Bifurcation of one–dimensional stochastic differential equation

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Abstract

We consider families of random dynamical systems induced by parametrized one dimensional stochastic differential equations. We give necessary and sufficient conditions on the invariant measures of the associated Markov semigroups which ensure a stochastic bifurcation. This leads to sufficient conditions on drift and diffusion coefficients for a stochastic pitchfork and transcritical bifurcation of the family of random dynamical systems.

Key words: one dimensional diffusion; random dynamical system; invariant measure; speed measure; backward equation; stochastic pitchfork bifurcation; stochastic transcritical bifurcation.

Introduction

Stochastic bifurcation theory studies qualitative changes in the asymptotic behaviour of trajectories of parametrized families of random dynamical systems. It has been the subject of several papers in the last decade. Bifurcation in a stochastic setting may be approached by different methods. The phenomenological approach studies the qualitative changes of the densities of invariant measures of the Markov semigroup generated by the random dynamical systems. Now according to Ledrappier [12], Le Jan [13] and Crauel [6], there is a one to one correspondence between invariant measures of the Markov semigroup and random invariant measures of the system which are measurable with respect to the history of the forward and backward equations. However, in general there are random invariant measures not belonging to these classes, and accordingly the pattern of sign changes of Lyapunov exponents which should announce the appearance of new bifurcating invariant measures, is richer than predicted by the phenomenological approach. Baxendale [5] gives an example in which the invariant density does not depend on the bifurcation parameter, while the top Lyapunov exponent changes sign. Crauel and Flandoli [8] exhibit an example, where the density changes from a one peak to a two peak function at a parameter value, while the random invariant measure remains stable. Therefore the phenomenological approach appears too narrow a concept. In our dynamical concept a bifurcation is understood as a qualitative change in the pattern of existing invariant measures of the system. For example, in the one dimensional situation a pitchfork bifurcation happens at a parameter value α_0 if for $\alpha < \alpha_0$ a fixed point x_0 of the motion is stable and $\mu = \delta_{x_0}$ is the only invariant measure, which becomes unstable at $\alpha = \alpha_0$ and two new stable invariant measures supported by $]-\infty, x_0[$ respectively $]x_0, \infty[$ appear. A transcritical bifurcation happens at a parameter value α_0 if for $\alpha < \alpha_0$ a fixed point x_0 of the motion is stable, and besides $\mu = \delta_{x_0}$ an unstable invariant measure ν supported by $]-\infty, x_0[$ are the only invariant measures, which change stability at $\alpha = \alpha_0$ and for $\alpha > \alpha_0$ the invariant measure ν is supported by $]x_0, \infty[$.

Mainly case studies of the bifurcation behaviour of families of one dimensional random dynamical systems have been performed by several authors. Arnold and Boxler [2] considered pitchfork and transcritical bifurcation for the systems induced by the explicitly solvable stochastic differential equations

$$dx_t = \left(\alpha x_t - x_t^3\right) dt + \sigma x_t \circ dW_t \quad (1) \qquad \text{and} \qquad dx_t = \left(\alpha x_t - x_t^2\right) dt + \sigma x_t \circ dW_t \quad (2)\,,$$

where $\sigma > 0$. They obtained a stochastic bifurcation scenario similar to the deterministic one. Arnold and Schmalfuß [3] add a nonlinear smooth perturbation to the drift term in (1) (see Example 3.8). Then they use a type of random fixed point theorem based on the negativity of Lyapunov exponents to state growth conditions on the perturbation under which the bifurcation pattern is preserved. In [14], Xu considers (1) and (2) with real noise and shows that under certain conditions this leads to bifurcation patterns differing from the deterministic ones. Crauel and Flandoli [8] prove that purely additive white noise completely destroys the deterministic (pitchfork) bifurcation scenario from our dynamical point of view.

In this paper we consider a general system of parametrized one dimensional stochastic differential equations

$$dx_t = b_\alpha(x_t)\, dt + \sigma_\alpha(x_t) \circ dW_t\,,$$

with zero as a fixed point and smoothness conditions for b_α and σ_α, $\alpha \in \mathbb{R}$. We give necessary and sufficient conditions on b_α and σ_α under which the invariant measures of the associated Markov semigroups induce bifurcations of the random dynamical system. Thereby we only consider pitchfork and transcritical bifurcations at $\alpha = 0$, remarking that, depending on the constellation of the bifurcating invariant measures, other notions of bifurcation are conceivable (see Theorem 2.4, 2.5 and Example 3.11).

The paper is organized as follows. In Section 1 we introduce invariant measures of random dynamical systems and invariant measures of the associated Markov semigroups. The well known pullback procedure (see e.g. Crauel [6]) leads to a one to one correspondence between these objects. In Section 2 we present necessary and sufficient conditions on the invariant measures of the Markov semigroups for a stochastic bifurcation at $\alpha = 0$. In Section 3 we obtain sufficient criteria on the spatial growth of drift and diffusion to ensure a pitchfork and a transcritical bifurcation at $\alpha = 0$.

Notations and preliminaries

For a topological space X, $C_b(X)$ denotes the sets of bounded continuous real valued functions. If X is an open subset of a Euclidean space and $k \in \mathbb{N}$, $\delta > 0$, $C^k(X)$ ($C^{k,\delta}(X)$) are the symbols used for the vector spaces of all k times continuously differentiable real valued functions (whose k–th derivative is locally δ–Hölder continuous). Borel sets on X will be denoted by $\mathcal{B}(X)$. One of our main objects of study will be random probability measures, i. e. probability measures on the product $(\Omega \times X, \mathcal{F} \otimes \mathcal{B})$, whose Ω–marginals equal $\mathbb{P}$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $(X, \mathcal{B})$ a measurable space. This set will be denoted by $\mathcal{M}^1_{\mathbb{P}} = \mathcal{M}^1_{\mathbb{P}}(\Omega \times X)$. It is well known that if $(X, \mathcal{B})$ is standard, then $\mu \in \mathcal{M}^1_{\mathbb{P}}$ has a $\mathbb{P}$-a.s. uniquely determined disintegration by a probability kernel $\omega \mapsto \mu_\omega$, so that $\mu(d\omega, dx) = \mu_\omega(dx)\mathbb{P}(d\omega)$ holds. A disintegration by a random Dirac measure with support given by a random variable a will sometimes be written δ_a. In particular δ_{x_0} denotes $\delta_{x_0} \times \mathbb{P}$ for $x_0 \in X$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\vartheta := (\vartheta_t)_{t\in\mathbb{R}}$, $\vartheta_t : \Omega \to \Omega$, $t \in \mathbb{R}$ a family of $\mathbb{P}$–preserving maps, such that $(\omega, t) \mapsto \vartheta_t\omega$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}), \mathcal{F}$ measurable and ϑ satisfies the *flow property*

$$\vartheta_0 = \mathrm{id}_\Omega\,, \qquad \vartheta_{t+s} = \vartheta_s \circ \vartheta_t\,, \qquad \text{for all } s, t \in \mathbb{R}. \tag{3}$$

$(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is called a *two sided–dynamical system.*

Let X be a d–dimensional C^∞–manifold and $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ a two–sided dynamical system. For $k \in \mathbb{N}$, a *local (continuous time) C^k random dynamical system* (RDS) over $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ on X is a measurable mapping

$$\varphi : D \to X\,, \quad (t, \omega, x) \mapsto \varphi(t, \omega)x\,,$$

defined on a measurable set $D \subset \mathbb{R} \times \Omega \times X$, with the following properties. For $\omega \in \Omega$

(i) the set $D(\omega) := \{(t, x) \in \mathbb{R} \times X : (t, \omega, x) \in D\} \subset \mathbb{R} \times X$ is non–void and open, and $\varphi(\omega) : D(\omega) \to X$ is k times differentiable with respect to x, and the derivatives are continuous with respect to (t, x),

(ii) for each $x \in X$ the set $D(\omega, x) := \{t \in \mathbb{R} : (t, \omega, x) \in D\} \subset \mathbb{R}$ is an open interval containing 0,

(iii) $\varphi(\omega)$ satisfies the *local cocycle property*:
$\varphi(0, \omega) = \mathrm{id}_X$ and for all $x \in X$, all $s \in D(\omega, x)$ and all $t \in D(\vartheta_s\omega, \varphi(s, \omega)x)$ we have

$$\varphi(t + s, \omega)x = \varphi(t, \vartheta_s\omega) \circ \varphi(s, \omega)x\,. \tag{4}$$

A local C^k random dynamical system is said to be *global*, if $D = \mathbb{R} \times \Omega \times X$.

Without loss of generality, we may and do assume that φ is a *perfect* (local) cocycle, i. e. (4) holds identically. This is a consequence of the perfection theorem of Arnold and Scheutzow [4], Arnold [1, 1.3.5].

1 Invariant measures of one–dimensional systems

In the following we denote by $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ the canonical two–sided dynamical system of the one–dimensional Wiener process. Here Ω is the set of real valued continuous functions on $\mathbb{R}$ pinched to 0 at 0, $\mathcal{F}$ the σ–algebra of Borel sets with respect to the compact open topology and $\mathbb{P}$ the measure for which the coordinate process $W = (W_t)_{t\in\mathbb{R}}$ yields two independent one–dimensional Brownian motions hooked up back to back at 0. For $t \in \mathbb{R}$, $\vartheta_t : \Omega \to \Omega$, $\omega \mapsto \big(s \mapsto \omega(t + s) - \omega(t)\big)$, yields the group of $\mathbb{P}$–preserving shifts satisfying the flow property (3) and which are $\mathbb{P}$–ergodic for $t \neq 0$.
We consider the following one–dimensional Stratonovich stochastic differential equation (SDE)

$$\begin{aligned} dx_t &= b(x_t)\, dt + \sigma(x_t) \circ dW_t && (5)\\ &= \Big(b(x_t) + \tfrac{1}{2}\sigma\sigma'(x_t)\Big)\, dt + \sigma(x_t)\, dW_t\,, && (6) \end{aligned}$$

with $b \in C^{1,\delta}(\mathbb{R})$, $\sigma \in C^{2,\delta}(\mathbb{R})$ for some $\delta > 0$. (6) is the equivalent Itô representation of (5). According to Arnold [1, 2.3.36] there exists a unique (up to indistinguishability) local C^1 random dynamical system (RDS) φ over ϑ on $\mathbb{R}$, such that $(\varphi(t,\cdot)x)_{t\in\mathbb{R}}$ is the unique maximal strong solution of (5) with initial value $x \in \mathbb{R}$. It is represented by

$$\varphi(t,\cdot)x = x + \int_0^t b\big(\varphi(s,\cdot)x\big)\, ds + \int_0^t \sigma\big(\varphi(s,\cdot)x\big) \circ dW_s \tag{7}$$

for all $t \in]\tau^-(\cdot,x), \tau^+(\cdot,x)[= D(\cdot,x)$, where $\tau^+(\omega,x)$ and $\tau^-(\omega,x)$, $\omega \in \Omega$, are the forward and backward explosion times of the orbit $\varphi(\cdot,\omega)x$ starting at time $t = 0$ in position x. An RDS φ is called *forward complete* (*backward complete*) on I, if $\mathbb{P}\{\tau^+(\cdot,x) = \infty\} = 1$ ($\mathbb{P}\{\tau^-(\cdot,x) = \infty\} = 1$) for all $x \in I$. In the one–dimensional case this is equivalent to $\mathbb{P}\{\tau^+(\cdot,x) = \infty \text{ for all } x \in I\} = 1$. Note that φ is global iff it is forward and backward complete.
We assume that

$$b(0) = 0 = \sigma(0)\,, \tag{8}$$

so that 0 is a fixed point of the diffeomorphisms $\varphi(t,\cdot)$ for all $t \in D(\cdot,\cdot)$.
The backward (local) cocycle over $\theta := \vartheta^{-1}$ corresponding to (7) is given by

$$\psi(t,\omega)x := \varphi(-t,\omega)x$$

for all $(t,\omega,x) \in D$. It is generated by the stochastic differential equation

$$dy_t = -b(y_t)\, dt + \sigma(y_t) \circ dW_{-t}\,, \tag{9}$$

as an elementary calculation shows.
We now assume the following *ellipticity condition*

(E) $\sigma(x) \neq 0$ for all $x \neq 0$.

As a consequence of our fixed point assumption (8) we can decompose $\mathbb{R}$ into invariant sets $I^+ := \mathbb{R}^+ \setminus \{0\}$, $I^- := \mathbb{R}^- \setminus \{0\}$ and $\{0\}$. Because of (E) φ and ψ are regular diffusions on each of the three sets, i.e. they live on the sets up to their explosion times $\tau^\pm$ and every point can be reached with positive probability from any other point in finite time. The separate dynamics on $I = I^+$, I^- are generated by the stochastic differential equation $dx_t = \pm b\big|_I(x_t)\, dt + \sigma\big|_I(x_t) \circ dW_t$, with the restrictions $b\big|_I$, $\sigma\big|_I$ of b, σ on the respective intervals. Thus φ and ψ induced by (5) respectively (9) restricted to $I^\pm$ coincide with the random dynamical systems induced by the above equation. This and the order preserving property of φ and ψ imply that the explosion of φ and ψ only happen to $\pm\infty$. Since many arguments are symmetrical with respect to I^+ and I^-, we often omit the superscripts $+$ and $-$ and simply write I. The restrictions of φ and ψ to I are again denoted by φ and ψ.

If φ is a global RDS, a measure $\mu \in \mathcal{M}^1_{\mathbb{P}}$ is called *φ–invariant*, if

$$\Theta_t \mu = \mu \quad \text{for all } t \in \mathbb{R}\,,$$

where $\Theta = (\Theta_t)_{t\in\mathbb{R}}$ is the induced skew-product flow defined on $\Omega \times \mathbb{R}$ by

$$\Theta_t(\omega,x) = (\vartheta_t\omega, \varphi(t,\omega)x), \quad t \in \mathbb{R}\,.$$

If φ is a local RDS, then let $E \subset \Omega \times \mathbb{R}$ be such that $E(\omega) = \{x \in \mathbb{R} : D(\omega,x) = \mathbb{R}\}$ is the *set of never exploding initial values*, which is known to be Θ–invariant. Restricting φ to E gives a global bundle RDS (see Arnold [1, 1.9.1]) for which we define the *φ–invariant measures* as above. For every φ–invariant μ we have $\mu(E) = 1$. In this sense φ–invariant measures are supported by E. We remark that the existence of a φ–invariant measure implies $E \neq \varnothing$ $\mathbb{P}$-a.s. In this case we tacitly assume that φ is restricted to E.
In terms of the disintegration of μ, φ–invariance may be expressed by the well known relation

$$\varphi(t,\omega)\mu_\omega = \mu_{\vartheta_t\omega} \quad \mathbb{P}\text{-a.s., } t \in \mathbb{R}\,. \tag{10}$$

The structure of φ–invariant measures in dimension one is very simple, as the following lemma of Arnold [1, 1.8.4.*(iv)*] shows.

Lemma 1.1 *Let φ be a local C^0 RDS with two–sided (continuous) time on $\mathbb{R}$ and μ an ergodic φ–invariant measure. Then there exists a real valued random variable $a : \Omega \to \mathbb{R}$ such that $\mu = \delta_a$.*

Remark 1.2 *(i)* For an ergodic φ–invariant measure $\mu = \delta_a$ equation (10) is equivalent to

$$\varphi(t, \cdot)a = a \circ \vartheta_t \quad \mathbb{P}\text{-a.s.}, \; t \in \mathbb{R}.$$

(ii) Ergodic φ– and ψ–invariant measures coincide, since for $t \in \mathbb{R}$ we have $\psi(t, \cdot)a = \varphi(-t, \cdot)a = a \circ \vartheta_{-t} = a \circ \vartheta_t^{-1} = a \circ \theta_t \quad \mathbb{P}$-a.s.

(iii) Because of the fixed point assumption at zero the Dirac measure δ_0 is φ–invariant. □

There is a close connection between φ–invariant measures and invariant measures of the Markov semigroups, denoted by $(P_t) = (P_t)_{t \geq 0}$ and $(\bar{P}_t) = (\bar{P}_t)_{t \geq 0}$ associated with the forward cocycle φ respectively the backward cocycle ψ. (P_t), $(\bar{P}_t)$ are defined on the sets $C_0(I)$ of continuous real valued functions vanishing at infinity. Their infinitesimal generators L, $\bar{L}$ are given for $f \in C^2(I)$ with compact support by

$$Lf = \tfrac{1}{2}\sigma^2 f'' + \left(b + \tfrac{1}{2}\sigma\sigma'\right) f' \qquad \text{and} \qquad \bar{L}f = \tfrac{1}{2}\sigma^2 f'' + \left(-b + \tfrac{1}{2}\sigma\sigma'\right) f'.$$

A σ–finite measure ν on $(I, \mathcal{B}(I))$ is called *invariant* with respect to (P_t), if ν satisfies $L^*\nu = 0$, where L^* denotes the formal adjoint operator of L, i.e. $\int f \, dL^*\nu = \int Lf \, d\nu$. A similar statement holds for $(\bar{P}_t)$ and $\bar{L}$.
Let $c \in I$ and $m(dx) = \rho(x)\, dx$ on $(I, \mathcal{B}(I))$ with

$$\rho(x) = \frac{2}{|\sigma(x)|} \exp\Big(2 \int_c^x \frac{b(y)}{\sigma^2(y)} \, dy\Big) . \tag{11}$$

Here we use the convention $\int_c^x \cdot = -\int_x^c \cdot$ for $x < c$, valid for Lebesgue integrals. The σ–finite measure m on $(I, \mathcal{B}(I))$ is called *speed measure* of φ. The speed measure of ψ is given by $\overline{m}(dx) = \overline{\rho}(x)dx$ with

$$\overline{\rho}(x) = \frac{2}{|\sigma(x)|} \exp\Big(2 \int_c^x \frac{-b(y)}{\sigma^2(y)} \, dy\Big) . \tag{12}$$

The speed measure depends on the real number $c \in I$. But the finiteness of m does not depend on c (see Karatzas and Shreve [11, p. 329]).

The following is well known.

Lemma 1.3 *The speed measure m of φ on $(I, \mathcal{B}(I))$ is invariant with respect to (P_t) i. e. m satisfies*

$$L^* m(f) = 0$$

for all $f \in C^2(I)$ with compact support. A similar statement holds for $\overline{m}$ with respect to $(\bar{P}_t)$ and $\bar{L}$.

A result of Ledrappier [12] (see also Le Jan [13] and Crauel [6]) gives a bijection between the invariant probability measures of the semigroup and the φ–invariant measures. Let

$$\mathcal{F}^+ = \sigma\big(\varphi(t, \cdot)x \,:\, t \geq 0,\, x \in \mathbb{R}\big) \qquad \text{and} \qquad \mathcal{F}^- = \sigma\big(\psi(t, \cdot)x \,:\, t \geq 0,\, x \in \mathbb{R}\big) .$$

If φ is forward complete, then the maps

$$\nu \mapsto \lim_{t \to \infty} \varphi(t, \vartheta_{-t} \cdot)\nu = \mu_\cdot \qquad \text{and} \qquad \mu \mapsto \mathbb{E}(\mu_\cdot) = \nu \tag{13}$$

are bijections between the invariant probability measures of the semigroup (P_t) and the $\mathcal{F}^-$–measurable φ–invariant measures. A similar statement holds, if φ is backward complete, for the invariant probability measures of $(\bar{P}_t)$ and the $\mathcal{F}^+$–measurable φ–invariant measures.
Note, forward completeness of φ on I in (13) is only needed in the left equality, the *pullback* relation.
Since we work under (E) there is at most one invariant measure ν of the semigroup on $(I, \mathcal{B}(I))$ (see e.g. Horsthemke and Lefever [10, p. 112]). Thus if ν is normalizable, the corresponding φ–invariant measure is ergodic, hence by Lemma 1.1 it is a one point measure.

We next show that ergodic $\mathcal{F}^-$– and $\mathcal{F}^+$–measurable φ–invariant measures cannot coexist. We need the following technical result.

Lemma 1.4 *Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$, $a < b$ and $f, g \in C(]a,b[)$ with $f, g > 0$. If $\int_a^b g(x)\,dx = \infty$ and $\int_a^b f(x)g(x)\,dx < \infty$, then $\int_a^b f(x)^{-1} g(x)\,dx = \infty$.*

Proof: Let $\mu(dx) = g(x)dx$ on $]a,b[$ and $A_\alpha = \{x \in]a,b[: f(x) > \alpha\}$ for $\alpha > 0$. Then for the μ–measure of A_α we have the estimate $\mu(A_\alpha) \leq \frac{1}{\alpha}\int_a^b f(x)g(x)\,dx < \infty$. Now $\int_a^b f(x)^{-1}g(x)\,dx = \int_a^b \mathbb{1}_{A_\alpha}(x) f(x)^{-1} g(x)\,dx + \int_a^b \mathbb{1}_{A_\alpha^c}(x) f(x)^{-1} g(x)\,dx \geq \frac{1}{\alpha}\int_a^b \mathbb{1}_{A_\alpha^c}(x)\,\mu(dx) = \infty$, since $\mu(A_\alpha^c) = \infty$. ∎

Lemma 1.5 *Let φ be the RDS induced by equation (5) and assume* (E). *If there exists an $\mathcal{F}^+$– or $\mathcal{F}^-$–measurable φ–invariant measure μ on $(I, \mathcal{B}(I))$, then it is unique.*

Proof: First note that our hypotheses of smoothness of σ and $\sigma(0) = 0$ imply

$$\int_I \frac{1}{|\sigma(x)|}\,dx \geq \Big|\int_0^c \frac{1}{|\sigma(x)|}\,dx\Big| = \infty \quad \text{for all } c \in I\,. \tag{14}$$

Suppose μ is φ–invariant and $\mu_\bullet$ is $\mathcal{F}^-$–measurable. By (E) and (13) there exists no other $\mathcal{F}^-$–measurable φ–invariant measure on $(I, \mathcal{B}(I))$.
Suppose there exists an $\mathcal{F}^+$–measurable φ–invariant measure on $(I, \mathcal{B}(I))$. The corresponding finite invariant measure of the semigroup $(\bar{P}_t)$ is the speed measure $\overline{m}$ of the backward equation. Hence we have $\int_I \frac{1}{|\sigma(x)|}\exp\big(2\int_c^x \frac{-b(y)}{\sigma^2(y)}\,dy\big)\,dx < \infty$, $c \in I$. By assumption and Lemma 1.4 with $g = \frac{1}{|\sigma|}$ and $f(\cdot) = \exp(\int_c^\cdot \frac{2b(y)}{\sigma^2(y)}\,dy)$ we have $\int_I \frac{1}{|\sigma(x)|}\exp\big(2\int_c^x \frac{b(y)}{\sigma^2(y)}\,dy\big)\,dx = \infty$ for $c \in I$. Thus the speed measure m is not normalizable. Since we assume (E) there is no finite invariant measure with respect to (P_t). This contradicts the assumption that the φ–invariant measure $\mu_\bullet$ is $\mathcal{F}^-$–measurable.
A similar argument is possible for $\mathcal{F}^+$–measurable φ–invariant measures. ∎

Lemma 1.6 *Let φ be the RDS induced by equation (5) and assume* (E).

(i) If the speed measure m is finite on $(I, \mathcal{B}(I))$, then either φ is forward complete, or φ explodes in finite positive time $\mathbb{P}$-a.s. *for every $x \in I$, i. e. $\mathbb{P}\{\tau^+(\cdot, x) < \infty\} = 1$ for all $x \in I$.*

In the second case the set of never exploding initial values E is $\mathbb{P}$-a.s. *empty, i. e.* $E(\cdot) = \varnothing$ $\mathbb{P}$-a.s., *hence there exists no φ–invariant measure on $(I, \mathcal{B}(I))$.*

Analogous statements hold, if we replace m by $\overline{m}$, τ^+ by τ^- and forward complete by backward complete.

(ii) Suppose the speed measure m is finite on $(I, \mathcal{B}(I))$. φ is forward complete on I if and only if $\overline{m}([I \setminus]-c, c[) = \infty$ for one (hence for all) $c \in I$.

A similar statement holds for backward completeness of φ on I.

(iii) The existence of an $\mathcal{F}^-$– or $\mathcal{F}^+$–measurable φ–invariant measure on $(I, \mathcal{B}(I))$ implies forward respectively backward completeness of φ on I.

Proof: Fix $c \in I$. Let l denote Lebesgue measure and let $h(\cdot) = 2\int_c^\cdot \frac{b}{\sigma^2}\,dl$. We only consider the case $I = I^+$. The case $I = I^-$ is similar.
(i) We will use Feller's test for explosion, which states in the present case $\mathbb{P}\{\tau^+(\cdot, x) = \infty\} = 1$ for every $x \in I^+$ if and only if $K(\cdot) := \int_c^\cdot \frac{1}{|\sigma(x)|} e^{-h(x)} \big(\int_c^x \frac{1}{|\sigma|} e^h\,dl\big)\,dx$ satisfies $|K(\infty)| = \infty = |K(0)|$.
(14) yields $\int_0^c \frac{1}{|\sigma|}\,dl = \infty$. By Lemma 1.4 it follows from the assumption $m(I^+) = \int_0^\infty \frac{1}{|\sigma|} e^h\,dl < \infty$ that the scale function $s(\cdot) := \int_c^\cdot \frac{1}{|\sigma|} e^{-h}\,dl$ satisfies $|s(0)| = \infty$. This implies $|K(0)| = \infty$.
Now if $|s(\infty)| = \infty$, then $|K(\infty)| = \infty$. So Feller's test for explosion gives forward completeness for φ.
If on the other hand $|s(\infty)| < \infty$, then K is bounded on I^+, hence $|K(\infty)| < \infty$. Since $|s(0)| = \infty$ and $|K(\infty)| < \infty$, it follows from Hackenbroch and Thalmaier [9, p. 343, 6.50 (iii) (3)] that $\mathbb{P}\{\tau^+(\cdot, x) < \infty\} = 1$ for all $x \in I^+$.

(ii) Since $\overline{m}([c,\infty[) = \int_c^\infty \frac{1}{|\sigma|} e^{-h}\, dl = s(\infty)$, the proof of *(i)* implies the assertion.

(iii) Let μ be an $\mathcal{F}^-$–measurable φ–invariant measure on $(I, \mathcal{B}(I))$. By (13) the speed measure m, with $m = m(I)\mathbb{E}\mu_\cdot$, is finite on $(I, \mathcal{B}(I))$. By *(i)*, φ is either forward complete or explodes $\mathbb{P}$-a.s. in finite time. But the last is a contradiction to the existence of φ–invariant measures, since $E = \varnothing$ by *(i)*. Similar for the other case. ∎

We next consider the Lyapunov exponents of the C^1 RDS φ induced by (5) with respect to φ–invariant measures, which give in the present case information about the exponential growth rate of the flow near random fixed points.

Let $\Phi(t,\cdot,x) = \frac{\partial}{\partial x}\varphi(t,\cdot)x$ for $(t,x) \in \mathbb{R} \times E(\cdot)$, the *linearization* of $\varphi(t,\cdot)$ at $x \in E(\cdot)$ for $t \in \mathbb{R}$. In case φ is represented by (7), Φ is induced by the following linear equation

$$d\Phi(t,\cdot,x) = b'\big(\varphi(t,\cdot)x\big)\Phi(t,\cdot,x)\, dt + \sigma'\big(\varphi(t,\cdot)x\big)\Phi(t,\cdot,x) \circ dW_t$$

with solution

$$\Phi(t,\cdot,x)v = v \exp\Big(\int_0^t b'\big(\varphi(s,\cdot)x\big)\, ds + \int_0^t \sigma'\big(\varphi(s,\cdot)x\big) \circ dW_s\Big)$$

for $t \in \mathbb{R}$ and $(x,v) \in E(\cdot) \times \mathbb{R}$.

For an RDS φ and $\mu \in \mathcal{M}^1_{\mathbb{P}}$ let

$$\beta^{\pm}(\varphi,\mu) := \int_{\Omega\times I} \sup_{0\le t\le 1} \log^+ |\Phi(t,\omega,x)^{\pm 1}|\, d\mu(\omega,x)\,.$$

Let μ be a φ–invariant measure such that $\beta^{\pm}(\varphi,\mu) < \infty$. The *Lyapunov exponent* of φ with respect to μ is defined by

$$\lambda_\varphi(\mu) := \lim_{t\to\infty} \frac{1}{t} \log|\Phi(t,\omega,x)|\,, \tag{15}$$

which exists for all $(\omega,x) \in \tilde{E} \subset E$, by the Multiplicative Ergodic Theorem of Oseledets (MET) (see Arnold [1, Theorem 4.2.6 (B)], where $\tilde{E}$ is Θ–invariant with $\mu(\tilde{E}) = 1$. $\lambda_\varphi(\mu)$ is independent of x and ω if μ is ergodic. Below we restrict ergodic measures μ to a sub–σ–algebra of $\mathcal{F}\otimes\mathcal{B}$ with respect to which (15) is measurable. Note that this does not affect the validity of (15). The Lyapunov exponents of φ and ψ with respect to μ are related by

$$\lambda_\psi(\mu) = \lim_{t\to\infty}\frac{1}{t}\log|\Phi(-t,\omega,x)| = \lim_{t\to\infty}\frac{1}{t}\log|\big(\Phi(t,\omega,x)\big)^{-1}| = -\lim_{t\to\infty}\frac{1}{t}\log|\Phi(t,\omega,x)| = -\lambda_\varphi(\mu)\,, \tag{16}$$

for $(\omega,x) \in \tilde{E}$. The second equality sign is justified by the MET.

A φ–invariant measure μ is called *stable* respectively *unstable*, if $\lambda_\varphi(\mu) < 0$ respectively $\lambda_\varphi(\mu) > 0$.

Lemma 1.7 *Let φ be the RDS induced by (5) and suppose* (E) *is fulfilled.*

(i) The Lyapunov exponent of the φ–invariant measure δ_0 satisfies $\lambda_\varphi(\delta_0) = b'(0)$.

(ii) An $\mathcal{F}^-$–measurable φ–invariant measure μ on $(I,\mathcal{B}(I))$ is stable. An $\mathcal{F}^+$–measurable φ–invariant measure ν on $(I,\mathcal{B}(I))$ is unstable. We have

$$\lambda_\varphi(\mu) = -2\int_I \Big(\frac{b(x)}{\sigma(x)}\Big)^2 \rho(x)\, dx < 0 \qquad \textit{and} \qquad \lambda_\varphi(\nu) = 2\int_I \Big(\frac{b(x)}{\sigma(x)}\Big)^2 \overline{\rho}(x)\, dx > 0\,.$$

Proof: *(i)* We have

$$\lambda_\varphi(\delta_0) = \lim_{t\to\infty}\frac{1}{t}\Big(\int_0^t b'\big(\varphi(s,\cdot)0\big)\, ds + \int_0^t \sigma'\big(\varphi(s,\cdot)0\big) \circ dW_s\Big) = b'(0) + \sigma'(0)\lim_{t\to\infty}\frac{W_t}{t} = b'(0)\,.$$

For the first part of *(ii)* see Arnold [1, Theorem 9.2.4]. The second part follows similary. ∎

For the following structure theorem of ergodic invarinat measures we have to restrict to φ–invariant measures satisfying an integrability condition. Denote by $B_\delta(x)$ the open interval around x with radius δ. For an RDS φ, $\mu \in \mathcal{M}^1_{\mathbb{P}}$, $t \in \mathbb{R}$ and $\delta > 0$ define

$$\gamma^\delta_t(\varphi,\mu) := \int_{\Omega\times I} \sup_{z\in B_\delta(x)} \log^+ |\Phi(-t,\omega,z)|\, d\mu(\omega,x)\,,$$

where Φ denotes the linearization of φ. The one–sided time restrictions of μ are defined by

$$\mu^+ := \mathbb{E}(\mu_\bullet|\mathcal{F}^+) \qquad \text{and} \qquad \mu^- := \mathbb{E}(\mu_\bullet|\mathcal{F}^-)\,.$$

For an RDS φ let

$$\mathcal{I}(\varphi) = \{\, \mu \in \mathcal{M}^1_{\mathbb{P}} \;\; : \;\; \text{for all } t > 0 \text{ there exists } \delta > 0 \text{ such that } \gamma^\delta_{\pm t}(\varphi,\mu^\pm) < \infty \text{ and } \beta^\pm(\varphi,\mu) < \infty \,\}\,.$$

The following Theorem is crucial for our treatment of bifurcations.

Theorem 1.8 *Let φ be a C^1 RDS with two–sided (continuous) time on $\mathbb{R}$. Then every ergodic φ–invariant measure in $\mathcal{I}(\varphi)$ is either $\mathcal{F}^+$– or $\mathcal{F}^-$–measurable. If an ergodic φ–invariant measure in $\mathcal{I}(\varphi)$ is both $\mathcal{F}^+$– and $\mathcal{F}^-$–measurable, it is a deterministic Dirac measure.*

Proof: This proof is based on results by Crauel [7, p. 16 ff]. Let μ be an ergodic φ–invariant measure. Let $h^{\mu^+}_t(\omega,x)$ the Radon-Nikodym density of the absolutely continuous part of $\varphi(t,\omega)^{-1}\mu^+_{\vartheta_t\omega}$ with respect to μ^+_ω for $(\omega,x)\in\Omega\times\mathbb{R}$ μ-a.s., $t\geq 0$. Let

$$\alpha_{\mu^+}(t) = -\int_E \log h^{\mu^+}_t \, d\mu^+\,,$$

be the relative entropy associated with μ^+, $t \geq 0$. Then we have $\alpha_{\mu^+}(t) = t\alpha_{\mu^+}(1)$. Put $\alpha_{\mu^+} = \alpha_{\mu^+}(1)$. We now prove that $\mu = \mu^+$ if and only if $\alpha_{\mu^+}(t) = 0$ for all $t \geq 0$.

Let first $\mu = \mu^+$. Then by φ–invariance we easily see that $h^{\mu^+}_t = 1$ μ^+-a.s. for $t \geq 0$.
To prove the converse, assume $\alpha_{\mu^+}(t) = 0$ for all $t \geq 0$. Then the inequality $\log(x) \leq x - 1$, valid for $x > 0$, with equality iff $x = 1$, shows that we must have $h^{\mu^+}_t = 1$ μ^+-a.s., hence $\varphi(t,\cdot)^{-1}\mu^+_{\vartheta_t\bullet} = \mu^+_\bullet$ $\mathbb{P}$-a.s., $t \geq 0$. By shifting on Ω with ϑ_{-t} and using the cocycle property of φ this is immediatly seen to yield $\mu^+_\bullet = \varphi(t,\vartheta_{-t}\bullet)\mu^+_{\vartheta_{-t}\bullet}$ $\mathbb{P}$-a.s. Since $\varphi(t,\vartheta_{-t}\bullet)\mu^+_{\vartheta_{-t}\bullet} \longrightarrow \mu_\bullet$, $t \to \infty$ $\mathbb{P}$-a.s. (see Crauel [7, p. 16]), this finally yields $\mu = \mu^+$.
Similarly, using ψ instead of φ, we obtain $\mu = \mu^-$ if and only if $\alpha_{\mu^-}(t) = 0$ for all $t \leq 0$, where $\alpha_{\mu^-}(t)$, $t \leq 0$, is analogous to $\alpha_{\mu^+}(t)$, $t \geq 0$.
Since $\beta^\pm(\varphi,\mu) < \infty$, the Lyapunov exponents of φ and ψ with respect to μ exists. If in addition $\mu \in \mathcal{I}(\varphi)$ we are allowed to use Theorem 5.1 of Crauel [7, p. 22] to get

$$\alpha_{\mu^+} \leq -\min\{0, \lambda_\varphi(\mu^+)\} = -\min\{0,\lambda_\psi(\mu)\} \qquad \text{and} \qquad \alpha_{\mu^-} \leq \min\{0,\lambda_\varphi(\mu^-)\} = \min\{0, -\lambda_\varphi(\mu)\}\,.$$

Now if $\lambda_\varphi(\mu) \geq 0$ then $\alpha_{\mu^+} = 0$. So the first part of the proof implies $\mu^+ = \mu$, hence $\mu_\bullet$ is $\mathcal{F}^+$–measurable. If on the other hand $\lambda_\varphi(\mu) \leq 0$, then $\alpha_{\mu^-} = 0$. So $\mu^- = \mu$, hence $\mu_\bullet$ is $\mathcal{F}^-$– measurable. If $\lambda_\varphi(\mu) = 0$, then $\alpha_{\mu^+} = \alpha_{\mu^-} = 0$, thus $\mu^- = \mu^+ = \mu$. Since $\mathcal{F}^+$ and $\mathcal{F}^-$ are independent, μ is deterministic, i. e. $\mu = \delta_{x_0}$ for some $x_0 \in E(\cdot)$, which has to be a fixed point of the cocycle φ. ∎

Remark 1.9 The first statement of Theorem 1.8 remains true for a general RDS on a one dimensional Riemannian C^r manifold, $r \geq 3$, and random invariant measures satisfying the integrability condition of the multiplicative ergodic theorem and the integrability condition of Theorem 5.1 in Crauel [7]. □

The following *integrability condition* on (b, σ), is needed for determining the set of φ–(ψ–)invariant measures.

(IC) If $m(I) < \infty$, $\overline{m}(I \setminus]-c, c[) = \infty$ for one $c \in I$ or if $\overline{m}(I) < \infty$, $m(I \setminus]-c, c[) = \infty$ for one $c \in I$, then the φ–invariant measure μ given by (13) is in $\mathcal{I}(\varphi)$.

Corollary 1.10 *Let φ be the RDS induced by (5) and suppose* (E) *and* (IC) *are fulfilled. Then every ergodic φ–invariant measure μ on $(I, \mathcal{B}(I))$ satisfies $\mu \in \mathcal{I}(\varphi)$ if and only if μ is $\mathcal{F}^+$– or $\mathcal{F}^-$–measurable.*

Proof: The first implication follows from Theorem 1.8. Now let μ be an ergodic $\mathcal{F}^-$–measurable φ–invariant measure on $(I, \mathcal{B}(I))$. Then φ is forward complete on I by Lemma 1.6*(iii)* and Lemma 1.6*(ii)* implies $\overline{m}(I \setminus]-c, c[) = \infty$, $c \in I$. We have $m(I) < \infty$ by pullback relation (13). Condition (IC) implies $\mu \in \mathcal{I}(\varphi)$. A similar argument holds for an $\mathcal{F}^+$–measurable φ–invariant measure. ∎

We can now summarize the results of our considerations in this section. Recall that due to Lemma 1.4 $m(I)$, $\overline{m}(I) < \infty$ is not simultaneously possible.

Theorem 1.11 *Let φ be the RDS induced by (5) and suppose* (E) *and* (IC) *are fulfilled, $I \in \{\mathbb{R}^+ \setminus \{0\}, \mathbb{R}^- \setminus \{0\}\}$. Then we have:*

(i) If $m(I) = \infty = \overline{m}(I)$, then there is no ergodic φ–invariant measure in $\mathcal{I}(\varphi)$ on $(I, \mathcal{B}(I))$.

(ii) If $m(I) < \infty$, $\overline{m}(I \setminus]-c, c[) = \infty$ for some (hence for all) $c \in I$, then there exists a stable $\mathcal{F}^-$–measurable φ–invariant ergodic measure μ on $(I, \mathcal{B}(I))$, which is unique in $\mathcal{I}(\varphi)$ and given by $\mu = \delta_{a^-}$ with an $\mathcal{F}^-$–measurable random variable a^-.

(iii) If $\overline{m}(I) < \infty$, $m(I \setminus]-c, c[) = \infty$ for some (hence for all) $c \in I$, then there exists an unstable $\mathcal{F}^+$–measurable φ–invariant ergodic measure μ on $(I, \mathcal{B}(I))$, which is unique in $\mathcal{I}(\varphi)$ and given by $\mu = \delta_{a^+}$ with an $\mathcal{F}^+$–measurable random variable a^+.

Proof: *(i)* Lemma 1.3 and (13) imply that there exists no $\mathcal{F}^+$– or $\mathcal{F}^-$–measurable φ–invariant measure on $(I, \mathcal{B}(I))$. Hence by Corollary 1.10 $\mathcal{I}(\varphi)$ contains no ergodic φ–invariant measure.
(ii) Because of Lemma 1.3 and condition (E) the normalized speed measure $m(I)^{-1}m$ is the unique invariant probability measure of the semigroup (P_t). Lemma 1.6*(ii)* and equation (13) yield the existence of an ergodic $\mathcal{F}^-$–measurable φ–invariant measure μ on $(I, \mathcal{B}(I))$. Lemma 1.2 finally gives the existence of a random variable a^- with values in I such that $\mu = \delta_{a^-}$. Hence a^- is $\mathcal{F}^-$–measurable. Condition (IC) and Corollary 1.10 imply $\mu \in \mathcal{I}(\varphi)$. Uniqueness in $\mathcal{I}(\varphi)$ now follows from Lemma 1.5. Due to Lemma 1.7, μ is stable.
Similarly we obtain *(iii)*. ∎

2 Bifurcation

We now study a parametrized family of SDE. For $\alpha \in \mathbb{R}$ consider

$$dx_t = b_\alpha(x_t)\, dt + \sigma_\alpha(x_t) \circ dW_t \,, \tag{17}$$

where $b_\alpha \in C^{1,\delta}(\mathbb{R})$, $\sigma_\alpha \in C^{2,\delta}(\mathbb{R})$ for some $\delta > 0$ and $b_\alpha(0) = 0 = \sigma_\alpha(0)$ for all $\alpha \in \mathbb{R}$. Throughout this section we assume that σ_α satisfies the ellipticity condition (E) and $(b_\alpha, \sigma_\alpha)$ satisfies the integrability condition (IC) for all $\alpha \in \mathbb{R}$.

Definition 2.1 *A family of RDS $(\varphi_\alpha)_{\alpha \in \mathbb{R}}$ on $\mathbb{R}$ undergoes a* stochastic pitchfork bifurcation *at $\alpha = 0$, if*

(i) for $\alpha \leq 0$, δ_0 is the only invariant measure of φ_α, which is stable for $\alpha < 0$ and the Lyapunov exponent of φ_0 with respect to δ_0 vanishes,

(ii) *for $\alpha > 0$ the system possesses besides δ_0, which is unstable, exactly two more ergodic invariant measures μ_α^1, μ_α^2 in $\mathcal{I}(\varphi_\alpha)$, described by $\mu_\alpha^i = \delta_{a_\alpha^i}$, $i = 1, 2$, with random variables $a_\alpha^1 > 0, a_\alpha^2 < 0$ $\mathbb{P}$-a.s. and μ_α^i, $i = 1, 2$, are stable.*

(iii) *We have $a_\alpha^i \longrightarrow 0$ in probability as $\alpha \downarrow 0$, $i = 1, 2$.*

Definition 2.2 *A family of RDS $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ on $\mathbb{R}$ undergoes a* stochastic transcritical bifurcation *at $\alpha = 0$, if*

(i) *for $\alpha < 0$, φ_α has exactly two ergodic invariant measures in $\mathcal{I}(\varphi_\alpha)$: δ_0, which is stable, and $\mu_\alpha = \delta_{a_\alpha}$ with a random variable $a_\alpha < 0$ $\mathbb{P}$-a.s., which is unstable, and $a_\alpha \longrightarrow 0$ in probability as $\alpha \uparrow 0$,*

(ii) *for $\alpha = 0$, δ_0 is the only invariant measure and the Lyapunov exponent of φ_0 with respect to δ_0 vanishes,*

(iii) *for $\alpha > 0$, φ_α has exactly two ergodic invariant measures in $\mathcal{I}(\varphi_\alpha)$: δ_0, which is unstable, and $\mu_\alpha = \delta_{a_\alpha}$ with $a_\alpha > 0$ $\mathbb{P}$-a.s., which is stable, and $a_\alpha \longrightarrow 0$ in probability as $\alpha \downarrow 0$.*

First we state a consequence of Theorem 1.8 for stochastic bifurcation.

Corollary 2.3 *Let $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ be a family of two–sided continuous time RDS on $\mathbb{R}$ and $(\mu_\alpha)_{\alpha\in\mathbb{R}}$ be a family of ergodic φ_α–invariant measures, such that $\mu_\alpha \in \mathcal{I}(\varphi_\alpha)$ for all $\alpha \in \mathbb{R}$. If $\alpha_0 \in \mathbb{R}$ is a zero of $\alpha \mapsto \lambda_\varphi(\mu_\alpha)$, then μ_{α_0} is a deterministic Dirac measure.*

Proof: If the Lyapunov exponent of φ_{α_0} with respect to μ_{α_0} vanishes, the proof of Theorem 1.8 gives $\mu_{\alpha_0} = (\mu_{\alpha_0})^+ = (\mu_{\alpha_0})^-$. Hence μ_{α_0} is $\mathcal{F}^+$– and $\mathcal{F}^-$–measurable. Now Theorem 1.8 gives the assertion. ■

Theorem 1.11 yields the following characterization of pitchfork and transcritical bifurcations.

Theorem 2.4 *Let $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ be the family of RDS induced by (17) and suppose* (E) *and* (IC) *are fulfilled for all $\alpha \in \mathbb{R}$. $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ undergoes a stochastic pitchfork bifurcation at $\alpha = 0$, if and only if*

(i) $\mathrm{sign}(b'_\alpha(0)) = \mathrm{sign}(\alpha)$ *for all $\alpha \in \mathbb{R}$,*

(ii) $m_\alpha(I) = \infty = \overline{m}_\alpha(I)$ *for $I = I^+, I^-$ for $\alpha \le 0$,*

(iii) $m_\alpha(I) < \infty$, $\overline{m}_\alpha(I \setminus]-c, c[) = \infty$, $c \in I$ *for $I = I^+, I^-$ for $\alpha > 0$,*

(iv) $\nu_\alpha := \frac{m_\alpha}{m_\alpha(I)} \longrightarrow \delta_0$ *weakly as $\alpha \downarrow 0$, for $I = I^+, I^-$,*

where m_α, $\overline{m}_\alpha$ are the speed measures of φ_α respectively ψ_α for $\alpha \in \mathbb{R}$ and the constant $c \in I$ in (iii) is independent of α.

Proof: Since $b_\alpha(0) = 0 = \sigma_\alpha(0)$ for all $\alpha \in \mathbb{R}$, 0 is a fixed point of $\varphi_\alpha(t, \omega)$ for all $(\alpha, t, \omega) \in \mathbb{R}\times\mathbb{R}\times\Omega$, hence the measure δ_0 is φ_α–invariant for all $\alpha \in \mathbb{R}$. The Lyapunov exponent $\lambda_\varphi(\delta_0)$ of δ_0 shows the required behaviour for all $\alpha \in \mathbb{R}$ iff *(i)* is true, due to Lemma 1.7.
In addition Theorem 1.11 implies that there is no ergodic φ_α–invariant measure on $I^\pm$ in $\mathcal{I}(\varphi_\alpha)$ for $\alpha \le 0$, and there are two ergodic $\mathcal{F}^-$–measurable φ_α–invariant measures in $\mathcal{I}(\varphi_\alpha)$, one on I^+ and one on I^-, for $\alpha > 0$, with the desired stability properties iff *(ii), (iii)* are fulfilled .
Finally, if $\mu_\alpha = \delta_{a_\alpha}$ is related to the invariant probability measure $\nu_\alpha = \frac{m_\alpha}{m_\alpha(I)}$ of the semigroup (P_t) via pullback relation (13), then we have $\nu_\alpha \longrightarrow \delta_0$ weakly iff $a_\alpha \longrightarrow 0$ in probability as $\alpha \downarrow 0$. This follows since $\mathbb{P}\{a_\alpha \ge \varepsilon\} = \nu_\alpha([\varepsilon, \infty[)$ for all $\varepsilon > 0$. A similar statement holds on I^-. This implies the remaining equivalence. ■

For a transcritical bifurcation we have the following theorem.

Theorem 2.5 *Let $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ be the family of RDS induced by (17) and suppose* (E) *and* (IC) *are fulfilled for all $\alpha \in \mathbb{R}$. $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ undergoes a stochastic transcritical bifurcation at $\alpha = 0$, if and only if* *(i)* $\mathrm{sign}(b'_\alpha(0)) = \mathrm{sign}(\alpha)$ *for all* $\alpha \in \mathbb{R}$,

(ii) $m_\alpha(I^+) = \infty = \overline{m}_\alpha(I^+)$ *and* $\overline{m}_\alpha(I^-) < \infty$, $m_\alpha(]-\infty, c]) = \infty$, $c \in I^-$ *for* $\alpha < 0$,

(iii) $m_0(I^\pm) = \infty = \overline{m}_0(I^\pm)$,

(iv) $m_\alpha(I^+) < \infty$, $\overline{m}_\alpha([c, \infty[) = \infty$, $c \in I^+$ *and* $m_\alpha(I^-) = \infty = \overline{m}_\alpha(I^-)$ *for* $\alpha > 0$,

(v) *(a)* $\overline{\nu}_\alpha := \frac{\overline{m}_\alpha}{\overline{m}_\alpha(I^-)} \longrightarrow \delta_0$ *weakly as* $\alpha \uparrow 0$ *and*

(b) $\nu_\alpha := \frac{m_\alpha}{m_\alpha(I^+)} \longrightarrow \delta_0$ *weakly as* $\alpha \downarrow 0$,

where m_α, $\overline{m}_\alpha$ *are the speed measures of* φ_α *respectively* ψ_α *for* $\alpha \in \mathbb{R}$ *and the constant* $c \in I$ *in (ii) and (iii) is independent of* α.

The proof is similar to the proof of Theorem 2.4.

Remark 2.6 There are more bifurcation scenarios obtainable, which depend only on the possible constellations of finiteness of the speed measures on the sets $I^\pm$. We will present one particular case in Example 3.11. □

3 Sufficient criteria for the finiteness of the speed measure

In the preceding section bifurcation was characterized by finiteness and continuity properties of the speed measures. In this section we shall go one step further. We give sufficient growth conditions on the diffusion and the drift coefficients under which the theorems of the preceding section apply and yield bifurcation. We shall finally show that our theory covers the examples known to date.

Throughout this section we shall assume the ellipticity condition (E). We restrict ourselves by assuming σ to be strictly increasing near $\pm\infty$, i. e. there exists $K > 0$, such that $\sigma'(x) > 0$ for all $|x| \geq K$. In the cases where σ is constant near $\pm\infty$ the following conditions can be changed in an obvious way (see Remark 3.2).

Condition 3.1 Conditions for the finiteness of the speed measure

(A1) There exists $\delta > 1$ and $K > 0$, such that for all $x \geq K$ we have

$$\frac{b\sigma' - \sigma^2\sigma''}{(\sigma')^2} \circ \sigma^{-1}(x) \leq -\delta \, \frac{x}{2\log x} \, .$$

(A2) There exists $\delta > 1$ and $\varepsilon > 0$, such that for all $0 < x \leq \varepsilon$ we have

$$\frac{b\sigma' - \sigma^2\sigma''}{(\sigma')^2} \circ \sigma^{-1}(x) \geq -\delta \, \frac{x}{2\log \frac{1}{x}} \, .$$

(A3) and **(A4)** are analogous conditions on I^- instead of I^+, (A3) corresponds to (A2).

Conditions for infiniteness of the speed measure

(B1) There exists $K > 0$, such that for all $x \geq K$ we have

$$\frac{b\sigma' - \sigma^2\sigma''}{(\sigma')^2} \circ \sigma^{-1}(x) \geq \frac{x}{2\log x} \, .$$

(B2) There exists $\varepsilon > 0$, such that for all $0 < x \leq \varepsilon$ we have

$$\frac{b\sigma' - \sigma^2\sigma''}{(\sigma')^2} \circ \sigma^{-1}(x) \leq \frac{x}{2\log \frac{1}{x}} \, .$$

(B3) and **(B4)** are analogous conditions on I^- instead of I^+, (B3) corresponds to (B2).

Remark 3.2 *(i)* In case $\sigma(x) = \sigma x,\ x \in \mathbb{R},\ \sigma > 0$ the growth condition stated above become more transparent. For example the inequality in (A1) reduces to

$$b(x) \leq -\,\delta\,\frac{\sigma^2}{2}\,\frac{x}{\log x}\,.$$

(ii) If σ is constant near infinity, then conditions (A1) and (B1) can replaced by:

(A1') There exists $\varepsilon > 0$ and $K > 0$, such that for all $x \geq K$ we have $b(x) \leq -\varepsilon$.

(B1') There exists $K > 0$, such that for all $x \geq K$ we have $b(x) \geq 0$.

Similar statements hold for the other conditions. □

Proposition 3.3 *The density ρ of the speed measure m of φ is*

(i) integrable at infinity, if the pair (b, σ) satisfies (A1),

(ii) integrable at zero on I^+, if the pair (b, σ) satisfies (A2),

(iii) not integrable at infinity, if the pair (b, σ) satisfies (B1),

(iv) not integrable at zero on I^+, if the pair (b, σ) satisfies (B2).

Similar statements holds for ρ on I^-, if we replace (A1) *by* (A4), (A2) *by* (A3), (B1) *by* (B4) *and* (B2) *by* (B3), *and for the density $\overline{\rho}$ of the speed measure $\overline{m}$ of ψ, if we replace (b, σ) by $(-b, \sigma)$.*

Proof: In the following statements $c_1, c_2, \ldots$ are positive constants and $c \in I^+$. Then

$$\begin{aligned}
2\int_c^x \frac{b}{\sigma^2}(y)\,dy &\leq c_1 + 2\int_K^{\sigma(x)} \frac{b}{\sigma'} \circ \sigma^{-1}(z)\,\frac{1}{z^2}\,dz \\
&\leq c_1 + \int_K^{\sigma(x)} -\frac{\delta}{z\log z}\,dz + \int_K^{\sigma(x)} \frac{\sigma''}{(\sigma')^2} \circ \sigma^{-1}(z)\,dz \\
&\leq c_2 + \int_e^{\sigma(x)\vee e} -\frac{\delta}{z\log z}\,dz + \log\sigma'(x) \\
&= c_2 - \delta\log\log\big(\sigma(x)\vee e\big) + \log\sigma'(x)
\end{aligned}$$

Therefore

$$\int_K^\infty \rho(x)\,dx \leq c_3\int_K^\infty \frac{\sigma'(x)}{\sigma(x)}\Big(\log\big(\sigma(x)\vee e\big)\Big)^{-\delta}\,dx\,.$$

Now if σ is bounded, the expression on the right hand side may be estimated by $\log\frac{\sigma(\infty)}{\sigma(K)}$ which is finite, where $\sigma(\infty) = \lim_{x\to\infty}\sigma(x)$, is well defined since σ is increasing near infinity. If σ is not bounded, then we may estimate further by

$$c_4\int_K^\infty \Big(\big(\log\sigma(x)\big)^{1-\delta}\Big)'\,dx < \infty\,.$$

A similar argument applies to the finiteness near 0 on I^+, and to the other cases. ■

Corollary 3.4 *(i) If (b, σ) satisfies* (A1) *and* (A2), *then $m(I^+) < \infty$.*
(ii) If (b, σ) satisfies (B1) *or* (B2), *then $m(I^+) = \infty$.*
Similar statements hold for m on I^- and for $\overline{m}$, if we replace (b, σ) by $(-b, \sigma)$.

In the following theorems we present the resulting sufficient conditions on drift and diffusion which lead to stochastic bifurcation of the family of RDS induced by (17).

Theorem 3.5 *Let* $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ *be the family of RDS induced by equation (17). Suppose* (E) *and* (IC) *are fulfilled for all* $\alpha \in \mathbb{R}$ *and the map* $\alpha \mapsto \rho_\alpha(x)$ *is upper semi continuous at* $\alpha = 0$ *for all* $x \in \mathbb{R}\setminus\{0\}$, *where* $\rho_\alpha(x)$ *is given by (11) with* b, σ *replaced by* b_α *and* σ_α.
Furthermore assume that for all $r > 0$ *there exists* $\alpha' > 0$ *such that* $\sup_{0\le\alpha\le\alpha'} m_\alpha(I\setminus]-r,r[) < \infty$, *where* m_α *denote the speed measure with density* ρ_α. *Then* $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ *undergoes a stochastic pitchfork bifurcation at* $\alpha = 0$, *if*

(i) $\mathrm{sign}(b'_\alpha(0)) = \mathrm{sign}(\alpha)$ *for all* $\alpha \in \mathbb{R}$,

(ii) $((b_\alpha, \sigma_\alpha))_{\alpha\in\mathbb{R}}$ *satisfies*

  (a) ((B1) *or* (B2)) *and* ((B3) *or* (B4)) *for* $\alpha < 0$,

  (b) (A1),(B2),(B3) *and* (A4) *for* $\alpha = 0$,

  (c) (A1),(A2),(A3) *and* (A4) *for* $\alpha > 0$,

  and there exists $\overline{\delta} > 1$ *such that* $\delta(\alpha) \ge \overline{\delta}$ *for all* $\alpha \ge 0$, *where* $\delta(\alpha)$ *is the constant appearing in* (A1),...,(A4) *for* $(b_\alpha, \sigma_\alpha)$ *in (b) and (c)*,

(iii) $(-b_\alpha, \sigma_\alpha)$ *satisfies* ((B1) *or* (B2)) *and* ((B3) *or* (B4)) *for all* $\alpha \le 0$, *and* (B1) *and* (B4) *for* $\alpha > 0$.

Proof: The existence of φ_α–invariant measures with the required stability follows from Proposition 3.3, Corollary 3.4 and Theorem 2.4. We only have to show that the normalized speed measures converge weakly to the Dirac measure at zero. This is equivalent to $a_\alpha \to 0$ in probability as $\alpha \downarrow 0$, where $\mu_\alpha = \delta_{a_\alpha}$ is the φ_α–invariant measure on I. We only consider the case $I = I^+$.
By Fatou's Lemma and since $\alpha \mapsto \rho_\alpha(x)$ is upper semi continuous at $\alpha = 0$ for all $x \in I^+$, we have

$$\limsup_{\alpha\downarrow 0} m_\alpha(I^+) \ge \liminf_{\alpha\downarrow 0} m_\alpha(I^+) \ge \int_{I^+} \liminf_{\alpha\downarrow 0} \rho_\alpha(x)\,dx \ge \int_{I^+} \rho_0(x)\,dx = \infty\,,$$

by assumption *(ii)(b)*. Hence $\lim_{\alpha\downarrow 0} m_\alpha(I^+) = \infty$. Then for every $\varepsilon > 0$ and $0 \le \alpha \le \alpha'$ we have

$$\mathbb{P}(\{a_\alpha \ge \varepsilon\}) = \frac{m_\alpha([\varepsilon,\infty[)}{m_\alpha(I^+)} = m_\alpha(I^+)^{-1}\int_\varepsilon^\infty \rho_\alpha(x)\,dx \le m_\alpha(I^+)^{-1} \sup_{0\le\alpha\le\alpha'} \int_\varepsilon^\infty \rho_\alpha(x)\,dx \xrightarrow{\alpha\downarrow 0} 0\,,$$

since $\sup_{0\le\alpha\le\alpha'} \int_\varepsilon^\infty \rho_\alpha(x)\,dx < \infty$. Thus $a_\alpha \to 0$ in probability as $\alpha \downarrow 0$. ∎

Theorem 3.6 *Let* $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ *be the family of RDS induced by equation (17). Suppose* (E) *and* (IC) *are fulfilled for all* $\alpha \in \mathbb{R}$ *and the maps* $\alpha \mapsto \rho_\alpha(x)$, $\alpha \mapsto \overline{\rho}(x)$ *are upper semi continuous at* $\alpha = 0$ *for all* $x \in I^+$ *respectively* I^-, *where* $\rho_\alpha(x)$, $\overline{\rho}_\alpha(x)$ *are given by (11) and (12) with* b, σ *replaced by* b_α *and* σ_α.
Furthermore assume that for all $r > 0$ *there exists* $\alpha' > 0$ *such that* $\sup_{0\le\alpha\le\alpha'} m_\alpha(I^+\setminus]0,r[) < \infty$ *and* $\sup_{-\alpha'\le\alpha\le 0} \overline{m}_\alpha(I^-\setminus]-r,0[) < \infty$, *where* m_α, $\overline{m}_\alpha$ *denote the speed measure with density* ρ_α *respectively* $\overline{\rho}_\alpha$. *Then* $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ *undergoes a stochastic transcritical bifurcation at* $\alpha = 0$, *if*

(i) $\mathrm{sign}(b'_\alpha(0)) = \mathrm{sign}(\alpha)$ *for all* $\alpha \in \mathbb{R}$,

(ii) $((b_\alpha, \sigma_\alpha))_{\alpha\in\mathbb{R}}$ *satisfies*

  (a) ((B1) *or* (B2)) *and* (B4) *for* $\alpha < 0$,

  (b) ((A1) *and* (B2)) *and* ((B3) *or* (B4)) *for* $\alpha = 0$,

  (c) ((A1) *and* (A2)) *and* ((B3) *or* (B4)) *for* $\alpha > 0$.

  and the constants $\delta(\alpha)$ *appearing in the conditions* (A1), (A2) *for* $(b_\alpha, \sigma_\alpha)$ *in (b) and (c) are bounded below by* $\overline{\delta} > 1$,

(iii) $((-b_\alpha, \sigma_\alpha))_{\alpha\in\mathbb{R}}$ *satisfies*

  (a) ((B1) *or* (B2)) *and* ((A3) *and* (A4)) *for* $\alpha < 0$,

  (b) ((B1) *or* (B2)) *and* ((B3) *and* (A4)) *for* $\alpha = 0$,

(c) (B1) *and* ((B3) *or* (B4)) *for* $\alpha > 0$.

and the constants $\delta(\alpha)$ *appearing in the conditions* (A3), (A4) *for* $(b_\alpha, \sigma_\alpha)$ *in (a) and (b) are bounded below by* $\overline{\delta} > 1$.

The proof, similar to that of Theorem 3.5, is omitted.

Example 3.7 (Arnold and Boxler [2])
Consider the explicitly solvable SDE

$$dx_t = \left(\alpha x_t - x_t^3\right) dt + \sigma x_t \circ dW_t \qquad \text{and} \qquad dx_t = \left(\alpha x_t - x_t^2\right) dt + \sigma x_t \circ dW_t\,.$$

Arnold and Boxler showed that the RDS induced by the first equation undergoes a stochastic pitchfork and the second a stochastic transcritical bifurcation. By monoton convergence we see that $\sup_{0 \le \alpha \le 1} m_\alpha(I \backslash]-r, r[) < \infty$. The conditions of Theorem 3.6 and 3.5 are now easily seen to be fulfilled. □

Example 3.8 (Arnold and Schmalfuß [3])
Consider the stochastic differential equations

$$dx_t = \left(\alpha x_t - x_t^3 + g(x_t)\right) dt + \sigma x_t \circ dW_t\,, \tag{18}$$

with $\alpha \in \mathbb{R}$, $\sigma > 0$ and $g \in C^1(\mathbb{R})$ satisfying

$$g(0) = 0\,, \qquad \frac{g(x)}{x} > 0,\, x \neq 0\,, \qquad |g'(x)| < 2x^2,\, x \neq 0\,.$$

Then $((b_\alpha, \sigma_\alpha))_{\alpha \in \mathbb{R}}$ with $b_\alpha(x) = \alpha x - x^3 + g(x)$ and $\sigma_\alpha(x) = \sigma x$ satisfy the assumptions of Theorem 3.5. Thus the family of RDS induced by equation (18) undergoes a stochastic pitchfork bifurcation at $\alpha = 0$.
□

Example 3.9 Now we consider the following equation for $n \in \mathbb{N}$, $n \geq 2$

$$dx_t = \Big(A(\alpha) x_t + \sum_{i=2}^{n} a_i(\alpha)\, x_t^i\Big) dt + \sigma x_t \circ dW_t\,, \tag{19}$$

with $\alpha \mapsto A(\alpha)$, $a_i(\alpha)$ continuous, $i = 2, \ldots, n$ and $\sigma > 0$. The family of RDS induced by (19) undergoes a stochastic pitchfork bifurcation at $\alpha = 0$, if

(i) n is odd and

(ii) $\text{sign}(A(\alpha)) = \text{sign}(\alpha)$, $\text{sign}(a_n(\alpha)) = -1$ for all $\alpha \in \mathbb{R}$. □

Example 3.10 Consider for $n, m \in \mathbb{N}$

$$dx_t = \Big(\sum_{i=1}^{n} a_i\, x_t^i\Big) dt + \sigma x_t^{2m+1} \circ dW_t\,,$$

with $\sigma > 0$. Condition (A1) is satisfied, if $n < 4m+1$ or if $n = 4m+1$ and $a_n < m\sigma^2$ or if $n > 4m+1$ and $a_n < 0$. Condition (A2) is satisfied, if with $i_0 = \min\{i : a_i \neq 0, i = 1, \ldots, n\}$ we have $i_0 < 4m+1$ or if $i_0 = 4m+1$ and $a_{i_0} > m\sigma^2$. Similar statements hold for the other conditions. □

Example 3.11 In this example we present another type of bifurcation scenario different from the two studied so far.

(i) Consider equation (17) with

$$b_\alpha(x) = \begin{cases} -\alpha x - x^2 & \text{if } \alpha \leq 0 \\ -\alpha x + x^2 & \text{if } \alpha > 0 \end{cases} \qquad \text{and} \qquad \sigma_\alpha(x) = \sigma x\,,$$

for x, $\alpha \in \mathbb{R}$ and $\sigma > 0$. Choose $c = \pm 1$ on $I^\pm$. So for the densities of the speed measures on $I^\pm$ we have

$$\rho_\alpha^\pm(x) = \begin{cases} k_\sigma^\pm \, |x|^{-\frac{2}{\sigma^2}\alpha - 1} \, e^{-\frac{2}{\sigma^2} x} & \text{if } \alpha \leq 0 \\ k_\sigma^\pm \, |x|^{-\frac{2}{\sigma^2}\alpha - 1} \, e^{\frac{2}{\sigma^2} x} & \text{if } \alpha > 0 \end{cases} \quad \text{and} \quad \overline{\rho}_\alpha^\pm(x) = \begin{cases} k_\sigma^\pm \, |x|^{\frac{2}{\sigma^2}\alpha - 1} \, e^{\frac{2}{\sigma^2} x} & \text{if } \alpha \leq 0 \\ k_\sigma^\pm \, |x|^{\frac{2}{\sigma^2}\alpha - 1} \, e^{-\frac{2}{\sigma^2} x} & \text{if } \alpha > 0 \, , \end{cases}$$

where $k_\sigma^\pm$ is a constant, depending on σ and the sign of c. Hence $m_\alpha(I^+) < \infty, \overline{m}_\alpha([2,\infty)) = \infty$, $m_\alpha(I^-) = \infty = \overline{m}_\alpha(I^-)$ for $\alpha < 0$ and $m_0(I^\pm) = \infty = \overline{m}_0(I^\pm)$ and $\overline{m}_\alpha(I^+) < \infty$, $m_\alpha([2,\infty)) = \infty$, $m_\alpha(I^-) = \infty = \overline{m}_\alpha(I^-)$. The family of RDS $(\varphi_\alpha)_{\alpha \in \mathbb{R}}$ undergoes in this case a kind of stochastic transcritical bifurcation (see Theorem 2.5).

(ii) Consider (17) with $b_\alpha(x) = \alpha x - x^2$ for $\alpha \leq 0$ and $b_\alpha(x) = \alpha x - x^3$ for $\alpha > 0$ and $\sigma_\alpha(x) = \sigma x$ for all $\alpha \in \mathbb{R}$ and $\sigma > 0$. The induced family of RDS $(\varphi_\alpha)_{\alpha \in \mathbb{R}}$ undergoes in this case a mixture of a stochastic transcritical and pitchfork bifurcation (see Theorems 2.4 and 2.5). □

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