On Malliavin's differentiability of BSDEs with time-delayed generators driven by Brownian motions and Poisson random measures

Łukasz Delong, Peter Imkeller

Abstract

We investigate Malliavin's differentiability of solutions of backward stochastic differential equations with time-delayed generators driven by Brownian motions and Poisson random measures, which are the components of a Lévy process. In this new type of equations, a generator at time t can depend on the past, up to time t, delayed values of a solution. For a time-delayed BSDE, we prove existence and uniqueness of a solution for a sufficiently small time horizon or for a sufficiently small Lipschitz constant of a generator. We study differentiability in Malliavin sense and derive equations which are satisfied by Malliavin derivatives. We consider differentiability both with respect to a continuous part of a Lévy process, which coincides with the notion of the classical Malliavin derivative for Hilbert-valued random variables, and with respect to a pure jump part, which leads to an increment quotient operator related to Picard difference operator.

Keywords: backward stochastic differential equation, time-delayed generator, Poisson random measure, Malliavin's calculus, canonical Lévy space, Picard difference operator.

1 Introduction

Backward stochastic differential equations was introduced in [16], and since than, they have been thoroughly studied in the literature, see [9] or [11] and references therein. One of the main line of research in the theory of BSDEs deals with Malliavin's differentiability of a solution. There exists an interesting connection between Malliavin's calculus and the structure of a solution of a backward stochastic differential equation. It is known that a solution of a BSDE driven by a Brownian motion and without a time-delay is differentiable in Malliavin sense and that Malliavin derivative satisfies a linear BSDE. Moreover, a solution can be interpreted in terms of Malliavin trace, see Proposition 5.3 in [9] or Theorem 3.3.1 in [11].

In this paper we study the equations with dynamics given by

$$Y(t) = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z(s)dW(s) - \int_t^T U(s,z)\tilde{M}(ds,dz),$$

which can be called backward stochastic differential equations with time-delayed generators driven by Brownian motions and Poisson random measures, which are the components of a Lévy process. In this new type of equations, a generator f at time s depends arbitrary on the past values of a solution $(Y_s, Z_s, U_s) = (Y(s+u), Z(s+u), U(s+u,.))_{-T\leq u\leq 0}$. Very recently, time-delayed BSDEs driven by Brownian motions and with Lipschitz continuous generators have been investigated for the the first time in [6], and in more depths in [7]. We would like to refer the interested reader to the accompanying paper [7], where results on existence of a unique solution, examples of multiple solutions and the lack of a solution are provided, and various properties, including a comparison principle, a measure solution, a property of a uniform boundedness and BMO martingale property, are studied. Compared to [7], in this work we consider an additional source of an uncertainty represented by a Poisson random measure. We would like to point out that all results from [7] can be extended and proved in the setting of this paper but these extensions are omitted. Our new goal is to investigate Malliavin's differentiability.

There are two contributions of this paper. First, we prove that that a unique solution exists provided that a Lipschitz constant of a generator is sufficiently small or a time horizon is sufficiently small. This is natural extension of Theorem 2.1 from [7]. Secondly and mainly, we establish Malliavin's differentiability of a solution of a time-delayed BSDE, both with respect to a continuous part of a Lévy process, which coincides with the notion of the classical Malliavin derivative for Hilbert-valued random variables, and with respect to a pure jump part, which leads to an increment quotient operator related to Picard difference operator. We prove that the well-known connection between (Z, U) and Malliavin trace of Y still holds in time-delayed equations. We adopt the definition of Malliavin derivative on the canonical

Lévy space following [21].

BSDEs without time-delays and driven by Poisson random measures have already been deeply investigated in the literature, see [3], [4] or [20], but contrary to the case with a single Brownian motion driver, the results on Malliavin's differentiability have not been established yet in a satisfactory way. To the best of our knowledge, only in [5], Malliavin's differentiability of a solution of a forward-backward SDE with jumps with respect to a Brownian part is considered but at the same time, differentiability with respect to a jump part is neglected. We would like to notice that for the first time differentiability of a solution of a BSDE with respect to a pure jump component of a Lévy process is considered. Moreover, the analysis is performed for a more general time-delayed equation.

We would like to point out that the relation between Malliavin's calculus and the structure of a solution is not only interesting from purely theoretical point. We believe that it is worth to take an effort and prove rather technical results of this paper as they can have real applications. Let us recall that for examle in mathematical finance, a hedging strategy in a complete market corresponds to Malliavin derivative of a wealth process, see [13]. Malliavin derivatives of Y also provides an efficient tool for estimating norms of (Z, U). In numerics, Malliavin's calculus may be applied to prove regularity of trajectories and convergence of discretization schemes, which have to used in order to solve BSDEs numerically, see [12].

This paper is structured as follows. Section 2 deals with existence and uniqueness of a solution. In section 3 we give definitions of the canonical Lévy space and Malliavin derivative and prove some technical lemmas. The main theorem concerning Malliavin's differentiability of a solution and the interpretation in terms of Malliavin trace are given in Section 4.

2 Existence and uniqueness of a solution

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ and a finite time horizon $T < \infty$. We assume that the filtration $\mathbb{F}$ is a natural filtration generated by a Lévy process $L := (L(t), 0 \leq t \leq T)$ and that $\mathcal{F}_0$ contains all sets of $\mathbb{P}$-measure zero. As usually, by $\mathcal{B}$ we denote Borel sets and λ stands for Lebesque measure.

It is well-known that a Lévy process satisfies Lévy-Itô decomposition

$$L(t) = at + \sigma W(t) + \int_0^t \int_{|z|\geq 1} zN(ds, dz) + \int_0^t \int_{0<|z|<1} z(N(ds, dz) - \nu(dz)ds),$$

for $0 \leq t \leq T$, with $a \in \mathbb{R}, \sigma \geq 0$, where $W := (W(t), 0 \leq t \leq T)$ denotes a Brownian motion and N denotes an independent random measure on $[0, T] \times (\mathbb{R} - \{0\})$. The random measure N

$$N(t,A)=\sharp\{0\le s\le t;\triangle L(s)\in A\},\quad 0\le t\le T, A\in\mathcal{B}(\mathbb{R}-\{0\}),$$

counts the number of jumps of a given size. It is called Poisson random measure as, for a fixed $t\in[0,T]$ and a set A such that its closure does not contain zero, $N(t,A)$ is a Poisson distributed random variable. The measure ν, defined on $[0,T]\times(\mathbb{R}-\{0\})$, is σ-finite measure and it is the compensator for the measure N. The compensated Poisson random measure (or martingale-valued measure) is denoted by $\tilde{N}(t,A)=N(t,A)-t\nu(A)$, $t\in[0,T], A\in\mathcal{B}(\mathbb{R}-\{0\})$. In this paper we deal with another random measure

$$\begin{aligned}\tilde{M}(t,A) &= \int_0^t\int_A z\tilde{N}(ds,dz)\\ &= \int_0^t\int_A zN(ds,dz)-\int_0^t\int_A z\nu(dz)ds,\quad 0\le t\le T, A\in\mathcal{B}(\mathbb{R}-\{0\}).\end{aligned}$$

It can be called compensated compound Poisson random measure as, for a fixed $t\in[0,T]$ and a set A such that its closure does not contain zero, $\int_0^t\int_A zN(ds,dz)$ is a compound Poisson distributed random variable. Finally, we introduce σ-finite measure

$$m(A)=\int_A z^2\nu(dz),\quad A\in\mathcal{B}(\mathbb{R}-\{0\}).$$

For details concerning Lévy processes, Poisson random measures and integration with respect to martingale-valued random measures we refer the reader to Chapter 2 and Chapter 4 of [2].

In this paper we study Malliavin's differentiability of a unique solution $(Y,Z,U):=(Y(t),Z(t),U(t,z))_{0\le t\le T,z\in(\mathbb{R}-\{0\})}$ of a backward stochastic differential equation with a time-delayed generator, which dynamics is given by

$$\begin{aligned}Y(t) &= \xi+\int_t^T f(s,Y_s,Z_s,U_s)ds\\ &\quad -\int_t^T Z(s)dW(s)-\int_t^T\int_{\mathbb{R}-\{0\}}U(s,z)\tilde{M}(ds,dz),\quad 0\le t\le T,\end{aligned}\qquad (2.1)$$

where the generator f depends on time-delayed, past, values of a solution denoted by $Y_s:=(Y(s+v))_{-T\le v\le 0}$, $Z_s:=(Z(s+v))_{-T\le v\le 0}$ and $U_s:=(U(s+v,.))_{-T\le v\le 0}$. We always set $Z(t)=U(t,.)=0$ and $Y(t)=Y(0)$ for $t<0$. We assume that the measure $\tilde{M}$, not $\tilde{N}$, is the driving factor, as we adopt a chaotic decomposition in terms of multiple integrals with respect to $\tilde{M}$, which gives the foundations for Malliavin's calculus on the canonical Lévy space, see the next section.

Let us introduce definitions of spaces.

Definition 2.1. *1. Let $L^2_{-T}(\mathbb{R})$ denote the space of measurable functions $z : [-T,0] \to \mathbb{R}$ satisfying*

$$\int_{-T}^{0} |z(t)|^2 dt < \infty.$$

2. Let $L^2_{-T,m}(\mathbb{R})$ denote the space of product measurable functions $u : [-T,0] \times (\mathbb{R} - \{0\}) \to \mathbb{R}$ satisfying

$$\int_{-T}^{0} \int_{\mathbb{R}-\{0\}} |u(t,z)|^2 m(dz) dt < \infty.$$

3. Let $L^{\infty}_{-T}(\mathbb{R})$ denote the space of bounded, measurable functions $y : [-T,0] \to \mathbb{R}$ satisfying

$$\sup_{t\in[-T,0]} \big|y(t)\big|^2 < \infty.$$

4. Let $\mathbb{L}^2(\mathbb{R})$ denote the space of $\mathcal{F}_T$-measurable random variables $\xi : \Omega \to \mathbb{R}$ satisfying

$$\mathbb{E}\big[\big|\xi\big|^2\big] < \infty.$$

5. Let $\mathbb{H}^2_T(\mathbb{R})$ denote the space of predictable processes $Z : \Omega \times [0,T] \to \mathbb{R}$ satisfying

$$\mathbb{E}\big[\int_0^T \big|Z(t)\big|^2 dt\big] < \infty.$$

6. Let $\mathbb{H}^2_{T,m}(\mathbb{R})$ denote the space of predictable processes $U : \Omega \times [0,T] \times (\mathbb{R} - \{0\}) \to \mathbb{R}$ satisfying

$$\mathbb{E}\big[\int_0^T \int_{\mathbb{R}-\{0\}} \big|U(t,z)\big|^2 m(dz) dt\big] < \infty.$$

7. Let $\mathbb{S}^2_T(\mathbb{R})$ denote the space of $\mathbb{F}$-adapted, product measurable processes $Y : \Omega \times [0,T] \to \mathbb{R}$ satisfying

$$\mathbb{E}\big[\sup_{t\in[0,T]} \big|Y(t)\big|^2\big] < \infty.$$

The spaces $\mathbb{H}^2_T(\mathbb{R})$,$\mathbb{H}^2_{T,m}(\mathbb{R})$ and $\mathbb{S}^2_T(\mathbb{R})$ are endowed with the norms

$$\begin{aligned} \big\|Z\big\|^2_{\mathbb{H}^2_T} &= \mathbb{E}\big[\int_0^T e^{\beta t}\big|Z(t)\big|^2 dt\big], \\ \big\|U\big\|^2_{\mathbb{H}^2_{T,m}} &= \mathbb{E}\big[\int_0^T \int_{\mathbb{R}-\{0\}} e^{\beta t}\big|U(t,z)\big|^2 m(dz) dt\big], \\ \big\|Y\big\|^2_{\mathbb{S}^2_T} &= \mathbb{E}\big[\sup_{t\in[0,T]} e^{\beta t}\big|Y(t)\big|^2\big], \end{aligned}$$

with some $\beta > 0$.

Predictability of Z means measurability with respect to the predictable σ-algebra, which we denote by $\mathcal{P}$, and predictability of U means measurability with respect to the product $\mathcal{P}\times\mathcal{B}(\mathbb{R}-\{0\})$. In the sequel let us simply denote $\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2_m(\mathbb{R})$ for $\mathbb{S}^2_T(\mathbb{R})\times\mathbb{H}^2_T(\mathbb{R})\times\mathbb{H}^2_{T,m}(\mathbb{R})$.

We start with establishing existence and uniqueness of a solution of the equation (2.1) under the following conditions:

(A1) the terminal value $\xi\in\mathbb{L}^2(\mathbb{R})$,

(A2) m is finite measure, $\int_{\mathbb{R}-\{0\}} z^2\nu(dz)<\infty$,

(A3) the generator $f:\Omega\times[0,T]\times L^\infty_{-T}(\mathbb{R})\times L^2_{-T}(\mathbb{R})\times L^2_{-T,m}(\mathbb{R})\to\mathbb{R}$ is product measurable, $\mathbb{F}$-adapted and Lipschitz continuous in the sense that for a probability measure α on $([-T,0]\times\mathcal{B}([-T,0]))$

$$\begin{aligned}&|f(t,y_t,z_t,u_t)-f(t,\tilde{y}_t,\tilde{z}_t,\tilde{u}_t)|^2\\ &\leq\ K\Big(\int_{-T}^0|y(t+v)-\tilde{y}(t+v)|^2\alpha(dv)+\int_{-T}^0|z(t+v)-\tilde{z}(t+v)|^2\alpha(dv)\\ &\quad+\int_{-T}^0\int_{\mathbb{R}-\{0\}}|u(t+v,z)-\tilde{u}(t+v,z)|^2m(dz)\alpha(dv)\Big),\end{aligned}$$

holds $\mathbb{P}\otimes\lambda$-a.e. $(\omega,t)\in\Omega\times[0,T]$ for any $(y_t,z_t,u_t),(\tilde{y}_t,\tilde{z}_t,\tilde{u}_t)\in L^\infty_{-T}(\mathbb{R})\times L^2_{-T}(\mathbb{R})\times L^2_{-T,m}(\mathbb{R})$.

(A4) $\mathbb{E}\big[\int_0^T|f(t,0,0,0)|^2dt\big]<\infty$,

(A5) $f(t,.,.,.)=0$ for $t<0$.

We remark that $f(t,0,0,0)$ in **(A4)** should be understood as a value of the generator $f(t,y_t,z_t,u_t)$ at $y(t+v)=z(t+v)=u(t+v,.)=0,-T\leq v\leq 0$. We would like to point out that the assumption **(A5)** in fact allows us to take $Y(t)=Y(0)$ and $Z(t)=U(t,.)=0$, for $t<0$, as a solution of (2.1). Finally, let us recall that under **(A2)** and for an integrand $U\in\mathbb{H}^2_m(\mathbb{R})$, a stochastic integral with respect to the martingale-valued measure $\tilde{M}$

$$\int_0^t\int_{\mathbb{R}-\{0\}}U(s,z)\tilde{M}(ds,dz),\quad 0\leq t\leq T,$$

is well-defined in Itô sense, see Chapter 4.1 in [2].

First let us notice that for $(Y,Z,U)\in\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2_m(\mathbb{R})$ the generator is well-defined and integrable as

$$
\begin{aligned}
\int_0^T |f(t,Y_t,Z_t,U_t)|^2dt &\leq 2\int_0^T |f(t,0,0,0)|^2dt + 2K\Big(\int_0^T\int_{-T}^0 |Y(t+v)|^2\alpha(dv)dt \\
&+ \int_0^T\int_{-T}^0 |Z(t+v)|^2\alpha(dv)dt + \int_0^T\int_{-T}^0\int_{\mathbb{R}-\{0\}} |U(t+v,z)|^2m(dz)\alpha(dv)dt\Big) \\
&= 2\int_0^T |f(t,0,0,0)|^2dt + 2K\int_{-T}^0\int_v^{T+v} |Y(w)|^2dw\alpha(dv) \\
&+2K\int_{-T}^0\int_v^{T+v} |Z(w)|^2dw\alpha(dv) + 2K\int_{-T}^0\int_v^{T+v}\int_{\mathbb{R}-\{0\}} |U(w,z)|^2m(dz)dw\alpha(dv) \\
&\leq 2\int_0^T |f(t,0,0,0)|^2dt + 2K\big(T\sup_{w\in[0,T]}|Y(w)|^2 \\
&+ \int_0^T |Z(w)|^2dw + \int_0^T\int_{\mathbb{R}-\{0\}} |U(w,z)|^2m(dz)dw\big) < \infty, \quad \mathbb{P}-a.s., \qquad (2.2)
\end{aligned}
$$

where we apply **(A3)**, Fubini's theorem, change the variables, use the assumption that $Z(t) = U(t,.) = 0$ and $Y(t) = Y(0)$ for $t < 0$ and the fact that the measure α integrates to 1.

The main theorem of this section is an extension of Theorem 2.1 from [7]. Although the extension is quite natural, the proof is given for completeness and convenience of the reader. The key result follows after a priori estimates

Lemma 2.1. *Let $(Y,Z,U) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_m(\mathbb{R})$ and $(\tilde{Y},\tilde{Z},\tilde{U}) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_m(\mathbb{R})$ denote solutions of (2.1) with corresponding parameters (ξ,f) and $(\tilde{\xi},\tilde{f})$ which satisfy the assumptions **(A1)-(A5)**. The following inequalities hold*

$$
\begin{aligned}
&\|Z-\tilde{Z}\|^2_{\mathbb{H}^2} + \|U-\tilde{U}\|^2_{\mathbb{H}^2_m} \\
&\quad\leq e^{\beta T}\mathbb{E}\big[|\xi-\tilde{\xi}|^2\big] + \frac{1}{\beta}\mathbb{E}\Big[\int_0^T e^{\beta t}|f(t,Y_t,Z_t,U_t) - \tilde{f}(t,\tilde{Y}_t,\tilde{Z}_t,\tilde{U}_t)|^2dt\Big], \quad (2.3)
\end{aligned}
$$

$$
\|Y-\tilde{Y}\|^2_{\mathbb{S}^2} \leq 8e^{\beta T}\mathbb{E}\big[|\xi-\tilde{\xi}|^2\big] + 8T\mathbb{E}\Big[\int_0^T e^{\beta t}|f(t,Y_t,Z_t,U_t) - \tilde{f}(t,\tilde{Y}_t,\tilde{Z}_t,\tilde{U}_t)|^2dt\Big] (2.4)
$$

Proof:
The inequality (2.3) follows by a straightforward extension of Lemma 3.2.1 from [11], by only adding an additional stochastic integral with respect to $\tilde{M}$. In order to prove the second inequality, first notice that for $t \in [0,T]$

$$
Y(t) - \tilde{Y}(t) = \mathbb{E}\Big[\xi - \tilde{\xi} + \int_t^T (f(s,Y_s,Z_s,U_s) - \tilde{f}(s,\tilde{Y}_s,\tilde{Z}_s,\tilde{U}_s))ds|\mathcal{F}_t\Big],
$$

and

$$
\begin{aligned}
& e^{\frac{\beta}{2}t}|Y(t)-\tilde{Y}(t)| \\
\leq \;& e^{\frac{\beta}{2}T}\mathbb{E}\big[|\xi-\tilde{\xi}||\mathcal{F}_t\big]+\mathbb{E}\big[\int_0^T e^{\frac{\beta}{2}s}|f(s,Y_s,Z_s,U_s)-\tilde{f}(s,\tilde{Y}_s,\tilde{Z}_s,\tilde{U}_s)|ds|\mathcal{F}_t\big],
\end{aligned}
$$

hold $\mathbb{P}$-a.s.. Doob's martingale inequality and Cauchy-Schwarz inequality yield the second estimate. The reader may also consult Proposition 2.2 in [3] or Proposition 3.3 in [4], where similar estimates for BSDEs with jumps are derived. □

Theorem 2.1. *Assume that* ***(A1)-(A5)*** *hold. For a sufficiently small time horizon T or for a sufficiently small Lipschitz constant K, the backward stochastic differential equation* (2.1) *has a unique solution* $(Y,Z,U)\in\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2_m(\mathbb{R})$.

Proof:
A classical procedure to prove existence and uniqueness of a solution of a stochastic differential equation is to construct Picard iterative sequence and show its convergence, see Theorem 2.1 in [9] or Theorem 3.2.1 in [11]. We follow the idea.

Let $Y^0(t)=Z^0(t)=U^0(t,z)=0$, $(t,z)\in[0,T]\times(\mathbb{R}-\{0\})$, and define recursively

$$
\begin{aligned}
Y^{n+1}(t)&=\xi+\int_t^T f(s,Y^n_s,Z^n_s,U^n_s)ds\\
&\quad-\int_t^T Z^{n+1}(s)dW(s)-\int_t^T\int_{\mathbb{R}-\{0\}}U^{n+1}(s,z)\tilde{M}(ds,dz),\quad 0\leq t\leq T. \quad (2.5)
\end{aligned}
$$

Step 1) Given $(Y^n,Z^n,U^n)\in\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2_m(\mathbb{R})$, the equation (2.5) has a unique solution $(Y^{n+1},Z^{n+1},U^{n+1})\in\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2_m(\mathbb{R})$.
Based on the inequality (2.2), we can conclude that

$$
\begin{aligned}
\mathbb{E}\big[\int_0^T|f(t,Y^n_t,Z^n_t,U^n_t)|^2dt\big] \;&\leq\; 2\mathbb{E}\big[\int_0^T|f(t,0,0,0)|^2dt\big]\\
&\quad+2K\big(T\|Y^n\|_{\mathbb{S}^2}+\|Z^n\|_{\mathbb{H}^2}+\|U^n\|_{\mathbb{H}^2_m}\big)<\infty.
\end{aligned}
$$

As in the case of BSDEs without time-delays, the martingale representation, see Theorem 13.49 in [10], provides a unique process $Z^{n+1}\in\mathbb{H}^2(\mathbb{R})$ and a unique, predictable process $\bar{U}^{n+1}$ satisfying

$$
\mathbb{E}\big[\int_0^T\int_{\mathbb{R}-\{0\}}|\bar{U}^{n+1}(t,z)|^2\nu(dz)dt\big]<\infty,
$$

such that

$$
\begin{aligned}
\xi+\int_0^T f(t,Y^n_t,Z^n_t,U^n_t)dt&=\mathbb{E}\big[\xi+\int_0^T f(t,Y^n_t,Z^n_t,U^n_t)dt\big]\\
&\quad+\int_0^T Z^{n+1}(t)dW(t)+\int_0^T\int_{\mathbb{R}-\{0\}}\bar{U}^{n+1}(t,z)\tilde{N}(dt,dz),\quad \mathbb{P}-a.s.
\end{aligned}
$$

For $(t,z)\in[0,T]\times(\mathbb{R}-\{0\})$ we define $U^{n+1}(t,z)=\frac{\bar{U}^{n+1}(t,z)}{z}\in\mathbb{H}^2_m(\mathbb{R})$ we have the required representation

$$\xi+\int_0^T f(t,Y^n_t,Z^n_t,U^n_t)dt=\mathbb{E}\big[\xi+\int_0^T f(t,Y^n_t,Z^n_t,U^n_t)dt\big]$$
$$+\int_0^T Z^{n+1}(t)dW(t)+\int_0^T\int_{\mathbb{R}-\{0\}}U^{n+1}(t,z)(t)\tilde{M}(dt,dz),\quad \mathbb{P}-a.s..$$

Finally, we take Y^{n+1} as a progressively measurable, càdlàg modification of

$$Y^{n+1}(t)(\omega)=\mathbb{E}\big[\xi+\int_t^T f(s,Y^n_s,Z^n_s,U^n_s))ds|\mathcal{F}_t\big],\quad \omega\in\Omega,t\in[0,T].$$

Similarly, as in Lemma 2.1, Doob's martingale inequality, Cauchy-Schwarz inequality and the estimates (2.2) yield that $Y^{n+1}\in\mathbb{S}^2(\mathbb{R})$.
Step 2) The convergence of a sequence (Y^n,Z^n,U^n) in $\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2_m(\mathbb{R})$.
The estimates (2.3) and (2.4) give the inequality

$$\big\|Y^{n+1}-Y^n\big\|^2_{\mathbb{S}^2}+\big\|Z^{n+1}-Z^n\big\|^2_{\mathbb{H}^2}+\big\|U^{n+1}-U^n\big\|^2_{\mathbb{H}^2_m}$$
$$\leq(8T+\frac{1}{\beta})\mathbb{E}\big[\int_0^T e^{\beta t}|f(t,Y^n_t,Z^n_t,U^n_t)-f(t,Y^{n-1}_t,Z^{n-1}_t,U^{n-1}_t)|^2dt\big].\quad(2.6)$$

By applying Lipschitz condition **(A3)**, Fubini's theorem, changing the variables and using the assumption $\forall n\geq 0$ $Y^n(s)=Y^n(0)$ and $Z^n(s)=U^n(s,.)=0$ for $s<0$, we can derive

$$
\begin{aligned}
&\mathbb{E}\big[\int_0^T e^{\beta t}|f(t,Y_t^n,Z_t^n,U_t^n)-f(t,Y_t^{n-1},Z_t^{n-1},U_t^{n-1})|^2dt\big]\\
&\leq K\mathbb{E}\big[\int_0^T e^{\beta t}\int_{-T}^0 |Y(t+v)-\tilde{Y}(t+v)|^2\alpha(dv)dt\\
&\quad+\int_0^T e^{\beta t}\int_{-T}^0 |Z(t+v)-\tilde{Z}(t+v)|^2\alpha(dv)dt\big]\\
&\quad+\int_0^T e^{\beta t}\int_{-T}^0\int_{\mathbb{R}-\{0\}} |U(t+v,z)-\tilde{U}(t+v,z)|^2m(dz)\alpha(dv)dt\big]\\
&= K\mathbb{E}\big[\int_{-T}^0 e^{-\beta v}\int_0^T e^{\beta(t+v)}|Y(t+v)-\tilde{Y}(t+v)|^2dt\alpha(dv)\\
&\quad+\int_{-T}^0 e^{-\beta v}\int_0^T e^{\beta(t+v)}|Z(t+v)-\tilde{Z}(t+v)|^2dt\alpha(dv)\big]\\
&\quad+\int_{-T}^0 e^{-\beta v}\int_0^T\int_{\mathbb{R}-\{0\}} e^{\beta(t+v)}|U(t+v,z)-\tilde{U}(t+v,z)|^2m(dz)dt\alpha(dv)\big]\\
&= K\mathbb{E}\big[\int_{-T}^0 e^{-\beta v}\int_v^{T+v} e^{\beta w}|Y(w)-\tilde{Y}(w)|^2dw\alpha(dv)\\
&\quad+\int_{-T}^0 e^{-\beta v}\int_v^{T+v} e^{\beta w}|Z(w)-\tilde{Z}(w)|^2dw\alpha(dv)\big]\\
&\quad+\int_{-T}^0 e^{-\beta v}\int_v^{T+v}\int_{\mathbb{R}-\{0\}} e^{\beta w}|U(w,z)-\tilde{u}(w,z)|^2m(dz)dw\alpha(dv)\big]\\
&\leq K\int_{-T}^0 e^{-\beta v}\alpha(dv)\big(T\|Y^n-Y^{n-1}\|_{\mathbb{S}^2}^2+\|Z^n-Z^{n-1}\|_{\mathbb{H}^2}^2+\|U^n-U^{n-1}\|_{\mathbb{H}^2m}^2\big).
\end{aligned}
\tag{2.7}
$$

From (2.6) and (2.7), we obtain

$$
\begin{aligned}
&\|Y^{n+1}-Y^n\|_{\mathbb{S}^2}^2+\|Z^{n+1}-Z^n\|_{\mathbb{H}^2}^2+\|U^{n+1}-U^n\|_{\mathbb{H}_m^2}^2\\
&\qquad\leq\delta(T,K,\beta,\alpha)\big(\|Y^n-Y^{n-1}\|_{\mathbb{S}^2}^2+\|Z^n-Z^{n-1}\|_{\mathbb{H}^2}^2+\|U^n-U^{n-1}\|_{\mathbb{H}_m^2}\big),
\end{aligned}
\tag{2.8}
$$

with

$$\delta(T,K,\beta,\alpha)=(8T+\frac{1}{\beta})K\int_{-T}^0 e^{-\beta v}\alpha(dv)\max\{1,T\}.$$

For $\beta=\frac{1}{T}$ we have

$$\delta(T,K,\beta,\alpha)\leq 9TKe\max\{1,T\}.$$

For a sufficiently small T or for a sufficiently small K, the inequality (2.8) is a contraction inequality and there exists a unique limit $(Y,Z,U)\in\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}_m^2(\mathbb{R})$ of a converging sequence (Y^n,Z^n,U^n), which satisfies the fix point equation

$$Y(t)=\mathbb{E}\big[\xi+\int_t^T f(s,Y_s,Z_s,U_s)ds|\mathcal{F}_t\big],\quad \mathbb{P}-a.s., 0\leq t\leq T.$$

Step 4) We define a solution $\bar{Y}$ of (4.1) as a progressively measurable, càdlàg modification of

$$\bar{Y}(t)(\omega) = \mathbb{E}\big[\xi + \int_t^T f(s, Y_s, Z_s, U_s)ds|\mathcal{F}_t\big], \quad \omega \in \Omega, t \in [0,T],$$

where (Y, Z, U) is the limit constructed in Step 3. □

We point out that in a general case of the generator satisfying assumptions **(A1)-(A5)**, existence and uniqueness of a solution don't hold with an arbitrary time horizon T and an arbitrary Lipschitz constant K, see [7] for examples.

3 Malliavin's calculus for canonical Lévy processes

There are various procedures to develop Malliavin's calculus for Lévy processes. In this paper we adopt the approach from [21] and we use a chaotic decomposition property in terms of multiple integrals with respect to the random measure $\tilde{M}$. A suitable canonical space need to be constructed on which Malliavin derivative with respect to a pure jump part of a Lévy process can be computed in a pathway sense.

In this section we give an overview of Malliavin's calculus on the canonical Lévy space, see [21] for details, and prove some technical result which are applied in the next section.

We assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is the product of two canonical spaces $(\Omega_W \times \Omega_N, \mathcal{F}_W \times \mathcal{F}_N, \mathbb{P}_W \times \mathbb{P}_N)$. The space $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ is the usual canonical space for a Brownian motion, with the space of continuous functions, σ-algebra generated by the topology of uniform convergence and Wiener measure. The canonical space for a pure jump Lévy process $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$ is the product space $\bigotimes_{k\geq 1}(\Omega_N^k, \mathcal{F}_N^k, \mathbb{P}_N^k)$ of the canonical spaces for compound Poisson processes on $[0,T]$ with intensities $\nu(S_k)$ and jump size distributions supported on S_k, where $(S_k)_{k\geq 1}$ forms a partition of $\mathbb{R} - \{0\}$ such that $0 < \nu(S_k) < \infty, k \geq 1$. As any trajectory of a compound Poisson process can be described by a finite sequence $((t_1, z_1), ..., (t_n, z_n))$, where $(t_1, ..., t_n)$ denotes the jump times and $(z_1, ..., z_n)$ denotes the corresponding sizes of the jumps, one can take $\Omega_N^k = \bigcup_{n\geq 0}([0,T] \times (\mathbb{R} - \{0\}))^n$, with $([0,T] \times (\mathbb{R} - \{0\})^0$ representing an empty sequence, the σ-algebra $\mathcal{F}_N^k = \bigvee_{n\geq 0} \mathcal{B}(([0,T] \times (\mathbb{R} - \{0\}))^n)$ and the measure $\mathbb{P}_N^k$ such that for $B = \cup_{n\geq 0} B_n, B_n \in \mathcal{B}(([0,T] \times (\mathbb{R} - \{0\}))^n)$, we have

$$P(B) = e^{-\nu(S_k)T} \sum_{n=0}^{\infty} \frac{(\nu(S_k))^n (dt \otimes \frac{\nu \mathbf{1}\{S_k\}}{\nu(S_k)})^{\otimes n}(B_n)}{n!}.$$

Consider the finite measure q defined on $[0,T]\times\mathbb{R}$

$$q(E)=\int_{E(0)}dt+\int_{E'}z^2\nu(dz)dt,\quad E\in\mathcal{B}([0,T]\times\mathbb{R}),$$

where $E(0)=\{t\in[0,T];(t,0)\in E\}$ and $E'=E-E(0)$, and the random measure Q on $[0,T]\times\mathbb{R}$

$$Q(E)=\int_{E(0)}dW(t)+\int_{E'}z\tilde{N}(dt,dz),\quad E\in\mathcal{B}([0,T]\times\mathbb{R}),$$

For a simple function $h_n=\mathbf{1}_{E_1\times\ldots\times E_n}$, with pairwise disjoints sets $E_1,...,E_n\in\mathcal{B}([0,T]\times\mathbb{R})$, a multiple two-parameter integral with respect to the random measure Q

$$I_n(h_n)=\int_{([0,T]\times\mathbb{R})^n}h((t_1,z_1),...(t_n,z_n))Q(dt_1,dz_1)\cdot...\cdot Q(dt_n,dz_n)$$

can be defined as

$$I_n(h_n)=Q(E_1)...Q(E_n).$$

The integral can be extended to the space $L^2_{T,q,n}(\mathbb{R})$ of product measurable, deterministic functions $h:([0,T]\times\mathbb{R})^n\to\mathbb{R}$ satisfying

$$\|h\|^2_{L^2_{T,q,n}}=\int_{([0,T]\times\mathbb{R})^n}|h_n((t_1,z_1),...,(t_n,z_n))|^2q(dt_1,dz_1)\cdot...\cdot q(dt_n,dz_n)<\infty.$$

The chaotic decomposition property yields that any $\mathbb{F}$-measurable random variable $H\in\mathbb{L}^2(\mathbb{R})$ has a unique representation

$$H=\sum_{n=0}^{\infty}I_n(h_n),\quad \mathbb{P}-a.s., \tag{3.1}$$

with symmetric, in its n pair variables (t,z), functions $h_n\in L^2_{T,q,n}(\mathbb{R})$. Moreover,

$$\mathbb{E}\big[H^2\big]=\sum_{n=0}^{\infty}n!\|h_n\|^2_{L^2_{T,q,n}}. \tag{3.2}$$

From this point it is possible to study two-parameter annihilation operators (Malliavin derivatives) and creation operators (Skorohod integrals).

Definition 3.1. *1. Let $\mathbb{D}^{1,2}(\mathbb{R})$ denote the space of $\mathbb{F}$-measurable random variables $H\in\mathbb{L}^2(\mathbb{R})$ with the representation $H=\sum_{n=0}^{\infty}I_n(h_n)$ satisfying*

$$\sum_{n=1}^{\infty}nn!\|h_n\|^2_{L^2_{T,q,n}}<\infty.$$

2. *Malliavin derivative* $DH : \Omega\times[0,T]\times\mathbb{R} \to \mathbb{R}$ *of a random variable* $H \in \mathbb{D}^{1,2}(\mathbb{R})$ *is a stochastic process defined as*

$$D_{t,z}H = \sum_{n=1}^{\infty} nI_{n-1}(h_n((t,z),\cdot), \quad q - a.e.(t,z) \in [0,T] \times \mathbb{R}, \mathbb{P} - a.s..$$

3. *Let* $\mathbb{L}^{1,2}(\mathbb{R})$ *denote the space of product measurable and* $\mathbb{F}$*-adapted processes* $G : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ *satisfying*

$$\mathbb{E}\big[\int_{[0,T]\times\mathbb{R}} |G(s,y)|^2 q(ds,dy)\big] < \infty,$$
$$G(s,y) \in \mathbb{D}^{1,2}(\mathbb{R}), \quad q - a.e.(s,y) \in [0,T] \times \mathbb{R},$$
$$\mathbb{E}\big[\int_{([0,T]\times\mathbb{R})^2} |D_{t,z}G(s,y)|^2 q(ds,dy)q(dt,dz)\big] < \infty.$$

In terms of the components of the representation of $G(s,y) = \sum_{n=0}^{\infty} I_n(g_n((s,y),.)$, *q-a.e.* $(s,y) \in [0,T] \times \mathbb{R}$, *the above conditions are equivalent to*

$$\sum_{n=1}^{\infty}(n+1)(n+1)!\|\hat{g}_n\|^2_{L^2_{T,q,n+1}} < \infty,$$

where $\hat{g}$ *denotes the symmetrization with respect to all* $n+1$ *pair variables. The space* $\mathbb{L}^{1,2}(\mathbb{R})$ *is Hilbert space endowed with the norm*

$$||G||^2_{\mathbb{L}^{1,2}} = \mathbb{E}\big[\int_{[0,T]\times\mathbb{R}} |G(s,y)|^2 q(ds,dy)\big]$$
$$+\mathbb{E}\big[\int_{([0,T]\times\mathbb{R})^2} |D_{t,z}G(s,y)|^2 q(ds,dy)q(dt,dz)\big].$$

4. *Skorohod integral with respect to the random measure* Q *of a process* $G : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ *with the representation* $G(s,y) = \sum_{n=0}^{\infty} I_n(g_n((s,y),.), q - a.e.(s,y) \in [0,T] \times \mathbb{R}$, *satisfying*

$$\sum_{n=0}^{\infty}(n+1)!\|\hat{g}_n\|^2_{L^2_{T,q,n+1}} < \infty,$$

is defined as

$$\int_{[0,T]\times\mathbb{R}} G(s,y)Q(ds,dy) = \sum_{n=0}^{\infty} I_{n+1}(\hat{g}_{n+1}), \quad \mathbb{P} - a.s.$$

The following practical rules of differentiation hold. Consider a random variable H defined on $\Omega_W \times \Omega_N$. The derivative $D_{t,0}H$ is a derivative with respect to the Brownian motion component of the Lévy process and we can apply the theory of the classical Malliavin's calculus for Hilbert space-valued random variables. If $\mathbb{P}^N$-a.s.

$\omega_N \in \Omega_N$ the random variable $H(.,\omega_N)$ is Brownian differentiable in the sense of the classical Malliavin's calculus, then we have the relation

$$D_{t,0}H(\omega_W,\omega_N) = D_tH(.,\omega_N)(\omega_W), \quad \lambda - a.e.t \in [0,T], \mathbb{P}^W \times \mathbb{P}^N - a.s., \tag{3.3}$$

where D_t denotes the classical Malliavin derivative on the canonical Brownian motion space, see Proposition 3.5 in [21]. In order to define $D_{t,z}F$ for $z \neq 0$, which is a derivative with respect to the pure jump part of the Lévy process, consider the following increment quotient operator

$$\Psi_{t,z}H(\omega_W,\omega_N) = \frac{H(\omega_W,\omega_N^{t,z}) - H(\omega_W,\omega_N)}{z}, \tag{3.4}$$

where $\omega_N^{t,z}$ transforms a sequence $\omega_N = ((t_1,z_1),(t_2,z_2),...)) \in \Omega_N$ into a new sequence $\omega_N^{t,z} = ((t,z),(t_1,z_1),(t_2,z_2),...)) \in \Omega_N$ by adding a jump of size z at time t into a trajectory. For $H \in \mathbb{L}^2(\mathbb{R})$ such that $\mathbb{E}\big[\int_0^T \int_{\mathbb{R}-\{0\}} \big|\Psi_{t,z}H\big|^2 m(dz)dt\big] < \infty$ we have the relation, see Proposition 5.5 in [21],

$$D_{t,z}H(\omega) = \Psi_{t,z}H(\omega), \quad \lambda \otimes m - a.e.(t,z) \in [0,T] \times (\mathbb{R} - \{0\}), \mathbb{P} - a.s.. \tag{3.5}$$

The operator (3.4) is closely related to Picard difference operator, introduced in [17], which is just the numerator of (3.4). It is possible to define Malliavin derivative in the way that it coincides with Picard difference operator, see [8]. We would like to point out once again that we adopt the exposition from [21] and define multiple two-parameter integrals with respect to the random measure $\tilde{M}$ which yields differentiation rules (3.3) and (3.5). This is the reason why we study time-delayed backward stochastic differential equations (2.1) driven by $\tilde{M}$ and not by $\tilde{N}$.

We prove now some technical results which are applied in the next section when dealing with the main theorem of this paper. The above lemmas are extensions of the classical Brownian Malliavin differentiation rules into the canonical Lévy space.

Lemma 3.1. *Assume that $H \in \mathbb{D}^{1,2}(\mathbb{R})$. Then, for $0 \le s \le T$, $\mathbb{E}\big[H|\mathcal{F}_s\big] \in \mathbb{D}^{1,2}(\mathbb{R})$ and*

$$D_{t,z}\mathbb{E}\big[H|\mathcal{F}_s\big] = \mathbb{E}\big[D_{t,z}H|\mathcal{F}_s\big]\mathbf{1}\{t \le s\}, \quad q - a.e.(t,z) \in [0,T] \times \mathbb{R}, \mathbb{P} - a.s..$$

Proof:
The proof is a straightforward extension of the proof of Proposition 1.2.8 from [15]. Details are left to the reader. □

The next result on commutativity of Lebesgue's integration and Malliavin's differentiability is commonly applied but we haven't found a direct proof.

Lemma 3.2. *Consider an integral* $\int_{[0,T]\times\mathbb{R}} G(s,y)\eta(ds,dy)$ *with respect to a finite measure* η *on* $[0,T]\times\mathbb{R}$ *and* $\mathbb{F}$*-adapted, product measurable integrand* $G:\Omega\times[0,T]\times\mathbb{R}$ *satisfying the conditions*

$$\begin{aligned} &\mathbb{E}\big[\int_{[0,T]\times\mathbb{R}} |G(s,y)|^2\eta(ds,dy)\big] < \infty, \\ &G(s,y)\in\mathbb{D}^{1,2}(\mathbb{R}), \quad \eta-a.e.(s,y)\in[0,T]\times\mathbb{R}, \\ &\mathbb{E}\big[\int_{([0,T]\times\mathbb{R})^2} |D_{t,z}G(s,y)|^2\eta(ds,dy)q(dt,dz)\big] < \infty. \end{aligned} \tag{3.6}$$

Then $\int_{[0,T]\times\mathbb{R}} G(s,y)\eta(ds,dy)\in\mathbb{D}^{1,2}(\mathbb{R})$ *and the following differential rule*

$$D_{t,z}\int_{[0,T]\times\mathbb{R}} G(s,y)\eta(ds,dy) = \int_{[0,T]\times\mathbb{R}} D_{t,z}G(s,y)\eta(ds,dy),$$

holds q*-a.e.* $(t,z)\in[0,T]\times\mathbb{R},\mathbb{P}$*-a.s..*

Proof:
We can assume that G is a predictable process. If it is not, then its unique predictable projection exists $G^{\mathcal{P}}$ and satisfies

$$\mathbb{E}\big[\int_{[0,T]\times\mathbb{R}} G(s,y)\eta(ds,dy)\big] = \mathbb{E}\big[\int_{[0,T]\times\mathbb{R}} (G(s,y))^{\mathcal{P}}\eta(ds,dy)\big],$$

see Theorem 5.16 in [10]. Take an arbitrary random variable $H\in\mathbb{L}^2(\mathbb{R})$ with a unique martingale representation $H=\mathbb{E}[H]+\int_{[0,T]\times\mathbb{R}} H(s,y)Q(ds,dy)$, in terms of Q, and notice

$$\begin{aligned} &\mathbb{E}\big[H\int_{[0,T]\times\mathbb{R}} G(s,y)\eta(ds,dy)\big] \\ &= \mathbb{E}[H]\mathbb{E}\big[\int_{[0,T]\times\mathbb{R}} G(s,y)\eta(ds,dy)\big] + \mathbb{E}\big[\int_{[0,T]\times\mathbb{R}} H(s,y)Q(ds,dy)\int_{[0,T]\times\mathbb{R}} G(s,y)\eta(ds,dy)\big] \\ &= \mathbb{E}[H]\mathbb{E}\big[\int_{[0,T]\times\mathbb{R}} G(s,y)\eta(ds,dy)\big] = \mathbb{E}[H]\mathbb{E}\big[\int_{[0,T]\times\mathbb{R}} (G(s,y))^{\mathcal{P}}\eta(ds,dy)\big] \\ &= \mathbb{E}\big[H\int_{[0,T]\times\mathbb{R}} (G(s,y))^{\mathcal{P}}\eta(ds,dy)\big], \end{aligned}$$

so that we can conclude that

$$\int_{[0,T]\times\mathbb{R}} G(s,y)\eta(ds,dy) = \int_{[0,T]\times\mathbb{R}} (G(s,y))^{\mathcal{P}}\eta(ds,dy), \quad \mathbb{P}-a.s..$$

We approximate the integrand G with simple functions. Let $\mathcal{S}$ denote the space of simple functions

$$G^{m,l}(s,y) = \sum_{k=1}^{l} G_{0,k}\mathbf{1}_{A_k}(y) + \sum_{i=1}^{m}\sum_{k=1}^{l} G_{i,k}\mathbf{1}_{(t_i,t_{i+1}]}(s)\mathbf{1}_{A_k}(y), \quad (s,y)\in[0,T]\times\mathbb{R},$$

with $0 = t_1 < t_2 < ... < t_m = T$, disjoint Borel sets $(A_k)_{k=1}^l$ forming a partition of $\mathbb{R}$, and measurable Malliavin's differentiable random variables $G_{i,k} \in \mathcal{F}_{t_i}$, $G_{i,k} \in \mathbb{D}^{1,2}(\mathbb{R})$, $i = 0, 1, ..., m, k = 1, ..., l$. Notice that $G^{m,l} \in \mathcal{S}$ is a predictable process and satisfies the assumptions of our lemma (3.6). Moreover, the space $\mathcal{S}$ forms a π-class, see 1.1 in [10] for the definition and Proposition 5.1 in [21] which states the multiplicative property of Malliavin derivative. Consider the space $\mathcal{H}$ containing predictable processes $\bar{G}$ satisfying the assumptions (3.6), for which there exists a sequence $G^{m,l} \in \mathcal{S}$ converging to $\bar{G}$ in the norm

$$d(G^{m,l}, \bar{G}) = \mathbb{E}\big[\int_{[0,T]\times\mathbb{R}} |G^{m,l}(s,y) - \bar{G}(s,y)|^2 \eta(ds,dy)\big] \\ +\mathbb{E}\big[\int_{([0,T]\times\mathbb{R})^2} |D_{t,z}G^{m,l}(s,y) - D_{t,z}\bar{G}(s,y)|^2 \eta(ds,dy) q(dt,dz)\big] \to 0, \quad (3.7)$$

with $(m,l) \to \infty$. It is obvious that

i) $\mathcal{H}$ is a linear space containing constant functions $1 \in \mathcal{H}$ and simple functions $\mathcal{S} \in \mathcal{H}$.

Next, we show that

ii) if $(F^n)_{n\geq 1} \in \mathcal{H}$ and $0 \leq F^n \uparrow F$ pointwise for η-a.e. $(s,y) \in [0,T] \times \mathbb{R}$, $\mathbb{P}$-a.s., and F satisfies the assumptions (3.6), then $F \in \mathcal{H}$.

Consider the unique representations $F(s,y) = \sum_{k=0}^{\infty} I_k(f_k((s,y),.))$ and $F^n(s,y) = \sum_{k=0}^{\infty} I_k(f_k^n((s,y),.))$, η-a.e. $(s,y) \in [0,T]\times\mathbb{R}$, which hold due to square integrability of F and F^n. The convergence $F^n \uparrow F$ under the assumptions of ii), the relation (3.2) and Lebesgue's dominated convergence theorem ($|F^n(.)| \leq |F(.)| + \epsilon$) implies that

$$\mathbb{E}\big[|F^n(s,y) - F(s,y)|^2\big] = \sum_{k=0}^{\infty} k! \|f_k^n((s,y),.) - f_k((s,y),.\|^2_{L^2_{T,q,k}} \to 0, \quad n \to \infty,$$

for η-a.e. $(s,y) \in [0,T] \times \mathbb{R}$, and also the convergence of the components

$$\lim_{n\to\infty} f_k^n((s,y),(s_1,y_1),...,(s_k,y_k)) = f_k((s,y),(s_1,y_1),...,(s_k,y_k)), \quad \forall k \geq 1,$$

pointwise $\eta \otimes q^{\otimes k}$-a.e. $(s,y) \times (s_1,y_1) \times ... \times (s_k,y_k) \in ([0,T] \times \mathbb{R})^{k+1}$. By Definition 3.1.2 of Malliavin derivative, the relation (3.2) and Lebesgue's dominated convergence theorem ($|f_k^n(.)| \leq |f_k(.)|+\epsilon$), we can prove that the assumptions of ii) implies the convergence in the norm

$$d(F^n, F) = \sum_{k=0}^{\infty} k! \int_{[0,T]\times\mathbb{R}} \|f_k^n((s,y),.) - f_k((s,y),.)\|^2_{L^2_{T,q,k}} \eta(ds,dy) \\ + \sum_{k=0}^{\infty} kk! \int_{[0,T]\times\mathbb{R}} \|f_k^n((s,y),.) - f_k((s,y),.\|^2_{L^2_{T,q,k}} \eta(ds,dy) \to 0, \quad n \to \infty.$$

If $(F^n)_{n\geq 1} \in \mathcal{H}$, then for every n we can find $F^{n,m,l} \in \mathcal{S}$ such that $d(F^{n,m,l}, F^n) \to 0$, $(m,l) \to \infty$. As $d(F^n, F) \to 0$, $n \to \infty$, then $d(F^{n,m,l}, F) \to 0$, $(n,m,l) \to \infty$, as well and F is predictable, which proves the claim ii).
We can conclude based on Monotone Class Theorem, see Theorem 1.4 in [10], that the property (3.7) holds for all predictable processes satisfying the assumptions of our lemma (3.6).
We can differentiate now the integral of a simple function in a straightforward way to obtain for q-a.e. $(t,z) \in [0,T] \times \mathbb{R}$

$$D_{t,z} \int_{[0,T]\times\mathbb{R}} G^{m,l}(s,y)\eta(ds,dy) = \int_{[0,T]\times\mathbb{R}} D_{t,z} G^{m,l}(s,y)\eta(ds,dy), \quad \mathbb{P} - a.s..$$

By (3.7) and Cauchy-Schwarz inequality, we obtain that the sequence of random variables $\int_{[0,T]\times\mathbb{R}} G^{m,l}(s,y)\eta(ds,dy)$ converges in $\mathbb{L}^2(\mathbb{R})$

$$\mathbb{E}\big[\big|\int_{[0,T]\times\mathbb{R}} G^{m,l}(s,y)\eta(ds,dy) - \int_{[0,T]\times\mathbb{R}} G(s,y)\eta(ds,dy)\big|^2\big] \to 0, \quad (m,l) \to \infty,$$

and the sequence $D_{t,z} \int_{[0,T]\times\mathbb{R}} G^{m,l}(s,y)\eta(ds,dy)$ converges in the following norm

$$\begin{aligned} \mathbb{E}\big[\int_{[0,T]\times\mathbb{R}} \big| D_{t,z} \int_{[0,T]\times\mathbb{R}} & G^{m,l}(s,y)\eta(ds,dy) \\ - \int_{[0,T]\times\mathbb{R}} & D_{t,z} G(s,y)\eta(ds,dy)\big|^2 q(dt,dz)\big] \to 0, \quad n \to \infty. \end{aligned}$$

The closability of Malliavin derivative yields the result, see Theorem 12.6 in [8]. □

Lemma 3.3. *Assume that* $G : \Omega \times [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ *is a predictable process and* $\mathbb{E}\big[\int_{[0,T]\times\mathbb{R}} |G(s,y)|^2 q(ds,dy)\big] < \infty$ *holds. Then*

$$G \in \mathbb{L}^{1,2}(\mathbb{R}) \text{ if and only if } \int_{[0,T]\times\mathbb{R}} G(s,y)Q(ds,dy) \in \mathbb{D}^{1,2}(\mathbb{R}).$$

Moreover, if $\int_{[0,T]\times\mathbb{R}} G(s,y)Q(ds,dy) \in \mathbb{D}^{1,2}(\mathbb{R})$ *then, for* q*-a.e* $(t,z) \in [0,T] \times \mathbb{R}$,

$$D_{t,z} \int_{[0,T]\times\mathbb{R}} G(s,y)Q(ds,dy) = G(t,z) + \int_{[0,T]\times\mathbb{R}} D_{t,z} G(s,y)Q(ds,dy), \quad \mathbb{P} - a.s.,$$

and $\int_{[0,T]\times\mathbb{R}} D_{t,z} G(s,y)Q(ds,dy)$ *is a stochastic integral in Itô sense.*

Proof:
Due to square integrability of G, for q-a.e $(s,y) \in [0,T] \times \mathbb{R}$, the chaotic decomposition property yields the unique representation $G(s,y) = \sum_{n=0}^{\infty} I_n(g_n((s,y),.)$, $g_n \in L^2_{T,q,n+1}$, $n \geq 0$. Square integrability and predictability of G implies that the stochastic integral $\int_{[0,T]\times\mathbb{R}} G(s,y)Q(ds,dy)$ is well-defined in Itô sense and the Skorohod integral, which coincides under the assumptions of our lemma with the Itô integral, see

Theorem 6.1 in [21], can be defined as the series expansion $\int_{[0,T]\times\mathbb{R}} G(s,y)Q(ds,dy) = \sum_{n=0}^{\infty} I_{n+1}(\hat{g}_n)$, see Definition 3.1.4. The Skorohod integral is Malliavin's differentiable if and only if $\sum_{n=1}^{\infty}(n+1)(n+1)!\|\hat{g}_n\|^2_{L^2_{T,q,n+1}} < \infty$, see Definition 3.1.2. This series converges if and only if $G \in \mathbb{L}^{1,2}(\mathbb{R})$, by Definition 3.1.3.
Based on Section 6 in [21], we can conclude that the required differential rule holds. To prove that the integral $\int_{[0,T]\times\mathbb{R}} D_{t,z}G(s,y)Q(ds,dy)$ is well-defined in Itô sense, it is sufficient to show that the integrand $(\omega,s,y)\mapsto D_{t,z}G(s,y)(\omega)$ is a predictable mapping on $\Omega\times[0,T]\times\mathbb{R}$, as square integrability is already satisfied due to $G\in\mathbb{L}^{1,2}(\mathbb{R})$. For q-a.e. $(s,y)\in[0,T]\times\mathbb{R}$, predictability of G implies that

$$G(s,y) = \sum_{n=0}^{\infty} I_n(g_n((s,y),.) = \sum_{n=0}^{\infty} I_n(g_n((s,y),.)\mathbf{1}_{[0,s)}^{\otimes n}(.)), \quad \mathbb{P}-a.s.,$$

and applying Definition 3.1.2 of Malliavin derivative yields

$$D_{t,z}G(s,y) = \sum_{n=0}^{\infty} nI_{n-1}(g_n((s,y),(t,z),.)\mathbf{1}_{[0,s)}^{\otimes n}((t,z),.)),$$
$$q\otimes q - a.e.(t,z)\times(s,y)\in([0,T]\times\mathbb{R})^2, \mathbb{P}-a.s.,$$

from which the required predictability of the integrand follows. By a by-product, let us notice that $(\omega,s,y,t,z)\mapsto D_{t,z}G(s,y)(\omega)$ is jointly measurable. □

4 Malliavin's differentiability of a solution

The main aim of this paper is to investigate Malliavin's differentiability of a solution of a backward stochastic differential equation with a time-delayed generator. In this section, additionally to **(A1)**-**(A5)**, we assume that

(A6) the generator f is of the following form

$$f(t,y_t,z_t,u_t)$$
$$:= \quad f(t,\int_{-T}^{0} y(t+v)\alpha(dv),\int_{-T}^{0} z(t+v)\alpha(dv),\int_{-T}^{0}\int_{\mathbb{R}-\{0\}} u(t+v,z)m(dz)\alpha(dv)),$$

with a product measurable and Lipschitz continuous function $f:[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$,

(A7) the terminal value is Malliavin's differentiable $\xi\in\mathbb{D}^{1,2}(\mathbb{R})$ and satisfies

$$\mathbb{E}\big[\int_{[0,T]\times\mathbb{R}} |D_{t,z}\xi|^2 q(dt,dz)\big] < \infty,$$
$$\lim_{\epsilon\downarrow 0}\mathbb{E}\big[\int_0^T\int_{|z|\leq\epsilon} |D_{t,z}\xi|^2 m(dz)dt\big] = 0,$$

(A8) for $t \in [0,T]$, the mapping $(y,z,u) \mapsto f(t,y,z,u)$ is continuously differentiable in (y,z,u), with uniformly bounded and continuous partial derivatives f_y, f_z, f_u, and we set $f_y(t,.,.,.) = f_z(t,.,.,.) = f_u(t,.,.,.) = 0$ for $t < 0$.

We point out that under **(A6)** the generator does not depend on $\omega \in \Omega$. This dependence is omitted for simplicity of notations and can be easily included. The assumption **(A8)** is classical when dealing with Malliavin's differentiability, see Proposition 5.3 in [9] and Theorem 3.3.1 in [11]. We also would like to remark that the generator in **(A6)** depends on $\int_{-T}^0 \int_{\mathbb{R}-\{0\}} u(t+v)m(dz)\alpha(dv)$, which corresponds to a standard form of dependence appearing in BSDEs without delays and with jumps, see Proposition 2.6 and Remark 2.7 in [3].

We can state our main theorem.

Theorem 4.1. *Consider a sufficiently small time horizon T and assume that the assumptions **(A1)**-**(A8)** hold.*
1. There exists a unique solution $(Y,Z,U) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_m(\mathbb{R})$ of the time-delayed BSDE

$$\begin{aligned} Y(t) &= \xi \\ &+ \int_t^T f(r, \int_{-T}^0 Y(r+v)\alpha(dv), \int_{-T}^0 Z(r+v)\alpha(dv), \int_{-T}^0 \int_{\mathbb{R}-\{0\}} U(r+v,z)m(dz)\alpha(dv))dr \\ &- \int_t^T Z(r)dW(r) - \int_t^T \int_{\mathbb{R}-\{0\}} U(r,y)\tilde{M}(dr,dy), \quad 0 \le t \le T. \end{aligned} \tag{4.1}$$

2. There exists a unique solution $(Y^{s,0}, Z^{s,0}, U^{s,0}) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_m(\mathbb{R})$ of the time-delayed BSDE

$$\begin{aligned} Y^{s,0}(t) &= D_{s,0}\xi + \int_t^T f^{s,0}(r)dr - \int_t^T Z^{s,0}(r)dW(r) \\ &- \int_t^T \int_{\mathbb{R}-\{0\}} U^{s,0}(r,y)\tilde{M}(dr,dy), \quad 0 \le s \le t \le T, \end{aligned} \tag{4.2}$$

with the generator

$$
\begin{aligned}
&f^{s,0}(r)\\
= \ & f_y(r, \int_{-T}^0 Y(r+v)\alpha(dv), \int_{-T}^0 Z(r+v)\alpha(dv), \int_{-T}^0 \int_{\mathbb{R}-\{0\}} U(r+v,y)m(dy)\alpha(dv))\\
&\cdot \int_{-T}^0 Y^{s,0}(r+v)\alpha(dv)\\
+ \ & f_z(r, \int_{-T}^0 Y(r+v)\alpha(dv), \int_{-T}^0 Z(r+v)\alpha(dv), \int_{-T}^0 \int_{\mathbb{R}-\{0\}} U(r+v,y)m(dy)\alpha(dv))\\
&\cdot \int_{-T}^0 Z^{s,0}(r+v)\alpha(dv)\\
+ \ & f_u(r, \int_{-T}^0 Y(r+v)\alpha(dv), \int_{-T}^0 Z(r+v)\alpha(dv), \int_{-T}^0 \int_{\mathbb{R}-\{0\}} U(r+v,y)m(dy)\alpha(dv))\\
&\cdot \int_{-T}^0 \int_{\mathbb{R}-\{0\}} U^{s,0}(r+v,y)m(dy)\alpha(dv), \qquad (4.3)
\end{aligned}
$$

and also there exists a unique solution $(Y^{s,z}, Z^{s,z}, U^{s,z}) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_m(\mathbb{R})$ *of the time-delayed BSDE*

$$
\begin{aligned}
Y^{s,z}(t) &= D_{s,z}\xi + \int_t^T f^{s,z}(r)dr - \int_t^T Z^{s,z}(r)dW(r)\\
&\quad - \int_t^T \int_{\mathbb{R}-\{0\}} U^{s,z}(r,y)\tilde{M}(dr,dy), \quad 0 \le s \le t \le T, z \neq 0, \quad (4.4)
\end{aligned}
$$

with the generator

$$
\begin{aligned}
&f^{s,z}(r)\\
= \ & (f(r, z\int_{-T}^0 Y^{s,z}(r+v)\alpha(dv) + \int_{-T}^0 Y(r+v)\alpha(dv),\\
& z\int_{-T}^0 Z^{s,z}(r+v)\alpha(dv) + \int_{-T}^0 Z(r+v)\alpha(dv),\\
& z\int_{-T}^0 \int_{\mathbb{R}-\{0\}} U^{s,z}(r+v,y)m(dy)\alpha(dv) + \int_{-T}^0 \int_{\mathbb{R}-\{0\}} U(r+v,x)m(dy)\alpha(dv))\\
& -f(r, \int_{-T}^0 Y(r+v)\alpha(dv), \int_{-T}^0 Z(r+v)\alpha(dv), \int_{-T}^0 \int_{\mathbb{R}-\{0\}} U(r+v,y)m(dy)\alpha(dv)))/z,\\
& \qquad (4.5)
\end{aligned}
$$

and we set

$$
Y^{s,z}(t) = Z^{s,z}(t) = U^{s,z}(t,y) = 0, \quad (y,z) \in (\mathbb{R}-\{0\}) \times \mathbb{R}, \mathbb{P}-a.s., t < s \le T. (4.6)
$$

Then $(Y,Z,U) \in \mathbb{L}^{1,2}(\mathbb{R}) \times \mathbb{L}^{1,2}(\mathbb{R}) \times \mathbb{L}^{1,2}(\mathbb{R})$ *and* $(Y^{s,z}(t), Z^{s,z}(t), U^{s,z}(t,y))_{0 \le s,t \le T, (y,z) \in (\mathbb{R}-\{0\}) \times \mathbb{R}}$ *is a version of* $(D_{s,z}Y(t), D_{s,z}Z(t), D_{s,z}U(t,y))_{0 \le s,t \le T, (y,z) \in (\mathbb{R}-\{0\}) \times \mathbb{R}}$.

Proof:
We follow the idea from the proofs of Proposition 5.3 in [9] and Theorem 3.3.1 in [11]. By C let us denote a finite constant which value may change from line to line.
Step 1) Existence of unique solutions of the equations (4.1), (4.2) and (4.4) for a sufficiently small time horizon T.
The existence of a unique solution $(Y, Z, U) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_m(\mathbb{R})$ of (4.1) follows from Theorem 2.1, as the assumptions **(A1)-(A5)** remain satisfied. Under the additional assumptions **(A7)** and **(A8)**, the time-delayed BSDEs (4.2) and (4.4), with the generators (4.3) and (4.5), fulfill the conditions of Theorem 2.1, in particular the corresponding generators are Lipschitz continuous in the sense of **(A3)**, and we can conclude that for $(s, z) \in [0, T] \times \mathbb{R}$ there exists a unique solution $(Y^{s,z}, Z^{s,z}, U^{s,z}) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_m(\mathbb{R})$ of (4.2) or (4.4) satisfying (4.6) .
Step 2) Consider a sequence (Y^n, Z^n, U^n) which converges to (Y, Z, U). Given that $(Y^n, Z^n, U^n) \in \mathbb{L}^{1,2}(\mathbb{R}) \times \mathbb{L}^{1,2}(\mathbb{R}) \times \mathbb{L}^{1,2}(\mathbb{R})$ we have that $(Y^{n+1}, Z^{n+1}, U^{n+1}) \in \mathbb{L}^{1,2}(\mathbb{R}) \times \mathbb{L}^{1,2}(\mathbb{R}) \times \mathbb{L}^{1,2}(\mathbb{R})$. Moreover, $\mathbb{E}\big[\int_{[0,T]} \sup_{t\in[0,T]} |D_{s,z}Y^n(t)|^2 q(ds, dz)\big] < \infty$ implies $\mathbb{E}\big[\int_{[0,T]\times\mathbb{R}} \sup_{t\in[0,T]} |D_{s,z}Y^{n+1}(t)|^2 q(ds, dz)\big] < \infty$.
We study Picard iterations

$$
\begin{aligned}
Y^{n+1}(t) &= \xi + \int_t^T f^n(r)dr \\
&\quad - \int_t^T Z^{n+1}(r)dW(r) - \int_t^T \int_{\mathbb{R}-\{0\}} U^{n+1}(r, y)\tilde{M}(dr, dy), \quad 0 \le t \le T, \quad (4.7)
\end{aligned}
$$

where we denote

$$
\begin{aligned}
&f^n(r) \\
&= f(r, \int_{-T}^0 Y^n(r+v)\alpha(dv), \int_{-T}^0 Z^n(r+v)\alpha(dv), \int_{-T}^0 \int_{\mathbb{R}-\{0\}} U^n(r+v, y)m(dy)\alpha(dv)).
\end{aligned}
$$

We first establish Malliavin's differentiability of $\int_t^T f^n(r)dr$ by applying Lemma 3.2. Notice that $Y^n(t) \in \mathbb{D}^{1,2}(\mathbb{R})$, λ-a.e. $t \in [-T, T]$, and similarly to (2.2), we can derive

$$
\begin{aligned}
&\int_0^T \mathbb{E}\big[\int_{-T}^0 |Y^n(r+v)|^2\alpha(dv)\big]dr = \mathbb{E}\big[\int_{-T}^0 \int_0^T |Y^n(r+v)|^2 dr\alpha(dv)\big] \\
&\mathbb{E}\big[\int_{-T}^0 \int_v^{T+v} |Y^n(w)|^2 dw\alpha(dv)\big] \le T\mathbb{E}\big[\sup_{w\in[0,T]} |Y^n(w)|^2)\big] < \infty
\end{aligned}
$$

together with

$$
\begin{aligned}
&\int_0^T \mathbb{E}\big[\int_{[0,T]\times\mathbb{R}} \int_{-T}^0 |D_{s,z}Y^n(r+v)|^2\alpha(dv)q(ds, dz)\big]dr \\
&\le \ T\mathbb{E}\big[\int_{[0,T]\times\mathbb{R}} \sup_{w\in[0,T]} |D_{s,z}Y^n(w)|^2 q(ds, dz)\big] < \infty.
\end{aligned}
$$

This yields that the assumptions of Lemma 3.2 are satisfied, λ-a.e. $r \in [0,T]$ $\int_{-T}^0 Y^n(r+v)\alpha(dv) \in \mathbb{D}^{1,2}(\mathbb{R})$, and we have

$$D_{s,z}\int_{-T}^0 Y^n(r+v)\alpha(dv) = \int_{-T}^0 D_{s,z}Y^n(r+v)\alpha(dv), \quad \mathbb{P}-a.s.,$$

for $q\otimes\lambda$-a.e. $(s,z,r)\in[0,T]\times\mathbb{R}\times[0,T]$. In the analogous way we derive that

$$D_{s,z}\int_{-T}^0 Z^n(r+v)\alpha(dv) = \int_{-T}^0 D_{s,z}Z^n(r+v)\alpha(dv), \quad \mathbb{P}-a.s.,$$
$$D_{s,z}\int_{-T}^0\int_{\mathbb{R}-\{0\}} U^n(r+v,y)m(dy)\alpha(dv) = \int_{-T}^0\int_{\mathbb{R}-\{0\}} D_{s,z}U^n(r+v,y)m(dy)\alpha(dv), \mathbb{P}-a.s.,$$

for $q\otimes\lambda$-a.e. $(s,z,r)\in[0,T]\times\mathbb{R}\times[0,T]$. We remark that from Proposition 5.4 and Proposition 5.5 in [21] follows the difference rule that if a random variable H is in the domain of $D_{t,z}, z\neq 0$, then $g(H)$, for Lipschitz continuous, deterministic, real function g is also in the domain of $D_{t,z}, z\neq 0$. By applying the above remark and the chain rule for $D_{t,0}$, see Theorem 2 in [18], we obtain that λ-a.e. $r\in[0,T]$ $f^n(r)\in\mathbb{D}^{1,2}(\mathbb{R})$, and for $q\otimes\lambda$-a.e. $(s,z,r)\in[0,T]\times\mathbb{R}\times[0,T]$

$$\begin{aligned}
&D_{s,0}f^n(r)\\
&= f_y(r,\int_{-T}^0 Y^n(r+v)\alpha(dv),\int_{-T}^0 Z^n(r+v)\alpha(dv),\int_{-T}^0\int_{\mathbb{R}-\{0\}} U^n(r+v,y)m(dy)\alpha(dv))\\
&\quad\cdot\int_{-T}^0 D_{s,0}Y^n(r+v)\alpha(dv)\\
&+ f_z(r,\int_{-T}^0 Y^n(r+v)\alpha(dv),\int_{-T}^0 Z^n(r+v)\alpha(dv),\int_{-T}^0\int_{\mathbb{R}-\{0\}} U^n(r+v,y)m(dy)\alpha(dv))\\
&\quad\cdot\int_{-T}^0 D_{s,0}Z^n(r+v)\alpha(dv)\\
&+ f_u(r,\int_{-T}^0 Y^n(r+v)\alpha(dv),\int_{-T}^0 Z^n(r+v)\alpha(dv),\int_{-T}^0\int_{\mathbb{R}-\{0\}} U^n(r+v,y)m(dy)\alpha(dv))\\
&\quad\cdot\int_{-T}^0\int_{\mathbb{R}-\{0\}} D_{s,0}U^n(r+v,y)m(dy)\alpha(dv),
\end{aligned}\tag{4.8}$$

and, for $z \neq 0$,

$$\begin{aligned}
&D_{s,z}f^n(r)\\
&= \quad (f(r, z\int_{-T}^{0} D_{s,z}Y^n(r+v)\alpha(dv) + \int_{-T}^{0} Y^n(r+v)\alpha(dv),\\
&\quad z\int_{-T}^{0} D_{s,z}Z^n(r+v)\alpha(dv) + \int_{-T}^{0} Z^n(r+v)\alpha(dv),\\
&\quad z\int_{-T}^{0}\int_{\mathbb{R}-\{0\}} D_{s,z}U^n(r+v,y)m(dy)\alpha(dv) + \int_{-T}^{0}\int_{\mathbb{R}-\{0\}} U^n(r+v,x)m(dy)\alpha(dv))\\
&\quad -f(r, \int_{-T}^{0} Y^n(r+v)\alpha(dv), \int_{-T}^{0} Z^n(r+v)\alpha(dv), \int_{-T}^{0}\int_{\mathbb{R}-\{0\}} U^n(r+v,y)m(dy)\alpha(dv)))/z.
\end{aligned} \tag{4.9}$$

We left it to the reader to check that

$$\mathbb{E}\big[\int_t^T |f^n(r)|^2 dr\big] < \infty,$$
$$\mathbb{E}\big[\int_t^T |D_{s,z}f^n(r)|^2 dr q(dt,dz)\big] < \infty,$$

and applying Lemma 3.2 again we derive (in the formal way) that $\xi + \int_t^T f^n(r)dr \in \mathbb{D}^{1,2}(\mathbb{R})$ with Malliavin derivative, for $0 \leq t \leq T$,

$$D_{s,z}\xi + \int_t^T D_{s,z}f^n(r)dr, \quad q-a.e.(s,z) \in [0,T]\times\mathbb{R}, \tag{4.10}$$

where $D_{s,z}f^n$ defined in (4.8) and (4.9). Moreover, based on Lemma 3.1 we can state that

$$Y^{n+1}(t) = \mathbb{E}\big[\xi + \int_t^T f^n(r)dr|\mathcal{F}_t\big] \in \mathbb{D}^{1,2}(\mathbb{R}), \quad 0 \leq t \leq T,$$

and from the equation(4.7) we conclude that

$$\int_t^T Z^{n+1}(r)dW(r) \in \mathbb{D}^{1,2}(\mathbb{R}), \quad 0 \leq t \leq T, \tag{4.11}$$

and

$$\int_t^T \int_{\mathbb{R}-\{0\}} U^{n+1}(r,y)\tilde{M}(dr,dy) \in \mathbb{D}^{1,2}(\mathbb{R}), \quad 0 \leq t \leq T. \tag{4.12}$$

Lemma 3.3 yields now that $(Z^{n+1}, U^{n+1}) \in \mathbb{L}^{1,2}(\mathbb{R}) \times \mathbb{L}^{1,2}(\mathbb{R})$.
We can differentiate the recursive equation (4.7) in the formal way and obtain for q-a.e.$(s,z) \in [0,T]\times\mathbb{R}$

$$\begin{aligned}
D_{s,z}Y^{n+1}(t) \quad &= \quad D_{s,z}\xi + \int_t^T D_{s,z}f^n(r)dr - \int_t^T D_{s,z}Z^{n+1}(r)dW(r)\\
&\quad - \int_t^T \int_{\mathbb{R}-\{0\}} D_{s,z}U^{n+1}(r,y)\tilde{M}(dr,dy), \quad s \leq t \leq T,
\end{aligned} \tag{4.13}$$

and

$$D_{s,z}Y^{n+1}(t) = D_{s,z}Z^{n+1}(t) = D_{s,z}U^{n+1}(t,y) = 0, \quad t < s, y \in (\mathbb{R} - \{0\}). \quad (4.14)$$

Notice that the time-delayed BSDE (4.13) with the generator (4.8) or (4.9) fulfills the assumptions of Theorem 2.1, in particular the corresponding generators are Lipschitz continuous in the sense of **(A3)**. We conclude that for q-a.e.$(s,z) \in [0,T] \times \mathbb{R}$ there exists a unique solution $(D_{s,z}Y^{n+1}, D_{s,z}Z^{n+1}, D_{s,z}U^{n+1}) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_m$ of (4.13) satisfying (4.14). By applying Lemma 2.1, with $\tilde{\xi} = 0$ and $\tilde{f} = f$, together with the estimate (2.7), we derive the inequality

$$\begin{aligned}
&\big\|D_{s,z}Y^{n+1}\big\|^2_{\mathbb{S}^2} + \big\|D_{s,z}Z^{n+1}\big\|^2_{\mathbb{H}^2} + \big\|D_{s,z}U^{n+1}\big\|^2_{\mathbb{H}^2_m} \\
&\quad \leq \; C\big(\mathbb{E}\big[|D_{s,z}\xi|^2\big] + \big\|D_{s,z}Y^n\big\|^2_{\mathbb{S}^2} + \big\|D_{s,z}Z^n\big\|^2_{\mathbb{H}^2} + \big\|D_{s,z}U^n\big\|^2_{\mathbb{H}^2_m}\big), \quad (4.15)
\end{aligned}$$

which yields that $\mathbb{E}\big[\int_{[0,T]\times\mathbb{R}} \sup_{t\in[0,T]} |D_{s,z}Y^{n+1}(t)|^2 q(ds,dz)\big] < \infty$ and, in particular, $Y^{n+1} \in \mathbb{L}^{1,2}(\mathbb{R})$.

Step 3) Integrability of the solution $Y^{s,z}(t), Z^{s,z}(t), U^{s,z}(t,y)$ with respect to the product measure q on $([0,T]\times\mathbb{R})^2$.

Take $(s,z) \in [0,T] \times \mathbb{R}$. Consider the unique solution $(Y^{s,z}, Z^{s,z}, U^{s,z}) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_m(\mathbb{R})$ of the equation (4.2) or (4.4). Lemma 2.1, with $\tilde{\xi} = 0$ and $\tilde{f} = f$, together with the estimates (2.7) and (2.8) yield the inequality

$$\begin{aligned}
&\big\|Y^{s,z}\big\|^2_{\mathbb{S}^2} + \big\|Z^{s,z}\big\|^2_{\mathbb{H}^2} + \big\|U^{s,z}\big\|^2_{\mathbb{H}^2_m} \\
&\quad \leq \delta(T,K,\beta,\alpha)\big(\mathbb{E}\big[|D_{s,z}\xi|^2\big] + \big\|Y^{s,z}\big\|^2_{\mathbb{S}^2} + \big\|Z^{s,z}\big\|^2_{\mathbb{H}^2} + \big\|U^{s,z}\big\|^2_{\mathbb{H}^2_m}\big).
\end{aligned}$$

For a sufficiently small T we obtain

$$\big\|Y^{s,z}\big\|^2_{\mathbb{S}^2} + \big\|Z^{s,z}\big\|^2_{\mathbb{H}^2} + \big\|U^{s,z}\big\|^2_{\mathbb{H}^2_m} \leq C\mathbb{E}\big[|D_{s,z}\xi|^2\big], \quad (4.16)$$

and we arrive at

$$\begin{aligned}
&\mathbb{E}\Big[\int_{([0,T]\times\mathbb{R})^2} |Y^{s,z}(t)|^2 q(dt,dy)q(ds,dz)\Big] < \infty, \\
&\mathbb{E}\Big[\int_{([0,T]\times\mathbb{R})^2} |Z^{s,z}(t)|^2 q(dt,dy)q(ds,dz)\Big] < \infty \\
&\mathbb{E}\Big[\int_{([0,T]\times\mathbb{R})^2} |U^{s,z}(t,y)|^2 q(dt,dy)q(ds,dz)\Big] < \infty.
\end{aligned}$$

Step 4) The convergence of (Y^n, Z^n, U^n) in $\mathbb{L}^{1,2}(\mathbb{R}) \times \mathbb{L}^{1,2}(\mathbb{R}) \times \mathbb{L}^{1,2}(\mathbb{R})$.

We already know that (Y^n, Z^n, U^n) converges in $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_m(\mathbb{R})$, see Theorem 2.1. We have to prove that the corresponding Malliavin derivatives converge. The convergence

$$\begin{aligned}
&\lim_{n\to\infty} \int_{[0,T]\times\mathbb{R}} \big(\big\|Y^{s,z} - D_{s,z}Y^{n+1}\big\|^2_{\mathbb{H}^2} \\
&\quad + \big\|Z^{s,z} - D_{s,z}Z^{n+1}\big\|^2_{\mathbb{H}^2} + \big\|U^{s,z} - D_{s,z}U^{n+1}\big\|^2_{\mathbb{H}^2_m}\big) q(ds,dz) = 0,
\end{aligned}$$

for $z=0$ can be proved in the similar way as in the case of a BSDE without a delay driven by a Brownian motion, see for example Theorem 3.3.1 in [11]. We only prove the convergence for $z\neq 0$.
Lemma 2.1, applied to the time-delayed BSDEs (4.4) and (4.13) with (4.14), yields the inequality

$$\begin{aligned}
&\big\|Y^{s,z}-D_{s,z}Y^{n+1}\big\|^2_{\mathbb{S}^2}+\big\|Z^{s,z}-D_{s,z}Z^{n+1}\big\|^2_{\mathbb{H}^2}+\big\|U^{s,z}-D_{s,z}U^{n+1}\big\|^2_{\mathbb{H}^2_m}\\
&\leq C\mathbb{E}\big[\int_s^T e^{\beta r}|f^{s,z}(r)-D_{s,z}f^n(r)|^2dr\big],\quad q-a.e.(s,z)\in[0,T]\times\mathbb{R}. \quad (4.17)
\end{aligned}$$

First, due to Lipschitz continuity of the generator f, assumed in **(A3)**, for $\lambda\otimes m\otimes\lambda$-a.e. $(s,z,r)\in[0,T]\times(\mathbb{R}-\{0\})\times[0,T]$, we have the following two estimates

$$\begin{aligned}
&|f^{s,z}(r)-D_{s,z}f^n(r)|^2\\
&\leq\ 2K\big(\int_{-T}^0|Y^{s,z}(r+v)|^2\alpha(dv)+\int_{-T}^0|Z^{s,z}(r+v)|^2\alpha(dv)\\
&\quad+\int_{-T}^0\int_{\mathbb{R}-\{0\}}|U^{s,z}(r+v,y)|^2m(dy)\alpha(dv)\big)\\
&\quad+2K\big(\int_{-T}^0|D_{s,z}Y^n(r+v)|^2\alpha(dv)+\int_{-T}^0|D_{s,z}Z^n(r+v)|^2\alpha(dv)\\
&\quad+\int_{-T}^0\int_{\mathbb{R}-\{0\}}|D_{s,z}U^n(r+v,y)|^2m(dy)\alpha(dv)\big), \quad (4.18)
\end{aligned}$$

and

$$\begin{aligned}
&|f^{s,z}(r)-D_{s,z}f^n(r)|^2\\
&\leq\ 2K\big(\int_{-T}^0|Y^{s,z}(r+v)-D_{s,z}Y^n(r+v)|^2\alpha(dv)\\
&\quad+\int_{-T}^0|Z^{s,z}(r+v)-D_{s,z}Z^n(r+v)|^2\alpha(dv)\\
&\quad+\int_{-T}^0\int_{\mathbb{R}-\{0\}}|U^{s,z}(r+v,y)-D_{s,z}U^n(r+v,y)|^2m(dy)\alpha(dv)\big)\\
&\quad+2K\big(\int_{-T}^0|Y(r+v)-Y^n(r+v)|^2\alpha(dv)+\int_{-T}^0|Z(r+v)-Z^n(r+v)|^2\alpha(dv)\\
&\quad+\int_{-T}^0\int_{\mathbb{R}-\{0\}}|U(r+v,y)-U^n(r+v,y)|^2m(dy)\alpha(dv)\big)/z. \quad (4.19)
\end{aligned}$$

Notice that

$$\begin{aligned}
&\int_{[0,T]\times(\mathbb{R}-\{0\})} \big(\big\|Y^{s,z}-D_{s,z}Y^{n+1}\big\|_{\mathbb{S}^2}^2 \\
&\qquad +\big\|Z^{s,z}-D_{s,z}Z^{n+1}\big\|_{\mathbb{H}^2}^2+\big\|U^{s,z}-D_{s,z}U^{n+1}\big\|_{\mathbb{H}_m^2}^2\big)q(ds,dz) \\
&= \lim_{\epsilon\downarrow 0}\int_0^T\int_{|z|>\epsilon}\big(\big\|Y^{s,z}-D_{s,z}Y^{n+1}\big\|_{\mathbb{H}^2}^2 \\
&\qquad +\big\|Z^{s,z}-D_{s,z}Z^{n+1}\big\|_{\mathbb{H}^2}^2+\big\|U^{s,z}-D_{s,z}U^{n+1}\big\|_{\mathbb{H}_m^2}^2\big)m(dz)ds. \qquad (4.20)
\end{aligned}$$

We prove that the convergence is uniform in n.
Choose a sufficiently small $\varepsilon>0$. By the assumption **(A7)** we can find $\bar{\epsilon}$ such that

$$\mathbb{E}\big[\int_0^T\int_{|z|\le\bar{\epsilon}}|D_{s,z}\xi|^2 m(dz)ds\big]<\varepsilon.$$

Take arbitrary $0<\epsilon_1<\epsilon_2\le\bar{\epsilon}$. By applying the inequality (4.17), the estimate (4.18) and by similar calculations as in (2.7) we can derive

$$\begin{aligned}
&\int_0^T\int_{\epsilon_1<|z|\le\epsilon_2}\big(\big\|Y^{s,z}-D_{s,z}Y^{n+1}\big\|_{\mathbb{S}^2}^2 \\
&\qquad +\big\|Z^{s,z}-D_{s,z}Z^{n+1}\big\|_{\mathbb{H}^2}^2+\big\|U^{s,z}-D_{s,z}U^{n+1}\big\|_{\mathbb{H}_m^2}^2\big)m(dz)ds \\
&\le C\int_0^T\int_{\epsilon_1<|z|\le\epsilon_2}\big(\mathbb{E}\big[\int_s^T e^{\beta r}|f^{s,z}(r)-D_{s,z}f^n(r)|^2dr\big]\big)m(dz)ds \\
&\le C\Big\{\int_0^T\int_{\epsilon_1<|z|\le\epsilon_2}\big(\big\|Y^{s,z}\big\|_{\mathbb{S}^2}^2+\big\|Z^{s,z}\big\|_{\mathbb{H}^2}^2+\big\|U^{s,z}\big\|_{\mathbb{H}_m^2}^2\big)m(dz)ds \\
&\qquad +\int_0^T\int_{\epsilon_1<|z|\le\epsilon_2}\big(\big\|D_{s,z}Y^n\big\|_{\mathbb{S}^2}^2+\big\|D_{s,z}Z^n\big\|_{\mathbb{H}^2}^2+\big\|D_{s,z}U^n\big\|_{\mathbb{H}_m^2}^2\big)m(dz)ds\Big\}. \qquad (4.21)
\end{aligned}$$

To estimate the first term in (4.21), notice that the inequality (4.16) yields

$$\begin{aligned}
&\int_0^T\int_{\epsilon_1<|z|\le\epsilon_2}\big(\big\|Y^{s,z}\big\|_{\mathbb{S}^2}^2+\big\|Z^{s,z}\big\|_{\mathbb{H}^2}^2+\big\|U^{s,z}\big\|_{\mathbb{H}_m^2}^2\big)m(dz)ds \\
&\qquad \le C\mathbb{E}\big[\int_0^T\int_{\epsilon_1<|z|\le\epsilon_2}|D_{s,z}\xi|^2m(dz)ds\big]<C\varepsilon. \qquad (4.22)
\end{aligned}$$

By applying the inequality (4.15), with $C<1$, we can estimate the second term in (4.21)

$$
\begin{aligned}
&\int_0^T\int_{\epsilon_1<|z|\le\epsilon_2}\big(\|D_{s,z}Y^n\|^2_{\mathbb{S}^2}+\|D_{s,z}Z^n\|^2_{\mathbb{H}^2}+\|D_{s,z}U^n\|^2_{\mathbb{H}^2_m}\big)m(dz)ds\\
\le\;& C\mathbb{E}\Big[\int_0^T\int_{\epsilon_1<|z|\le\epsilon_2}|D_{s,z}\xi|^2m(dz)ds\Big]\\
&+C\int_0^T\int_{\epsilon_1<|z|\le\epsilon_2}\big(\|D_{s,z}Y^{n-1}\|^2_{\mathbb{S}^2}+\|D_{s,z}Z^{n-1}\|^2_{\mathbb{H}^2}+\|D_{s,z}U^{n-1}\|^2_{\mathbb{H}^2_m}\big)m(dz)ds\\
<\;& \frac{C\varepsilon}{1-C}\\
&+C^n\int_0^T\int_{\epsilon_1<|z|\le\epsilon_2}\big(\|D_{s,z}Y^0\|^2_{\mathbb{S}^2}+\|D_{s,z}Z^0\|^2_{\mathbb{H}^2}+\|D_{s,z}U^0\|^2_{\mathbb{H}^2_m}\big)m(dz)ds. \quad (4.23)
\end{aligned}
$$

Compare with the estimate (2.7) to conclude that $C<1$ holds indeed for T sufficiently small. Choosing $Y^0=Z^0=U^0=0$ and combining (4.22) and (4.23) gives the uniform convergence of (4.20).
Next, by applying the inequality (4.17), the estimate (4.19) and similar calculations as in (2.7) and (2.8) we can derive

$$
\begin{aligned}
&\int_0^T\int_{|z|>\epsilon}\big(\|Y^{s,z}-D_{t,z}Y^{n+1}\|^2_{\mathbb{S}^2}\\
&\quad+\|Z^{s,z}-D_{s,z}Z^{n+1}\|^2_{\mathbb{H}^2}+\|U^{s,z}-D_{s,z}U^{n+1}\|^2_{\mathbb{H}^2_m}\big)m(dz)ds\\
\le\;& C\int_0^T\int_{|z|>\epsilon}\Big(\mathbb{E}\Big[\int_s^T e^{\beta r}|f^{s,z}(r)-D_{s,z}f^n(r)|^2dr\Big]\Big)m(dz)ds\\
\le\;& \delta(T,K,\beta,\alpha)\Big\{\int_0^T\int_{|z|>\epsilon}\big(\|Y^{s,z}-D_{t,z}Y^n\|^2_{\mathbb{S}^2}\\
&\quad+\|Z^{s,z}-D_{s,z}Z^n\|^2_{\mathbb{H}^2}+\|U^{s,z}-D_{s,z}U^n\|^2_{\mathbb{H}^2_m}\big)m(dz)ds\\
&\quad+\big(\|Y^n-Y\|^2_{\mathbb{S}^2}+\|Z^n-Z\|^2_{\mathbb{H}^2}+\|U^n-U\|^2_{\mathbb{H}^2_m}\big)\int_{|z|>\epsilon}\nu(dz)\Big\},
\end{aligned}
$$

with $\delta:=\delta(T,K,\beta,\alpha)<1$ for T sufficiently small.
Due to convergence of (Y^n,Z^n,U^n), for an arbitrary sufficiently small $\varepsilon>0$ we can find N sufficiently large such that for all $n\ge N$

$$
\big(\|Y^n-Y\|^2_{\mathbb{S}^2}+\|Z^n-Z\|^2_{\mathbb{H}^2}+\|U^n-U\|^2_{\mathbb{H}^2_m}\big)\int_{|z|>\epsilon}\nu(dz)<\varepsilon.
$$

We derive the recursion for $n \geq N$

$$\begin{aligned}
&\int_0^T \int_{|z|>\epsilon} \big(\big\|Y^{s,z} - D_{s,z}Y^{n+1}\big\|^2_{\mathbb{S}^2} \\
&\quad + \big\|Z^{s,z} - D_{s,z}Z^{n+1}\big\|^2_{\mathbb{H}^2} + \big\|U^{s,z} - D_{s,z}U^{n+1}\big\|^2_{\mathbb{H}^2_m}\big) m(dz)ds \\
< \; & \delta\Big\{\int_0^T \int_{|z|>\epsilon} \big(\big\|Y^{s,z} - D_{s,z}Y^{n}\big\|^2_{\mathbb{S}^2} \\
&\quad + \big\|Z^{s,z} - D_{s,z}Z^{n}\big\|^2_{\mathbb{H}^2} + \big\|U^{s,z} - D_{s,z}U^{n}\big\|^2_{\mathbb{H}^2_m}\big) m(dz)ds\Big\} + \delta\varepsilon \\
< \; & \delta^{n-N} \int_0^T \int_{|z|>\epsilon} \big(\big\|Y^{s,z} - D_{s,z}Y^{N}\big\|^2_{\mathbb{S}^2} \\
&\quad + \big\|Z^{s,z} - D_{s,z}Z^{N}\big\|^2_{\mathbb{H}^2} + \big\|U^{s,z} - D_{s,z}U^{N}\big\|^2_{\mathbb{H}^2_m}\big) m(dz)ds + \frac{\delta\varepsilon}{1-\delta},
\end{aligned}$$

and finally we conclude that

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^T \int_{|z|>\epsilon} \big(\big\|Y^{s,z} - D_{s,z}Y^{n+1}\big\|^2_{\mathbb{S}^2} \\
&\quad + \big\|Z^{s,z} - D_{s,z}Z^{n+1}\big\|^2_{\mathbb{H}^2} + \big\|U^{s,z} - D_{s,z}U^{n+1}\big\|^2_{\mathbb{H}^2_m}\big) m(dz)ds = 0.
\end{aligned}$$

The convergence of

$$\begin{aligned}
&\lim_{n\to\infty} \int_{[0,T]\times(\mathbb{R}-\{0\})} \big(\big\|Y^{s,z} - D_{s,z}Y^{n+1}\big\|^2_{\mathbb{S}^2} \\
&\quad + \big\|Z^{s,z} - D_{s,z}Z^{n+1}\big\|^2_{\mathbb{H}^2} + \big\|U^{s,z} - D_{s,z}U^{n+1}\big\|^2_{\mathbb{H}^2_m}\big) q(ds,dz) = 0,
\end{aligned}$$

now follows by interchanging the limits in (4.20).
Step 4) As the space $\mathbb{L}^{1,2}(\mathbb{R})$ is Hilbert space and Malliavin derivative is a closed operator, see Theorem 12.6 in [8], the claim that $(Y, Z, U) \in \mathbb{L}^{1,2}(\mathbb{R}) \times \mathbb{L}^{1,2}(\mathbb{R}) \times \mathbb{L}^{1,2}(\mathbb{R})$ and $(Y^{s,z}(t), Z^{s,z}(t), U^{s,z}(t,y))_{0\leq s,t\leq T,(y,z)\in(\mathbb{R}-\{0\})\mathbb{R}}$ is a version of the derivative $(D_{s,z}Y(t), D_{s,z}Z(t), D_{s,z}U(t,y))_{0\leq s,t\leq T,(y,z)\in(\mathbb{R}-\{0\})\mathbb{R}}$ has been proved. □

The last lemma shows that the relation that a solution (Z, U) can be interpreted in terms of Malliavin trace of Y still holds for BSDEs with time-delayed generators.

Lemma 4.1. *Under the assumptions of Theorem 4.1, we have that*

$$\begin{aligned}
\big((D_{t,0}Y(t))^{\mathcal{P}}\big)_{0\leq t\leq T} &\text{ is a version of } \big(Z(t)\big)_{0\leq t\leq T}, \\
\big((D_{t,z}Y(t))^{\mathcal{P}}\big)_{0\leq t\leq T, z\in(\mathbb{R}-\{0\})} &\text{ is a version of } \big(U(t,z)\big)_{0\leq t\leq T, z\in(\mathbb{R}-\{0\})},
\end{aligned}$$

where $(\cdot)^{\mathcal{P}}$ denotes a predictable projection of a process.

Proof:
The solution of (4.1) satisfies

$$\begin{aligned} Y(s) & \\ = \; & Y(0) - \int_0^s f(r, \int_{-T}^0 Y(r+v)\alpha(dv), \int_{-T}^0 Z(r+v)\alpha(dv), \int_{-T}^0 \int_{\mathbb{R}-\{0\}} U(r+v,y)m(dy)dv)dr \\ & + \int_0^s Z(r)dW(r) + \int_0^s \int_{\mathbb{R}-\{0\}} U(r,y)\tilde{M}(dr,dy), \quad 0 \le s \le T. \end{aligned} \tag{4.24}$$

By differentiating (4.24), see Lemma 3.3, we obtain, q-a.e. $(u,z) \in [0,T] \times \mathbb{R}$,

$$\begin{aligned} D_{u,0}Y(s) &= Z(u) - \int_u^s D_{u,0}f(r)dr + \int_u^s D_{u,0}Z(r)dW(r) \\ &\quad + \int_u^s \int_{\mathbb{R}-\{0\}} D_{u,0}U(r,y)\tilde{M}(dr,dy), \quad 0 \le u \le s \le T, \end{aligned}$$

and for $z \neq 0$

$$\begin{aligned} D_{u,z}Y(s) &= U(u,z) - \int_u^s D_{u,z}f(r)dr + \int_u^s D_{u,z}Z(r)dW(r) \\ &\quad + \int_u^s \int_{\mathbb{R}-\{0\}} D_{u,z}U(r,y)\tilde{M}(dr,dy), \quad 0 \le u \le s \le T, \end{aligned}$$

where the derivative operators $D_{u,z}$ are defined according to (4.3) and (4.5). As the mappings $s \mapsto \int_u^s D_{u,z}f(r)dr, s \mapsto \int_u^s D_{u,z}Z(r)dW(r)$ are $\mathbb{P}$-a.s. continuous and the mapping $s \mapsto \int_u^s \int_{\mathbb{R}-\{0\}} D_{u,z}U(r,y)\tilde{M}(dr,dy)$ is $\mathbb{P}$-a.s. càdlàg, see Theorems 4.2.12 and 4.2.14 in [2], taking the limit $s \downarrow u$ yields

$$\begin{aligned} D_{u,0}Y(u) &= Z(u), \quad \lambda - a.e. u \in [0,T], \mathbb{P} - a.s., \\ D_{u,z}Y(u) &= U(u,z) \quad \lambda \otimes m - a.e.(u,z) \in [0,T] \times (\mathbb{R} - \{0\}), \mathbb{P} - a.s.. \end{aligned}$$

As $Y \in \mathbb{S}^2(\mathbb{R})$ has $\mathbb{P} - a.s.$ càdlàg $\mathbb{F}$-adapted trajectories we have the representation, $0 \le u \le T$,

$$Y(u) = \sum_{n=0}^{\infty} I_n(g_n((u,0),.) = \sum_{n=0}^{\infty} I_n(g_n((u,0),.)\mathbf{1}_{[0,u]}^{\otimes n}(.)), \quad g_n \in L^2_{T,q,n+1}, n \ge 0,$$

with càdlàg mappings $u \mapsto g_n((u,0),.)$. By Definition 3.1.2 of Malliavin derivative we arrive at

$$D_{u,z}Y(u) = \sum_{n=0}^{\infty} nI_{n-1}(g_n((u,0),(u,z),.)\mathbf{1}_{[0,u]}^{\otimes n}((u,z),.)), \quad q - a.e.(u,z) \in [0,T] \times \mathbb{R}.$$

For $\delta_{\{0\}} \times m$-a.e. $z \in \mathbb{R}$, we conclude that the mapping $(u,\omega) \mapsto D_{u,z}Y(u)(\omega)$ is $\mathbb{F}$-adapted and measurable and have a progressively measurable (optional) modification. Moreover, notice that the optional process $D_{u,z}Y(u)$ and its unique predictable projection $(D_{u,z}Y(u))^{\mathcal{P}}$ are modifications of each other, see Theorem 5.5 in

[10]. Finally, we remark that there exists $\mathcal{P}\times\mathcal{B}(\mathbb{R})$ measurable version of $(\omega,u,z)\mapsto (D_{u,z}Y(u)(\omega))^{\mathcal{P}}$, see Lemma 2.2 in [1]. This completes the proof. □

Acknowledgements: This paper was written while the first author was staying at Humboldt University Berlin. Łukasz Delong acknowledges the financial support from AMaMeF programme.

References

[1] Ankirchner, S., Imkeller, P. (2008) *Quadratic hedging of weather and catastrophe risk by using short term climate predictions.* Preprint.

[2] Applebaum, D. (2004) *Lévy Processes and Stochastic Calculus.* Cambridge University Press, Cambridge.

[3] Barles, G., Buckdahn, R., Pardoux, E. (1997) *Backward stochastic differential eqautions and integral-partial differential equations.* Stochastics and Stochastic Reports 60, 57-83.

[4] Becherer, D. (2006) *Bounded solutions to backward SDE's with jumps for utility optimization and indifference pricing.* The Annals of Applied Probability, 16, 4, 2027-2054.

[5] Bouchard, B., Elie, R. (2008) *Discrete time approximation of decuopled Forward-Backward SDE with jumps.* Stochastic Processes and their applications 118, 53-75.

[6] Buckdahn, R., Imkeller, P. (2008) *Backward stochastic differential equations with time delayed generator.* Preprint.

[7] Delong, Ł, Imkeller, P. (2009) *Backward stochastic differential equations with time-delayed generators - new results and counterexamples.* Preprint.

[8] Di Nunno, G., Øksendal, B., Proske, F. (2009) *Malliavin Calculus for Lévy processes with Applications to Finance.* Springer-Verlag.

[9] El Karoui, N., Peng, S., Quenez, M.C., (1997) *Backward stochastic differential equations in finance.* Mathematical Finance, 7, 1, 1-71.

[10] He, S., Wang, J., Yan, J. (1992) *Semimartingale Theory and Stochastic Calculus.* CRC Press Inc.

[11] Imkeller, P. (2008) *Malliavin's Calculus and Applications in Stochastic Control and Finance.* IM PAN Lectures Notes, Warsaw.

[12] Imkeller, P., Reis, G., (2009) *Path regularity and explicit truncations order for BSDE with drivers of quadratic growth.* Preprint

[13] Karatzas, I, Ocone, D. L. (1992) *A Generalized Clark Representation Formula with Application to Optimal Portfolios* Stochastics 34, 187-220

[14] Mohammed, S.E.A. (1984) *Stochastic functional Differential Equations.* Pitman.

[15] Nualart, D. (1995) *The Malliavin Calculus and Related Topics.* Springer-Verlag.

[16] Pardoux, E. Peng, S. (1990) *Adapted solution of a backward stochastic differential equation.* Systems Control Letters 14, 55-61.

[17] Picard, J. (1996) *On the existence of smooth densities for jump processes.* Probability Theory and Related Fields 105, 481-511.

[18] Petrou, E. (2008) *Malliavin calculus in Lévy spaces and applications to finance.* Electronic Journal of Probability 13, 852-879.

[19] Protter, P. (1992) *Stochastic Integration and Differential Equations.* Springer-Verlag.

[20] Royer, M. (2006) *Backward stochastic differential equations with jumps and related non-linear expectations.* Stochastic Processes and their Applications, 116, 1358-1376.

[21] Solé, J. L., Utzet, F., Vives, J. (2007) *Canonical Lévy process and Malliavin calculus.* Stochastic Processes and their Applications, 117, 165-187.