

A family $(A(\omega), \omega \in \Omega)$ of closed subsets of $\mathbf{R}^d$ is called *measurable* if for any $x \in \mathbf{R}^d$ the function $\omega \mapsto d(A(\omega), x) = \inf\{|x - y| : y \in A(\omega)\}$ is measurable. Motivated by the needs of section 3.2, we shall define random attractors for more general systems of attracted sets. Let $\mathcal{D}$ be a system of measurable closed and nonempty sets $\omega \mapsto D(\omega)$. In addition we suppose that D fulfills the following filtering property: if D' is a measurable set with closed and nonempty images and $D'(\omega) \subset D(\omega)$ for $\omega \in \Omega$ and $D \in \mathcal{D}$ then $D' \in \mathcal{D}$. Such a system is briefly named *universe*. We hasten to emphasize that the system of compact random sets uniformly bounded in ω is a universe, the one the reader may imagine if we speak of universes. We call it *universe of compact sets*. As we shall see in section 3.2, it is however not the only one which matters for us.

For a given universe $\mathcal{D}$ a measurable set $A \in \mathcal{D}$ with compact images is called a *random attractor* for the random dynamical system $(\phi_t)_{t\geq 0}$ if A is ϕ-*invariant*, i.e. for $\omega \in \Omega$ we have

$$\phi_t(\omega)A(\omega) = A(\theta_t\omega),$$

and absorbs sets from $\mathcal{D}$, i.e.

$$\lim_{t\to\infty} \operatorname{dist}(\phi(\theta_{-t}\omega)D(\theta_{-t}\omega), A(\omega)) = 0$$

for any $D \in \mathcal{D}$, see Flandoli, Schmalfuss [8], where *dist* denotes the semi-Hausdorff distance

$$\operatorname{dist}(A, B) = \sup_{x\in A} \inf_{y\in B} |x - y|.$$

Note that a random attractor is unique. We remark that the more intuitive relationship

$$\lim_{t\to\infty} \operatorname{dist}(\phi_t(\omega)\, B, A(\theta_t(\omega)) = 0$$

holds only for convergence in probability.

The following theorem is a version of Crauel, Flandoli [7], Flandoli, Schmalfuss [8] or Schmalfuss [19]:

Theorem 0.1 *Let $\mathcal{D}$ be a universe of measurable sets. Suppose that $x \to \phi_t(\omega)x$ is continuous. In addition we suppose that there exists a compact measurable set $B \in \mathcal{D}$ such that*

$$\phi_t(\theta_{-t}\omega)D(\theta_{-t}\omega) \subset B(\omega)$$

for $t \geq t(\omega, D)$ and any $D \in \mathcal{D}$. Then there exists a random attractor with respect to $\mathcal{D}$.

The other important example of universes is given by the *tempered* random sets. A random variable $R > 0$ is tempered if

$$\lim_{t\to\pm\infty} \frac{1}{|t|} \log^+ R(\theta_t\omega) = 0 \tag{7}$$

for $\omega \in \Omega$, see Arnold [1], p.164. Note that (7) is equivalent to

$$\lim_{t\to\pm\infty} e^{-c|t|} R(\theta_t\omega) = 0 \quad \text{for any} \quad c > 0.$$

A measurable set D is called *tempered* if $D(\omega)$ is contained in a ball with center zero and tempered radius $R(\omega), \omega \in \Omega$. Then the system of measurable sets with compact and nonempty tempered images forms the universe of *tempered sets*. The universe which matters in section 3.2 consists of tempered sets with a simple additional condition and will be described precisely later on.

Of course, the universe of compact sets is contained in the tempered one. Let us briefly point out that the difference is not very big from the point of view of random dynamical systems, however. Temperedness of R may be paraphrased by stating that the Lyapunov exponent of the stationary process $t \mapsto R(\theta_t\omega)$ is zero. But if it is not zero, then we automatically have

$$\limsup_{t\to\pm\infty} \frac{1}{|t|} \log^+ R(\theta_t\omega) = +\infty.$$

A function $k : \mathbf{R}^m \to \mathbf{R}$ is said to be *subexponentially growing* if there exists $c > 0$ such that $z \mapsto \frac{k(z)}{\exp(c|z|)}$ is bounded on $\mathbf{R}^m$.

1 The conjugacy of flows

Let $f_0, \cdots, f_m$ be $C^\infty-$vector fields on $\mathbf{R}^d$. Suppose that $f_1, \cdots, f_m$ are globally Lipschitz. We consider the flow $\phi = (\phi_t)_{t\geq 0}$ generated by the stochastic differential equation

$$dx_t = f_0(x_t)\, dt + \sum_{i=1}^m f_i(x_t) \circ dW_t^i. \tag{8}$$

We shall assume that ϕ is forward complete, i.e. that ϕ_t is a $C^\infty-$diffeomorphism for all $t \geq 0$. Bismut and Michel [3], [4], in a control theoretic study decomposed ϕ into two components, one stemming from the pure diffusion part of (8), and one given by the modified drift part of (8). This decomposition was later on taken up by Ocone and Pardoux [16] in a framework of stochastic differential equations with non-adapted coefficients. It rests upon a formula of the type of Itô-Ventzell (see Ventzell [23], or Sznitman [22]). We shall be interested in a diffusion part possessing a stationary state. Properly interpreted in the framework of random diffeomorphisms, this state will then yield a conjugacy relation between the flow ϕ and a flow associated with the modified drift part. In particular, it will yield that the flow associated with our stochastic differential equation is related to the flow of a random differential equation by a stationary change of coordinates.

To produce this stationary state, we shall provide the pure diffusion part of (8) with an auxiliary drift determined by the stationary solution of a Langevin equation in dimension m.This auxiliary drift has to be taken into account in the drift term of (8) in return, of course. For the applications to the existence of global attractors we have in mind, however, it will do no harm, as will be discussed later on.

So from now on we fix $\mu > 0$, and consider the stochastic differential equation

$$\begin{aligned} dz_t &= dW_t - \mu\, z_t\, dt, \\ dx_t &= [f_0(x_t) + \mu \sum_{i=1}^m f_i(x_t)\, z_t^i]\, dt + \sum_{i=1}^m f_i(x_t) \circ dz_t^i \end{aligned} \tag{9}$$

with values in $\mathbf{R}^m \times \mathbf{R}^d$. Although the Ornstein-Uhlenbeck process z depends on the parameter μ, this will not be made explicit by a sub- or superscript. In this setting the result of Bismut and Michel [3] can be seen to decompose the flow ϕ of the second component of (9), which of course is just (8), in the following way. Let $\rho = (z, \psi)$ be the flow of the sde

$$\begin{aligned} dz_t &= dW_t - \mu\, z_t\, dt, \\ dy_t &= \sum_{i=1}^{m} f_i(y_t) \circ dz_t^i. \end{aligned} \tag{10}$$

Now still denote the stationary solution of the Langevin part of (9) by $z = (z_t)_{t \in \mathbf{R}}$, and let $\chi_t(x), t \geq 0, x \in \mathbf{R}^d$, be the solution of the random differential equation

$$dy_t = (\frac{\partial \psi_t}{\partial x})^{-1}(y_t)[f_0(\psi_t(y_t)) + \mu \sum_{i=1}^{m} f_i(\psi_t(y_t))\, z_t^i]dt, \tag{11}$$

$$y_0 = x. \tag{12}$$

Then the Itô-Ventzell formula yields

$$\phi_t(x) = \psi_t(\chi_t(x)), \quad t \geq 0, \quad x \in \mathbf{R}^d. \tag{13}$$

In general, ψ may of course not be a stationary process with respect to the canonical shift on Wiener space. We aim at a modification of (13) with a stationary diffeomorphism instead of ψ. For this reason we shall give a more general version of (13) in which we should also admit the possibility that the set of stationary states does not just consist of one point, but is given by a spread-out invariant measure. This leads us to invariant measures associated with the original stochastic differential equation considered as transporting diffeomorphisms of $\mathbf{R}^d$.

Let $D(\mathbf{R}^d)$ denote the set of diffeomorphisms of $\mathbf{R}^d$, endowed with the topology induced by the space $C^{1,0}(\mathbf{R}^d)$ (see Kunita [9], p. 114, or Arnold [1], pp. 552-555). Let $(Q_t)_{t \in \mathbf{R}}$ be the group of linear operators defined by

$$Q_t f(\rho) = E(f(\Phi_t^\rho)), \quad f \in C_b(D(\mathbf{R}^d)),$$

where $(\Phi_t^\rho)_{t \in \mathbf{R}}$ is given by the solution of (the second line of) (10) starting in $\rho \in D(\mathbf{R}^d)$, and still with the stationary Ornstein-Uhlenbeck process z. Remarking that $D(\mathbf{R}^d)$ is a Polish space, we assume that $(Q_t)_{t \geq 0}$ possesses an invariant measure γ on $D(\mathbf{R}^d)$. We shall later on study special cases and conditions under which this is guaranteed.

The *pull back* of ν according to Ledrappier [12], Le Jan [13] and Crauel [5] yields a random invariant mesure γ which is *Markovian* and is given by the formula

$$\nu = \lim_{t\to\infty} \Phi_t^{\cdot} \circ \theta_{-t}(\gamma).$$

Its invariance is expressed in the fact that

$$\nu_t = \nu \circ \theta_t = \Phi_t^{\cdot}(\nu).$$

We then have the following extension of the formula of Bismut and Michel [3].

Theorem 1.1 *For $\rho \in D(\mathbf{R}^d)$ let $\chi_t^\rho(x), t \in \mathbf{R}_+, x \in \mathbf{R}^d$, be the solution of the random differential equation*

$$\begin{aligned} dy_t &= (\frac{\partial \Phi_t^\rho}{\partial x})^{-1}(y_t)[f_0(\Phi_t^\rho(y_t)) + \mu \sum_{i=1}^m f_i(\Phi_t^\rho(y_t))\, z_t^i]\, dt, \\ y_0 &= \rho(x). \end{aligned} \tag{14}$$

Then we have for $t \in \mathbf{R}_+$

$$\phi_t(x) = \int_{D(\mathbf{R}^d)} \Phi^\rho(\chi_t^\rho(\rho^{-1}(x))\ \nu_t(d\rho).$$

Proof:

The Itô-Ventzell formula yields for $x \in \mathbf{R}^d, t \in \mathbf{R}, \rho \in D(\mathbf{R}^d)$

$$\phi_t(x) = \Phi_t^\rho(\chi_t^\rho(\rho^{-1}(x))),$$

$P-$a. s.. Using the measurability of the parametrized process in all variables and the invariance property of ν we obtain the equation

$$\begin{aligned} \int \rho(\chi_t^\rho(\rho^{-1}(x))\ \nu_t(d\rho) &= \int \Phi_t^\rho(\chi_t^\rho(\rho^{-1}(x))\ \nu_0(d\rho) \\ &= \int \phi_t(x)\, \nu_0(d\rho) \\ &= \phi_t(x). \end{aligned}$$

□

One might ask at this point why an *exterior* process such as the Ornstein-Uhlenbeck process enters our considerations. Couldn't we just subtract from the diffusion part of

(8) an auxiliary drift of the form $\sum_{i=1}^m g_i(x_t)\, dt$ with g_i appropriately chosen such that the existence of an invariant measure is guaranteed? In general, this should indeed be a reasonable way to go. The reason why in this paper we still stick with the Ornstein-Uhlenbeck process is this. In the examples we consider in section 3, the diffusion terms are exclusively linear. In this case, the introduction of an auxiliary Langevin equation makes it possible to remain inside the framework of linear vector fields for the modified diffusion flows. This will be a consequence of the following considerations, in which we shall make a considerable restriction of generality. We shall assume that the vector fields commute, as is the case in all the examples studied later on and in fact in all the prominent examples known from the literature of systems perturbed by white noise. This makes the Doss-Sussmann representation of solutions of stochastic differential equations enter the scene. We remark at this point that for vector fields generating a Lie algebra which is nilpotent of order 2, our arguments can be made rigorous as well, via the introduction of a stationary Lévy area, besides the stationary Ornstein-Uhlenbeck process with which we work in the simpler setting. An argument of this type works most likely also in the case of nilpotent Lie algebras. Lyons [15] indicates that at least for non-commuting linear vector fields the solution is a smooth function of the stationary pair consisting of Ornstein-Uhlenbeck process and Lévy area. We conjecture that also in this case we can find a stationary diffeomorphism for a large class of systems, so that the subsequent results extend to this setting as well.

So assume from now on that $[f_i, f_j] = 0$ for $1 \le i, j \le m$. Let $u : \mathbf{R}^m \times \mathbf{R}^d \to \mathbf{R}^d$ be the smooth field which satisfies

$$\begin{aligned} \frac{\partial u}{\partial z^i}(z, x) &= f_i(u(z, x)), \\ u(0, x) &= x, \end{aligned} \tag{15}$$

$1 \le i \le m, (z, x) \in \mathbf{R}^m \times \mathbf{R}^d$. Existence and uniqueness of u are consequences of our hypotheses on $f_1, \cdots, f_m$, see Spivak [21] Chapters 6, 7. We continue to denote the stationary solution of the m-dimensional Langevin equation with parameter μ by $z = (z_t)_{t \in \mathbf{R}}$. Then the following key result follows easily from the Doss-Sussmann representation.

Let z_t be the stationary Ornstein-Uhlenbeck process introduced in (6). Then there exists an $\mathbf{F}^0_{-\infty}$ measurable random variable denoted by z_0 such that $z_t = z_0 \circ \theta_t$. In particular $z_0 = z_t|_{t=0}$.

Theorem 1.2 *Let $\Phi = u(z_0, \cdot)$. Then $\Phi \in D(\mathbf{R}^d)$ and $\Phi_t = \Phi \circ \theta_t, t \geq 0$, solves the $D(\mathbf{R}^d)-$valued stochastic differential equation*

$$d\Phi_t = \sum_{i=1}^{m} f_i(\Phi_t) \circ dz_t^i.$$

Finally, Φ is tempered.

Proof:

The first part is a direct consequence of (15) and the stationarity of z. Temperedness of Φ follows easily from our assumption on the vector fields $f_1, ..., f_m$: it implies that Φ is integrable. □

Theorem 1.2 allows us to exemplify Theorem 1.1 in the case of commuting vector fields.

Theorem 1.3 *Let $\Phi = u(z_0, \cdot)$, $(\chi_t(x))_{t\geq 0}$ be the flow on $\mathbf{R}^d$ generated by the random differential equation*

$$dy_t = g(\theta_t \cdot, y_t)\, dt,$$

where for $\omega \in \Omega, y \in \mathbf{R}^d$

$$g(\omega, y) = (\frac{\partial \Phi}{\partial x})^{-1}(\omega)[f_0(\Phi(\omega)\, y) + \mu \sum_{i=1}^{m} f_i(\Phi(\omega)\, y)\, z_0^i(\omega)].$$

Then we have

$$\phi_t(\omega, x) = \Phi(\theta_t\omega)\chi_t(\omega, \Phi^{-1}(\omega)x) \tag{16}$$

$x \in \mathbf{R}^d, t \geq 0$, for the flow ϕ of solutions of (8).

Proof:

In our setting, the Doss-Sussmann representation just states

$$\phi_t(\Phi(x)) = u(z_t, \chi_t(x)),$$

$t \geq 0$. This is another way of writing (16). □

Remark: Note that the random isomorphisms appearing in the preceding theorem are tempered.

We specialize our main result to the case of linear vector fields.

Corollary 1.1 *Let $f_0 \in C^\infty(\mathbf{R}^d), A_1, \cdots, A_m \in \mathbf{R}^{d\times d}$ be such that $[A_i, A_j] = 0$ for $1 \leq i, j \leq m$ and*

$$dx_t = f_0(x_t)\, dt + \sum_{i=1}^m A_i\, x_t \circ dW_t^t$$

is forward complete. Denote its flow by ϕ. Let z be the stationary solution of the Langevin equation

$$dz_t = dW_t - \mu\, z_t\, dt$$

for some $\mu > 0$, and let $\Phi = \exp(A_1\, z_0^1 + \cdots + A_m\, z_0^m)$, and $(\chi_t(x))_{t\geq 0}$ the flow of the random differential equation

$$dy_t = g(\theta_t \cdot, y_t)\, dt,$$

where

$$g(\omega, y) = \Phi^{-1}(\omega)\, f_0(\Phi(\omega)y) + \mu \sum_{i=1}^m A_i\, y\, z_0^i(\omega),$$

$\omega \in \Omega, y \in \mathbf{R}^d$. Then we have

$$\phi_t(x) = \Phi \circ \theta_t\; \chi_t(\Phi^{-1}\, x),$$

$t \geq 0, x \in \mathbf{R}^d$.

Proof:

This is a combination of Theorems 1.3 and 1.1. □

Finally, we specialize Theorem 1.3 to the case of affine vector fields.

Corollary 1.2 *Let $f_0 \in C^\infty(\mathbf{R}^d), A_1, \cdots, A_m \in \mathbf{R}^{d\times d}, b_1, \cdots, b_m \in \mathbf{R}^d$ be such that $[A_i, A_j] = 0$ for $1 \leq i, j \leq m$ and*

$$dx_t = f_0(x_t)\, dt + \sum_{i=1}^m (A_i\, x_t + b_i) \circ dW_t^i$$

is forward complete. Let z be the stationary solution of the Langevin equation

$$dz_t = dW_t - \mu \, z_t \, dt$$

for some $\mu > 0$, let $\Phi = \exp(A_1 \, z_0^1 + \cdots + A_m \, z_0^m)$, and Ψ the affine mapping given by

$$\Psi(x) = \Phi(x) + \sum_{i=1}^{m} A_i^{-1} (\exp(A_i \, z_0^i) - I) \, b_i.$$

(Here and in the following, we denote by A^{-1} the pseudo-inverse of A.) Finally, let $(\chi_t(x))_{t \geq 0}$ be the flow of the random differential equation

$$dy_t = g(\theta_t \cdot, y_t) \, dt,$$

where

$$g(\omega, y) = \Psi^{-1}(\omega) \, [f_0(\Psi(\omega) y) + \mu \sum_{i=1}^{m} A_i \, \Psi(\omega) \, y \, z_0^i(\omega)],$$

$\omega \in \Omega, y \in \mathbf{R}^d$. Then we have

$$\phi_t(x) = \Psi \circ \theta_t \;\, \chi_t(\Psi^{-1} \, x),$$

$t \geq 0, x \in \mathbf{R}^d$.

Proof:

Note that $\Psi \circ \theta_t = \Phi_t + \sum_{i=1}^{m} A_i^{-1} \, (\exp(A_i \, z_t^i) - I) \, b_i$. The remaining proof is identical to the one of the preceding corollary. □

2 Random attractors via flow decomposition

Attractors of conjugate flows are related in very simple way, as will now be made precise. Consequently, section 1 gives us the opportunity to obtain random attractors of flows ϕ generated by stochastic differential equations from attractors of flows χ related with random non-autonomous ordinary differential equations. We work in the framework of the preceding section. We fix $f_0, \cdots, f_m \in C^\infty(\mathbf{R}^d)$, such that $f_1, \cdots, f_m$ are globally Lipschitz, $[f_i, f_j] = 0$ for $1 \leq i, j \leq m$, and such that

$$dx_t = f_0(x_t) \, dt + \sum_{i=1}^{m} f_i(x_t) \circ dW_t^i \tag{17}$$

is forward complete. We denote the flow generated by this equation by ϕ. Furthermore, we suppose that z is the stationary solution of the Langevin equation

$$dz_t = dW_t - \mu\, z_t\, dt \tag{18}$$

with values in $\mathbf{R}^m$, and write $\Phi = u(z_0, \cdot)$ for the stationary solution of the sde with values in $D(\mathbf{R}^d)$ given by

$$d\Phi_t = \sum_{i=1}^m f_i(\Phi_t) \circ dz_t^i, \tag{19}$$

where u is the solution of the system of partial differential equations

$$\begin{aligned} \frac{\partial u}{\partial z^i}(z, x) &= f_i(u(z, x)), \\ u(0, x) &= x, \end{aligned}$$

$1 \leq i \leq m, (z, x) \in \mathbf{R}^m \times \mathbf{R}^d$. Finally, χ denotes the flow generated by the non-autonomous random differential equation

$$dy_t = g(\theta_t \cdot, y_t)\, dt, \tag{20}$$

where

$$g(\omega, y) = (\frac{\partial \Phi}{\partial x})^{-1}(\omega)\, [f_0(\Phi(\omega) y) + \mu \sum_{i=1}^m f_i(\Phi(\omega)\, y)\, z_0^i(\omega)], \tag{21}$$

$\omega \in \Omega, y \in \mathbf{R}^d$. Attractors of ϕ and χ are related by the following theorem.

Theorem 2.1 *There is a one-to-one correspondence between random attractors of ϕ and random attractors of χ. If $(A(\omega), \omega \in \Omega)$ is a random attractor of χ, then $(\Phi(\omega)\, A(\omega), \omega \in \Omega)$ is a random attractor of ϕ. If $(B(\omega), \omega \in \Omega)$ is a random attractor of ϕ, then $(\Phi^{-1}(\omega)\, B(\omega), \omega \in \Omega)$ is a random attractor of χ attracting tempered sets.*

Proof:

We show the first one of two obviously parallel statements. Let $(A(\omega), \omega \in \Omega)$ be an attractor of χ. Then by definition for $\omega \in \Omega, t \geq 0$

$$\chi_t \circ \theta_{-t}(\omega)\ A(\theta_{-t}(\omega)) = A(\omega).$$

Therefore by Theorem 1.3

$$\phi_t \circ \theta_{-t}(\omega)(\Phi(\theta_{-t}\omega)\, A(\theta_{-t}\omega)) = \Phi(\omega)\ \chi_t \circ \theta_{-t}(\omega)\, (A(\theta_{-t}\omega)) = \Phi(\omega)\, A(\omega)$$

for $\omega \in \Omega$. Hence $(\Phi(\omega)\, A(\omega), \omega \in \Omega)$ is a ϕ-invariant random compact set. It is equally simple to show that if a tempered family $(D(\omega), \omega \in \Omega)$ is attracted by χ, then $(\Phi(\omega)\, D(\omega), \omega \in \Omega)$ is attracted by ϕ. Recall hereby from Theorem 1.2 that Φ is tempered. This completes the proof. $\square$

Once arrived at the random differential equation on which the flow χ is based, we may consider our problem of finding a random attractor as a perturbation problem for a deterministic differential equation given by the drift vector field alone. We first give a general condition for the existence of a random attractor of χ, which may not yet look very practical, in terms of Lyapunov type functions. We shall see that the following class of such functions behaves particularly well. A function $U : \mathbf{R}^d \to \mathbf{R}_+$ will be said to *preserve temperedness* if $U^{-1}(D)$ is tempered for tempered D. It is easy to see that any function U of polynomial growth preserves temperedness. In the examples of section 3 all Lyapunov functions will be of this type.

For further use, we denote

$$h(z, y) = (\frac{\partial u}{\partial x})^{-1}(z, y)\, (f_0(u(z, y)) + \mu \sum_{i=1}^{m} f_i(u(z, y))\, z^i),$$

for $(z, y) \in \mathbf{R}^m \times \mathbf{R}^d$.

Theorem 2.2 *Let $U : \mathbf{R}^d \to \mathbf{R}_+$ be a C^1-function such that $\lim_{|x|\to\infty} U(x) = \infty$. Suppose that for any $M > 0$ $\sup_{|y|\le M} |h(z, y)|$ is subexponentially growing in $z \in \mathbf{R}^m$ and there exists a subexponentially growing function $k : \mathbf{R}^m \to \mathbf{R}_+$ such that we have*

$$\limsup_{|y|\to\infty} \sup_{z\in\mathbf{R}^m} \frac{\langle \nabla \ln U(y), h(z, y)\rangle}{k(z)} \le 1, \tag{22}$$

$$\int_{\mathbf{R}^m} k(z)\, P_{z_0}(dz) < 0. \tag{23}$$

Then χ has a random attractor which attracts compact sets. If U preserves temperedness, then χ has a random attractor for tempered sets.

Proof:

(22) just says that for $\epsilon > 0$ there exists $M > 0$ such that for $|y| > M$ we have

$$\langle \nabla \ln U(y), h(z, y)\rangle \le k(z) + \epsilon$$

for all $z \in \mathbf{R}^m$. Choose $\epsilon > 0$ such that

$$\int_{\mathbf{R}^m} k(z)\, P_{z_0}(dz) < -\epsilon,$$

and let $l(z) = \sup_{|y| \leq M} |h(z, y)| \cdot \sup_{|y| \leq M} |\nabla U(y)|, z \in \mathbf{R}^m$. Then l is subexponentially growing, and, by an eventual passage to the function $k' = k + \epsilon$ which by choice of ϵ still fulfills (23), we may assume without loss of generality that for $y \in \mathbf{R}^d$ we have

$$\langle \nabla U(y), h(z, y) \rangle \leq k(z)\, U(y) + l(z).$$

Hence for $x \in \mathbf{R}^d, t \geq 0$ we may write

$$\begin{aligned} U(\chi_t(x)) &= U(x) + \int_0^t \langle \nabla U(\chi_s(x)), h(z_0 \circ \theta_s, \chi_s(x)) \rangle\, ds \\ &\leq U(x) + \int_0^t [k(z_0 \circ \theta_s)\, U(\chi_s(x)) + l(z_0 \circ \theta_s)]\, ds. \end{aligned}$$

Hence a standard comparison argument easily gives

$$\begin{aligned} U(\chi_t \circ \theta_{-t}(x)) &\leq U(x)\, \exp(\int_{-t}^0 k(z_0 \circ \theta_s)\, ds) \\ &+ \int_{-t}^0 \exp(\int_v^0 k(z_0 \circ \theta_u)\, du)\, l(z_0 \circ \theta_v)\, dv. \end{aligned} \tag{24}$$

Since k is subexponential, and by stationarity, the ergodic theorem of Birkhoff may be applied to give

$$\lim_{t \to \pm\infty} \frac{1}{t} \int_0^t k(z_0 \circ \theta_u)\, du = \int_{\mathbf{R}^m} k(z)\, P_{z_0}(dz) < 0. \tag{25}$$

Since also l is subexponential, (25) implies that the limit of the right hand side of (24) as $t \to \infty$ exists and is given by

$$Y = \int_{-\infty}^0 \exp(\int_v^0 k(z_0 \circ \theta_u)\, du)\, l(z_0 \circ \theta_v)\, dv. \tag{26}$$

Now let for $\omega \in \Omega$

$$B(\omega) = U^{-1}([0, 2Y(\omega)]).$$

Then a well known argument (see Crauel, Flandoli [7], or Keller, Schmalfuss [11], or Schenk-Hoppé [17]) shows that $(B(\omega), \omega \in \Omega)$ is a random absorbing set, from which a random attractor may be obtained using Theorem 0.1. Since the inverse images of compact sets by U are compact, this yields the assertion for the compact universe.

Now note that Y is tempered, in addition. Indeed, for $c > 0, t \in \mathbf{R}, \omega \in \Omega$, we can estimate $e^{-c|t|} Y(\theta_t \omega)$ by

$$e^{-c|t|} \int_{-\infty}^{0} e^{\int_{t+v}^{0} k(z(\theta_u \omega) - Ek(z_0) du - \int_t^0 k(z(\theta_u \omega) - Ek(z_0) du - vEk(z_0) + \log^+ l(\theta_{t+v}\omega)} dv.$$

For any $\epsilon > 0$, $\omega \in \Omega$ there exists a $t(\epsilon, \omega) > 0$ so that for $|t| > t(\epsilon, \omega)$, $t < 0$ we have

$$\log^+ l(\theta_t \omega) < \frac{\epsilon}{3}|t|, \qquad \left| \int_t^0 k(z(\theta_u \omega)) - Ek(z_0) du \right| < \frac{\epsilon}{3}|t|.$$

If we choose $\epsilon < \min(\frac{c}{2}, -Ek(z_0))$ then we have the asserted convergence for $t \to -\infty$. For $t \to \infty$ we have the same convergence, see Arnold [1], p. 164f. If U preserves temperedness, the measurable set B is obviously tempered, and Theorem 0.1 applies again. □

Let us now suppose we know a Lyapunov function of the deterministic system

$$dy_t = f_0(y_t)\, dt, \tag{27}$$

say $V : \mathbf{R}^d \to \mathbf{R}_+$, i.e. there exists $\alpha > 0$ such that

$$\limsup_{|y| \to \infty} \langle \nabla \ln V(y), f_0(y) \rangle \leq -\alpha. \tag{28}$$

The question we shall discuss is: under which additional conditions on V does the perturbed system (20) still have a Lyapunov function? Indeed, we shall discuss additional conditions under which V itself remains a Lyapunov function. One of these conditions will contain a statement saying that the additional drift caused by our decomposition and given by $\mu \, (\frac{\partial \Phi_t}{\partial x})^{-1} \sum_{i=1}^m f_i(\Phi_t y_t)\, z_t^i \, dt$ is negligible in a sense to be made precise below. As a consequence, the drift intensity μ of the auxiliary OU process may be chosen arbitrarily large. This fact turns out to be favourable for the treatment of the remaining drift given by $(\frac{\partial \Phi_t}{\partial x})^{-1} f_0(\Phi_t \, y_t)\, dt$.

Let us remark that by choosing μ large we may ensure that the stationary diffeomorphisms Φ are close to the identity. This justifies the notion *perturbation* of f_0 for the vector field of (20). This feature is easily checked. Indeed, we have $\Phi = u(z_0, .)$ and $u(0, .) = I$, the identity. It follows from the definition of z that

$$z_t = \int_{-\infty}^{0} \exp(\mu u)\, dW_{t+u}, \quad t \in \mathbf{R},$$

is a stationary solution of the Langevin equation. Hence as $\mu \to \infty$, z_0 converges to 0 in L^2. Hence Φ approaches the identity as $\mu \to \infty$, even $P-$a.s., along a suitable sequence.

We shall now state our prototypical theorem for the existence of random attractors of differential equations *perturbed* by diffusive noise. The criterion we formulate essentially says that the perturbation of the vector field f_0 described by h is not strong enough to affect a given Lyapunov function. It will later on be seen to be fulfilled in particular situations, for example for the noisy Duffing-van der Pol equation, for which it was proved in Keller, Schmalfuss [11].

Theorem 2.3 *Let V be a Lyapunov function of*

$$dy_t = f_0(y_t)\, dt.$$

Suppose that there exists $\mu > 0$ and a subexponentially growing function $k : \mathbf{R}^m \to \mathbf{R}_+$ such that

$$\limsup_{|y|\to\infty} \sup_{z\in\mathbf{R}^m} \frac{|\langle \nabla \ln V(y), f_0(y) - h(z,y)\rangle|}{|\langle \nabla \ln V(y), f_0(y)\rangle|\, k(z)} \leq 1, \tag{29}$$

$$\int_{\mathbf{R}^m} k(z)\, P_{z_0}(dz) < \alpha, \tag{30}$$

where α is the constant of (28). Then χ has a global random attractor for the compact sets. If V preserves temperedness, then χ has a random attractor for the tempered sets.

Proof:

By choice of k and definition of V, (29) and (30) imply (22) and (23) of Theorem 2.2.The theorem applies and finishes the proof. □

A particular situation in which the hypotheses of Theorem (2.3) are simple to check, and the freedom in the choice of the drift parameter μ comes into play quite practically, arises if the auxiliary drift stemming from the introduction of the OU process pushes into directions in which the gradient of V is essentially smaller. This situation will be seen to be given in the case of the noisy Duffing-van der Pol equation in section 3.

Corollary 2.1 *Let V be a Lyapunov function of*

$$dy_t = f_0(y_t)\,dt,$$

and denote

$$l(z,y) = (\frac{\partial u}{\partial x})^{-1}(z,.)\ \sum_{i=1}^{m} f_i(u(z,y))\,z^i,$$

$z \in \mathbf{R}^m, y \in \mathbf{R}^d$. *Suppose that there are subexponentially growing functions k_1 and k_2 such that*

$$\limsup_{|y|\to\infty}\ \sup_{z\in\mathbf{R}^m} |\langle \nabla \ln V(y), \frac{l(z,y)}{k_1(z)}\rangle| = 0, \tag{31}$$

$$\limsup_{|y|\to\infty} \frac{\langle \nabla \ln V(y), f_0(y) - (\frac{\partial u}{\partial x})^{-1}(z,y)\,f_0(u(z,y))\rangle}{|\langle \nabla \ln V(y), f_0(y)\rangle|\,k_2(z)} \leq 1, \tag{32}$$

$$\lim_{z\to 0} k_2(z) = 0. \tag{33}$$

Then χ has a global random attractor for the compact sets. If V preserves temperedness, then χ has a random attractor for the tempered sets..

Proof:

Let α be the constant of (28) again. By the remark made before Theorem 2.3 and dominated convergence, (33) allows us to choose μ large enough to ensure

$$\int_{\mathbf{R}^m} k(z)\,P_{z_0}(dz) < \alpha.$$

Next, using (31), we may choose $\delta > 0$ small enough so that

$$\limsup_{|y|\to\infty}\ \sup_{z\in\mathbf{R}^m} |\langle \nabla \ln V(y), \frac{l(z,y)}{k_1(z)}\rangle| \leq \delta,$$

as well as

$$\int_{\mathbf{R}^m} (\delta\,\mu\,k_1(z) + k_2(z))\,P_{z_0}(dz) < \alpha.$$

Then obviously the function $k = \delta\,\mu\,k_1 + k_2$ fulfills (30) of Theorem (2.3). And (31) and (32) imply (29). Hence Theorem 2.3 applies. □

In case of purely additive noise, the hypotheses of Corollary 2.1 are easily seen to be satisfied.

Corollary 2.2 *Let V be a Lyapunov function of*

$$dy_t = f_0(y_t)\,dt,$$

such that $\lim_{|y|\to\infty} |\nabla \ln V(y)| = 0$. *Assume that for some* $b_1, \cdots, b_m \in \mathbf{R}^d$ *we have* $f_i = b_i, 1 \le i \le m$, *and that there exists a subexponentially growing function* $k : \mathbf{R}^m \to \mathbf{R}_+$ *such that*

$$\limsup_{|y|\to\infty} \sup_{z\in\mathbf{R}^m} \frac{|\langle \nabla \ln V(y), f_0(y) - f_0(y + \sum_{i=1}^m b_i\, z^i)\rangle|}{|\langle \nabla \ln V(y), f_0(y)\rangle|\, k(z)} \le 1, \tag{34}$$

$$\lim_{z\to 0} k(z) = 0. \tag{35}$$

Then χ has a global random attractor for the compact sets. If V preserves temperedness, then χ has a random attractor for the tempered sets..

Proof:

We have $u(z,x) = x + \sum_{i=1}^m z^i\, b_i, (z,x) \in \mathbf{R}^m \times \mathbf{R}^d$. So (32) evidently is a consequence of (34). It therefore remains to verify condition (31) of the preceding corollary. Indeed, we have

$$l(z,y) = \langle \nabla \ln V(y), \sum_{i=1}^m b_i\, z^i\rangle,$$

$(z,y) \in \mathbf{R}^m \times \mathbf{R}^d$. Hence (31) is an easy consequence of the hypothesis $\lim_{|y|\to\infty} |\nabla \ln V(y)| = 0$ and the exponential boundedness of $z \mapsto \sum_{i=1}^m b_i\, z^i$.□

3 Examples

In this section we shall discuss a number of examples in which the criteria for the existence of global attractors of the preceding section apply. All the examples describe well known dynamical systems perturbed by some noise, and have been extensively studied in the literature.

We start by considering the Duffing-van der Pol oscillator with multiplicative noise on the position variable. It has been studied in Keller, Schmalfuss [11].

3.1 The Duffing-van der Pol oscillator with multiplicative noise on the position

The system we investigate comes from the second order noisy differential equation

$$\ddot{y}_t - \beta\,\dot{y}_t + y_t^3 + y_t^2\,\dot{y}_t + y_t - \sigma\,y_t \circ \dot{W}_t = 0,$$

with parameters $\beta \in \mathbf{R}, \sigma \neq 0$. In the usual two-dimensional setting it becomes

$$\begin{aligned} dy_1(t) &= y_2(t)\,dt, \\ dy_2(t) &= [-y_1(t) + \beta\,y_2(t) - y_1(t)^3 - y_1(t)^2\,y_2(t)]dt + \sigma\,y_1(t) \circ dW_t, \end{aligned}$$

and, after a transformation into new coordinates, the *Lienard coordinates* (see Schenk-Hoppé [17]), given by

$$x_1 = y_1, \quad x_2 = y_2 - \beta\,y_1 + \frac{1}{3}y_1^3,$$

$y = (y_1, y_2) \in \mathbf{R}^2$, it takes the form

$$dx_t = f_0(x_t)\,dt + A\,x_t \circ dW_t, \tag{36}$$

where

$$A = \begin{bmatrix} 0 & 0 \\ \sigma & 0 \end{bmatrix}, \quad f_0(x) = \begin{bmatrix} \beta\,x_1 - \frac{1}{3}x_1^3 + x_2 \\ -x_1 - x_1^3 \end{bmatrix},$$

$x \in \mathbf{R}^2$. Let us first look at the deterministic, *unperturbed* system

$$dy_t = f_0(y_t)\,dt. \tag{37}$$

Then, the function

$$V(y) = \frac{7}{24}y_1^4 + \frac{1}{2}y_1^2 + \frac{1}{4}y_2^2 + \frac{1}{2}(y_1 - y_2)^2,$$

$y \in \mathbf{R}^2$, is seen to be a Lyapunov function of (37). Indeed, we have

$$\nabla V(y) = \begin{bmatrix} \frac{7}{6}y_1^3 + \frac{3}{2}y_1 - y_2 \\ \frac{3}{2}y_2 - y_1 \end{bmatrix}, \tag{38}$$

and therefore

$$\langle \nabla V(y), f_0(y) \rangle = -\frac{7}{18}y_1^6 + (\frac{1}{3} + \frac{7}{6}\beta)y_1^4 + (2\beta + 1)\,y_1^2 + (\frac{1}{2} - \beta)\,y_1\,y_2 - y_2^2, \tag{39}$$

$y \in \mathbf{R}^2$. From (39) it is easy to see that, setting for $y \in \mathbf{R}^2$

$$\kappa(y) = y_1^6 + y_2^2,$$

we have

$$\limsup_{|y|\to\infty} \frac{\langle \nabla V(y), f_0(y)\rangle}{\kappa(y)} \leq -\alpha, \tag{40}$$

for some positive constant α. (40) is indeed sharper than the inequality

$$\limsup_{|y|\to\infty} \langle \nabla \ln V(y), f_0(y)\rangle \leq -\alpha \tag{41}$$

for some $\alpha > 0$, which is needed to show that V is a Lyapunov function of (37).

Now let $\mu > 0$ be given and decompose the flow corresponding to (36) in the manner described in section 1. We shall now verify the hypotheses of Corollary 2.1, starting with (31). Note that (38) implies for $y \in \mathbf{R}^2, z \in \mathbf{R}$

$$|\langle \nabla V(y), A\, y\, z\rangle| \leq |z|\, |\sigma y_1 (\frac{3}{2} y_2 - y_1)| \leq c_1\, |z|\, [|y_1|^3 + |y_2|^{\frac{3}{2}}]$$

with some positive constant c_1, and hence clearly

$$\lim_{|y|\to\infty} \sup_{z\in\mathbf{R}} \frac{|\langle \nabla \ln V(y), A\, y\, z\rangle|}{\langle \nabla \ln V(y), f_0(y)\rangle\, k(z)} = 0,$$

with the subexponential function $k(z) = |z|, z \in \mathbf{R}$. This entails (31).

To verify the crucial condition (32) of Corollary 2.1, let us split the drift vector field into its linear and non-linear part. We have, setting for $y \in \mathbf{R}^2$

$$B = \begin{bmatrix} \beta & 1 \\ -1 & 0 \end{bmatrix}, \quad g_0(y) = \begin{bmatrix} -\frac{1}{3} y_1^3 \\ -y_1^3 \end{bmatrix},$$

the equation

$$f_0(y) = B\, y + g_0(y).$$

For the linear part, a rather crude estimate already works. Indeed, for $z \in \mathbf{R}, y \in \mathbf{R}^2$ we have

$$|\langle \nabla V(y), (B - e^{-Az}\, B\, e^{Az}) y\rangle| \leq |\nabla V(y)|\, |y|\, k(z), \tag{42}$$

with $k(z) = |B - e^{-Az}\, B\, e^{Az}|, z \in \mathbf{R}$. Now clearly k is subexponential and $\lim_{z\to 0} k(z) = 0$. Moreover,

$$|\nabla V(y)|\, |y| \leq c_2\, \kappa(y),$$

for $y \in \mathbf{R}^2$, with some constant c_2, as is seen by elementary estimates. Hence we obtain altogether for the contribution of the linear part of f_0, observing the inequality (40)

$$\limsup_{|y|\to\infty} \sup_{z\in\mathbf{R}} \frac{\langle \nabla \ln V(y), (B - e^{-Az}\, Be^{Az})\, y\rangle}{|\langle \nabla \ln V(y), f_0(y)\rangle|\, k(z)} \tag{43}$$
$$= \limsup_{|y|\to\infty} \frac{\langle \nabla V(y), (B - e^{-Az}\, Be^{Az})\, y\rangle}{|\langle \nabla V(y), f_0(y)\rangle|\, k(z)} \leq 1,$$

with k subexponential, $\lim_{z\to 0} k(z) = 0$.

It remains to consider the non-linear part of f_0. Note that in the case considered, our stationary diffeomorphisms are just given by

$$\Phi = e^{Az_0} = \begin{bmatrix} 1 & 0 \\ \sigma\, z_0 & 1 \end{bmatrix},$$

hence

$$\Phi^{-1} = \begin{bmatrix} 1 & 0 \\ -\sigma\, z_0 & 1 \end{bmatrix}.$$

Hence for $z \in \mathbf{R}, y \in \mathbf{R}^2$ we have

$$e^{-Az} g_0(e^{Az}\, y) = \begin{bmatrix} -\frac{1}{3}\, y_1^3 \\ \sigma\, z\, \frac{1}{3}\, y_1^3 - y_1^3 \end{bmatrix},$$

and therefore

$$e^{-Az}\, g_0(e^{Az}\, y) - g_0(y) = \begin{bmatrix} 0 \\ \sigma\, z\, \frac{1}{3}\, y_1^3 \end{bmatrix}.$$

In view of (38), this implies for $z \in \mathbf{R}, y \in \mathbf{R}^2$

$$\begin{aligned} |\langle \nabla \ln V(y), e^{-Az} g_0(e^{Az}\, y) - g_0(y)\rangle| &\leq |\sigma\, z|\, |\frac{1}{3}\, y_1^3 (\frac{3}{2} y_2 - y_1)| \\ &\leq c_3\, |z| (\kappa(y) + 1) \\ &\leq c_4\, |z| (|\langle \nabla V(y), f_0(y)\rangle| + 1), \end{aligned} \tag{44}$$

where we have used elementary estimates along with (40), and c_3, c_4 are some positive constants only depending on σ. Now let $l(z) = c_3\, |z|, z \in \mathbf{R}$. Then we see that (44) implies

$$\limsup_{|y|\to\infty} \sup_{z\in\mathbf{R}} \frac{\langle \nabla \ln V(y), (e^{-Az}\, g_0(e^{Az})\, y) - g_0(y)\rangle}{|\langle \nabla \ln V(y), f_0(y)\rangle|\, l(z)} \leq 1, \tag{45}$$

where l is subexponential and $\lim_{z\to 0} l(z) = 0$. Now combine (45) and (43) to obtain (32) of Corollary 2.1. Hence the Duffing-van der Pol oscillator possesses a global attractor according to Theorem 2.1 which attracts the tempered sets.

Let us next consider similar models with different sources of noise (see also Schenk-Hoppé [17]).

3.2 The Duffing-van der Pol oscillator with multiplicative noise on the velocity

In this subsection we shall investigate the system related to the second order equation

$$\ddot{y}_t - \beta\,\dot{y}_t + y_t^3 + y_t^2\,\dot{y}_t + y_t - \sigma\,\dot{y}_t \circ d\dot{W}_t = 0,$$

with parameters $\beta \in \mathbf{R}, \sigma \neq 0$. We now make a clearer distinction between the y-coordinates and the x-coordinates (the Lienard coordinates) than in the preceding subsection. In the y-coordinates we obtain the following two-dimensional sde

$$dy_t = g_0(y_t)\,dt + A\,y_t \circ dW_t, \tag{46}$$

where

$$g_0(y) = \begin{bmatrix} y_2 \\ -y_1 + \beta\,y_2 - y_1^3 - y_1^2\,y_2 \end{bmatrix}, \quad y \in \mathbf{R}^2,$$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & \sigma \end{bmatrix}.$$

Note that this time we may not pass to Lienard coordinates without changing the linearity in the diffusion term. So we stick to y-coordinates in (46), but may transform into x-coordinates if the calculations with Lyapunov functions require to do so. Denote therefore

$$t(y) = \begin{bmatrix} y_1 \\ y_2 - \beta\,y_1 + \frac{1}{3}y_1^3 \end{bmatrix}, \quad t^{-1}(x) = \begin{bmatrix} x_1 \\ x_2 + \beta\,x_1 - \frac{1}{3}x_1^3 \end{bmatrix},$$

$x, y \in \mathbf{R}^2$. The Jacobian of t is given by

$$Dt(y) = \begin{bmatrix} 1 & 0 \\ y_1^2 - \beta & 1 \end{bmatrix}, \quad y \in \mathbf{R}^2.$$

Let us first look at the deterministic system

$$dy_t = g_0(y_t)\, dt, \tag{47}$$

which, in x-coordinates, may be described by

$$dx_t = f_0(x_t)\, dt,$$

as in the preceding section, with

$$Dt(y)\, g_0(y) = f_0(t(y)), \quad \text{or} \quad f_0(x) = (Dt)(t^{-1}(x))\, g_0(t^{-1}(x)), \tag{48}$$

$x, y \in \mathbf{R}^2$. We know a Lyapunov function of the system in x-coordinates from the preceding section, where it was defined as V. Let

$$U(y) = V(t(y)), \quad y \in \mathbf{R}^2.$$

Then U is a Lyapunov function for (47), since we have

$$\langle \nabla_y U(y), g_0(y) \rangle = \langle \nabla_x V(t(y)), Dt(y) g_0(y) \rangle = \langle \nabla_x V(t(y)), f_0(t(y)) \rangle, \tag{49}$$

$y \in \mathbf{R}^2$. We aim at deriving conditions on the coefficients of our system under which U is a Lyapunov function of the stochastic perturbation of (47). More precisely, we shall apply the criteria of Theorem 2.3. This time, our stationary random isomorphism is given by

$$\Phi = \begin{bmatrix} 1 & 0 \\ 0 & e^{\sigma z_0} \end{bmatrix},$$

and

$$u(z, y) = \begin{bmatrix} 1 & 0 \\ 0 & e^{\sigma z} \end{bmatrix} y = \begin{bmatrix} y_1 \\ e^{\sigma z}\, y_2 \end{bmatrix}, \quad z \in \mathbf{R}, y \in \mathbf{R}^2.$$

Hence

$$(\frac{\partial u}{\partial x})^{-1}(z, .) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\sigma z} \end{bmatrix}.$$

According to (49), we may write

$$\begin{aligned} &\langle \nabla_y U(y), g_0(y) - (\frac{\partial u}{\partial x})^{-1}(z, .)\, g_0(u(z, y)) \rangle \\ &\quad = \langle \nabla_x V(t(y)), Dt(y)[g_0(y) - (\frac{\partial u}{\partial x})^{-1}(z, .)\, g_0(u(z, y))] \rangle. \end{aligned}$$

So we have to estimate for large $x \in \mathbf{R}^2$

$$\langle \nabla_x V(x), (Dt)(t^{-1}(x))[g_0(t^{-1}(x)) - (\frac{\partial u}{\partial x})^{-1}(z,.)\, g_0(u(z,t^{-1}(x)))]\rangle$$
$$= \langle \nabla_x V(x), [f_0(x) - (\frac{\partial u}{\partial x})^{-1}(z,.)\, f_0(u(z,x))]\rangle.$$

The linear part may be treated exactly as in (42) and (43). Let us give the arguments for the nonlinear part. It is given by

$$n(z,x) = \begin{bmatrix} -\frac{1}{3}x_1^3 \\ -x_1^3 \end{bmatrix} - \begin{bmatrix} e^{\sigma z}\,(-\frac{1}{3}x_1^3) \\ e^{-\sigma z}(-x_1^3) \end{bmatrix} = \begin{bmatrix} (e^{\sigma z}-1)\,\frac{1}{3}x_1^3 \\ (e^{-\sigma z}-1)\,x_1^3 \end{bmatrix}.$$

Hence we have

$$\langle \nabla_x V(x), n(z,x)\rangle = [\frac{7}{18}x_1^6 + \frac{1}{2}x_1^4 - \frac{1}{3}x_2 x_1^3](e^{\sigma z}-1) + [\frac{3}{2}x_2 x_1^3 - x_1^4](e^{-\sigma z}-1),$$

and, by elementary algebra using in particular the inequality

$$|x_2 x_1^3| \le \frac{1}{2}[x_2^2 + x_1^6]$$

we arrive at

$$|\langle \nabla_x V(x), n(z,x)\rangle| \le k(z)\,\kappa(x),$$

where κ is defined as in the preceding section, and

$$k(z) = c_1[|e^{\sigma z}-1| + |e^{-\sigma z}-1|],$$

$z \in \mathbf{R}$, with a constant $c_1 > 0$. We may summarize the computations just executed in the following result. There exists a subexponential function k such that $\lim_{z\to 0} k(z) = 0$ and such that

$$\limsup_{|x|\to\infty} \sup_{z\in\mathbf{R}} \frac{|\langle \nabla_x \ln V(x), f_0(x) - (\frac{\partial u}{\partial x})^{-1}(z,.)\, f_0(u(z,x))\rangle|}{|\langle \nabla_x \ln V(x), f_0(x)\rangle|\, k(z)} \le 1,$$

and consequently

$$\limsup_{|y|\to\infty} \sup_{z\in\mathbf{R}} \frac{|\langle \nabla_y \ln U(y), g_0(y) - (\frac{\partial u}{\partial x})^{-1}(z,.)\, g_0(u(z,y))\rangle|}{|\langle \nabla_y \ln U(y), g_0(y)\rangle|\, k(z)} \le 1. \qquad (50)$$

Let us now investigate the auxiliary drift term. Again we work in Lienard coordinates. For $t(y) = x \in \mathbf{R}^2$ with big enough absolute value we have by arguments already discussed

$$
\begin{aligned}
|\langle \nabla_y U(y), Ay\rangle| &= |\langle \nabla_x V(t(y)), Dt(y)Ay\rangle| \\
&= \langle \nabla_x V(x), (Dt)(t^{-1}(x))\, At^{-1}(x)\rangle| \\
&= |\langle \nabla_x V(x), \begin{bmatrix} 0 \\ \sigma(x_2 + \beta x_1 - \frac{1}{3}x_1^3) \end{bmatrix}\rangle| \\
&= \sigma|\frac{3}{2}x_2^2 + (\frac{3}{2}\beta - 1)x_1x_2 - \beta x_1^2 - \frac{1}{2}x_2x_1^3 + \frac{1}{3}x_1^4| \\
&\leq \sigma\, c_2\, \kappa(x)
\end{aligned}
$$

with a constant c_2 independent of σ. Hence there are subexponential functions k_1 and $k_2(z) = \mu\, c_2\, \sigma\, |z|, z \in \mathbf{R}$, such that $\lim_{z\to 0} k_1(z) = 0$, and such that

$$
\limsup_{|y|\to\infty} \sup_{z\in\mathbf{R}} \frac{|\langle \nabla_y \ln U(y), g_0(y) - h(z,y)\rangle|}{|\langle \nabla_y \ln U(y), g_0(y)\rangle|\, (k_1(z) + k_2(z))} \leq 1, \tag{51}
$$

where we recall

$$
h(z,y) = (\frac{\partial u}{\partial x})^{-1}(z,.)\, g_0(u(z,y)) + \mu\, z\, A\, y,
$$

$z \in \mathbf{R}, y \in \mathbf{R}^2$. It remains to choose σ small enough to ensure

$$
\int_{\mathbf{R}} (k_1 + k_2)(z)\, P_{z_0}(dz) < \alpha, \tag{52}
$$

to see that for small enough σ our system has a global random attractor for tempered sets, due to Theorem 2.3.

Remark:1. A similar analysis is possible in case

$$
A = \begin{bmatrix} 0 & 0 \\ \rho & \sigma \end{bmatrix}
$$

with $\rho, \sigma \in \mathbf{R}$.

2. The statement *for small enough* σ above could be made more precise in terms of a function of the parameters β and σ of the system.

3.3 The Duffing-van der Pol oscillator with additional additive noise

Let

$$
A_1 = \begin{bmatrix} 0 & 0 \\ \sigma_1 & 0 \end{bmatrix}, \quad A_2 = 0, \quad b_1 = 0, \quad b_2 = \begin{bmatrix} 0 \\ \sigma_2 \end{bmatrix},
$$

where $\sigma_1, \sigma_2 \neq 0$. Let $W = (W^1, W^2)$ be a two-dimensional Wiener process and f_0 as in the preceding subsection. We consider the stochastic differential equation

$$dx_t = f_0(x_t)\, dt + \sum_{i=1}^{2} (A_i\, x_t + b_i) \circ dW_t^i. \tag{53}$$

Let V be the Lyapunov function of the unperturbed system given above. Choose $\mu > 0$ as the drift of an auxiliary 2-dimensional OU process. To verify that the additional drift is small, note first that the stationary diffeomorphism in the case considered is given by

$$\Phi\, x = e^{A_1 z_0^1}\, x + b_2\, z_0^2,$$

$x \in \mathbf{R}^2$. Hence

$$\begin{aligned} & |\langle \nabla \ln V(y), (\frac{\partial u}{\partial x})^{-1}(y) \sum_{i=1}^{2} [A_i\, u(z,y) + b_i]\, z^i \rangle| \\ = \;& |\langle \nabla \ln V(y), e^{-A_1 z^1} [A_1\, (e^{A_1 z^1} y + b_2\, z^2)\, z^1 + b_2\, z^2] \rangle| \\ \leq \;& |\langle \nabla \ln V(y), z^1\, A_1\, y \rangle| + |\nabla \ln V(y)|\, |e^{-A_1 z^1}\, (z^1 z^2 + z^2)\, b_2|. \end{aligned}$$

The first term in the above estimate has been treated in the preceding subsection. For the second one, just note that $\lim_{|y|\to\infty} |\nabla \ln V(y)| = 0$, and that the second factor is an exponentially bounded function of $z \in \mathbf{R}^2$. So we can conclude

$$\lim_{|y|\to\infty} \sup_{z \in \mathbf{R}^2} \frac{|\langle \nabla \ln V(y), (\frac{\partial u}{\partial x})^{-1}(y) \sum_{i=1}^{2} [A_i\, (u(z,y) + b_i]\, z^i \rangle|}{|\langle \nabla \ln V(y), f_0(y) \rangle|\, k(z)} = 0, \tag{54}$$

with the subexponential function $k(z) = |e^{-A_1 z^1}\, (z^1 z^2 + z^2)\, b_2|$, $z = (z^1, z^2) \in \mathbf{R}^2$. This proves (31).

To verify (32) of Corollary 2.1, we again split the linear and non-linear part of f_0. The estimate for the linear part just follows similar arguments as the ones above. For the non-linear part, we note that in the notation of the preceding subsection

$$e^{-A_1 z^1}\, g_0(e^{A_1 z^1} y + b_2\, z^2) = e^{-A_1 z^1}\, g_0(e^{A_1 z^1} y),$$

since b_2 only depends on the second coordinate. Hence also in this case the necessary estimates have already been performed in the preceding example. Hence the existence of a global attractor for tempered sets follows from Theorem 2.1.

3.4 The noisy damped harmonic oscillator in a double well potential

Let $U(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2, x \in \mathbf{R}$. We consider the noisy damped harmonic oscillator

$$\ddot{x}_t + \gamma\,\dot{x}_t - U'(x_t) = \sigma \circ \dot{W}_t,$$

with $\gamma > 0, \sigma \in \mathbf{R}$. In the two-dimensional setting we thus obtain the stochastic differential equation

$$dx_t = f_0(x_t)\,dt + b_1 \circ dW_t, \tag{55}$$

where

$$f_0(x) = \begin{bmatrix} x_2 \\ -\gamma\,x_2 + x_1 - x_1^3 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 0 \\ \sigma \end{bmatrix},$$

$x \in \mathbf{R}^2$. Let us first define a Lyapunov function for the deterministic system

$$dy_t = f_0(y_t)\,dt. \tag{56}$$

Let

$$V(y) = y_1^4 + \gamma^2\,y_1^2 + \gamma\,y_1\,y_2 + 2\,y_2^2,$$

$y = (y_1, y_2) \in \mathbf{R}^2$. Then

$$\nabla V(y) = \begin{bmatrix} 4y_1^3 + 2\gamma^2\,y_1 + \gamma\,y_2 \\ \gamma\,y_1 + 4y_2 \end{bmatrix},$$

and consequently

$$\langle \nabla V(y), f_0(y) \rangle = -\gamma\,y_1^4 + \gamma\,y_1^2 - 3\gamma\,y_2^2 + (4+\gamma^2)\,y_1\,y_2. \tag{57}$$

It is therefore easy to see that

$$\limsup_{|y|\to\infty} \langle \nabla \ln V(y), f_0(y) \rangle \leq -\gamma, \quad \lim_{|y|\to\infty} |\nabla \ln V(y)| = 0.$$

This identifies V as a Lyapunov function of (56) for all $\gamma > 0$. To be able to apply Corollary 2.2 to show that (55) possesses a global attractor for $\gamma > 0, \sigma \in \mathbf{R}$, we just have to note that

$$f_0(y) - f_0(y + b_1\,z^1) = \sigma \begin{bmatrix} -1 \\ \gamma \end{bmatrix}.$$

Then arguments as used before apply.

Remark:A similar analysis is possible for any potential fiunction U such that $\lim_{|x|\to\infty} U(x) = \infty$ with a possible restriction for the attraction of tempered sets.

3.5 The Lorenz system with multiplicative noise

We consider a stochastic perturbation of the well known Lorenz system (see Leonov and Boichenko [14]) given by the following stochastic differential equation

$$\begin{aligned} dy_t &= f_0(y_t)\,dt + f_1(y_t)\circ dW_t && (58)\\ f_0(y) &= \begin{bmatrix} -d\,y_1 + dy_2 - ay_2y_3 \\ r\,y_1 - y_2 - y_1y_3 \\ -b\,y_3 + y_1y_2 \end{bmatrix}, \qquad f_1(y) = \sigma\,Ay, \quad A = \mathrm{id}, \\ &\text{with the parameters} \quad d,\ b,\ r > 0, \qquad a,\ \sigma \in \mathbf{R}. && (59) \end{aligned}$$

The deterministic version of equation (58) covers a number of physical models exhibiting chaotic behavior, see Leonov and Boichenko [14]. In particular, for $\alpha = 0$ this system has been introduced by E. N. Lorenz as an approximation of the Boussinesq equation describing heat convection.

The stationary diffeomorphisms Φ, Φ^{-1} for $A = \sigma\,\mathrm{id}$ are given by

$$\Phi = e^{\sigma z_0}\mathrm{id}, \quad \Phi^{-1} = e^{-\sigma z_0}\mathrm{id}$$

and we have

$$u(z,y) = e^{\sigma z}\,y, \quad z \in \mathbf{R}, y \in \mathbf{R}^3.$$

Let δ be a constant such that $a + \delta > 0$. Then the function

$$V(y_1,y_2,y_3) = \frac{1}{2}\left(y_1^2 + \delta\,y_2^2 + (a+\delta)\left(y_3 - \frac{d+\delta\,r}{a+\delta}\right)^2\right). \qquad (60)$$

is a Lyapunov function for the deterministic version of (58), see Leonov and Boichenko [14]. Indeed, we have

$$\begin{aligned} \langle \nabla V(y), f_0(y)\rangle &= -dy_1^2 - \delta y_2^2 - b(a+\delta)y_3^2 + b(d+\delta r)y_3 \\ &\le -dy_1^2 - \delta y_2^2 - \frac{1}{2}b(a+\delta)\left(y_3 - \frac{d+\delta r}{a+\delta}\right)^2 + \frac{1}{2}b\frac{(d+\delta r)^2}{a+\delta} \\ &= -\alpha V(y) + \beta && (61) \end{aligned}$$

where $\alpha = 2\min(d, 1, \frac{1}{2}b)$ and $\beta = \frac{1}{2}b\frac{(d+\delta r)^2}{a+\delta}$.

We now aim at verifying the hypotheses of Theorem 2.3. This is done in three steps.

Firstly, a simple calculation shows that

$$|\langle \nabla V(y), f_0(y) - (\frac{\partial u}{\partial x})^{-1}(z,.)\, f_0(u(z,y))\rangle| = (d+\delta r)\,|e^{\sigma z} - 1|\,|y_1 y_2|.$$

Note that the expectation of $e^{\sigma z_0}$ is finite.

Secondly, for the term

$$|\langle \nabla V(y), \mu A y z\rangle|$$

the following estimate works:

$$\begin{aligned}
|\langle \nabla V(y), Ay\rangle| &= \left| y_1^2 + \delta y_2^2 + (a+\delta) y_3 \left(y_3 - \frac{d+\delta\, r}{a+\delta}\right)\right| \\
&= \left| y_1^2 + \delta y_2^2 + (a+\delta)\left(y_3 - \frac{d+\delta r}{a+\delta}\right)^2 + (d+\delta r)\left(y_3 - \frac{d+\delta r}{a+\delta}\right)\right| \\
&\le y_1^2 + \delta y_2^2 + (a+\delta+d+\delta r)\left(y_3 - \frac{d+\delta r}{a+\delta}\right)^2 + d + \delta r \\
&\le \gamma_1 V(y) + \gamma_2,
\end{aligned}$$

$\gamma_1 = 2\frac{a+\delta+d+\delta r}{a+\delta}$ and $\gamma_2 = d + \delta r$.

Finally, we obtain for sufficiently large $|y|$

$$|\langle \nabla V(y), f_0(y)\rangle| \ge \alpha V(y) - \beta > 0.$$

We now check the assumptions of Theorem 2.3. We have for sufficiently large $|y|$

$$\begin{aligned}
&\frac{|\langle \nabla V(y), f_0(y) - h(z,y)\rangle|}{|\langle \nabla V(y) f_0(y)\rangle|} \qquad (62)\\
&\le \frac{(\max(1,\delta^{-1})|e^{\sigma z} - 1|(d+\delta r) + \mu\gamma_1|\sigma z|)V(y) + \mu\gamma_2|\sigma||z|}{\alpha V(y) - \beta} \\
&= \frac{\max(1,\delta^{-1})|e^{\sigma z} - 1|(d+\delta r) + \mu\gamma_1|\sigma||z|}{\alpha - \frac{\beta}{V(y)}} + \frac{\mu\gamma_2|\sigma||z|}{\alpha V(y) - \beta}
\end{aligned}$$

where h is defined in Theorem 2.3. If we choose $|\sigma|$ sufficiently small then the integral of

$$k(z) = \frac{2(\max(1,\delta^{-1})|e^{\sigma z} - 1|(d+\delta r) + \mu\gamma_1|\sigma||z|}{\alpha} + \mu\gamma_2|\sigma||z|$$

with respect to the law of z_0 is less than α. On the other hand the right hand side of (62) is less than $k(z)$ if $|y|$ is sufficiently large such that

$$0 < \frac{1}{\alpha - \frac{\beta}{V(y)}} \leq \frac{2}{\alpha}, \qquad \frac{1}{\alpha V(y) - \beta} \leq 1.$$

We may now apply Theorem 2.3 to get the existence of a random attractor for tempered sets for small enough σ.

3.6 The Lorenz system with partially multiplicative noise

We now study (58) where the noise is described by the matrix A defined as follows

$$\begin{bmatrix} 0 & \sigma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \sigma \in \mathbf{R}. \tag{63}$$

The stationary homeomorphisms for this system are obviously given by

$$\Phi = \begin{bmatrix} 1 & \sigma z_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \Phi^{-1} = \begin{bmatrix} 1 & -\sigma z_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

u by

$$u(z,y) = \begin{bmatrix} y_1 + \sigma\, z\, y_2 \\ 0 \\ 0 \end{bmatrix}, \quad z \in \mathbf{R}, y \in \mathbf{R}^3.$$

We first show that

$$\tilde{V}(y) = \frac{1}{2}(y_2^2 + (y_3 - r)^2)$$

is a Lyapunov type function for our system which ensures the existence of a non-random closed noncompact absorbing set. In fact, we have:

$$\begin{aligned} &\langle \nabla \tilde{V}(y), f_0(y) \rangle \leq -y_2^2 - \frac{b}{2}(y_3 - r)^2 + \frac{b}{2}r^2 = -\tilde{\alpha}\tilde{V}(y) + \tilde{\beta}, \\ &\langle \nabla \tilde{V}(y), f_0(y) - (\frac{\partial u}{\partial x})^{-1}(z,.)\, f_0(u(z,y)) \rangle = 0, \\ &\langle \nabla \tilde{V}(y), f_1(y) \rangle = 0, \end{aligned} \tag{64}$$

where $\tilde{\alpha} = \min(1, \frac{b}{2})$ and $\tilde{\beta} = \frac{b}{2}r^2$. These estimates lead us to the following auxiliary result.

Lemma 3.1 *The set $\tilde{B} = \tilde{V}^{-1}([0, \frac{b}{\tilde{\alpha}} r^2])$ is an absorbing closed but noncompact set. In particular, this set is independent of ω. In addition $\tilde{B}$ is positively invariant with respect to the flow χ generated by (58), (63).*

Indeed, the assertion is a straightforward consequence of (64).

Lemma 3.1 essentially states that the second and third components of the system become bounded by a non-random constant as time elapses. Let us next use this fact to construct a random attractor of our system. For the Lyapunov function defined in (60) we have

$$|\langle \nabla V(y), (\frac{\partial u}{\partial x})^{-1}(z,.)\, f_1(u(z,y))\rangle| = \mu|\sigma z y_1 y_2| \le |\sigma||z||y_1| + \mu^2|\sigma||z||y_2|$$

and

$$\begin{aligned} |\langle \nabla V(y), f_0(y) - (\frac{\partial u}{\partial x})^{-1}(z,.)\, f_0(u(z,y))\rangle| &= |\sigma\, z|[|r y_1^2 + (d-1-\sigma r z)\, y_1 y_2 + d y_2^2 \\ &\quad -y_3(y_1^2 + y_2^2 + \sigma z\, y_1 y_2)|] \end{aligned}$$

Therefore we can find constants c_1, c_2 only depending on b, r, d, δ such that for y in $\tilde{C} = \{y \in \mathbf{R}^3 : d(y, \tilde{B}) \le 1\}$, where d denotes the Euclidean distance from a point to a closed set,

$$\begin{aligned} &|\langle \nabla V(y), f_0(y) - (\frac{\partial u}{\partial x})^{-1}(z,.)\, f_0(u(z,y)) - \mu z A y\rangle| \\ &\quad \le c_1[|\sigma z|(1+|\sigma z|)(1+|y_3|))V(y) + \mu|\sigma z||y_1|] \\ &\quad \le c_2[|\sigma z|(1+|\sigma z|)V(y) + \mu|\sigma z|\,|y_1|]. \end{aligned} \tag{65}$$

Now choose $k(z) = \frac{c_2}{\alpha}|\sigma z|(1+|\sigma z|), z \in \mathbf{R}$. Then by (61) and definition of V, the following inequality holds for any $y \in \tilde{C}$

$$\sup_{z \in \mathbf{R}} \frac{|\langle \nabla \ln V(y), f_0(y) - h(z,y)\rangle|}{|\langle \nabla \ln V(y), f_0(y)\rangle| k(z)} \le \frac{\alpha V(y) + \alpha\, \mu\, |y_1|}{\alpha V(y) - \beta},$$

so that an analogue of (29), in which the lim sup is taken over $\tilde{C}$, obviously follows. Hence the proof of Theorem 2.2 yields an absorbing set B which absorbs tempered subsets of the deterministic set $\tilde{B}$. By Lemma 3.1 we can assume that this set is contained in $\tilde{B}$.

Now we introduce a universe of random sets as announced above. The system of tempered sets in $\mathbf{R}^3$ such that the (y_2, y_3)-projection is uniformly bounded in ω will be suitable. Let $D(\omega)$ be such a set. Then there exists a $t(D) \geq 0$ independent of ω such that

$$\chi_t(\theta_{-t}\omega, D(\theta_{-t}\omega)) \subset \tilde{B}$$

for any $t \geq t(D)$. Since

$$\omega \mapsto \chi_{t(D)}(\theta_{-t(D)}\omega, D(\theta_{-t(D)}\omega))$$

is tempered (in particular with respect to the y_1-direction) and the (y_2, y_3)-projection is contained in $\tilde{B}$ it follows that $B(\omega)$ absorbs $\chi_t(\theta_{-t}\omega, D(\theta_{-t}\omega))$:

$$\chi_t(\theta_{-t}\omega, D(\theta_{-t}\omega)) \subset \chi_{t-t(D)}(\theta_{-t+t(D)}\omega, \chi_{t(D)}(\theta_{-t}\omega, D(\theta_{-t}\omega)) \cap \tilde{B}) \subset B(\omega)$$

for sufficiently large t. Hence Theorem 0.1 applies and yields the existence of a global random attractor for tempered sets.

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